Recent Results and Questions in Combinatorial Game Complexities ⋆

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Abstract. Aim: Present a systematic development of part of the theory of combinatorial games from the ground up. Approach: Computational complexity. Combinatorial games are completely determined; the questions of interest are efficiencies of strategies. Methodology: Divide and conquer. Isolate the various difficulties separating hard from easy games, and attack them individually. Presentation: Informal; examples of games sampled from various levels illustrate the theory, with emphasis on formulating and motivating new and old research problems.

1 Introduction

Roughly speaking, *combinatorial games* are two-player perfect-information games (no hidden information) without chance moves (no dice). Outcome is (lose, win) or (tie, tie) or (draw, draw) for the two players, who play alternately. A tie is an end position which is neither a losing nor a winning position, such as in tic-tac-toe. Every draw is a non-end position from which no player can force a win, but always has a nonlosing next move (a *dynamic* tie; so a tie can be considered to be a static draw!). We assume throughout *normal* play, i.e., the player making the last move wins and the opponent loses.

Whereas, in the realm of existential questions, tractabilities and efficiencies are by and large linearly ordered, from polynomial to exponential, for problems with an unbounded number of alternating quantifiers, such as games, the notion of a "tractable" or "efficient" computation is much more complex. (Which is more tractable: a game that ends after four moves, but it's undecidable who wins [41], or a game that takes an Ackermann number of moves to finish but the winner can play randomly, having to pay attention only near the end [20]?)

One of the simplest combinatorial games is Nim: given a finite number of heaps containing finitely many tokens. A move consists of selecting a heap and removing from it a positive number of tokens, possibly the entire heap. The player making the last move wins.

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The strategy depends on the XOR (eXclusive OR) operation, also known as addition over GF(2). Suppose there are 3 heaps, of sizes 1, 2, 3. Can you win? Well, ... write the sizes in binary and add without carry (XOR), and you get $1 \oplus 2 \oplus 3 = 0$. Since the sum is 0, the *second* player can win, i.e., the opponent of the player beginning to play from the given position. If the sum is nonzero, the first player can win. This is a special case of a general rule (§2).

Note that also chess is a combinatorial game. The motivating question is why is Nim so simple, and chess, a member in the same family, so seemingly hard?

Among the main mathematical differences between Nim and chess:

- Nim-type games are *acyclic*, chess contains *cycles*: positions may be repeated.
- No interaction between tokens in Nim (which may be viewed as a board game—see "game-graph" below and §2); interaction in chess.
- Nim is *impartial*: the set of moves is the same for both players for all game positions; chess is *partizan*: the black player cannot move a white piece.
- Nim's *termination set* has size 1; chess has large termination set size.

We attempt to bridge the gap between Nim and chess by adopting a Divide and Conquer strategy: attack the differences separately, one-at-a-time.

With any combinatorial game Γ we associate a digraph $G = (V, E)$, called the *game-graph* of Γ, where V is the set of positions of Γ and $(a, b) \in E$ if and only if there is a move from position a to position b. Fig. 1a shows the game-graph of a Nim-pile of size 4, Fig. 1b depicts the game-graph of 2 Nim-piles, each of size 2, and Fig. 1c shows an arbitrary game-graph. The meaning of the P- and N-labels will become clear in the next section.

We use game positions and vertices synonymously; the same for moves and edges of Γ. The set of *followers* of a position u is $F(u) = \{v \in V : (u, v) \in E\}$. A *leaf* is a vertex u with $F(u) = \emptyset$, and the set of *predecessors* is $F^{-1}(u) = \{v \in V : (v, u) \in E\}$.

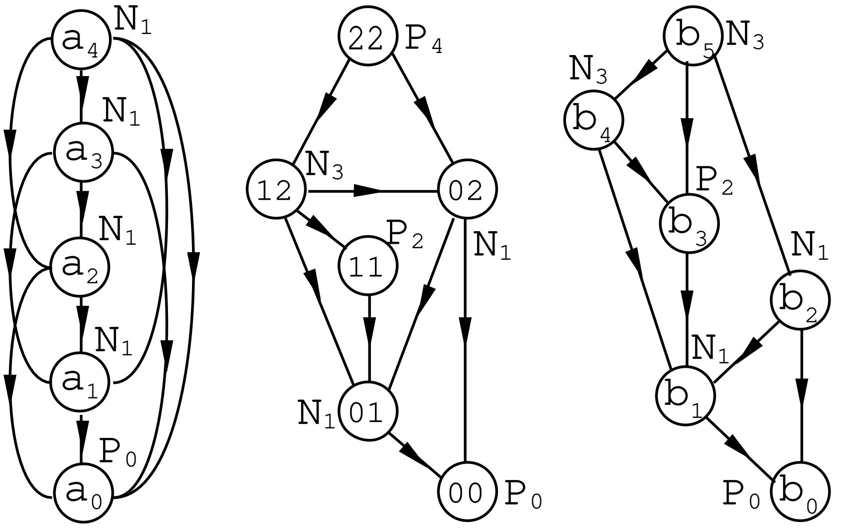
Combinatorial games are treated in several current books, notably [6], [3], [33], [40], and a forthcoming book [16]. At the end of [33] there is a games bibliography containing 400 items, which has been upgraded to 666 items in [40]. An updated version of the bibliography is maintained in the "Dynamic Surveys" section of *Electronic J. of Combinatorics*; it is also accessible from my home page.

In the sequel, exercises are solvable tasks so the readers can test their grasp of the material; problems are research questions.

2 The Classical Theory

The aim in this section is to present a polynomial strategy for Nim-type games: impartial acyclic games. In particular, no draw is possible.

Example. Place one token on each of the four starred vertices of the Directed Acyclic Graph (DAG) "Beat Doug" (Fig. 2). A move consists of selecting a token and moving it, along a directed edge, to a neighboring vertex. Tokens can coexist peacefully on the same vertex. For the given position, what's the time needed to:



(**a**)
Game − graph of
1 Nim − pile

(**b**)
Game − graph of
2 Nim − piles

(**c**)
Random
game − graph

Fig. 1. Some game-graphs.

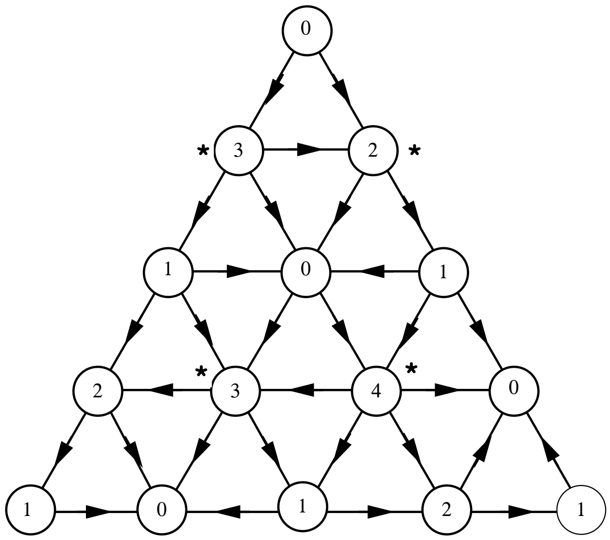


Fig. 2. Beat Doug on this DAG. The numbers inside the vertices are the g-values.

(a) compute who can win;
(b) compute an optimal next move;
(c) consummate the win?

