Rulesets for Beatty Games

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Abstract

We describe a ruleset for a 2-pile subtraction game with P-positions $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ for any irrational $1 < \alpha < 2$, and β such that $1/\alpha + 1/\beta = 1$. We determine the α's for which the game can be represented as a finite modification of t-Wythoff ([18], [9]) and describe this modification.

1 Introduction

The game *t-Wythoff* (see [18] and [9]) is a two-player game played on two piles of tokens where each player can either (a) remove any positive amount of tokens from one pile or (b) remove $x > 0$ tokens from one pile and $y > 0$ from the other provided that $|x - y| < t$ where $t \geq 1$ is a parameter of the game. The player first unable to move loses (*normal* play).

The case $t = 1$, in which the second type of moves is to remove the same amount of tokens from both piles, is the classical *Wythoff* game [22], a modification of the game Nim. From among the extensive literature on Wythoff's game, we mention just three: [3], [9], [23].

We restrict attention to invariant subtraction games, such as t-Wythoff. An *invariant* subtraction game is a subtraction game in which every move can be made from every game position, provided only that every pile retains a nonnegative number of tokens after the move. Invariant vector games were defined formally in [15], and further explored in [6]. Furthermore, we assume

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that the piles are unordered. Additional references on invariant subtraction games are, for example, [19] and [21].

In every finite acyclic impartial game, every position is either an N-position – a position from which the **N**ext player can win, or a P-position – a position from which the **P**revious player can win. Throughout the paper we consider normal play and thus $(0,0)$ is always a P-position. It is known that the P-positions of t-Wythoff are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, where α is given by the continued fraction $[1; t, t, t, \ldots]$ and β is such that $1/\alpha + 1/\beta = 1$.

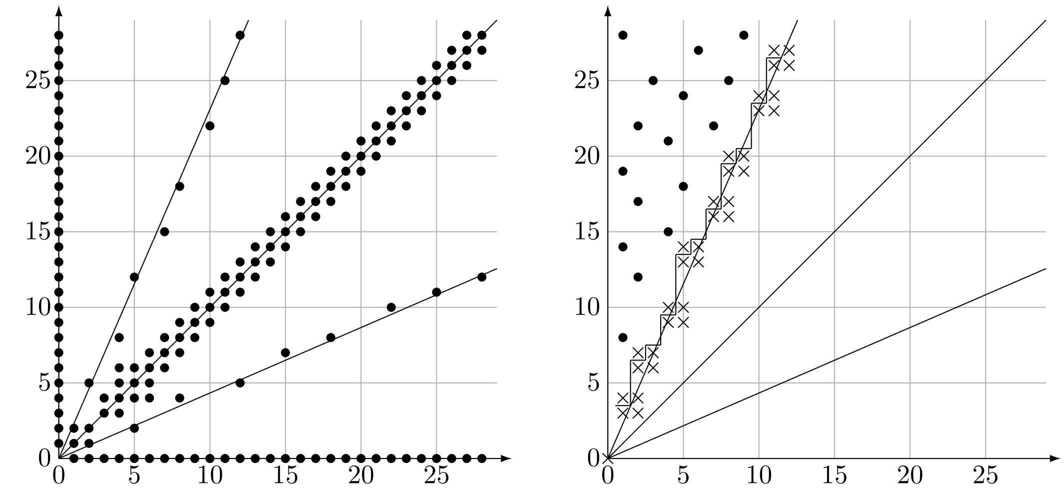
In [6] it was conjectured that for every irrational $1 < \alpha < 2$, the set $P_\alpha = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ constitutes the set of P-positions of some invariant game. The conjecture was proven in [20]. We dub such games *Beatty games*. Note that even though the proof given in [20] is constructive, the ruleset is rather complicated, especially compared to the one of t-Wythoff. For special cases of α, simpler rulesets appear in the literature. For example, see [6] for a ruleset for the case $\alpha = [1; 1, q, 1, q, 1, q, \ldots]$ $(q \geq 1)$ or [21] for $\alpha = [1; q, 1, q, 1, \ldots]$ $(q \geq 1)$. For modifications of Wythoff's game having P_α as their set of P-positions, see [17], [4], [14]. We use here the standard continued fraction notation $[a_0, a_1, a_2, \ldots]$ for $a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \ldots}}$.

The aim of this paper is to suggest compact rulesets for all Beatty games. That is, for every irrational $1 < \alpha < 2$, find a compact ruleset whose corresponding P-positions are P_α. The term "compact" is a little vague. In this paper we give two different meanings for "compact": The first is an invariant game whose moves are precisely those of t-Wythoff, except for some finite modification. We call such a game *MTW* (Modified t-Wythoff), defined precisely in Section 5, Definition 1(i). We will prove the following theorem:

Theorem 1. *Let $1 < \alpha < 2$ be irrational. Then, there exists an MTW game whose P-positions are P_α if and only if*

$$\alpha^2 + b\alpha - c = 0 \quad \textit{for some } b, c \in \mathbb{Z} \textit{ such that } b - c + 1 < 0. \tag{1}$$

A consequence of this theorem is that for almost all α, *there is no* MTW ruleset. In fact, it will follow from the proof, that the N-positions of the form $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ require infinitely many new moves. This brings us to the second meaning of "compact": We show (see Theorem 2) that by adding the moves from these N-positions to $(0,0)$ (together with finitely many additional



(a) A ruleset

(b) The set $\mathscr{F}$ – forbidden subtractions

Figure 1: A Beatty game with $\alpha = [1; 2, 3, 4, \ldots]$

moves) we obtain a ruleset for *every* irrational $1 < \alpha < 2$. Asymptotically, the moves of t-Wythoff are located on three lines: the x-axis, the y-axis and the $x = y$ diagonal. The moves described in Theorem 2 are located on 5 lines: the three lines mentioned above, together with the two lines: $\alpha x = \beta y$ and $\beta x = \alpha y$. This is illustrated in Figure 1(a). Hence the second meaning we give to "a compact ruleset" is that asymptotically the moves are located on a finite number of lines (we also prefer to keep this number as small as possible).

This paper is structured as follows:

Section 2 describes the framework and introduces some notation.

In Section 3 we present the set $\mathscr{F}$ – the set of subtractions which connect one P-position to another. This set plays a crucial role in Theorem 1 and Theorem 2, as a move can be added to the game if and only if it is not in $\mathscr{F}$. An example for the set $\mathscr{F}$, for $\alpha = [1; 2, 3, 4, \ldots]$, is shown in Figure 1(b)–both crosses and dots (the differences between them will be explained presently).

In Section 4 we prove Theorem 2. We start with this theorem as it gives a more general result, and some of the techniques used to prove it are also used in the proof of Theorem 1.

Section 5 is dedicated to the proof of Theorem 1.

In Section 6 we present a detailed analysis for two special cases of MTW rulesets.

2 Preliminaries

A position in the game is denoted by a pair (X, Y) where X and Y are the sizes of the piles. A move, that allows a player to take $x \geq 0$ tokens from one pile and $y \geq 0$ tokens from the other is denoted by a pair (x, y). We use the convention that $X \leq Y$ and $x \leq y$. Note that, potentially, there can be two results of playing the move (x, y) from the position (X, Y): $(X - x, Y - y)$ and $(X - y, Y - x)$.

