TRIOMINEERING, TRIDOMINEERING AND L-TRIDOMINEERING

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Abstract. The game of Domineering is a combinatorial game that has been solved for several boards, including the standard 8×8 board. We create new partizan – and some impartial – combinatorial games by using triominoes instead of and along with dominoes. We analyze these games for some small boards providing a dictionary of values and prove properties that permit expressing some connected boards as sums of smaller subboards.

1. Introduction

The game of Domineering was invented by Göran Andersson around 1973, according to [2], [3], and [6]. The two players in Domineering alternately tile a board using a regular domino (a 2×1 tile). Left is usually called *Vertical* and Right *Horizontal*. They place their tiles, without overlapping, vertically and horizontally, respectively. The player making the last move wins. The game is *partizan*, since the set of moves is different for each player. Conway [3] and Berlekamp, Conway, and Guy [1] have computed the value of Domineering for several boards, not necessarily rectangular. D.M. Breuker, J.W.H.M Uiterwijk and H.J. van den Herik [2] have determined who wins the game of Domineering for additional boards. In particular, they showed that the first player can win on the classical 8×8 board, which was the original game presented by Andersson.

We create new combinatorial games by admitting the use of larger tiles and provide a dictionary of values for each game. In these dictionaries, we represent unavailable squares –squares that cannot be used during the play –by black squares. We also formulate and prove some general properties of these games.

The basic theory of combinatorial games can be found in [1] and [3]. A concise 18-page summary with the basic results in combinatorial game theory can be found in [4]. We recall that for a partizan game G, Left can win if $G > 0$, Right can win if $G < 0$, the second player can win if $G = 0$, and the first player can win if $G || 0$ (G is *fuzzy* with (incomparable with) 0).

2. The Games

2.1. **Triomineering.** In our first game, we substitute the domino by a "straight" triomino; that is, a 3×1 tile. We call this game Triomineering. Its rules are exactly the same as for Domineering: the two players, Vertical and Horizontal, tile alternately vertically and horizontally, respectively. Overlapping is not permitted. The player making the last move wins.

Figure 1 gives the values of Triomineering for boards up to six squares, including the 35 boards with 6 squares, excluding their negatives obtained by a right angle turn. Figure 2 displays 59 boards with seven squares and their values. According to [1] Ch. 5, there are 108 boards with seven squares.

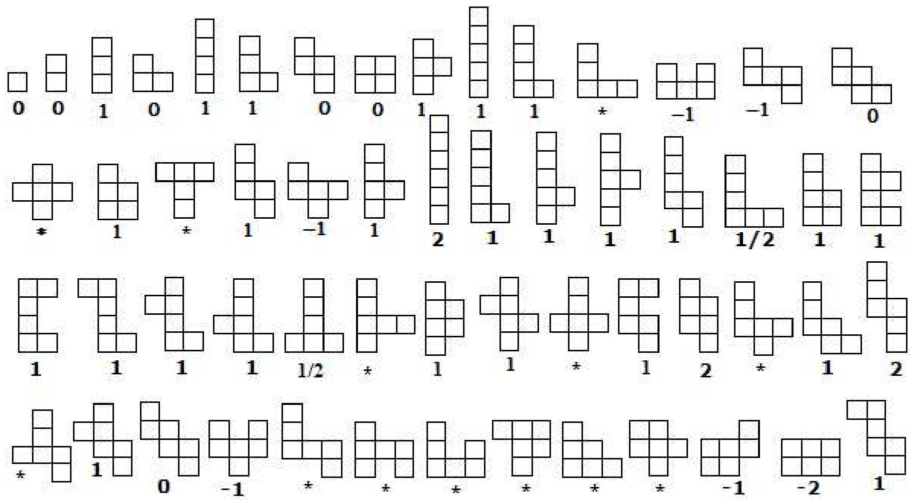

FIGURE 1. Triomineering values for boards up to 6 squares.

