Virus Versus Mankind

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Humanity is but a passing episode in the eternal life of the virus

Abstract. We define a two-player virus game played on a finite cyclic digraph $G = (V, E)$. Each vertex is either occupied by a single virus, or is unoccupied. A move consists of transplanting a virus from some u into a selected neighborhood $N(u)$ of u, while devouring every virus in $N(u)$, and replicating in $N(u)$, i.e., placing a virus on all vertices of $N(u)$ where there wasn't any virus. The player first killing all the virus wins, and the opponent loses. If there is no last move, the outcome is a draw. Giving a minimum of the underlying theory, we exhibit the nature of the games on hand of examples. The 3-fold motivation for exploring these games stems from complexity considerations in combinatorial game theory, extending the hitherto 0-player and solitaire cellular automata games to two-player games, and the theory of linear error correcting codes.

Keywords: two-player cellular automata games, generalized Sprague-Grundy function

1 Introduction

The virus is engaged mainly in the following two activities:

- feed: devour and get devoured, or
- replicate.[1]

Man tries to employ genetic engineering to destroy the virus totally, winning if he succeeds. The two activities of the virus, however, have a single mission in mind: bring about the speedy demise of mankind. But if it realizes that it fails in this, it will strive to bring about its own death, making itself the last move, rather than give man the satisfaction of destroying the virus. If the virus

[1] Actually, mankind is similarly active. The first of these activities, however, is then termed more explicitly as warfare: kill and get killed.

succeeds in this, it won, having successfully committed suicide. Both agree that if there is no last move, i.e., there is always a virus, then the outcome is a draw.

These activities can be modeled as follows. A finite cyclic digraph $G = (V, E)$ represents the universe, where V denotes the location of the virus: $w(u) = 0$ if u isn't occupied by a virus, $w(u) = 1$ if a virus is at u. There is at most one virus at u for all $u \in V$. Both players alternate in selecting a virus, i.e., a vertex u with $w(u) = 1$ and *firing* it, that is, "complementing" it together with a selected neighborhood $N(u)$ of vertices. By complementing we mean that $w(u)$ switches to 0, and $w(v)$ reverses its parity for every vertex $v \in N(u)$. In biological language, the virus at u moves away from u, devouring every virus in $N(u)$, and replicating in v for all $v \in N(u)$ for which $w(v) = 0$ prior to the move. However, if u is a leaf, the virus at u can neither devour nor replicate, so it dies of frustration: it's dead wood, though $w(u)$ remains 1. The player making the last move wins (after which all vertices, except possibly leaves, have *weight* $w = 0$), and the opponent loses. If there is no last move, i.e., there is always a vertex with weight $w = 1$, the outcome is a draw. A precise definition of the games follows.

Given a finite digraph $G = (V, E)$, also called *groundgraph*. Order V in some way, say

$$V = \{z_0, \dots, z_{n-1}\}.$$

This ordering, with $|V| = n$, is assumed throughout. The set of *followers* of a vertex z_i is denoted by $F(z_i) = \{z_j \in V : (z_i, z_j) \in E\}$. If $F(z_i) = \emptyset$, then u is a *leaf*.

Let $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ denote the following game graph of the virus game played on $G = (V, E)$. The digraph $\mathbf{G}$ is also called the *virus graph* or *game graph* of G. Any vertex in $\mathbf{G}$ can be described in the form of a binary vector $\mathbf{u} = (u^0, \dots, u^{n-1})$, where $u^k = 1$ if $w(z_k) = 1$, $u^k = 0$ if $w(z_k) = 0$. In particular, $\Phi = (0, \dots, 0)$ is a leaf of $\mathbf{V}$, and $|\mathbf{V}| = 2^n$.