To answer these questions, we say, informally, that a position (vertex) $u \in V$ is labeled N (Next player win) if the player about to move from it can win; otherwise it's a P-position (Previous player win). Denoting by $\mathcal{P}$ the set of all P-positions, by $\mathcal{N}$ the set of all N-positions, we have, for any acyclic game,

$$u \in \mathcal{P} \quad \text{if and only if} \quad F(u) \subseteq \mathcal{N} \ , \tag{1}$$

$$u \in \mathcal{N} \quad \text{if and only if} \quad F(u) \cap \mathcal{P} \neq \emptyset \ . \tag{2}$$

Let S be any finite subset of nonnegative integers. Define $\operatorname{mex} S$ (Minimum EXcluded value of S) by

$$\operatorname{mex} S = \min \mathbb{Z}^0 \setminus S = \text{ smallest nonnegative integer not in } S \ .$$

For any digraph $G = (V, E)$, we then define the *Sprague-Grundy* function $g : V \to \mathbb{Z}^0$ (for short: g-function), by $g(u) = \operatorname{mex} g(F(u))$, where for any set T, $g(T) = \{g(t) : t \in T\}$. Note that $\operatorname{mex}\ \emptyset = 0$.

The g-function exists uniquely on every finite DAG, and can be computed in $O(|V| + |E|)$ steps using the following algorithm.

Algorithm SG *for computing the Sprague-Grundy function of a given finite DAG.*

1. Label all leaves by 0.
2. If u is any unlabeled vertex for which $F(u)$ has been labeled, label u with $\operatorname{mex} \ell\big(F(u)\big)$ ($\ell(u)$ stands for the label of u). Repeat 2.
3. Put $g(u) \leftarrow \ell(u)$ for all u for which $\ell(u)$ has been computed. End.

Exercise 1. Prove that as long as there is an unlabeled vertex, there is an unlabeled vertex u such that $F(u)$ is labeled.

For any nonnegative integer h, write $h = \sum_{i \geq 0} h^i 2^i$ for the binary encoding of h ($h^i \in \{0, 1\}$). If a and b are nonnegative integers, then their *Nim-sum* $a \oplus b = c$ is defined by

$$c^i \equiv a^i + b^i \pmod 2, \ c^i \in \{0, 1\} \quad (i \geq 0) \ .$$

Clearly the Nim-sum, also called *exclusive or, XOR*, or *addition over* GF(2), is associative and $a \oplus a = 0$ for every a. We use the notation

$$\sum_{i=1}^{m}{}' a_i = a_1 \oplus a_2 \oplus \cdots \oplus a_m \ .$$

To solve our "Beat Doug" game, it is convenient to view each of the four tokens as residing on a different copy of the digraph. More generally, we can have, say, m distinct DAGs $G_1, \ldots, G_m$, with one token on each.

Definition 1. *Let $\{\Gamma_1, \ldots, \Gamma_m\}$ be a finite disjoint collection of games with game-graphs $\{G_1 = (V_1, E_1), \ldots, G_m = (V_m, E_m)\}$, which may have cycles or loops, or may be infinite. Then the sum-game $\Gamma = \mathcal{S}(\Gamma_1, \ldots, \Gamma_m)$ is the 2-player game in which a position has the form $(u_1, \ldots, u_m)$ with $u_i \in V_i$ for all i, and a move consists of selecting some Γ_i and making a legal move $u_i \to v_i$ in it $((u_i, v_i) \in E_i)$.*

The *sum-graph* $G = \mathcal{S}(G_1, \ldots, G_m)$ is the digraph $G = (V, E)$ defined as follows:

$$V = \{(u_1, \ldots, u_m)\colon\ u_i \in V_i, \quad i \in \{1, \ldots, m\}\} \ .$$

If $\boldsymbol{u} = (u_1, \ldots, u_m)$, $\mathbf{v} = (v_1, \ldots, v_m) \in V$, then $(\boldsymbol{u}, \mathbf{v}) \in E$ if there is some $j \in \{1, \ldots, m\}$ such that $v_j \in F(u_j)$, that is, $(u_j, v_j) \in E_j$, and $u_i = v_i$ for all $i \neq j$.

Note that the sum-graph has size exponential in the input size (of a single DAG). The contribution of the classical theory is to provide a polynomial strategy despite this exponential size. It is encapsulated in the following two statements.

Theorem 1. *Let $G = (V, E)$ be a finite DAG, g the Sprague-Grundy function. Then*

$$\mathcal{P} = \{u \in V : g(u) = 0\}, \qquad \mathcal{N} = \{u \in V : g(u) \neq 0\} \ .$$

Theorem 2. *Let $G = (V, E)$ be the sum-graph of the finite DAGs $G_1, \ldots, G_m$, and let $\sigma(\boldsymbol{u}) = \sum_{i=1}^{\prime m} g(u_i)$ for all $\boldsymbol{u} = (u_1, \ldots, u_m) \in V$. Then σ is the unique g-function of G.*

These two theorems imply directly,

Corollary 1. *Let $G = (V, E)$ be the sum-graph of the disjoint finite DAGs $G_1, \ldots, G_m$ $(m \in \mathbb{Z}^+)$. Then the P- and N-labels of G for normal play are given by*

$$\mathcal{P} = \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = 0\}, \qquad \mathcal{N} = \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) \neq 0\} \ .$$

These labels can be computed polynomially in the size of the standard graph encodings of $G_1, \ldots, G_m$.

It is illustrative to prove Corollary 1 without using Theorems 1 and 2. In view of (1) and (2), it suffices to show that: (i) every move from a position $\boldsymbol{u}$ with $\sigma(\boldsymbol{u}) = 0$ results in a position $\mathbf{v}$ with $\sigma(\mathbf{v}) \neq 0$, and (ii) there is a move from a position $\boldsymbol{u}$ with $\sigma(\boldsymbol{u}) \neq 0$ to a follower $\mathbf{v}$ with $\sigma(\mathbf{v}) = 0$.

(i) A move from $\boldsymbol{u}$ with $\sigma(\boldsymbol{u}) = 0$ means to select some u_j and move to a follower, say u'_j. By the property of the g-function, $g(u'_j) \neq g(u_j)$. But then $\sigma(\boldsymbol{u})$ is necessarily changed (see Table 1).

(ii) Consider the leftmost 1-bit of $\sigma(\boldsymbol{u})$. It is in some column c (the 2nd bit-column from the left in Table 1). The $g(u_i)$ must therefore have an odd number

Table 1. A typical Nim-sum.

$$
\begin{array}{rl}
g(u_1) & = 1\ 1\ 0\ 1 \cdots 0\ 1 \\
g(u_2) & = 0\ 0\ 1\ 1 \cdots 0\ 0 \\
\vdots & \\
g(u_j) & = 1\ 1\ 0\ 0 \cdots 1\ 1 \\
\vdots & \\
g(u_m) & = 0\ 1\ 0\ 0 \cdots 1\ 1 \\
\hline
\sigma(\boldsymbol{u}) & = 0\ 1\ 1\ 0 \cdots 0\ 1
\end{array}
$$

of 1-bits in column c, so there is at least one, for example $g(u_j)$ in Table 1. What we wish to do is to change the bits of $g(u_j)$ as follows: change the 1-bit in column c to 0, and complement the bit in every column of $g(u_j)$ in which $\sigma(\boldsymbol{u})$ has a 1. Replacing $g(u_j)$ by this value t, clearly results in Nim-sum 0. The remaining question is only whether there is a follower $u'_j \in F(u_j)$ such that $g(u'_j) = t$. The answer is positive: the leftmost change of $g(u_j)$ was to replace a 1-bit by 0, hence $t < g(u_j)$. By the mex property of g there is indeed $u'_j \in F(u_j)$ with $g(u'_j) = t$, and the proof is complete.

Our Beat Doug problem is now solved with a polynomial strategy. Since $2 \oplus 3 \oplus 3 \oplus 4 = 6$, the given position is in $\mathcal{N}$. Moving $4 \to 2$ is a unique winning move.

More generally, the questions we asked at the beginning of this section can now be answered. Who can win is answered directly by Corollary 1. Computing an optimal next move is suggested by the above proof of Corollary 1. The time for all of these and (c) is only $O(|V| + |E|)$, the complexity of Algorithm SG.

We note that Nim is a special case of this theory: the game graph of Nim is derived from $m \le n$ copies of the complete digraph $G = (V, E)$ with $V = \{u_1, \ldots, u_n\}$, and $(u_j, u_i) \in E$ if and only if $i < j$. For this case we have $g(u_i) = i$, so the rule for Nim given in §1 is seen to be a special case of Corollary 1.