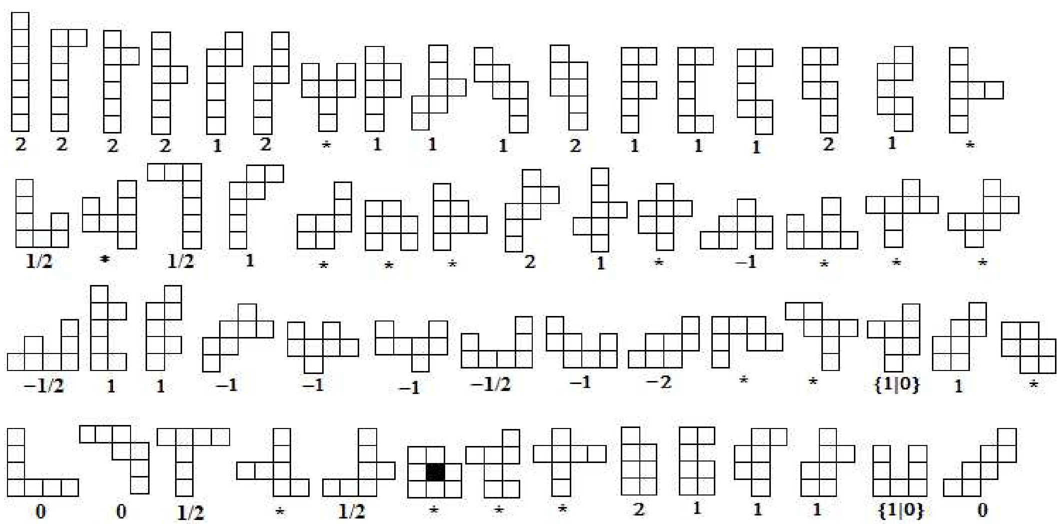

FIGURE 2. Triomineering values for some boards with 7 squares.

TABLE 1. Values of Triomineering for small rectangular boards.

	1	2	3	4	5	6
1	0	0	-1	-1	-1	-2
2	0	0	-2	-2	-2	-4
3	1	2	± 2	$\{3 \mid -3/2\}$	$\{4 \mid -1, -1*\}$	$\{4 \mid 0 \mid\mid\mid -1/2 \mid\mid -1 \mid -2\}$
4	1	2	$\{3/2 \mid -3\}$	$\pm 5/2$	$\{3 \mid -2, -2*\}$	$\{3 \mid -3/2 \mid\mid\mid -7/4 \mid\mid -3 \mid -4\}$
5	1	2	$\{1, 1* \mid -4\}$	$\{2, 2* \mid -3\}$	± 2	$\{-3 \mid\mid -3, -3* \mid -8\}$
6	2	4	$\{2 \mid 1 \mid\mid 1/2 \mid\mid\mid 0 \mid -4\}$	$\{4 \mid 3 \mid\mid 7/4 \mid\mid\mid 3/2 \mid -3\}$	$\{8 \mid 3, 3* \mid\mid 3\}$	F

Table 1 depicts the value of Triomineering for some rectangular boards. We have omitted "messy" values – those that take considerable space to express. Instead, we have used F to indicate that the first player wins, V to indicate that Vertical wins, and H to indicate that Horizontal wins. This notation is also used in the tables (below) containing the values of

Tridomineering and L-Tridomineering for small rectangular boards. Note that, curiously, the 3×3 board and the 5×5 board have the same value.

Definition 2.1.1. A subset of a rectangular board is called *2-wide* if it has a row with exactly two consecutive squares and no row or column with three consecutive squares.

Clearly any 2-wide board is a 0-game, since no player is able to move.

Definition 2.1.2. Two boards F and G placed next to each other are said to be *concatenated horizontally* or simply *concatenated*, if one can place a horizontal domino so that it covers one square from F and one from G.

We denote any concatenation of G and F by GF.

Definition 2.1.3. Let G and F be subsets of rectangular boards, GF a concatenation of them. If each column of GF intersects only one of the two boards, then G is *smoothly left-aligned* to F, or simply *left-aligned*.

The definition evidently depends on how the boards are concatenated. For example, a_1 can be left-aligned to a_2 in Figure 3 by moving them together in a parallel fashion. But if a_2 is lowered by two units, followed by concatenation with a_1, then a_1 is not left-aligned to a_2. Also, b_1 cannot be left-aligned to b_2 in a parallel fashion, but it can be if b_2 is first lowered by one unit (so the right arm of b_1 meets the left arm of b_2).

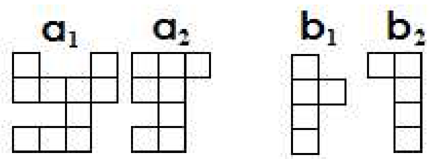

FIGURE 3. Aligned and non-aligned boards.