Let $\mathbb{V}$ denote the set of all possible subtraction moves:

$$\mathbb{V} = \{(x, y) \in \mathbb{Z}^2_{\geq 0} : x \leq y,\ 0 < y\}.$$

The ruleset of any invariant game (played on two unordered piles) is a subset of $\mathbb{V}$. For example, the ruleset of t-Wythoff is

$$\mathrm{Wy}(t) = \{(0, y) : y > 0\} \cup \{(x, y) : 0 < x \leq y \text{ and } y - x < t\} \subseteq \mathbb{V}.$$

The set Wy(0) is the ruleset of Nim, while Wy(1) is the ruleset of the classical Wythoff game.

In this paper, β always denotes $\alpha/(\alpha - 1)$ (so that $1/\alpha + 1/\beta = 1$). Throughout we assume $0 < \alpha < \beta$, which implies $1 < \alpha < 2 < \beta$.

For $x \in \mathbb{R}$, we write $x = \lfloor x \rfloor + \{x\}$ where $\lfloor x \rfloor \in \mathbb{Z}$ and $0 \leq \{x\} < 1$.

Every continued fraction alluded to in the sequel is a *simple* continued fraction (with numerators 1, denominators positive integers): $[a_0; a_1, a_2, \ldots] = a_0 + 1/(a_1 + 1/(a_2 + \cdots))$. See, e.g., [16, ch. 10].

3 Forbidden subtractions

When suggesting a candidate for a ruleset $\mathcal{V} \subseteq \mathbb{V}$ whose P-positions should be $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$, one must check two things: (a) No P-position has a P-position follower and (b) Every N-position has a P-position follower. These two requirements are, in a sense, contrary: (a) bounds $\mathcal{V}$ from above while (b) bounds $\mathcal{V}$ from below.

This section deals with (a). In order to check whether (a) holds, construct the set $\mathscr{F} \subseteq \mathbb{V}$ of forbidden subtractions – those subtractions that connect one P-position to another. Then one simply checks that $\mathcal{V} \cap \mathscr{F} = \emptyset$. We have

$\mathscr{F} = \mathscr{F}_1 \cup \mathscr{F}_2$ where $\mathscr{F}_1 = \{(\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor) : n > m \geq 0\}$ and $\mathscr{F}_2 = \{(\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor, \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor) : \lfloor \alpha n \rfloor > \lfloor \beta m \rfloor, m > 0\}$. See Figure 1(b) for an example of $\mathscr{F}$. The subtractions of $\mathscr{F}_1$ are represented as $\times$ while those of $\mathscr{F}_2$ are represented as $\bullet$.

Throughout this paper we will frequently use the following observation:

Observation 1. *Let $n, m, k \in \mathbb{Z}_{\geq 0}$ such that $n - m = k$. Then for any positive real θ, $\lfloor \theta n \rfloor - \lfloor \theta m \rfloor = \lfloor \theta k \rfloor + a$ where $a = 1$ if $\{\theta n\} < \{\theta k\}$, and $a = 0$ otherwise.*

This follows from $\lfloor \theta n \rfloor - \lfloor \theta m \rfloor - \lfloor \theta k \rfloor = -\{\theta n\} + \{\theta m\} + \{\theta k\}$. So the right-hand side must be an integer; in fact, it is either 0 or 1.

In general, the structure of $\mathscr{F}_2$ is much more complicated than that of $\mathscr{F}_1$. See [14] for a detailed analysis of $\mathscr{F}_2$. Fortunately, this kind of detailed analysis is not necessary here. Instead, Proposition 1 below will suffice. We precede the proposition with the following geometric interpretation: the forbidden subtractions of $\mathscr{F}_2$ all lie above the line $\beta x = \alpha y$, see Figure 1(b). We will use this proposition to verify that the moves we add to $\mathcal{V}$ are not in $\mathscr{F}_2$.

Proposition 1. *If $(\lfloor \alpha k \rfloor, y) \in \mathscr{F}_2$ then $y \geq \lfloor \beta k \rfloor + 1$. In addition, if $(\lfloor \alpha k \rfloor + 1, y) \in \mathscr{F}_2$ then $y \geq \lfloor \beta k \rfloor + 2$.*

Proof. Assume that $(\lfloor \alpha k \rfloor, y) \in \mathscr{F}_2$. Then there are $n > m > 0$ such that $\lfloor \alpha n \rfloor - \lfloor \beta m \rfloor = \lfloor \alpha k \rfloor$ and $\lfloor \beta n \rfloor - \lfloor \alpha m \rfloor = y$.

We have $\lfloor \beta m \rfloor = \lfloor \alpha n \rfloor - \lfloor \alpha k \rfloor \leq \lfloor \alpha (n - k) \rfloor + 1$. Therefore,

$$\begin{aligned} y = \lfloor \beta n \rfloor - \lfloor \alpha m \rfloor &= (\lfloor \beta n \rfloor - \lfloor \beta k \rfloor) - \lfloor \alpha m \rfloor + \lfloor \beta k \rfloor \geq \\ &\geq \lfloor \beta (n - k) \rfloor - \lfloor \alpha m \rfloor + \lfloor \beta k \rfloor \geq && \text{as } m, n - k > 0 \\ &\geq \lfloor \alpha (n - k) \rfloor - \lfloor \beta m \rfloor + 2 + \lfloor \beta k \rfloor \geq \lfloor \beta k \rfloor + 1. \end{aligned}$$

The second assertion is proven similarly. □

Now, consider the set $\mathscr{F}_1$. Note that one can write $\mathscr{F}_1 = \bigcup_{k=1}^{\infty} \mathscr{F}_1^k \cup P_\alpha$ where $\mathscr{F}_1^k := \{(\lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor, \lfloor \beta n \rfloor - \lfloor \beta m \rfloor) : m > 0, n - m = k\}$. Fix $k \geq 1$ and consider the set $\mathscr{F}_1^k$. Write $x = \lfloor \alpha n \rfloor - \lfloor \alpha m \rfloor = \lfloor \alpha k \rfloor + a$ where $a \in \{0, 1\}$ (see Observation 1). Similarly, write $y = \lfloor \beta n \rfloor - \lfloor \beta m \rfloor = \lfloor \beta k \rfloor + b$, $b \in \{0, 1\}$.

Geometrically, the values of a and b are determined by the position of the point $(u, v) = (\{\alpha n\}, \{\beta n\})$ in $[0, 1)^2$ with respect to $p_k := (\{\alpha k\}, \{\beta k\})$.

Namely, divide $[0,1)^2$ into four open rectangles $R_{00}^{p_k}, R_{01}^{p_k}, R_{10}^{p_k}, R_{11}^{p_k}$ as shown in Figure 2. For example, $R_{11}^{p_k} = \{(u,v) : u < \{\alpha k\}, v < \{\beta k\}\}$. Then, $(\{\alpha n\}, \{\beta n\}) \in R_{ij}^{p_k}$ if and only if $a = i$ and $b = j$. The constraint $m > 0$ guarantees that $\{\alpha n\} \neq \{\alpha k\}$ and $\{\beta n\} \neq \{\beta k\}$.