In the sequel, exercises are solvable tasks so the readers can test their grasp of the material; problems are research questions.

Exercise 2. Stars and Buttons. Place 8 stars and 8 buttons on the 8 rows of a chessboard, so that the stars are west of the buttons (see Fig. 3). Sam plays the stars, Betty the buttons. Movement is horizontal within each row. A move of Sam is to select a star and slide it to any other square within the same row, east or west, but such that Sam's buttons remain west of Betty's buttons. Betty moves analogously with the buttons.
Is the game position depicted in Fig. 3 a draw? If not, who can win? What's the general rule?

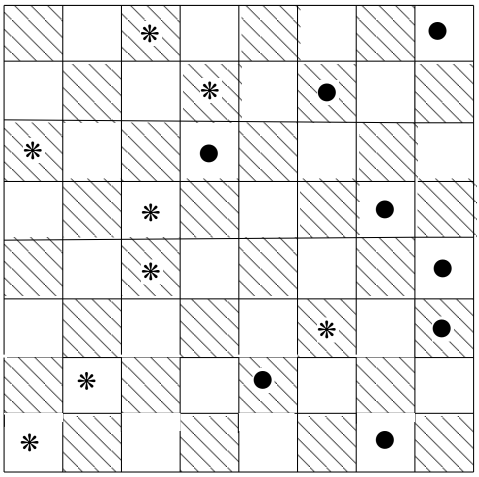


Fig. 3. Stars and Buttons.

Unfortunately, the strategy of classical games is not very robust: slight perturbations in various directions can make the analysis considerably more difficult. We point out just one of these, problems caused by succinctness.

In Corollary 1, the phrase "size of the standard graph encodings" means $\Omega(|V| + |E|)$ for a DAG $G = (V, E)$. What happens if the input is succinct, i.e., it has size $\Omega(\log(|V| + |E|))$? This is indeed the case for many games. In fact, Nim is succinct; it has a polynomial strategy only since its g-values form an arithmetic sequence, so Algorithm SG with its $O(|V| + |E|)$ complexity can be skipped.

2.1 Octal Games

A family of succinct games is the class of *octal games* introduced by Guy and Smith. We begin with a finite (usually single) linear chain of beads or tokens, and delete from it, and/or split the chain into two, according to rules encoded by an octal number $.a_1a_2\ldots$, where a_k specifies the conditions under which k tokens can be removed (Table 2).

Thus Nim is $.333\ldots$. Another example is Kayles: .137: remove 1 only if it's the entire chain, 2 if it's an entire chain or it lies at one end of the chain, 3 if it's the entire chain, or lies at one end of the chain, or if it's somewhere in the middle of the chain, which is thus split into two. Kayles has g-values which are periodic with period 34 and preperiod 68.

Any octal game is clearly succinct. Periodicity of its g-values makes it polynomial. An octal $.a_1a_2\ldots$ is *finite*, if there is N such that $a_i = 0$ for all $i \geq N$.

Table 2.

Value of a_k	Conditions for removal of block of k consecutive tokens
0	Not permitted
1	If block is entire row
2	If block lies at one end of row but is not a complete row
3	If 1 or 2 hold
4	If block is strictly within row
5	If 1 or 4 hold
6	If 2 or 4 hold
7	If 1 or 2 or 4 hold

A few hundred octal games with finite octal have been shown to be polynomial by exhibiting periodicity of the g-values.

Exercise 3. Split-Nim, or Lasker's Nim. Adjoin to the standard Nim-game move the option of splitting a heap into two nonempty heaps, without removing any token. Find a strategy for this game.

Problem 1. Are all octal games polynomial? Is there an intractable octal game? Does the subset of octal games with finite octal contain only periodic games?

A related game is *Grundy's game*, in which the only move is to split a chain into two nonempty chains of *unequal* size.

Problem 2. Is Grundy's game ultimately periodic? In [3, Ch. 4] it is conjectured that the answer is positive.

Exercise 4. The only move is to split a chain into two nonempty parts, which may have equal size. Who wins? What's a winning strategy?

2.2 Poset Games

Poset games are games played on partially ordered sets. As an example, consider chomp [29], [31], in which two players alternately move on a given $n \times m$ matrix of 1's (see Fig. 4). For a technical reason there is a single 0 at the origin. A move consists of pointing to some 1, say at location (i, j), and removing the entire north-east sector (i.e., replacing all the 1s by 0s inside the sector). The player removing the last 1 wins. The input size is $\log(mn)$, which is succinct. In addition, this game is not the sum of totally ordered sets, as Nim; rather it is the product of 2 Nim-piles.

Neither polynomiality nor intractability are known for general m, n. However, it is easy to see that player I can win: If taking the element (m, n) (the "largest" element) is an opening winning move, then player I can make it and win. If it is a losing move, then there is a winning answer, say taking element (i, j). Then

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} & & & & j & & \\ & & & & \downarrow & & \end{array} \\
 & \begin{pmatrix}
1 & 1 & 1 & \dots & 1 & 1 & 1 \\
1 & 1 & 1 & \dots & 1 & 1 & 1 \\
 & \vdots & & & & & \\
1 & 1 & 1 & \dots & 1 & 1 & 1 \\
1 & 1 & 1 & \dots & 1 & 1 & 1 \\
0 & 1 & 1 & \dots & 1 & 1 & 1
\end{pmatrix}
\end{array}
$$

$i \longrightarrow$ (row 2)

Fig. 4. An initial position of chomp.

player I's first move is to take $(i, j) \dots$ This argument holds in general for poset games with a largest element, but it is nonconstructive. Incidentally, computer simulations of chomp suggest that more often than not, an opening winning move is *not* to take the largest element.

Problem 3. Give a constructive, preferably polynomial, strategy for chomp.

For the superset game, define $A_n^k = \{B \subseteq \{1, \dots, n\} : 0 < |B| \le k\}$. A move in this 2-player game consists of pointing at an as yet unremoved subset and removing it, together with all sets *containing* it. For normal play we then clearly have $A_n^0 \in \mathcal{P}$ if and only if $n \equiv 0 \pmod 1$, $A_n^1 \in \mathcal{P}$ if and only if $n \equiv 0 \pmod 2$. In [30] it is shown that $A_n^2 \in \mathcal{P}$ if and only if $n \equiv 0 \pmod 3$. It is therefore conjectured there that $A_n^k \in \mathcal{P}$ if and only if $n \equiv 0 \pmod{k+1}$.

Incidentally, note that $A_n^n \in \mathcal{N}$ by the above nonconstructive argument, and so if the conjecture is true, then the unique winning move *is* to remove the largest element $\{1, \dots, n\}$, much unlike the observed behavior of chomp. We mention that at the end of [22], the g-values of the first few positions of A_n^k have been computed.

Problem 4. Prove or disprove the Gale-Neyman conjecture.

In von Neumann's Hackendot, played on a forest, a player points to an as yet unremoved vertex, and removes the unique path from that vertex to the root of the tree the vertex belongs to. This removal breaks up the tree into a forest, in general. The game is an N-position when begun on a tree, by the above nonconstructive argument. An interesting polynomial strategy for the game is given in [50]. See also [3], ch. 17.

2.3 A New Class of Heap Games

Another family of succinct games, proposed and analysed in [15], is played on two heaps of tokens and depends on two parameters $s, t \in \mathbb{Z}^+$. There are two types of moves: I. Take any positive number of tokens from a single heap, possibly the entire heap. II. Take $k > 0$ and $l > 0$ from the two heaps, say $0 < k \leq l$. This move is constrained by the condition $0 < k \leq l < sk + t$, which is equivalent to $0 \leq l - k < (s-1)k + t, \quad k \in \mathbb{Z}^+$.

The case $s = t = 1$ is known as *Wythoff's game* [52], [53], [10]. For $s = t = 2$, the first few P-positions are listed in Table 3. The reader will do well to try and construct the next few entries of the table before reading on.