Proposition 2.1.4. *Let G be left-aligned to F, and F left-aligned to H, with F being 2-wide. Then in Triomineering,*

(a) $GF + FH \leq GFH$,
(b) *If* $GF = G$ *then* $GFH = G + FH$,
(c) *If* $FH = H$ *then* $GFH = H + GF$.

Proof. (a) It suffices to show that Horizontal can win as second player in $GF + FH - GFH$ (see Figure 4(a)). If Vertical begins by playing exclusively on G or H, then Horizontal can respond by playing exclusively on $-G$ or $-H$, respectively, and conversely. Since G is left-aligned to F, Vertical has no moves using squares from *both* of these boards, and the same holds for F and H. The only way Vertical can use squares from two boards, is on $-H$ and $-F$ or $-G$ and $-F$. Horizontal can counter these moves by using squares from H and F or G and F, respectively. These are the only options of Vertical since F is 2-wide, so no player can move exclusively on F or move using squares from all three boards. Hence Horizontal can win.

(b) We first prove that $GFH \leq G + FH$ by showing that $GFH - G - FH \leq 0$ (see Figure 4(b)). If Vertical, as first player, plays exclusively on G or H, then Horizontal can respond by playing exclusively on $-G$ or $-H$, respectively, and conversely. Further, if Vertical plays using squares

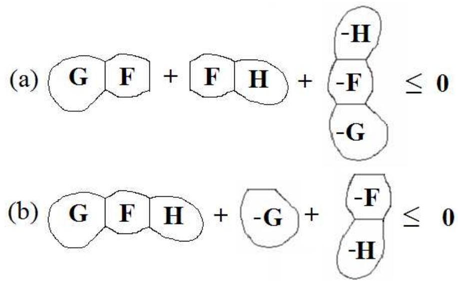

FIGURE 4. Illustrating the proof of Proposition 2.1.4.

from $-H$ and $-F$, then Horizontal can move using squares from H and F, respectively. Since these are the only options of Vertical, Horizontal can win.

To complete the proof, we use the result just proved and 2.1.4 (a):

$$GFH \leq G + FH = GF + FH \leq GFH,$$

and the result follows.

(c) The proof is the same as for (b). ■

Corollary 2.1.5. *Let G be left-aligned to $\square\square$, and $\square\square$ left-aligned to H. Then*

(a) $G\square\square + \square\square H \leq G\square\square H$,
(b) *If* $G\square\square = G$ *then* $G\square\square H = G + \square\square H$,
(c) *If* $\square\square H = H$ *then* $G\square\square H = H + G\square\square$.

Proof. These are special cases of Proposition 2.1.4 (a) – (c) with $F = \square\square$. ■

Remarks.

(1) The condition that the boards must be aligned is necessary: Figure 5(a) depicts a case where F is not left-aligned to H, with F 2-wide. Proposition 2.1.4 (a) does not hold, for otherwise $* + \{0|-1\} + 1 = \{1*|*\} \leq 0$, which is false since $\{1*|*\}$ is positive.

(2) The condition that F must be 2-wide is also necessary. Figure 5(b) exhibits a case where $F = 0$ is not 2-wide and the proposition does not hold, since

$$\{1|-3\} + \{1|1/2\} = \{2|3/2||-2|-5/2\} \not\leq \{2|3/2||-2|-3\}.$$

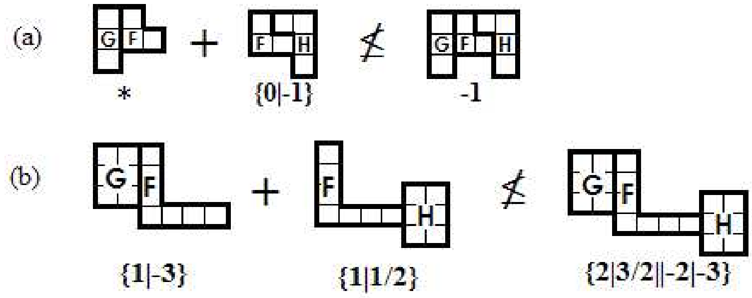

FIGURE 5. Boards alignment and 2-wideness are necessary.

(3) If $F \neq 0$, then Proposition 2.1.4 does not hold. Consider $F = G = H =$ the 3×1 tile. Then $GFH = \pm 2$ (Table 1), and $GF = FH = 2$. Clearly, $2 + 2 \not\leq \pm 2$, so (a) does not hold.