The following proposition provides a criterion for testing whether the subtraction $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ is in $\mathscr{F}_1$. Let $D = \{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{\geq 1}\} \subseteq [0,1)^2$ and let E be its topological closure.

Proposition 2. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $a, b \in \{0,1\}$. Then, the subtraction $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b)$ is in $\mathscr{F}_1$ if and only if either $a = b = 0$ or $E \cap R_{ab}^{p_k} \neq \emptyset$.*

Proof. In this proof we will omit the p_k from $R_{ab}^{p_k}$ and simply write R_{ab} instead. The case $a = b = 0$ is trivial so we assume otherwise. Assume that $E \cap R_{ab} \neq \emptyset$. Since R_{ab} is open, $D \cap R_{ab} \neq \emptyset$. Since D has no isolated points, $|D \cap R_{ab}| = \aleph_0$ and thus one can choose $(\{\alpha n\}, \{\beta n\}) \in D \cap R_{ab}$ with $n > k$. Choosing $m = n - k$, we obtain the requested result. For the second direction note that if $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b) \in \mathscr{F}_1$ then necessarily $(\lfloor \alpha k \rfloor + a, \lfloor \beta k \rfloor + b) \in \mathscr{F}_1^k$. The second part of the proof is identical. $\square$

Therefore we have to study the set $D = \{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{\geq 1}\} \subseteq [0,1)^2$. The structure of D (or more accurately, of its topological closure, E) hinges on the rational dependence of α, β and 1. We thus seek solutions (A, B, C) of the equation

$$A\alpha + B\beta + C = 0, \qquad \text{where } A, B, C \in \mathbb{Z}. \tag{2}$$

It is easy to see that the equation has a non-trivial solution if and only if α is the root of a quadratic polynomial with integer coefficients. In fact, if (A, B, C) is a solution then α will satisfy $A\alpha^2 + (B + C - A)\alpha - C = 0$. Note

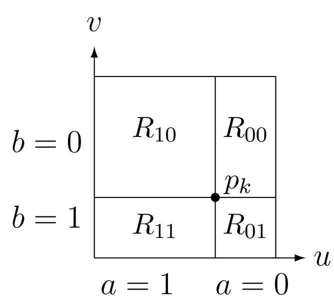


Figure 2: Determining a, b

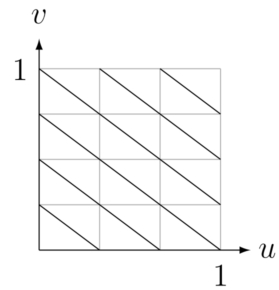


Figure 3: The set E for $A = 3$ and $B = 4$

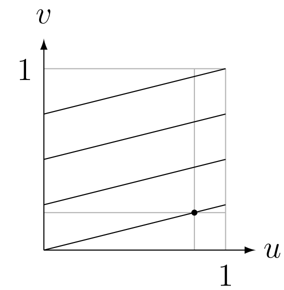


Figure 4: Proof of Lemma 1 from [6]

that we can choose A, B, C such that $\gcd(A, B, C) = 1$ and $A > 0$. These restrictions make the solution unique.

The following proposition is a result of Kronecker's theorem (see, for example, [16, Ch. 23]).

Proposition 3. *If (2) has no non-trivial solution, then E is the entire* $[0, 1)^2$. *Otherwise,* $E = \{(u, v) \in [0, 1)^2 : Au + Bv \in \mathbb{Z}\}$.

An example for the set E, where $A = 3$ and $B = 4$ is shown in Figure 3.

As an example of how Proposition 3 and the above discussion may be used, we give here a short proof of Lemma 1 from [6]. The lemma states that for $\alpha = 1 + (\sqrt{t^2 + 4t} - t)/2$, and n such that $\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor = 1$, we have $\lfloor \beta(n+1) \rfloor - \lfloor \beta n \rfloor = 2$. The α of Lemma 1 satisfies $1 \cdot \alpha - t \cdot \beta + (2t - 1) = 0$. Proposition 3 implies that the points of D all lie on t segments, as shown in Figure 4. Moreover, one can easily check that the point $p_1 = (\{\alpha\}, \{\beta\})$ lies on the bottom segment. Recall that $\lfloor \alpha(n + 1) \rfloor - \lfloor \alpha n \rfloor = \lfloor \alpha \rfloor + a = 1 + a$ and $\lfloor \beta(n+1) \rfloor - \lfloor \beta n \rfloor = \lfloor \beta \rfloor + b = 2 + b$. Here $a = 0$ and since $R_{01}^{p_1} \cap D = \emptyset$, we must have $b = 0$. This completes the proof. We remark that Figure 7 in [2] is similar to Figure 4. Also other figures there bear a resemblance to figures here.

4 A ruleset for an arbitrary α

Let $1 < \alpha < 2$ be an arbitrary irrational number. In this section we construct an invariant game with a rather simple "one-line" ruleset for which the set of P-positions is P_α. An illustration for such a "one-line" ruleset is given in Example 1 on page 10.

We will construct the set of moves, $\mathcal{V}_\alpha$, gradually. As we add moves to the game, we must verify that the moves we add are not in $\mathscr{F}$ – this will guarantee that no P-position has a P-position follower. Moreover, we will have to add enough moves such that every N-position will have a P-position follower, while keeping the game invariant.

The description of the rulesets we suggest (for an arbitrary α) appears in Theorem 2, which is presented in two parts: Theorem 2(a) deals with the case $\beta > 3$, and Theorem 2(b) deals with the case $2 < \beta < 3$.

4.1 $\beta > 3$

For the sake of simplicity, we assume first that $\beta > 4$.

Denote $t = \lfloor \beta \rfloor - 1$. Partition the N-positions as follows: N_1 is the set of N-positions of the form $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$, N_2 is the set of N-positions of the form $(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1)$, and N_3 is the set of all other N-positions.

Lemma 1. *The ruleset* $\mathrm{Wy}(t) \setminus \{(2, \lfloor \beta \rfloor)\}$ *does not intersect* $\mathscr{F}$ *and allows the players to move from any position in* N_3 *to a* P*-position.*

Proof. Propositions 1 and 2 imply that the only move of $\mathrm{Wy}(t)$ which might be in $\mathscr{F}$ is $(2, \lfloor \beta \rfloor)$ so this move is excluded.

Let (x, y) be an N_3-position $(x \leq y)$. Let n be the maximal integer for which $y - x = \lfloor \beta n \rfloor - \lfloor \alpha n \rfloor + m$ for some $m \geq 0$. As the difference $(\lfloor \beta(n+1) \rfloor - \lfloor \alpha(n+1) \rfloor) - (\lfloor \beta n \rfloor - \lfloor \alpha n \rfloor)$ is at most $t + 1$, we have $m \leq t$.

If $x \leq \lfloor \alpha n \rfloor$ then either $x = \lfloor \alpha k \rfloor$ or $x = \lfloor \beta k \rfloor$ for some $k \in \mathbb{Z}_{\geq 0}$. In both cases one can move to $(\lfloor \alpha k \rfloor, \lfloor \beta k \rfloor)$ using a Nim move.