Table 3. The first few P-positions for $s = t = 2$.

n	A_n	B_n
0	0	0
1	1	4
2	2	8
3	3	12
4	5	18
5	6	22
6	7	26
7	9	32
8	10	36
9	11	40
10	13	46
11	14	50
12	15	54
13	16	58

The set of all P-positions, as a function of the parameters s, t, is computed next.

Theorem 3. *$\mathcal{P} = \bigcup_{i=0}^{\infty}\{(A_i, B_i)\}$, where*

$$A_n = mex\{A_i, B_i : 0 \leq i < n\},\ B_n = sA_n + tn \quad (n \in \mathbb{Z}^0)\ . \tag{3}$$

The proof, omitted here, is based on (1) and (2) and the observation that A and B are *complementary* sets, where $A = \bigcup_{n=1}^{\infty} A_n$, $B = \bigcup_{n=1}^{\infty} B_n$.

The *statement* of Theorem 3 enables one to decide whether any given position (x, y) is a P-position or N-position, and the *proof*, analogously to the proof of Corollary 1 above, indicates a winning move from any N-position. These two things together constitute a winning strategy for the game. It turns out that the computation of a next move is polynomial. The complementarity of A and B implies that either $x = A_n$ or $x = B_n$ for some n. Hence Table 1 has to be computed only up to the appearance of x. Moreover, it is not hard to see that $n \leq x$, and if $x = A_n$, then $x/2 < n$, so the table has to be computed up to at most $\Omega(x)$, which implies a strategy computation linear in x. Since the input size of the problem is $\Omega(\log(|x| + |y|))$, this "linear" strategy is actually exponential!

Do the games have a polynomial strategy other than the exhibited exponential one? To answer this question we introduce a numeration system $\mathcal{U}$, whose basis elements are defined by the recurrence: $u_{-1} = 1/s$, $u_0 = 1$, $u_n = (s+t-1)u_{n-1} + su_{n-2}$ $(n \geq 1)$. The digits for $\mathcal{U}$ are $d_i \in \{0, \ldots, s+t-1\}$ such that $d_{i+1} = s+t-1 \Rightarrow d_i < s$ $(i \geq 0)$. Every positive integer m has a unique representation over $\mathcal{U}$ (special case of Theorem 2 of [14]). Moreover, the greedy algorithm of repeatedly dividing m or its remainder by the largest u_i not exceeding this remainder, yields this unique representation. The case $s = t = 1$ specializes to a binary representation known as the Zeckendorf representation [55].

Example. We consider the case $s = t = 2$. Then $u_1 = 4$, $u_2 = 14$, $u_3 = 50$, $u_4 = 178, \ldots$. The representations of the integers 1 to 60 in this special quaternary numeration system are displayed in Table 4.[1]

Is there a connection between Tables 3 and 4? If we scan the first few entries of both, we may be tempted to conclude that all the entries under A_n in Table 3 have representations ending in no 0 in Table 4. But then 14 is a counterexample, whose representation ends in two 0s. Also it appears that all the B_n have representations ending in a single 0. But 50, with representation 1000 is a counterexample, in fact, the only counterexample in the range of the two tables.

Let $V_0 < V_1 < \ldots$ be the set of all positive integers whose representation over $\mathcal{U}$ ends in an even number of 0s, and let W_k be the "left shift" of V_k, i.e., the representation of W_k is obtained from that of V_k by shifting the latter one place to the left. In particular, the representation of W_k ends in an odd number of 0s. For any $s, t \in \mathbb{Z}^+$ we have,

Theorem 4. *For all* $n \in \mathbb{Z}^0$, $(V_n, W_n) = (A_n, B_n)$.

Thus $(1, 4)$ of Table 3 has representation $(1, 10)$, and $(6, 22)$ has representation $(12, 120)$: 10 is the "left shift" of 1, 120 the left shift of 12. It is not hard to see that Theorem 4 implies a *linear strategy* for these games.

In [5] it is shown that a sequence $\{A_n\}$ is *spectral*, i.e., there exist reals α and γ such that $A_n = \lfloor n\alpha + \gamma \rfloor$ if and only if $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \leq 1$ for all $i, m, n \geq 1$. In addition to the quest for generalizing Wythoff's game

[1] Some of my best friends are nonsemitic, among them referees and readers of my articles. A number of them have commented to me that in a table such as Table 4, the basis elements 1, 4, 14, 50 should be written from left to right rather than from right to left. I disagree. A quaternary number $n = d_m \ldots d_0$ such as $213 = 3\times1+1\times4+2\times14 = 35$ in decimal, now easily readable from the table, would be reversed, so its value would be wrong! There is a discrepancy in nonsemitic languages, often ignored, between text, including mathematical formulas, and "digital" numbers. Though all of these are both written and read from left to right, the basis elements of the latter, which are usually implicit but here explicit, nevertheless increase from right to left. (There is an even greater discrepancy when embedding formulas and digital numbers in semitic language texts, but it is well-known and acknowledged. Moreover, word processors have already learned to overcome it; human beings still have difficulties with it.)

Table 4. A quaternary representation of the first few integers in $\mathbb{Z}^+$.

50	14	4	1	n	14	4	1	n
	2	0	3	31			1	1
	2	1	0	32			2	2
	2	1	1	33			3	3
	2	1	2	34		1	0	4
	2	1	3	35		1	1	5
	2	2	0	36		1	2	6
	2	2	1	37		1	3	7
	2	2	2	38		2	0	8
	2	2	3	39		2	1	9
	2	3	0	40		2	2	10
	2	3	1	41		2	3	11
	3	0	0	42		3	0	12
	3	0	1	43		3	1	13
	3	0	2	44	1	0	0	14
	3	0	3	45	1	0	1	15
	3	1	0	46	1	0	2	16
	3	1	1	47	1	0	3	17
	3	1	2	48	1	1	0	18
	3	1	3	49	1	1	1	19
1	0	0	0	50	1	1	2	20
1	0	0	1	51	1	1	3	21
1	0	0	2	52	1	2	0	22
1	0	0	3	53	1	2	1	23
1	0	1	0	54	1	2	2	24
1	0	1	1	55	1	2	3	25
1	0	1	2	56	1	3	0	26
1	0	1	3	57	1	3	1	27
1	0	2	0	58	2	0	0	28
1	0	2	1	59	2	0	1	29
1	0	2	2	60	2	0	2	30

($s = t = 1$), a motivation was to investigate sequences satisfying $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \leq 2$. Vera Sòs told me that she has also been interested in this question. We have shown that our sequences are spectral if and only if $s = 1$.

Exercise 5. For parameters $s, t \in \mathbb{Z}^+$, define sequences $\{Q_n\}$, $\{A'_n\}$, $\{B'_n\}$ ($n \in \mathbb{Z}^0$):

(i) $Q_n = Q_m$ if $n = tQ_m + sm$ and Q_m has already occurred precisely once; else $Q_n = \text{mex}\{Q_m : 0 \leq m < n\}$.

(ii) $A'_n =$ smallest k such that $Q_k = n$.

(iii) $B'_n =$ largest k such that $Q_k = n$.

Prove that for every $s, t \in \mathbb{Z}^+$, $A'_n = A_n$, $B'_n = B_n$ for all $n \in \mathbb{Z}^0$, where A_n, B_n are defined in (3).

Problem 5. Is it true that for the subfamily $s = t$ of the sequences $\{A_n\}$ defined in (3), we have $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \leq s$? If this is true, does also the converse hold, namely does $|(A_{n+i} - A_n) - (A_{m+i} - A_m)| \leq s$ imply (3) with $s = t$? Investigate the full family of these sequences (any $s, t \in \mathbb{Z}^+$).

Note that a problem in game theory led to a problem on sequences which is of independent interest.