(4) Proposition 2.1.4 holds also for playing "straight" n-polyomineering if we require F to be $(n-1)$-wide. The proof of this claim is entirely analogous to the above. For increasing n, there is a growing set of $(n-1)$-wide boards F, for each of which Proposition 2.1.4 holds. For Domineering, however, Proposition 2.1.4 holds if and only if F is the 1×1 tile (see also Proposition 2.2.1 below).

The last remark shows that for increasing size of the (smallest) tiling polyomino, the power of Proposition 2.1.4 increases, as it permits to express as sums a growing variety of boards that are not disjoint. This can be observed already for Triomineering. In Figure 6 we have applied Proposition 2.1.4(b) to two different cases: The first one with F being the 2×2 board, G the 3×3 board, and H the 3×1 tile. In the second case, we take F to be a 2×2 board with one square removed. Note that F need not to be rectangular. From Figure 6, we see that

$$\{2|1||-2|-3\} = \pm 2 + \{0|-1\}$$
$$\pm(2)^* = \pm 2 + *$$

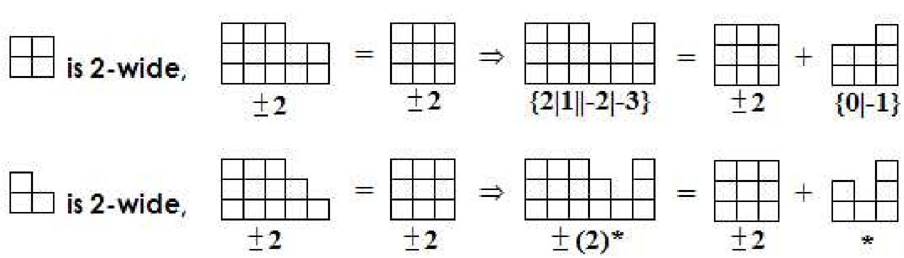

FIGURE 6. Two sample applications of Proposition 2.1.4 (b).

In general we cannot divide a "connected" board into pieces so that the value of the original board equals the sum of the values of the smaller boards, but here we can.

2.2. **Tridomineering.** Here Vertical and Horizontal alternate in tiling with either a domino or a straight triomino. The player making the last move wins. A dictionary of values for boards of up to six squares is depicted in Figure 7. Table 2 presents some of the values of rectangular boards.

Proposition 2.2.1. *Let G be left-aligned to $\square$, and $\square$ left aligned to H. Then for Tridomineering we have,*

(a) $G\square + \square H \leq G\square H$,
(b) *If* $G\square = G$ *then* $G\square H = G + \square H$,
(c) *If* $\square H = H$ *then* $G\square H = H + G\square$.

Proof. Same as Proposition 2.1.4 with F replaced by $\square$. ■

This proposition holds also for Domineering; see [3] Ch. 10.

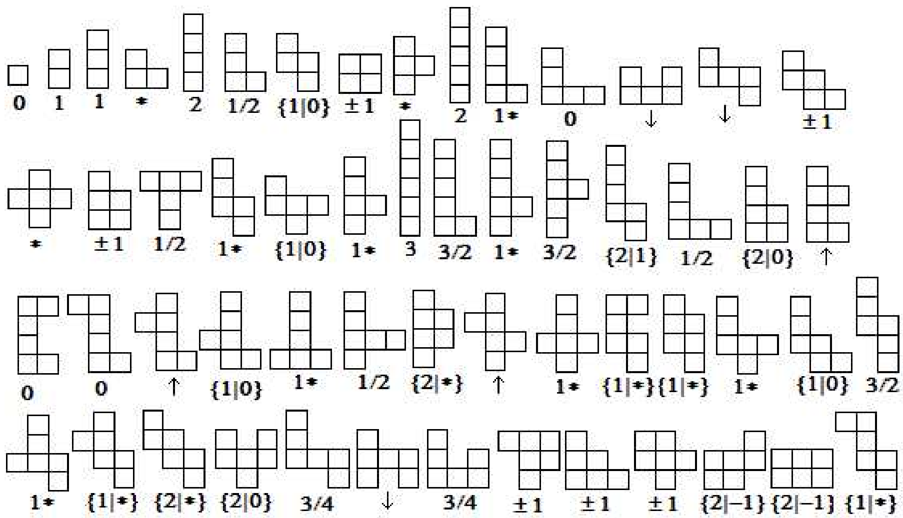

FIGURE 7. Tridomineering values for boards up to 6 squares.