Assume now that $x > \lfloor \alpha n \rfloor$. Consider the move $(x - \lfloor \alpha n \rfloor, y - \lfloor \beta n \rfloor)$ from the N-position (x, y) to the P-position $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$. Note that $(y - \lfloor \beta n \rfloor) - (x - \lfloor \alpha n \rfloor) = m$ and $0 \leq m \leq t$. So as long as $m \neq t$ and $(x - \lfloor \alpha n \rfloor, y - \lfloor \beta n \rfloor) \neq (2, \lfloor \beta \rfloor)$, this is a valid move.

If $m = t$, it follows from the maximality of n that the difference $(\lfloor \beta(n+1) \rfloor - \lfloor \alpha(n+1) \rfloor) - (\lfloor \beta n \rfloor - \lfloor \alpha n \rfloor)$ is exactly $t+1$ (where $\lfloor \beta(n+1) \rfloor - \lfloor \beta n \rfloor = t+2$ and $\lfloor \alpha(n+1) \rfloor - \lfloor \alpha n \rfloor = 1$). Therefore, $y - x = \lfloor \beta(n+1) \rfloor - \lfloor \alpha(n+1) \rfloor - 1$. Thus, $(x, y) = (\lfloor \alpha(n+1) \rfloor + j, \lfloor \beta(n+1) \rfloor - 1 + j)$ for $j \geq 1$ ($j = 0$ gives an N_1-position). Then one can move to $(\lfloor \alpha(n+1) \rfloor, \lfloor \beta(n+1) \rfloor)$ (note that the move in this case is $(j - 1, j)$ and it is valid since $\beta > 4$).

The last case we have to consider is $(x - \lfloor \alpha n \rfloor, y - \lfloor \beta n \rfloor) = (2, \lfloor \beta \rfloor)$. There are three possibilities for $(\lfloor \alpha(n+1) \rfloor, \lfloor \beta(n+1) \rfloor)$: $(x - 1, y)$, $(x, y + 1)$ and $(x - 1, y + 1)$. The first is disposed of by a Nim move. In the second (x, y) is an N_1-position, and in the third it is an N_2-position. $\square$

For N_1-positions, we simply add the following moves to the game:

$$F_\alpha = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1) : n \in \mathbb{Z}_{\geq 1}\},$$

which allow the player to move directly to $(0,0)$ (note that as $\beta > 3$, none of these moves is in $\mathscr{F}$).

For N_2-positions, we could add the moves $\{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : n \in \mathbb{Z}_{\geq 1}\}$ as we did with N_1, but it is possible to solve this by adding finitely many moves instead. Take $n_0 \geq 2$ such that $\{\alpha n_0\} > 1 - \{\alpha\}$, and add the moves:

$$\{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : 1 < n < n_0\} \cup$$

$$\cup \{(\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor), (\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor - 1)\}.$$

Consider the N_2-position $(x,y) = (\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1)$. We may assume that $n \geq n_0$ (otherwise we move to $(0,0)$). If $\{\alpha n\} > 1 - \{\alpha\}$ then $\lfloor \alpha(n+1) \rfloor = \lfloor \alpha n \rfloor + 2$. Therefore $x = \lfloor \beta k \rfloor$ for some k, and so (x,y) is solved by a Nim move. If $\{\alpha n\} < \{\alpha n_0\}$, then we can move to $(\lfloor \alpha(n - n_0) \rfloor, \lfloor \beta(n - n_0) \rfloor)$.

We now resume the case $\beta > 3$.

Theorem 2(a). For $\beta > 3$, there exists a finite set of moves, S_α, such that the P-positions of the invariant game defined by

$$\mathcal{V}_\alpha = \big(\mathrm{Wy}(\lfloor \beta \rfloor - 1) \setminus \{(2, \lfloor \beta \rfloor)\}\big) \cup F_\alpha \cup S_\alpha$$

are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$.

Proof. (i) Assume first $\beta > 4$. Choose n_0 as above and let

$$\begin{aligned} S_\alpha = \quad & \{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : 1 < n < n_0\} \cup \\ & \cup \{(\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor), (\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor - 1)\}. \end{aligned}$$

The addition of the moves $F_\alpha \cup S_\alpha$ clearly leaves the game invariant.

(ii) Now let $3 < \beta < 4$. Notice that the only place we used the fact that $\beta > 4$ was to prove that the move $(j-1, j)$ was valid in the case $m = t$, $(x,y) = (\lfloor \alpha(n+1) \rfloor + j, \lfloor \beta(n+1) \rfloor - 1 + j)$. If $\lfloor \beta \rfloor = 3$ and $j = 3$, then $(j-1, j) = (2,3) = (2, \lfloor \beta \rfloor)$ is not a valid move.

Notice that in this case $(x,y) = (\lfloor \alpha(n+1) \rfloor + j, \lfloor \beta(n+1) \rfloor - 1 + j) = (\lfloor \alpha n \rfloor + 4, \lfloor \beta n \rfloor + 6)$, so the move $(4,6)$ takes care of this case. Observe that one can choose $n_0 = 2$ as $1 - \{\alpha\} < 2/3 < \{2\alpha\}$ and then $(4,6)$ is already in $\mathcal{V}_\alpha$. $\square$

Example 1. Let $\alpha = [1; 2, 3, 4, \ldots] \approx 1.43313$ and $\beta = [3; 3, 4, 5, \ldots] \approx 3.30879$. We have $n_0 = 2$ and $S_\alpha = \{(4,5),(4,6)\}$. Therefore, the possible moves are:

(a) Remove $x > 0$ tokens from one pile.

(b) Remove x tokens from one pile and y tokens from the other where $|x - y| < 2$. The move $x = 2$, $y = 3$ is not allowed.

(c) Remove 4 tokens from one pile and 6 tokens from the other.

(d) Remove $\lfloor \alpha n \rfloor$ tokens from one pile and $\lfloor \beta n \rfloor - 1$ tokens from the other.

The ruleset is shown in Figure 1(a) in the introduction.

4.2 $2 < \beta < 3$

This case is slightly more complicated, since:

1. Wy(1) is not enough here and we need Wy(2), which in turn has much more subtractions that are in $\mathscr{F}$ and thus should be excluded (previously we had only one: $(2, \lfloor \beta \rfloor)$).

2. Roughly speaking, the P-positions are more dense in the $\lfloor \beta n \rfloor$ direction and we cannot add all the moves of F_α (since $F_\alpha \cap \mathscr{F}_1 \neq \emptyset$).

Start with the ruleset Wy(2), and for now ignore the fact that adding some of them is illegal. Consider the N-position (x, y). Take the maximal n such that $y - x = \lfloor \beta n \rfloor - \lfloor \alpha n \rfloor + m$ for $m \geq 0$. We have $m \in \{0, 1\}$. As in Section 4.1, we may assume $x > \lfloor \alpha n \rfloor$. But then one can move to $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$.