Our discussions of succinct games might have led the reader to believe that succinct games are more difficult than their nonsuccinct variants. This, however, is not always the case: given a finite (undirected) graph G. A move in *vertex Kayles* is to label an as yet unlabeled vertex not adjacent to any labeled vertex. The player first unable to play loses, and the opponent wins. This is clearly a nonsuccinct variation of the polynomial Kayles. A partizan variation of vertex Kayles is called *bigraph vertex Kayles*. Both versions have been proved Pspace-hard in [47].

Problem 6. There is a large "no-man's-land" of games lying in between the polynomial 0.137 and the Pspace-hard vertex Kayles. Reduce this boundary, i.e., decide whether vertex Kayles is polynomial for graphs with various restrictions on their degrees.

3 Cyclic Games

Here we will learn to win or at least to draw, whenever possible, in a game of marbles, which may have cycles (Fig. 5).

If neither of the two players can force a win in a finite number of moves but both can continue to move indefinitely, the outcome is defined as a Draw for both players, labeled by D. The set of all D-positions is denoted by $\mathcal{D}$. In addition to (1) and (2), we have,

$$u \in \mathcal{D} \text{ if and only if } F(u) \cap \mathcal{P} = \emptyset \text{ and } F(u) \cap \mathcal{D} \neq \emptyset\ .$$

The theory of impartial games with draws was developed by Smith [49]. See also Fraenkel and Perl [21], Fraenkel and Tassa [23], Conway [6, Ch. 11],

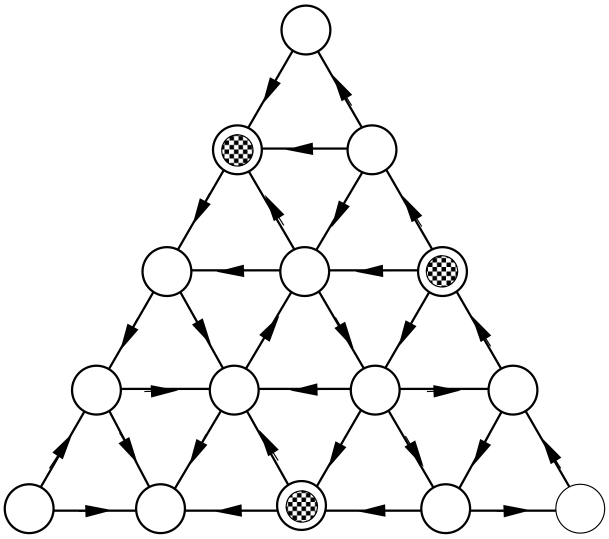


Fig. 5. A game of marbles.

Berlekamp, Conway and Guy [3, Ch. 12], Fraenkel and Yesha [27] and Fraenkel and Yesha [28].

Any digraph which *may* contain cycles or loops is called *cyclic*. It turns out that the question of the existence of the g-function on cyclic digraphs is NP-complete. However, the following generalized g-function γ exists uniquely on any finite cyclic digraph.

Given a digraph $G = (V, E)$. The *Generalized Sprague-Grundy function*, also called γ*-function*, is a mapping $\gamma: V \to \mathbb{Z}^0 \cup \{\infty\}$, where the symbol ∞ indicates a value larger than any natural number. If $\gamma(u) = \infty$, we also say that $\gamma(u)$ is infinite. We wish to define γ also on certain subsets of vertices. Specifically: $\gamma\big(F(u)\big) = \big\{\gamma(v) < \infty: v \in F(u)\big\}$. If $\gamma(u) = \infty$ and if we denote the set $\gamma\big(F(u)\big)$ by K for brevity, then we also write $\gamma(u) = \infty(K)$. Next we define equality of $\gamma(u)$ and $\gamma(v)$: if $\gamma(u) = k$ and $\gamma(v) = \ell$ then $\gamma(u) = \gamma(v)$ if one of the following holds: (a) $k = \ell < \infty$; (b) $k = \infty(K)$, $\ell = \infty(L)$ and $K = L$. We also use the notations

$$V^f = \big\{u \in V: \gamma(u) < \infty\big\}, \qquad V^\infty = V \setminus V^f \ ,$$

$$\gamma'(u) = \operatorname{mex} \gamma\big(F(u)\big) \ . \tag{4}$$

We need some device to tell the winner where to move to. This device is a counter function, as used in the following definition. For realizing an optimal strategy, we will normally select a follower of least counter function value with specified γ-value. The need for a counter function is illustrated in Fig. 6. A token on vertex 4 can be moved to a P-position on vertex 3 or 5. The former is winning, the latter only nonlosing. If only local information is available, or this subgraph is embedded in a larger one, we may not know to which P-follower to move. We remark that the counter function also enables one to prove assertions by induction.

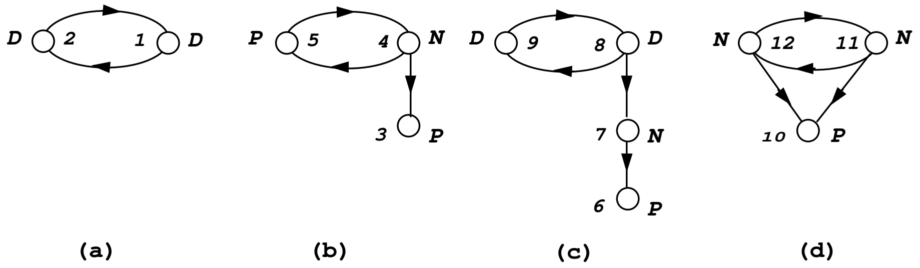


Fig. 6. Four cyclic game-graphs.

Definition 2. *Given a cyclic digraph* $G = (V, E)$*. A function* $\gamma: V \to \mathbb{Z}^0 \cup \{\infty\}$ *is a* γ*-function with counter function* $c: V^f \to J$*, where* J *is any infinite well-ordered set, if the following three conditions hold:*

A. *If* $\gamma(u) < \infty$*, then* $\gamma(u) = \gamma'(u)$*.*
B. *If there exists* $v \in F(u)$ *with* $\gamma(v) > \gamma(u)$*, then there exists* $w \in F(v)$ *satisfying* $\gamma(w) = \gamma(u)$ *and* $c(w) < c(u)$*.*
C. *If* $\gamma(u) = \infty$*, then there is* $v \in F(u)$ *with* $\gamma(v) = \infty(K)$ *such that* $\gamma'(u) \notin K$*.*

REMARKS

- In **B** we have necessarily $u \in V^f$; and we may have $\gamma(v) = \infty$ as in **C**.
- To make condition **C** more accessible, we state it also in the following equivalent form:
  C'. If for every $v \in F(u)$ with $\gamma(v) = \infty$ there is $w \in F(v)$ with $\gamma(w) = \gamma'(u)$, then $\gamma(u) < \infty$.
- If condition **C'** is satisfied, then $\gamma(u) < \infty$, and so by **A**, $\gamma(w) = \gamma'(u) = \gamma(u)$.
- To keep the notation simple, we write $\infty(0)$, $\infty(1)$, $\infty(0,1)$ etc., for $\infty(\{0\})$, $\infty(\{1\})$, $\infty(\{0,1\})$, etc.

We next provide an algorithm for computing γ. Initially a special symbol ν is attached to the label $\ell(u)$ of every vertex u, where $\ell(u) = \nu$ means that u has no label. We also introduce the notation $V_\nu = \{u \in V: \ell(u) = \nu\}$.

Algorithm GSG *for computing the Generalized Sprague-Grundy function for a given finite cyclic digraph* $G = (V, E)$.

1. (Initialize labels and counter.) Put $i \leftarrow 0$, $m \leftarrow 0$, $\ell(u) \leftarrow \nu$ for all $u \in V$.

2. (Label and counter.) As long as there exists $u \in V_\nu$ such that no follower of u is labeled i and every follower of u which is either unlabeled or labeled ∞ has a follower labeled i, put $\ell(u) \leftarrow i$, $c(u) \leftarrow m$, $m \leftarrow m + 1$.