TABLE 2. Values of Tridomineering for small rectangular boards.

	1	2	3	4	5
1	0	-1	-1	-2	-2
2	1	±1	$\{2\|-1\}$	$\{2\|0\|\|-1^*\}$	$\{0,\{3\|0\}\|-1,\{0\|-2\}\}$
3	1	$\{1\|-2\}$	±2	$\{2\|-1\|\|-2\uparrow\}$	$\{-1,\{3\|-1\}\|-7/4,\{-1\|-4\}\}$
4	2	$\{1^*\|\|0\|-2\}$	$\{2\downarrow\|\|1\|-2\}$	$\pm1^*$	F
5	2	$\{1,\{2\|0\}\|0,\{0\|-3\}\}$	$\{7/4,\{4\|1\}\|1,\{1\|-3\}\}$	F	F

2.3. **L-Tridomineering.** In this game we adjoin a third tiling piece in addition to the domino and straight triomino: an L-shaped triomino; that is, a 2×2 tile with one square removed. The L-triomino adds a total of 4 new moves to the set of moves of each player, since it can be rotated and placed in 4 different positions on a rectangular board. A dictionary of values of this game for small boards is exhibited in Figures 8 and 9, and Table 3 depicts the values of L-Tridomineering for some rectangular boards.

TABLE 3. Values of L-Tridomineering for small rectangular boards.

	1	2	3	4	5
1	0	-1	-1	-2	-2
2	1	±1	$\{2\|-1\}$	$\{1^*,\{2\|0\}\|-1^*\}$	$\{0,\{3\|0\}\|0,\{0\|-2\}\}$
3	1	$\{1\|-2\}$	±2	H	H
4	2	$\{1^*\|\{0\|-2\},-1^*\}$	V	F	F
5	2	$\{0,\{2\|0\}\|0,\{0\|-3\}\}$	V	F	F

As can be seen in the dictionaries of Figures 8 and 9, the value *2 is attained several times on small boards of L-Tridomineering. The first time is on a board of only 6 squares. For domineering, on the other hand, it is not so easy to construct a board with value *2. Such a board was recently constructed by G.C. Drummond-Cole, see [5]. It appears that the values of L-Tridomineering are hotter than those of our preceding games. If this is indeed the case in general, it may be due to the L-shaped triomino, which can be used by both players. Thus the game resembles more an impartial game, every nonzero value of which is hot.

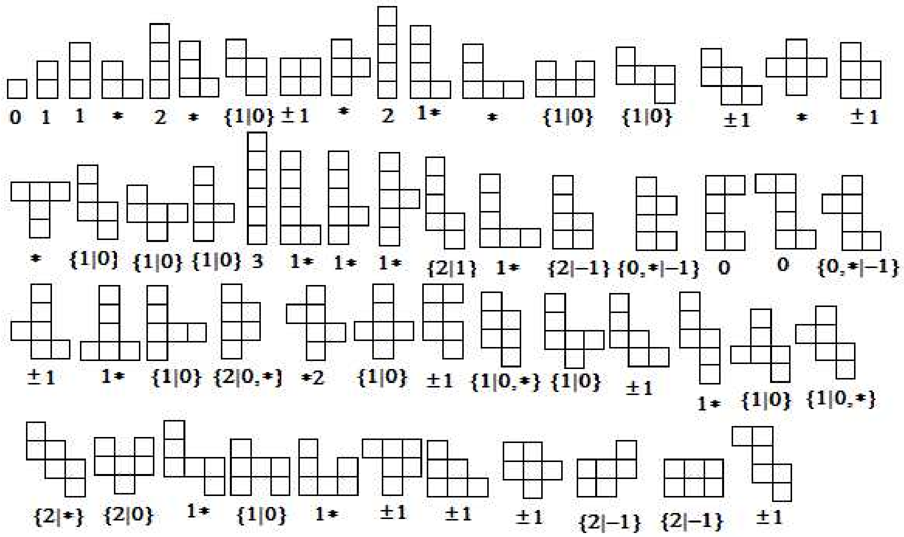

FIGURE 8. L-Tridomineering values for boards up to 6 squares.