Now we have to exclude the moves in $\mathrm{Wy}(2) \cap \mathscr{F}$. Note that this is a finite set of moves. In fact, we will exclude the larger set:

$$G_\alpha = \left\{ (\lfloor \alpha k \rfloor + z, \lfloor \beta k \rfloor + w) : \begin{matrix} k \geq 1,\ z, w \in \{0,1\} \text{ and} \\ \lfloor \beta k \rfloor + w - \lfloor \alpha k \rfloor - z < 2 \end{matrix} \right\}.$$

We have to make sure that for each excluded move in G_α, there is an alternative move. Let (x, y), n, m be as before and suppose that $x = \lfloor \alpha n \rfloor + \lfloor \alpha k \rfloor + z$ and $y = \lfloor \beta n \rfloor + \lfloor \beta k \rfloor + w$ for $k \geq 1$ and $z, w \in \{0, 1\}$. Note that $(\lfloor \alpha(n+k) \rfloor, \lfloor \beta(n+k) \rfloor)$ is also of the form $(\lfloor \alpha n \rfloor + \lfloor \alpha k \rfloor + a, \lfloor \beta n \rfloor + \lfloor \beta k \rfloor + b)$

for $a, b \in \{0, 1\}$. Figure 5 shows the 8 possible relative positions of the N-position (x, y) and the P-position $(\lfloor \alpha(n+k) \rfloor, \lfloor \beta(n+k) \rfloor)$. Note that we can rule out (a), (d), (f), (g) and (h) since they all contradict the maximality of n. (b) is solved by the Nim move $(0, 1)$, so we are left with (c) and (e). Once again the N-positions that require special treatment are $N_1 \cup N_2$.

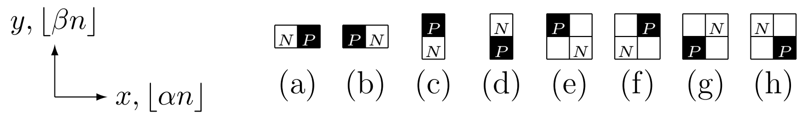


Figure 5: Relative positions of the
N-position and $(\lfloor \alpha(n+k) \rfloor, \lfloor \beta(n+k) \rfloor)$

We handle N_2 similarly to what we did in Section 4.1. Let n_0' be the smallest n such that $\lfloor \beta n \rfloor - \lfloor \alpha n \rfloor \geq 2$ (smaller n's do not correspond to N-positions). Find $n_0 \geq n_0'$ such that $\{\alpha n_0\} > 1 - \{\alpha\}$ and $\lfloor \beta n_0 \rfloor - \lfloor \alpha n_0 \rfloor \geq 3$. Then add the moves:

$$\{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : n_0' \leq n < n_0\} \cup$$

$$\{(\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor), (\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor - 1)\}.$$

As we mentioned before, N_1 is slightly more complicated here, as we cannot simply add all of F_α. Fortunately, we cannot add $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ only when $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1) = (\lfloor \alpha(n-1) \rfloor + 1, \lfloor \beta(n-1) \rfloor + 1)$ and in this case we can play the move $(1, 1)$. Thus, we add the moves: $F_\alpha \setminus (F_\alpha + (1, 2))$ (where $F_\alpha + (1, 2)$ is the set $\{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor + 1) : n \in \mathbb{Z}_{\geq 1}\}$).

The above discussion proves:

Theorem 2(b). For $2 < \beta < 3$, there exists a finite set of moves, S_α, such that the P-positions of the invariant game defined by

$$\mathcal{V}_\alpha = (\mathrm{Wy}(2) \setminus G_\alpha) \cup (F_\alpha \setminus (F_\alpha + (1, 2))) \cup S_\alpha$$

are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$.

Proof. Choose n_0' and n_0 as above and let

$$\begin{aligned} S_\alpha = \quad & \{(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1) : n_0' \leq n < n_0\} \cup \\ & \cup \{(\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor), (\lfloor \alpha n_0 \rfloor + 2, \lfloor \beta n_0 \rfloor - 1)\}. \end{aligned}$$

$\square$

5 Modified t-Wythoff (MTW)

A disadvantage of the description of the ruleset described in Section 4 is that it involves α explicitly. We can ask the following question: Can we describe a ruleset that doesn't involve α?

Of course, cardinality considerations imply that this is not possible for all α, as we have $\aleph$ different α's but only $\aleph_0$ finite descriptions of rulesets.

Therefore this will be possible only for a subset of the α's.

We start with two examples:

Example 2. Let $t \geq 1$. For $\alpha = [1; t, t, t, \ldots]$, the ruleset of t-Wythoff satisfies the requirement. That is, α is not mentioned explicitly in the ruleset.

Example 3. In [6], the authors give the following set of moves for $\alpha = [1; 1, q, 1, q, \ldots]$: Wythoff moves (Wy(1)) except for the moves: $(2, 2)$, $(4, 4)$, $\ldots$, $(2q - 2, 2q - 2)$; but with the move $(2q + 1, 2q + 2)$ added.

Note that this representation has no "α-dependent" moves. Instead, it is a finite modification of t-Wythoff. In light of the last example we make the following definition:

Definition 1. (i) A ruleset $\mathcal{V}$ is said to be *MTW* (Modified t-Wythoff) if it is of the form $\mathcal{V} = \mathrm{Wy}(t) \bigtriangleup S$ where $\bigtriangleup$ denotes the symmetric difference, $S \subseteq \mathbb{V}$ is a finite subset of moves and $t \geq 1$.

(ii) Let $1 < \alpha < 2$ be irrational. We say that α is *MTW*, if there exists an MTW ruleset whose corresponding P-positions are $\{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$.

In this section we prove Theorem 1 that was stated in the introduction, restated here:

Theorem 1. *Let $1 < \alpha < 2$ be irrational. Then, there exists an MTW game whose P-positions are P_α if and only if*

$$\alpha^2 + b\alpha - c = 0 \quad \textit{for some } b, c \in \mathbb{Z} \textit{ such that } b - c + 1 < 0. \tag{1}$$

It is easy to see that (1) holds if and only if (2) (see page 6) has a solution with $A = 1$ and $B < 0$.

Proof of Theorem 1. We first prove that if α is MTW then (2) has a solution with $A = 1$ and $B < 0$. Let $\mathcal{V}$ be an MTW ruleset for P_α. First, consider the case where (2) has a solution with $A > 1$ and $B < 0$. Figure 6(a) shows the set $\{(\{\alpha n\}, \{\beta n\}) : n \in \mathbb{Z}_{\geq 0}\}$ in such a case. We focus on n's for which the point $(\{\alpha n\}, \{\beta n\})$ is very close to the point $(1/A, 0)$. Formally, take a sequence $\{n_i\}_{i=1}^{\infty}$ such that $\{\alpha n_i\} \to 1/A$ and $\{\beta n_i\} \to 0$ as $i \to \infty$.

For these n_i's, consider the N-position $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$. There must be a move in $\mathcal{V}$ to a P-position $(\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor)$. Let $k_i = n_i - m_i \geq 1$. Note that there can be two moves that take $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$ to $(\lfloor \alpha m_i \rfloor, \lfloor \beta m_i \rfloor)$: $(\lfloor \alpha n_i \rfloor - \lfloor \alpha m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \beta m_i \rfloor)$ and $(\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \alpha m_i \rfloor)$.

For the second type, we have

$$(\lfloor \beta n_i \rfloor - 1 - \lfloor \alpha m_i \rfloor) - (\lfloor \alpha n_i \rfloor - \lfloor \beta m_i \rfloor) \approx (\beta - \alpha)(n_i + m_i) \to \infty.$$

Hence this move can be in $\mathcal{V}$ only for finitely many n_i's.