The field consisting of two elements $\{0, 1\}$ under Nim-addition $\oplus$ (also called exclusive or — XOR) and and ordinary multiplication is known as GF(2), the Galois field of two elements. Thus $\mathbf{V}$ is an abelian group under $\oplus$ with identity Φ. Every nonzero element has order 2. Moreover, $\mathbf{V}$ is a vector space over GF(2) satisfying $1\mathbf{u} = \mathbf{u}$ for all $\mathbf{u} \in \mathbf{V}$. For $i \in \{0, \dots, n-1\}$, define unit vectors $\mathbf{z}_i = (z_i^0, \dots, z_i^{n-1})$ with $z_i^j = 1$ if $i = j$; $z_i^j = 0$ otherwise. They span the vector space.

For defining $\mathbf{E}$, let $\mathbf{u} \in \mathbf{V}$ and let $0 \le k \le n-1$. For $0 \le q = q(k) \le |F(z_k)|$, let $F^q(z_k) \subseteq F(z_k)$ be any subset of $F(z_k)$ satisfying

$$|F^q(z_k)| = q. \tag{1}$$

Define,

$$\begin{aligned}(\mathbf{u}, \mathbf{v}) \in \mathbf{E} \text{ if there exists } k \text{ such that } u^k = 1 \text{ and}\\ \mathbf{v} = \mathbf{u} \oplus \mathbf{z}_k \oplus \{\mathbf{z}_\ell : z_\ell \in F^q(z_k)\},\end{aligned}$$

for some $F^q(z_k)$ satisfying (1).

Informally, an edge $(\mathbf{u}, \mathbf{v})$ reflects the firing of u^k in $\mathbf{u}$ (with $u^k = 1$), i.e., the complementing of the weights of z_k and $F^q(z_k)$. Such an edge exists for every $F^q(z_k)$ satisfying (1). Note that if $z_k \in G$ is a leaf, then there is no move from $\mathbf{z}_k$, since then $\{z_\ell \in F^q(z_k)\} = \emptyset$.

Remark The strategy of a cellular automata game on a finite *acyclic* digraph is the same as that of a sum-game on $F^q(z_k)$, i.e., a game without complementation of the 1s of $F^q(z_k)$. This follows from the fact that $a \oplus a = 0$ for any nonnegative integer a.

For $s \in \mathbb{Z}^+$, an *s-game* on a digraph $G = (V, E)$ is a virus game on G satisfying $q \leq s$ for all $k \in \{0, \ldots, n-1\}$ (q as in (1)). An *s-regular game* is an s-game such that for all k we have $q = s$ if $|F(z_k)| \geq s$, otherwise $q = |F(z_k)|$. Thus if $s \geq \max_{u \in V} |F(u)|$, then firing u in an s-regular game entails complementing all of $F(u)$, for all $u \in V$, in addition to complementing u, unless u is a leaf.

The emphasis here is on exhibiting examples of two-player virus games. Their theory and complexity are explored in [11], where also formal proofs are given. Motivation for defining this particular family of games and how they fit into the general theory and complexity of combinatorial games are given in the epilogue (§7).

2 Initial Examples

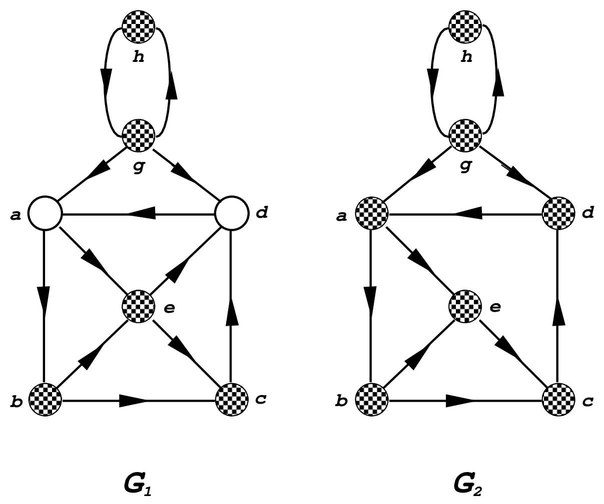

Fig. 1. A two-player virus game. © 2000

In Figures 1 and 2, the shaded vertices have weight 1, the rest have weight 0.