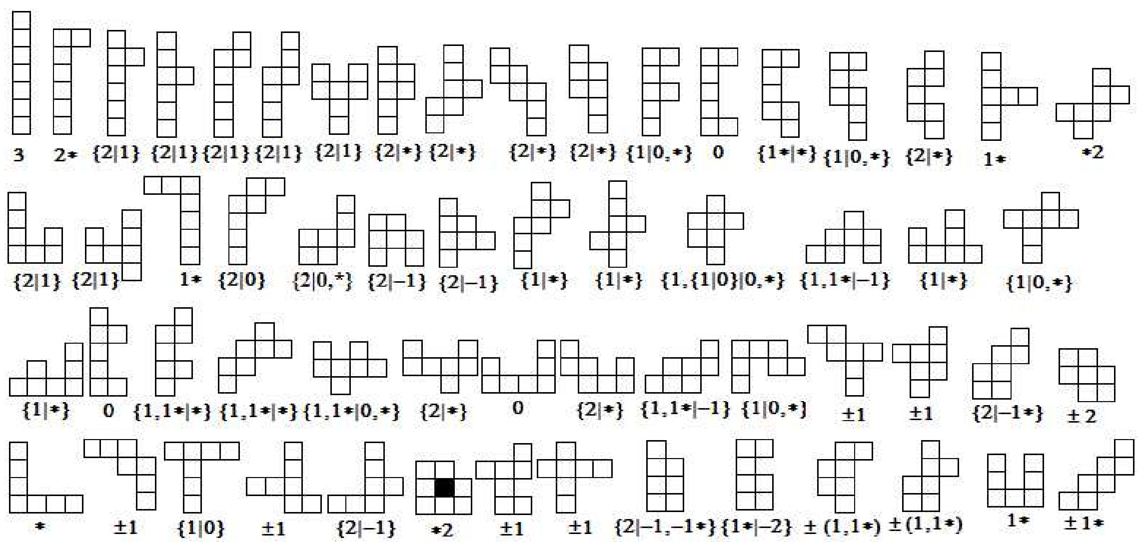

FIGURE 9. L-Tridomineering values for some boards with 7 squares.

L-Tridomineering also satisfies Proposition 2.2.1 of Tridomineering.

3. COMPUTATION OF THE VALUES

The values presented in this paper were computed using Aaron Siegel's *Combinatorial Game Suite software* (CGS) [8]. For CGS to understand the rules of the newly created games, it was necessary to write a Java plug-in for each game. The plug-ins were designed so CGS creates all possible positions that can be reached in a game without considering efficiency. This approach differs from that of [2] where the authors cut off some positions using an $\alpha - \beta$ search technique to determine the winner. However, we computed also the values of the games, needed for playing sums.

To improve the efficiency, we could have written a code that deletes dominated positions and replaces reversible ones early in the search, rather than finding all the possible positions reached in the game and then simplifying by using domination and reversibility. Due to memory constraints, we restricted the computation to boards not exceeding 6×6. We did not present here all the values that were computed. For those too long to write down, we only present the winner, according to whether G is positive, negative, zero, or fuzzy with 0. For example, the value corresponding to the game of Triomineering played on the 6×6 board is

$$\pm(\{2|1||1/2\},\{\{6|5||9/2|||2|1||1/2\},\{6|5||9/2|||4|0\}|\{4|0|||{-1/2}||{-1}|{-2}\},\{7/2||3|2|||{-1/2}||{-1}|{-2}\}\})$$

which is fuzzy with 0, and so the first player wins.

4. Some families of values

Note that square boards have values of the form $\{A| - A\} = \pm A$, where A is any game, since the set of moves for Horizontal and Vertical are the same by symmetry. We also note that Tables 1-3 are antisymmetric, since the value of a game on a board B is the negative of the game on board B', where B' is the rotation of B by $90°$ about its center (clockwise or counterclockwise).

Finally, in addition to the dictionaries of values already provided, we present in Figure 10 some patterns of boards that have a clearly discernible pattern of values. One can prove the validity of some of these patterns by using the properties that were pointed out in Propositions 2.1.4 and 2.2.1. For example, to prove by induction the validity of the first sequence of Triomineering , we use Proposition 2.1.4 (a) and (b), and the well-known fact that $*+*+\cdots+*$ is 0 if we add an even number of stars, and $*$ if we add an odd number of them.