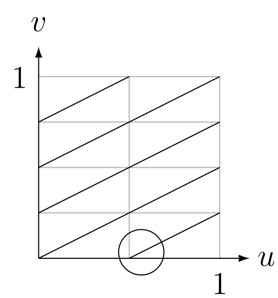
For the first type, we have $(\lfloor \alpha n_i \rfloor - \lfloor \alpha m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \beta m_i \rfloor) = (\lfloor \alpha k_i \rfloor + a, \lfloor \beta k_i \rfloor + b - 1)$ where $a, b \in \{0, 1\}$.

If $(a, b) = (0, 1)$ then this move is a P-position and therefore cannot be in $\mathcal{V}$.

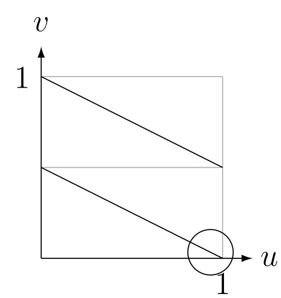
Assume that $(a, b) = (1, 1)$. We have $\{\alpha n_i\} < \{\alpha k_i\}$. Since $(\{\alpha n_i\}, \{\beta n_i\})$ was chosen to be close to $(1/A, 0)$ we may assume $\{\alpha n_i\} > 1/A$. Hence $R_{10}^{p_{k_i}} \cap E \neq \emptyset$. Proposition 2 implies that $(\lfloor \alpha k_i \rfloor + 1, \lfloor \beta k_i \rfloor + 0)$ connects two P-positions, which means that this move cannot be in $\mathcal{V}$.

For the other two cases ($b = 0$) we must have $\{\beta n_i\} > \{\beta k_i\} > 0$ and this is impossible as $\{\beta n_i\} \to 0$ (as there can be only finitely many different k_i's).

This completes the proof for the case that (2) has a solution with $A > 1$, $B < 0$. We will now prove the remaining two cases: (a) (2) has a solution with $B > 0$ and (b) (2) has no non-trivial solution. In both cases, we can



(a) $A = 2$, $B = -4$



(b) $A = 1$, $B = 2$

Figure 6: Proof of Theorem 1($\Rightarrow$)

choose a sequence $\{n_i\}_{i=1}^{\infty}$ such that $(\{\alpha n_i\}, \{\beta n_i\}) \to (1,0)$ as $i \to \infty$ (see, for example, Figure 6(b)).

We consider the N-position $(\lfloor \alpha n_i \rfloor, \lfloor \beta n_i \rfloor - 1)$. As in the first case, we only have to consider the moves: $(\lfloor \alpha n_i \rfloor - \lfloor \alpha m_i \rfloor, \lfloor \beta n_i \rfloor - 1 - \lfloor \beta m_i \rfloor) = (\lfloor \alpha k_i \rfloor + a, \lfloor \beta k_i \rfloor + b - 1)$. But, as $(\{\alpha n_i\}, \{\beta n_i\}) \to (1,0)$, for all but finitely many n_i's, we have $\{\alpha n_i\} > \{\alpha k_i\}$ and $\{\beta n_i\} < \{\beta k_i\}$. For these n_i's we have $(a,b) = (0,1)$, so the move is $(\lfloor \alpha k_i \rfloor, \lfloor \beta k_i \rfloor)$, but this is a P-position so this move cannot be in $\mathcal{V}$.

Second direction Assume that (2) has a solution with $A = 1$ and $B < 0$ and denote $k = -B$. We show how to construct an MTW ruleset for α.

We assume first that $\beta > 3$. Note that the set of moves given in Section 4.1 has only one component that is not a finite modification of $\mathrm{Wy}(\lfloor \beta \rfloor - 1)$: F_α. Let $\mathcal{V}'_\alpha$ be the set of moves suggested there, without adding F_α. The set $\mathcal{V}'_\alpha$ satisfies: (a) it is a finite modification of $\mathrm{Wy}(\lfloor \beta \rfloor - 1)$, (b) $\mathcal{V}'_\alpha \cap \mathscr{F} = \emptyset$ and (c) it allows the players to move from N_2- and N_3-positions to P-positions.

Proposition 3 implies that $E = \{(u,v) \in [0,1)^2 : u - kv \in \mathbb{Z}\} = \{(u,v) \in [0,1)^2 : u = \{kv\}\}$. Therefore, the set E consists of k segments as illustrated in Figure 7 (for $B = -4$).

To handle N_1-positions, find two points $p_r = (\{\alpha r\}, \{\beta r\})$ and $p_s = (\{\alpha s\}, \{\beta s\})$ in D such that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$ (see Figure 7) and add the moves $(\lfloor \alpha r \rfloor, \lfloor \beta r \rfloor - 1)$, $(\lfloor \alpha s \rfloor + 1, \lfloor \beta s \rfloor)$. Note that since $R_{10}^{p_s} \cap D = \emptyset$, Proposition 2 implies that $(\lfloor \alpha s \rfloor + 1, \lfloor \beta s \rfloor) \notin \mathscr{F}_1$. We can use these moves to take care of $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ for $n \geq \max\{r,s\}$: if $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ is an N_1-position with $(\{\alpha n\}, \{\beta n\}) \in R_{00}^{p_r}$ then we can move to $(\lfloor \alpha(n-r) \rfloor, \lfloor \beta(n-r) \rfloor)$ and if $(\{\alpha n\}, \{\beta n\}) \in R_{11}^{p_s}$ then we can move to $(\lfloor \alpha(n-s) \rfloor, \lfloor \beta(n-s) \rfloor)$. Now add a move from F_α for each $n < \max\{r,s\}$.

We now turn to the case $\beta < 3$. Here the only infinite component is $F_\alpha \setminus (F_\alpha + (1,2))$. We solve it by finding two points p_r, p_s as before, except

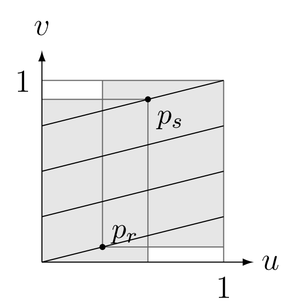


Figure 7: The set E with the points p_r and p_s

now we have one additional restriction: the move that corresponds to p_r must not be in $\mathscr{F}$. This translates to $p_r \notin R_{00}^{p_1}$. It is easy to see that this additional requirement can also be satisfied. □

6 Explicit rulesets

We can now analyze more carefully the finite modification whose existence is stated in Theorem 1($\Leftarrow$). We assume $A = 1$, $B < 0$ and denote $k = -B$. Fix $k > 1$ (for $k = 1$ we get t-Wythoff). The α's which satisfy (1) such that $-k = B = b - c + 1$ are now parametrized by $c \geq 2k - 1$. Figure 8 demonstrates how the points $(\{\alpha\}, \{\beta\})$ (where $k = 3$, $c \geq 5$) can be obtained by intersecting the curve induced by $1/\alpha + 1/\beta = 1$ and the k segments induced by $\{\alpha\} - k\{\beta\} \in \mathbb{Z}$, and shows how $c \pmod k$ determines the segment on which the point lies.