3. (∞-label.) For every $u \in V_\nu$ which has no follower labeled i, put $\ell(u) \leftarrow \infty$.
4. (Increase label.) If $V_\nu \neq \emptyset$, put $i \leftarrow i+1$ and return to 2; otherwise end.

A realization of Algorithm GSG performs an endorder traversal of the digraph for each finite ℓ-value. Letting $\gamma_{\max} = \max\{\gamma(u) : u \in V^f\}$, we clearly have $\gamma_{\max} < |V|$. Hence the number of steps of the algorithm is bounded by $O((|V|+|E|)|V|)$, which is $O(|E||V|)$ for a connected digraph.

Exercise 6. Compute the γ-values of the digraph in Fig. 6.

The Generalized Nim-sum of two nonnegative integers is their Nim-sum (§2). The Generalized Nim-sum of a nonnegative integer a and $\infty(L)$, for any finite subset $L \subset \mathbb{Z}^0$, is defined by

$$a \oplus \infty(L) = \infty(L) \oplus a = \infty(L \oplus a),$$

where $L \oplus a = \{\ell \oplus a : \ell \in L\}$. The Generalized Nim-sum of $\infty(L_1)$ and $\infty(L_2)$, for any finite subsets L_1, L_2, is defined by

$$\infty(L_1) \oplus \infty(L_2) = \infty(L_2) \oplus \infty(L_1) = \infty(\emptyset).$$

The reason for the last definition is that no sum of games containing the components $\infty(L_1)$ and $\infty(L_2)$ has a follower with finite Nim-sum.

Finally, the generalized Nim-sum of $m \geq 2$ summands is $\sum_{i=1}^{\prime m} a_i = a_1 \oplus \cdots \oplus a_m$ which is well-defined since the Generalized Nim-sum is clearly associative.

We use the notations $\oplus$, Σ' and σ for both the Nim-sum and the Generalized Nim-sum, since the latter is clearly a generalization of the former, so no confusion can arise.

Consistent with the notation (4), we write

$$\sigma'(\boldsymbol{u}) = \operatorname{mex} \sigma\big(F(\boldsymbol{u})\big).$$

Analogously to Theorems 1 and 2 and Corollary 1 (§2), we have,

Theorem 5. *Let $G = (V, E)$ be a finite digraph. Then*

$$\mathcal{P} = \{u \in V : \gamma(u) = 0\}, \quad \mathcal{D} = \{u \in V : \gamma(u) = \infty(L), \quad 0 \notin L\},$$
$$\mathcal{N} = \{u \in V : 0 < \gamma(u) < \infty\} \cup \{u \in V : \gamma(u) = \infty(L), \quad 0 \in L\},$$

and

$$\text{for every } u \in \mathcal{P} \text{ and for every } v \in F(u) \cap \mathcal{N} \text{ there is } w \in F(v) \cap \mathcal{P} \text{ with } c(w) < c(u). \tag{5}$$

The three labels and c can be computed polynomially in the size of the standard encoding of G.

Definition 3. *A counter function c such that $\gamma(u) < \gamma(v)$ implies $c(u) < c(v)$ is called a monotonic counter function.*

Theorem 6. *Let $G = (V, E)$ be the sum-graph of the finite cyclic digraphs $G_1 = (V_1, E_1), \ldots, G_m = (V_m, E_m)$, and let $\sigma(\boldsymbol{u}) = \sum_{i=1}^{\prime m} \gamma(u_i)$ for every $\boldsymbol{u} = (u_1, \ldots, u_m) \in V$ with $u_i \in V_i$ for all i. Then σ is the unique γ-function of G with counter function $c(\boldsymbol{u}) = \sum_{i=1}^{m} c_i(u_i)$ satisfying (5), where the c_i are monotonic counter functions for γ in G_i $(1 \le i \le m)$.*

Corollary 2. *Let $G = (V, E)$ be the game-graph of the sum of the finite cyclic digraphs $G_1, \ldots, G_m$ $(m \in \mathbb{Z}^+)$. Then the P-, N- and D-labels of G are given by*

$$\mathcal{P} = \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = 0\}, \quad \mathcal{D} = \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = \infty(K), \quad 0 \notin K\}$$
$$\mathcal{N} = \{\boldsymbol{u} \in V : 0 < \sigma(\boldsymbol{u}) < \infty\} \cup \{\boldsymbol{u} \in V : \sigma(\boldsymbol{u}) = \infty(K), \quad 0 \in K\},$$

and (5) *is satisfied for the counter function c defined in Theorem* 5*. The labels and c can be computed polynomially in the size of the standard encodings of $G_1, \ldots, G_m$.*

Example. Put a token on vertex 11 of Fig. 6. It is clearly an N-position, since a move to 10 wins. Adjoin to the token on 11 a token on 7. By inspection it should be clear that the sum-game is a D-position. Indeed, $\gamma(7) = 1$, $\gamma(11) = \infty(0)$, so by Corollary 2, the γ-value of the sum is $\infty(0) \oplus 1 = \infty(1)$, which doesn't contain 0, so is a D-position. But a token on 11 and one on 5 (with $\gamma(5) = 0$) gives γ-value $\infty(0) \oplus 0 = \infty(0)$, so this is an N-position and a unique winning move is $11 \to 10$.

The marble game is now solved polynomially. The marbles are on vertices labeled by γ-values 1, 2 and $\infty(2,3)$. Thus the γ-value of the sum is $1 \oplus 2 \oplus \infty(2,3) = 3 \oplus \infty(2,3) = \infty(1,0)$, so it is an N-position, and moving the token on the ∞-value to the vertex labeled 3 is a unique winning move.

4 Annihilation Games

Another step in the direction from Nim to chess is to permit interactions between tokens, which are completely absent in the former, but very much present in the latter. In general, interactions between tokens give rise to Pspace-complete and Exptime-complete games. See e.g., Even and Tarjan [11], Fraenkel *et al.* [17], Fraenkel and Yesha [26], Fraenkel and Lichtenstein [19], Fraenkel and Goldschmidt [18], Lichtenstein and Sipser [37], Kasai *et al.* [35], Reisch [42], [43], Robson [44], [45], [46], Adachi *et al.* [2], Goldstein and Reingold [32]. Some of these games are even Expspace-complete. On the other hand, there are also games which are complete in P: Adachi *et al.* [1]. They have a nontrivial lower polynomial bound.

In this section, however, we consider a particular interaction called *annihilation*, which "only just" preserves polynomiality.

We play an annihilation game on a finite digraph G, with at most one token on each vertex. The two players alternate in selecting a token and pushing it to a neighboring vertex. When a token is moved onto a vertex occupied by another token, both tokens are removed from the game (annihilation move). For example, consider the game "Beat Anne" (Fig. 7), where the starred vertices contain tokens. Who wins or draws? Note that since there is no leaf, an odd number of tokens on any component would be a trivial draw.

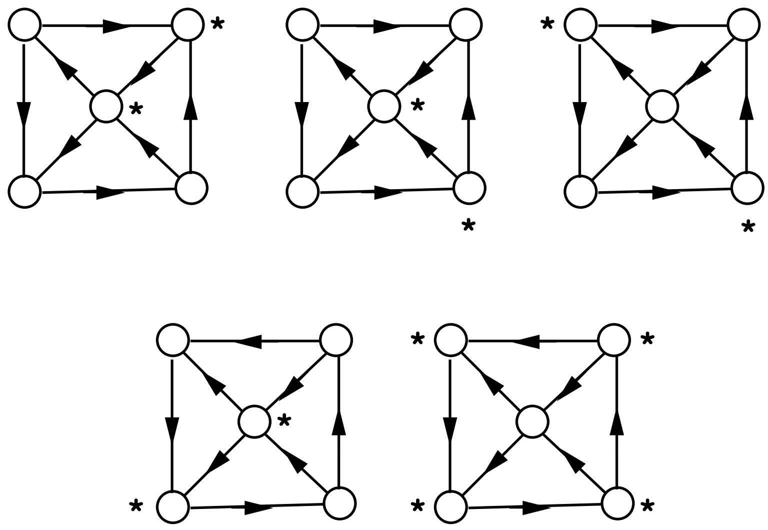


Fig. 7. Beat Anne on this *Ann*ihilation game.