The last two lines of Figure 10 constitute a sequence of values for L-Tridomineering. The values obtained are,

$$*, \downarrow, \underbrace{\{\downarrow | -1\}, -1, -1^*, -1\downarrow}, \underbrace{\{-1\downarrow | -2\}, -2, -2^*, -2\downarrow}, \underbrace{\{-2\downarrow | -3\}, -3, -3^*, -3\downarrow} \cdots$$

The emerging pattern is a four-term block of the form $\{(1-a)\downarrow | -a\}, -a, -a^*, -a\downarrow$ for $a \in \mathbb{Z}_{\geq 1}$ increasing in steps of 1.

We remark, in passing, that the "double cross" in Fig 10 (second board from left in first row of L-Tridomineering) has value $\{\{0|-1\}, *|*, 0, \{0|-1\}\} = \{*|0, \{0|-1\}\}$ (by domination) $= \{*|0\}$ (by reversibility) $= \downarrow$. Note that, perhaps a bit counterintuitively, the best opening move for Right is to tile with a domino the middle 2 squares.

5. Lemineering, an Impartial Game

Motivated by L-Tridomineering, we create a game in which each of the two players is only allowed to use the L-triomino. Since the sets of moves for each player are identical, Lemineering is an *impartial game*, so the Sprague-Grundy theory tells us that its values are *nimbers*. So for each short game of Lemineering G, we have,

$$G = \{*a_1, *a_2, \cdots, *a_n| *a_1, *a_2, \cdots, *a_n\} = \text{mex}\{*a_1, *a_2, \cdots, *a_n\} ,$$

where for any set $S \subsetneq \mathbb{Z}_{\geq 0}$, $\text{mex}\, S = \min \mathbb{Z}_{\geq 0} \setminus S =$ smallest nonnegative integer not in S.