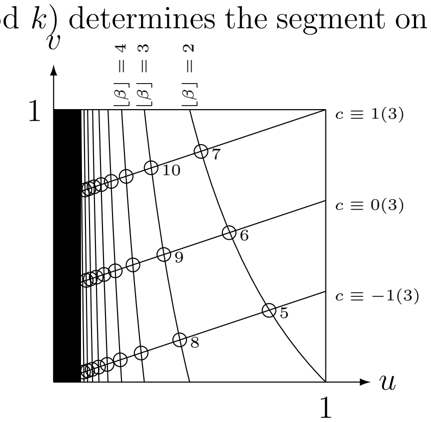


Figure 8: The point $(\{\alpha\}, \{\beta\})$ for $k = 3$ and different c's

Example 4. Consider the case $k = 3$ and $c = 5$. We have $b = 1$ and therefore $\alpha^2 + \alpha - 5 = 0$. Hence, $\alpha = (\sqrt{21} - 1)/2 = [1; 1, 3, 1, 3, \ldots] \approx 1.79129$ and $\beta = (\sqrt{21} + 9)/6 \approx 2.26376$. Indeed $\{\alpha\} - 3\{\beta\} = 0 \in \mathbb{Z}$ and $\{\beta\} < 1/3$.

It is easy to see that $\lfloor\beta\rfloor = \lfloor(c+1)/k\rfloor$. So we can write $c = k\lfloor\beta\rfloor + \tilde{c} - 1$ where $0 \leq \tilde{c} < k$. Table 1 shows the minimal values of r and s such that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$ for $k = 3$ (see the proof of Theorem 1($\Leftarrow$)). It can be seen from the table that for c large enough, the values of r and s depend strongly on $\tilde{c}$:

$$r = \begin{cases} 1, & \tilde{c} = 0 \\ 3, & \tilde{c} = 1, 2 \end{cases}, \qquad s = \begin{cases} 2\lfloor\beta\rfloor, & \tilde{c} = 0 \\ 5, & \tilde{c} = 1 \\ 4, & \tilde{c} = 2 \end{cases}.$$

For small c's this analysis does not hold. We therefore focus only on c large enough, as for small c, each case can be investigated individually anyway.

Note that "large enough c" is equivalent to "small enough $\{\alpha\}$". In particular, we will assume that we deal with the case $\beta > 3$ ($\{\alpha\} < 1/2$).

In the rest of this section, we analyze two special cases: $\tilde{c} = 0$ and $\gcd(\tilde{c}, k) = 1$.

c	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19
$\lfloor\beta\rfloor$	2	2	2	3	3	3	4	4	4	5	5	5	6	6	6
$\tilde{c}$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
r	4	2	6	1	9	4	1	5	4	1	5	3	1	3	3
s	3	7	1	6	2	5	8	2	1	10	2	4	12	5	4
c	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34
$\lfloor\beta\rfloor$	7	7	7	8	8	8	9	9	9	10	10	10	11	11	11
$\tilde{c}$	0	1	2	0	1	2	0	1	2	0	1	2	0	1	2
r	1	3	3	1	3	3	1	3	3	1	3	3	1	3	3
s	14	5	4	16	5	4	18	5	4	20	5	4	22	5	4

Table 1: The minimal values for r, s such that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$ for $k = 3$

6.1 Case I: $\tilde{c} = 0$

Note that we have $\alpha = [1; t, tk, t, tk, \ldots]$ and $\beta = [1 + t; tk, t, tk, t, \ldots]$ where $t = \lfloor\beta\rfloor - 1$. This case can be thought of as a generalization of the case $[1; 1, k, 1, k, \ldots]$ described in [6]. That being said, in this paper we only deal with the case $t \geq 2$.

In this case, the point $(\{\alpha\}, \{\beta\})$ lies on the bottom segment. From the continued fraction of β we learn that the smallest i for which $\{\beta i\} > (k-1)/k$ (that is, the point is on the top segment) is $i = t(k-1)+1$. Choose $r = 1$ and $s = t(k-1)+2$. Since $t \geq 2$, $(\{\alpha s\}, \{\beta s\})$ is on the top segment and $\{\alpha s\} \geq \{\alpha\}$ which guarantee that $D \subseteq R_{00}^{p_r} \cup R_{11}^{p_s}$. For an illustration, see Figure 9.

Let $i \leq s$. Observe that $\{\beta i\} \geq \{\beta\}$. It follows that $p_i \in R_{00}^{p_1} \cup R_{10}^{p_1}$ and thus one of the two moves $(\lfloor\alpha\rfloor, \lfloor\beta\rfloor - 1)$ and $(\lfloor\alpha\rfloor + 1, \lfloor\beta\rfloor - 1)$ takes care of the N_1-position $(\lfloor\alpha i\rfloor, \lfloor\beta i\rfloor - 1)$. Note that both moves are already contained in $\mathrm{Wy}(t)$, so it remains to add the move $(\lfloor\alpha s\rfloor + 1, \lfloor\beta s\rfloor)$ to handle N_1-positions with $p_i \in R_{11}^{p_s}$.

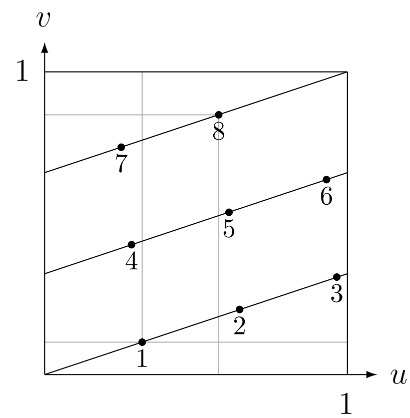


Figure 9: The points $(\{\alpha i\}, \{\beta i\})$ for $1 \leq i \leq 8$, $k = 3$, $c = 11$

It is easy to verify that

$$\lfloor \alpha s \rfloor = (t+1)(k-1) + 2, \quad \lfloor \beta s \rfloor = (t+1)(t(k-1)+2).$$

Recall that in Theorem 2(a) we added moves to deal with N_2-positions. Fortunately, in this special case, whenever $(x - \lfloor \alpha n \rfloor, y - \lfloor \beta n \rfloor) = (2, \lfloor \beta \rfloor)$ (see the proof of Lemma 1), we have $(x, y) \neq (\lfloor \alpha(n+1) \rfloor + 1, \lfloor \beta(n+1) \rfloor - 1)$. This is due to the fact that $R_{01}^{p_1} \cap D = \emptyset$. Thus, these additional moves are not necessary.

We proved the following:

Proposition 4. *Let $\alpha = [1; t, tk, t, tk, \ldots]$ for $t > 1$ and $k > 1$. The P-positions of the game defined by the moves*

$$\mathrm{Wy}(t) \setminus \big\{(2, t+1)\big\} \cup \big\{\big((t+1)(k-1)+3, (t+1)(t(k-1)+2)\big)\big\}$$

are P_α.

Example 5. Consider the ruleset of t-Wythoff with $t = 2$ where the move $(2, 3)$ is excluded but $(9, 18)$ is permitted. It follows from Proposition 4 that the P-positions are P_α for $\alpha = \sqrt{12} - 2 = [1; 2, 6, 2, 6, \ldots]$. Here $k = 3$ and $c = 8$.