If the digraph G is acyclic and walk-finite, then the annihilation game on G has the same strategy as the non-annihilation sum-game (Definition 1, §2) on G. This can be seen, informally, from the fact that $g(u) \oplus g(u) = 0$. If the winner moved an additional token onto u, then the loser will be the first to move a token from u to a follower v. The winner can then make the same move with another token from u, and so on until both tokens reach a leaf. So annihilating two tokens on u affects at most the length of play, not the strategy. If the loser made that move, then by the argument used in the proof of Corollary 1 (§2), the winner can move a token not on u, so again we may as well annihilate two tokens on u.

The situation is completely changed if G is cyclic, however. The position $\mathbf{u}$ consisting of 1 token each on vertices z_0 and z_2 in Fig. 8(a) is evidently a D-position in the non-annihilation game. But in the annihilation game it is a P-position in normal play: a token moved by player I is moved on by player II, who wins by annihilating both tokens. Thus $\gamma(\mathbf{u}) = 0$. Also a position consisting of 1 token on each of 2 adjacent vertices has γ-value 1. On the other hand, the position consisting of one token each on z_0 and z_2 in Fig. 8(b) is a D-position in both the annihilation and non-annihilation games; its γ-value is ∞. Also a position consisting of 1 token each on 2 adjacent vertices of Fig. 8(b) has γ-value $\infty(0)$.

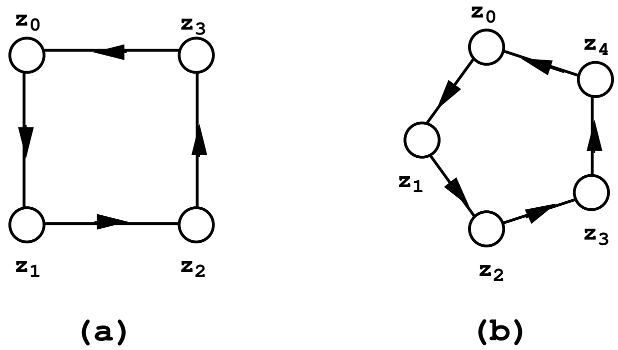


Fig. 8. Annihilation on simple cyclic digraphs.

For a formal treatment of annihilation games see Fraenkel and Yesha [27], [25], Fraenkel [13]. Misère play of annihilation games has been considered by Ferguson [12]. Here we shall only illustrate the theory.

A *groundgraph* $G = (V, E)$ (the digraph on which an annihilation game is played), together with the *even* component $\mathbf{G}^0$ (pertaining to an even number of tokens on G) of its *annihilation graph*, is displayed in Fig. 9. The odd component is not shown. It should be clear that the annihilation graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ contains 2^n vertices, half of them on $\mathbf{G}^0$, where $n = |V|$. The vertices of $\mathbf{V}$ form a vector space of dimension n over $GF(2)^n$.

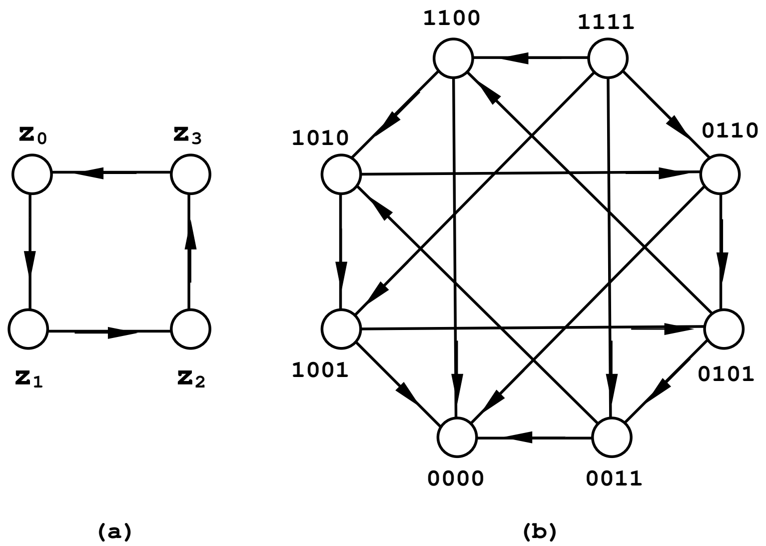


Fig. 9. A groundgraph (a) and the even component of its annihilation graph (b).

An $O(n^6)$ strategy can nevertheless be restored. The first key observation is that γ on $\mathbf{G}$ is, essentially, additive. The second key observation is that $\mathbf{V}_0$, $\mathbf{V}^f$ and γ can be computed by restricting attention to the linear span of vectors of weight at most 4, where $\mathbf{V}_0$ and $\mathbf{V}^f$ are subspaces on which γ vanishes and is finite, respectively, and the weight of a vector is the number of its 1-bits. So instead of having to deal with an annihilation graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ of 2^n vertices, it suffices to consider a polynomial fragment of it containing only $O(n^4)$ vertices.

It turns out that γ is a homomorphism from $\mathbf{V}^f$ onto $GF(2^t)$ for suitable $t \leq \lceil \log_2 n \rceil$ with kernel $\mathbf{V}_0$ and quotient space $\mathbf{V}^f/\mathbf{V}_0$, and that the triangularization over $GF(2)$ of a matrix of size $n \times O(n^4)$ (involving $O(n^6)$ operations) produces bases for $\mathbf{V}_0$ and $\mathbf{V}^f$. The result of this computation for our example is depicted in Fig. 10.

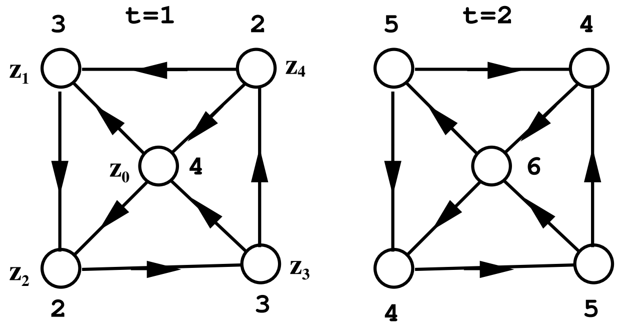


Fig. 10. The γ-values for the annihilation game can be deduced easily from the displayed numbers.

We Nim-add the values of occupied vertices. If the sum in a component is $< 2^t$, then it is the true γ-values for the annihilation game with these tokens on that component. If the sum is $\geq 2^t$, then that component contributes ∞ to the generalized Nim-sum (§3). Thus a token on a vertex labeled 4 and one on the vertex labeled 6 in Fig. 10 Nim-adds to $2 < 2^2$, so this is the true γ-value. But a congruent position on the left component gives $2 \oplus 4 = 6 > 2^1$, so the γ-value is ∞ (actually $\infty(0,1)$). For the game "Beat Anne" (Fig. 7) we thus have (Nim-adding from left to right and from top to bottom), $2 \oplus 3 \oplus 0 \oplus \infty(0,1) \oplus 0 = 1 \oplus \infty(0,1) = \infty(0,1)$. Thus this is an N-position by Theorem 5 (§3). The unique winning move is to slide the token on the center vertex of the lower left component to its north-west-follower.

The fact that we used only a small portion of the exponentially large annihilation graph to reconstruct the γ-function has a curious ramification. We don't have a counter function for all $\mathbf{u} \in \mathbf{V}^f$, so a player in an N-position always has a nonlosing next move, but may not be able to consummate his win. The third key observation is that a counter function $\widetilde{u} = \{\mathbf{u}_1, \ldots, \mathbf{u}_h\}$ for a position $\mathbf{u}$ with $\Sigma' \mathbf{u}_i = \mathbf{u}$, can be simulated, where the $\mathbf{u}_i$ have weight ≤ 4. However, a follower

$\widetilde{v}$ of $\widetilde{u}$ may actually be the counter value of a *predecessor* $\mathbf{v} = F^{-1}(\mathbf{u})$ of $\mathbf{u}$. But after $O(n^6)$ steps, a follower must nevertheless be encountered and a win is guaranteed.