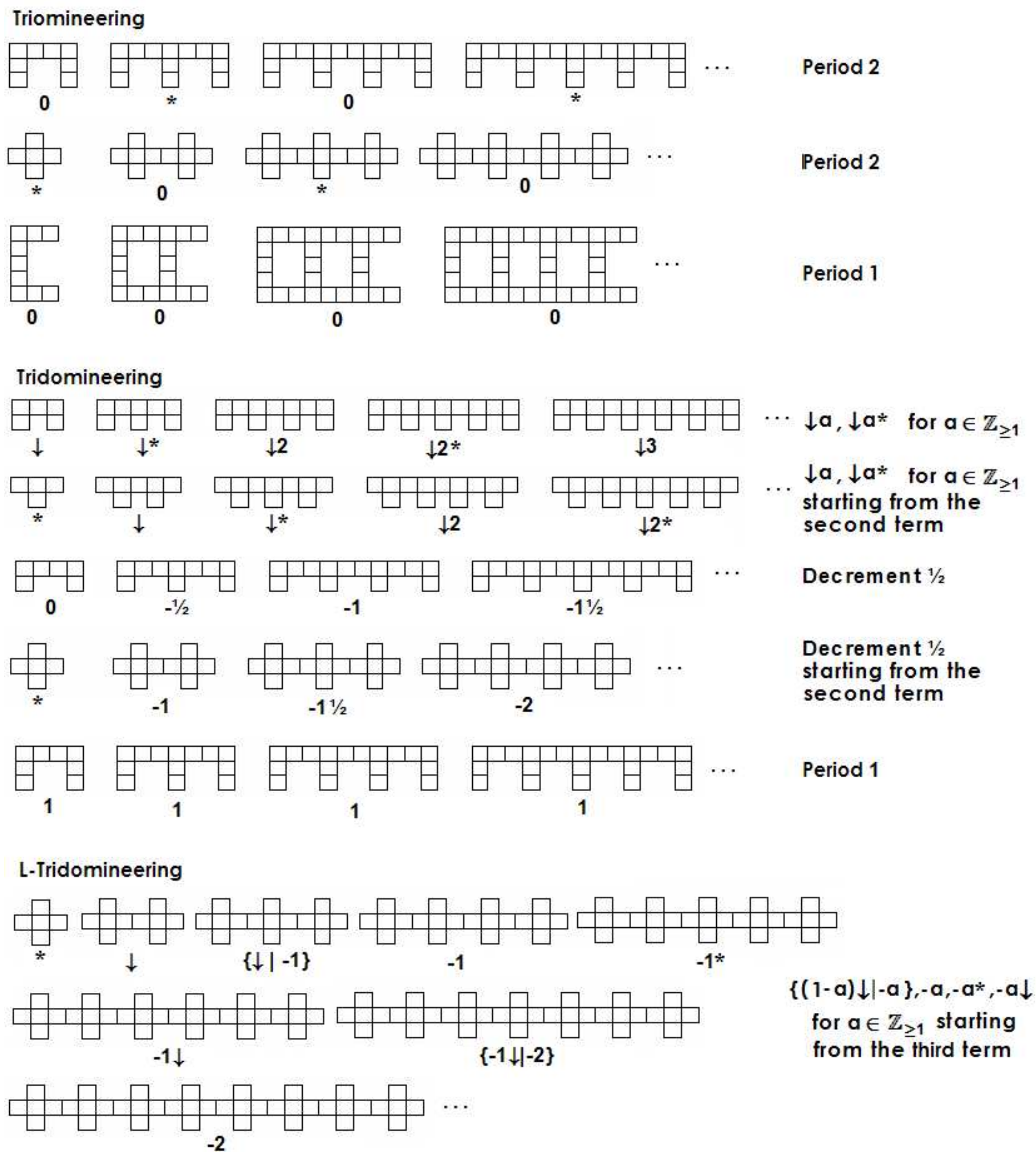

Figure 10. Families of patterns in these games.

We present the value of Lemineering for small rectangular boards in Table 4. Note that Table 4 is symmetric, since every nimber is its own negative.

Table 4. Sprague-Grundy values for Lemineering.

	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	1	2	0	3	1
3	0	2	0	1	2	2
4	0	0	1	0	1	0
5	0	3	2	1	0	1
6	0	1	2	0	1	1

There is also an impartial version of Domineering called Cram. In this game, both players are allowed to place horizontal and vertical dominoes. Winning strategies for this game have been found for some rectangular boards [7]. A similar analysis can be done to find winning strategies for the impartial versions of Triomineering and Tridomineering. For example, on an $m \times n$ board, where both m and n are even, the second player can win the impartial version of Triomineering and Tridomineering by reflecting the first players move through both axes. For Tridomineering, if one of m or n is odd and the other is even, the first player can win by tiling the middle two squares and then applying the symmetry strategy. If $m, n > 1$ are odd and we play the impartial version of Triomineering or Tridomineering, the first player can win by tiling the middle three squares and then playing by symmetry. Unfortunately, these symmetry arguments do not work for Lemineering, since we are only allowed to use the L-shaped triomino; so the first player could place an L-triomino in the middle of the board, breaking all existing symmetry.

6. Concluding Remarks

- We proved some properties of Triomineering (Proposition 2.1.4) which generalize those of Domineering (Proposition 2.2.1). This may lead one to think that there could be some isomorphism between Triomineering and Domineering. We believe, however, that this is not the case. In particular, we propose that there is no position in Triomineering with value
$$\pm(0, \{\{2|0\}, 2+_2 |\{2|0\}, -_2\}),$$
which is the value of a 4×4 board in Domineering.
- It is natural to generalize the results of this paper to larger polyominoes. Results of the form of Proposition 2.1.4 grow stronger with increasing size of the smallest participating polyomino.
- The games presented here were analyzed for *normal* play; that is, the player making the last move wins. One can also analyze these games for *misère* play, where the player making the last move loses, but then the obvious usefulness of sums is lost.

References

[1] E.R. Berlekamp, J.H. Conway, and R.K. Guy, *Winning Ways for your Mathematical Plays. Vol. 1*, Academic Press, 1982.
[2] D.M. Breuker, J.W.H.M Uiterwijk, and H.J. van den Herik, “Solving 8×8 Domineering,” *Theoretical Computer Science* **230** (2000) 195–206.
[3] J.H. Conway, *On Numbers and Games*, Academic Press, 1976.
[4] ———, “All Games Bright and Beautiful,” *The American Mathematical Monthly* **84** (1977) 417-434.
[5] G.C. Drummond-Cole, “Position of Value *2 in Generalized Domineering and Chess,” `http://www.math.sunysb.edu/~blafard` (July 2004).
[6] M. Gardner, “Mathematical Games” column, *Scientific American* 230(2) (February 1974) 106–108.
[7] ———, “Cram, bynum and quadraphage,” in *Knotted Doughnuts and Other Mathematical Entertainments*, Ch. 16, W.H. Freeman and Company, 1986.
[8] A. Siegel, *Combinatorial Game Suite*, `http://cgsuite.sourceforge.net/` (June 2004).

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