6.2 Case II: $\tilde{c}$ and k are coprime

Note that as long as $i\{\alpha\} < 1$ (recall that we assumed that $\{\alpha\}$ is small enough), the point $(\{\alpha i\}, \{\beta i\})$ is on the $i \cdot \tilde{c} \pmod{k}$ segment (where 0 is the bottom segment). The smallest i on the top segment is therefore $1 \le d < k$ such that $d \equiv -\tilde{c}^{-1} \pmod{k}$. Choose s to be the second time it happens: $s = k + d$, and choose $r = k$ so that $r < s$ and $(\{\alpha r\}, \{\beta r\})$ is on the bottom segment.

We have,

$$\begin{aligned} \lfloor \alpha r \rfloor &= r = k, & \lfloor \beta r \rfloor &= k \lfloor \beta \rfloor + \tilde{c} = c + 1, \\ \lfloor \alpha s \rfloor &= s = k + d, & \lfloor \beta s \rfloor &= c + ((c+1)d + 1)/k. \end{aligned}$$

Proposition 5. *Suppose that $c \ge 2k^2$ and $\gcd(c+1, k) = 1$. Let $1 \le d < k$ be such that $d \equiv -(1+c)^{-1} \pmod{k}$. Then, the P-positions of the game defined by the moves*

$$\begin{aligned} &\mathrm{Wy}(\lfloor \beta \rfloor - 1) \setminus \big\{(2, \lfloor \beta \rfloor)\big\} \cup \\ \cup\, &\big\{(k+2, c), (k+2, c+1), \\ &(k+d+1, c + ((c+1)d+1)/k)\big\} \cup \\ \cup\, &\{(i+1, \lfloor i(c+1)/k \rfloor - 1) : 1 \le i \le k\} \cup \\ \cup\, &\{(i, \lfloor i(c+1)/k \rfloor - 1) : 1 \le i \le k\} \end{aligned}$$

are P_α.

Proof. Note that the fact that $c \ge 2k^2$ implies that $\{\alpha\} < 1/(2k-1)$ and therefore $i\{\alpha\} < 1$ for all $1 \le i \le k + d$. As explained above, in this case we can choose $r = k$ and $s = k + d$ (see Theorem 1($\Leftarrow$)).

The basic moves are $\mathrm{Wy}(\lfloor \beta \rfloor - 1) \setminus \big\{(2, \lfloor \beta \rfloor)\big\}$. As in the proofs of Theorem 1($\Leftarrow$) and Theorem 2(a) we have to deal with N_1- and N_2-positions.

Let $n \in \mathbb{Z}_{\ge 1}$ and consider the two positions $(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor - 1)$ and $(\lfloor \alpha n \rfloor + 1, \lfloor \beta n \rfloor - 1)$. If $n \le k$ we can move directly to $(0, 0)$ for both positions.

For the N_1-position, we use the move $(\lfloor \alpha r \rfloor, \lfloor \beta r \rfloor - 1) = (k, c)$ or $(\lfloor \alpha r \rfloor + 1, \lfloor \beta r \rfloor - 1) = (k+1, c)$ if $p_n \in R_{00}^{p_r}$ or $p_n \in R_{10}^{p_r}$ respectively. Otherwise $p_n \in R_{11}^{p_s}$ and we use the move $(\lfloor \alpha s \rfloor + 1, \lfloor \beta s \rfloor) = (k+d+1, c+((c+1)d+1)/k)$.

For the N_2-position, we use the move $(k+1, c)$ if $p_n \in R_{00}^{p_r}$, $(k+2, c)$ if $p_n \in R_{10}^{p_r}$ and $(k+2, c+1)$ if $p_n \in R_{11}^{p_r}$. $\square$

Example 6. For $k = 3$ and $c = 19$ we have $\alpha = (\sqrt{301} - 15)/2$, $\lfloor \beta \rfloor = 6$, $\tilde{c} = 2$ and $d = 1$. The ruleset that corresponds to P_α is the ruleset of t-Wythoff with $t = 5$ where the move $(2, 6)$ is excluded but the following moves are permitted:

$$(2, 12), (3, 12), (3, 19), (4, 19), (5, 19), (5, 20), (5, 26).$$

7 Epilogue

Here we wind-up up and motivate.

7.1 Wrapup

In this paper we presented "compact" rulesets for Beatty games. A Beatty game is an invariant subtraction game, played on two unordered piles of tokens, whose P-positions are $P_\alpha = \{(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) : n \in \mathbb{Z}_{\geq 0}\}$ for some irrational $1 < \alpha < 2$ and β such that $1/\alpha + 1/\beta = 1$.

Theorem 1 shows that the α's for which there exists a finite modification of t-Wythoff (for some $t \geq 1$) are exactly the algebraic integers of degree 2 with one constraint – the minimal polynomial must satisfy $f(1) < 0$. The theorem also explains how to construct such a ruleset for an α with this property. For some special cases, we explicitly described the ruleset (see Proposition 4 and Proposition 5).

In Theorem 2 we described a compact ruleset for a game with these P-positions – this time for *any* irrational $1 < \alpha < 2$. The meaning of "compact" here was that all the moves lie (asymptotically) on only 5 lines (see Figure 1(a)).

We remark that in [13] the notion of "k-invariance" was defined, and simple rulesets were formulated for the case where the P-positions are complementary Beatty sequences and the rulesets are for 2-invariant games. The moves are located on the same slopes as in Figure 1(a) above. The results of the present paper suggest that the answer to the Problem at the end of [13] is negative.

7.2 Background and motivation

The conventional problems in combinatorial games are, given a game, find its P-positions, or even its Sprague-Grundy function. Is it useful to invert these problems? In [8] it is stated: "At the BIRS 2011 workshop on combinatorial games, A.S. Fraenkel posed the following intriguing problem: Find nice (short/simple) rules for a(n as yet unknown) 2-player combinatorial game for which the set **P** of P-positions are a given pair of complementary Beatty sequences [1]." Indeed, in [10] the following problem was solved: Given a set **P** of candidate P-positions, find a game with compact, succinct game rules whose set of P-positions is **P**. Here and below we are concerned with 2-pile subtraction games, so $\mathbf{P} = (A, B)$. In all references below the sequences A and B are disjoint, but in [10] even the case where A and B are not disjoint was tackled. In [11] the set $\mathbf{P}$ consists of $m \geq 3$ sequences which were elsewhere conjectured to be the only ones which are distinct and pairwise complementary. (Fraenkel conjecture, [7], p. 19.) An extension is given in [12]. In [8] itself, rulesets for interesting sets $\mathbf{P}$ were constructed. The same holds for [21].

As was noted above, in [20] it was proved that for any pair of complementary Beatty sequences, there always exists an invariant game having this pair as its P-positions, proving the conjecture enunciated in [6] about existence of invariant Beatty games. See also [5], [2]. In this subsection we cited studies where rulesets for such games (invariant and variant) games were actually *constructed*. In the present paper, we characterized the set of α for which there exist such rulesets, and we also constructed them.

We could have presented this subsection in the Introduction, which would then have become more cumbersome due to the need to explain and define the jargon which we have used here freely. Besides, the contents of the paper stands independently of this subsection.

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