Example. Suppose that the position $\mathbf{u}$ consists of one token each on the vertices z_1, z_2, z_3, z_4 of the left component of Fig. 10. Then $\mathbf{u} = (z_0, z_1, z_2, z_3, z_4) = (0, 1, 1, 1, 1)$, which is conveniently denoted by $\mathbf{u} = 15$. It has representation $\widetilde{u} = \{5, 10\}$, with $\mathbf{u}_1 = 5, \mathbf{u}_2 = 10$. Suppose player I moves $\mathbf{u} \to \mathbf{v} = 6$ (annihilating the tokens on z_1, z_4). The move from z_4 to z_1 means that the representation component 5 became 12, i.e., $\widetilde{v} = (12, 10) = (\mathbf{v_1}, \mathbf{u}_2)$. Now player I will naturally want to make the winning annihilation move $6 \to \emptyset$. The corresponding follower is obtained by annihilating $\mathbf{v_1} = 12$, since 12 was the counter component player II produced by his move. The resulting representation is thus 10, which is a *predecessor* of 6 rather than a follower. We therefore pretend that play began from the predecessor 10 rather than from 15. Then the follower 6 that player 2 moved to has representation 6, and now player I can consummate his win.

Kalmár [34] and Smith [49] defined a *strategy in the wide sense* to be a strategy which depends on the present position and on all its antecedents, from the beginning of play. Having defined this notion, both authors concluded that it seems logical that it suffices to consider a *strategy in the narrow sense*, which is a strategy that depends only on the present position (the terminology *Markov strategy* suggests itself here). They then promptly restricted attention to strategies in the narrow sense.

Let us define a *strategy in the broad sense* to be a strategy that depends on the present position v and on *all* its predecessors $u \in F^{-1}(v)$, whether or not such u is a position in the play of the game. This notion, if anything, seems to be even less needed than a strategy in the wide sense.

Yet for computing a winning move in an annihilation game, the only strategy that's polynomial that we were able to produce was a strategy in the wide sense.

Problem 7. Is there an efficient *Markov* strategy for annihilation games?

Problem 8. Is there a strategy for annihilation games better than $O(n^6)$?

5 Partizan Games

A game is *impartial* if both players have the same set of moves for all game positions. Otherwise the game is *partizan*. Nim-like games are impartial. Chess-like games are partizan, because if Betty plays the black pieces, Willy will not let her touch the white pieces.

In this section we shall refer to partizan games simply as games. The following two inductive definitions are due to Conway [6], [7], [8], who introduced the powerful theory of partizan games.

(i) If M^L and M^R are any two sets of games, there exists a game $M = \{M^L \mid M^R\}$. All games are constructed in this way.

(ii) If LE and RI are any two sets of numbers and no member of LE is $\geq$ any member of RI, then there exists a number $\{LE \mid RI\}$. All numbers are constructed in this way.

Thus the numbers constitute a subclass of the class of games.

The first games and numbers are created by putting $M^L = M^R = LE = RI = \emptyset$. Some samples are given in Fig. 11, where L (Left) plays to the south-west and R (Right) to the south-east. If, as usual, the player first unable to move is the loser and his opponent the winner, then the examples suggest the following statements:

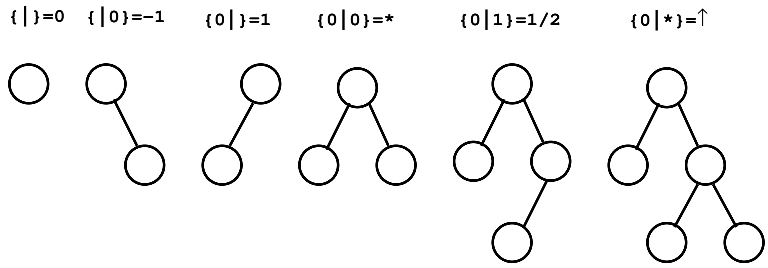


Fig. 11. A few early partizan games.

$M > 0$ if L can win,
$M < 0$ if R can win,
$M = 0$ if player II can win,
$M \parallel 0$ if player I can win (for example in $*$).

We shall in fact define $>$, $<$, $=$, $\parallel$ by these conditions. The relations can be combined as follows:

$M \geq 0$ if L can win as player II,
$M \leq 0$ if R can win as player II,
$M \mid\!\rhd 0$ if L can win as player I,
$M \lhd\!\mid 0$ if R can win as player I.

These definitions turn out to be consistent with the following inductive definitions of $\leq$ and $\parallel$. Let $x = \{x^L \mid x^R\}$. Then

$x \leq y$ if and only if $x^L \lhd\!\mid y$ for all x^L and $x \lhd\!\mid y^R$ for all y^R,
$x = y$ if and only if $x \leq y$ and $y \leq x$,
$x \parallel y$ (x is *fuzzy* with y) if and only if $x \not\leq y$ and $y \not\leq x$.

One can also provide a consistency proof of both definitions. This enables one to prove many properties of games in a simple manner. For example, define $-M = \{-M^R \mid -M^L\}$. Then $G - G = 0$ (player II can win in $G - G$ by imitating the moves of player I in the other game). Also:

$$(x \leq y \lhd\!\mid z \quad \text{or} \quad x \lhd\!\mid y \leq z) \quad \Rightarrow \quad x \lhd\!\mid z$$

(consider the game $z - x = (z - y) + (y - x)$, in which L can win as first player), and

$$x^L \lhd | \, x \lhd | \, x^R$$

(clearly L can win as player I in $x - x^L$ and in $x^R - x$).

If x is a number, then $x^L < x < x^R$.

Most important is that a sum of games simply becomes a *sum*, defined by:

$$M + H = \{M^L + H, M + H^L \mid M^R + H, M + H^R\} \ .$$

Consider for example the game of Domineering [6, Ch. 10], [3, Ch. 5], played with dominoes each of which covers precisely two squares of an $n \times n$ chessboard. The player L tiles vertically, R horizontally. After Domineering is played for a while, the board may break up into several parts, whence the game becomes a sum of these parts. The values of several small configurations are given in Fig. 12.

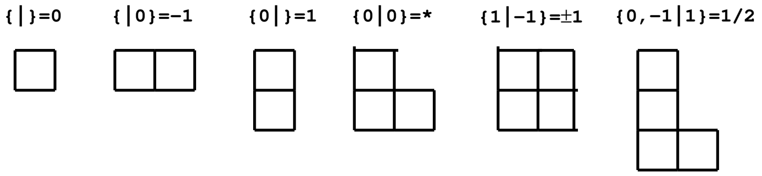


Fig. 12. A few values of Domineering.

Since the relation between the value of a game M and 0 determines the strategy for playing it, computing the value of M is fundamental to the theory. In this direction the *Simplicity Theorem* is helpful: Let $x = \{x^L \mid x^R\}$ be a game. If there exists a *number* z such that

$$x^L \lhd | \, z \lhd | \, x^R$$

and no option of z satisfies this relation, then $x = z$. Thus $\{-1 \mid 99\} = 0$. There also exist algorithms for computing values associated with games that are not numbers. Partizan games with possible draws are discussed in [36], [9], [48], [24], [3, Ch. 11]

Sums of partizan games are Pspace-complete [39], even if the component games have the form $\{a \| \{b | c\}\}$ with $a, b, c \in \mathbb{Z}$ [54], [38]. Berlekamp and his students have been engaged in a striking application of partizan games to certain endgames of Go [4]. Computations with partizan games are facilitated using Wolfe's efficient computational tools [51].

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