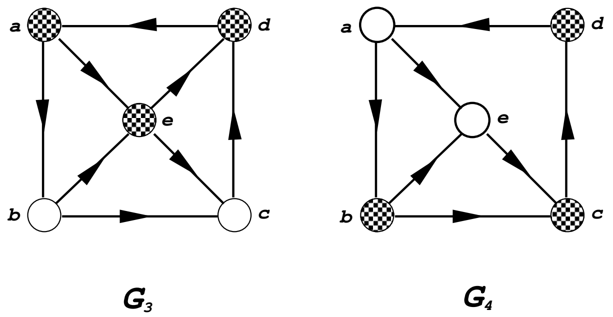

Fig. 2. Adding two more components. © 2000

Example 1 We play a 3-regular game on the *sum* of the two components G_1, G_2 of Figure 1, for which we compute the so-called *γ-values* 3, 2 respectively (more about the γ-function follows in the next sections). A sum of games is a finite collection of disjoint component games, where a move consists of selecting any component and making a move in it. Thus a move in the sum of two-player cellular automata games consists of selecting a component, and firing a vertex in it. Then the γ-value of the sum is given by $3 \oplus 2 = 1$. Since this value is finite but > 0, the first player can win, namely by moving to a position with γ-value 0. This is can be affected in one of three ways, all involving moves in G_1:

- Fire vertex b, which complements the weights of b, c, e, namely it makes $w(b) = w(c) = w(e) = 0$, leaving the other weights unchanged. Then the γ-value of the resulting position of G_1 changes to 2, leading to $2 \oplus 2 = 0$.
- Fire c, complementing the weights of c and d, again transforming the γ-value of the position of G_1 to 2.
- Fire e, complementing the weights of c, d, e, resulting in a new position of γ-value 2.

Example 2 Play a 2-regular game on the two components of Figure 2. The γ-value of the position of each is 1, so the γ-value of the sum is $1 \oplus 1 = 0$. This implies that whatever move the first player will do, the second player can win.

Example 3 We now wish to play a 3-regular game on all four components G_1, G_2, G_3, G_4. The value of the sum is $3 \oplus 2 \oplus 1 \oplus 1 = 1$. Thus the first player can win, either by selecting one of the above three moves, or by moving to a position with γ-value 0 in one of G_3 or G_4. This can be affected by firing a or d or e in G_3, or by firing b or c or d in G_4.

An elaboration on these three examples follows in §4.

3 Background

For explaining the above examples, we need some basic facts. Let $G = (V, E)$ be any finite cyclic digraph. A digraph is cyclic if it *may* contain cycles. The

Generalized Sprague-Grundy function γ is a mapping $\gamma: V \to \mathbb{Z}^0 \cup \{\infty\}$ which satisfies certain conditions, where the symbol ∞ indicates a value larger than any natural number. If $\gamma(u) = \infty$, we also say that $\gamma(u)$ is infinite. We wish to define γ also on certain subsets of vertices. Specifically: $\gamma\big(F(u)\big) = \big\{\gamma(v) < \infty: v \in F(u)\big\}$. If $\gamma(u) = \infty$ and if we denote the set $\gamma\big(F(u)\big)$ by K for brevity, then we also write $\gamma(u) = \infty(K)$. Next we define equality of $\gamma(u)$ and $\gamma(v)$: if $\gamma(u) = k$ and $\gamma(v) = \ell$ then $\gamma(u) = \gamma(v)$ if one of the following holds: (a) $k = \ell < \infty$; (b) $k = \infty(K)$, $\ell = \infty(L)$ and $K = L$. We also use the notations

$$V_i = \{u \in V : \gamma(u) = i\} \ (i \in \mathbb{Z}^0), \quad V^f = \big\{u \in V: \gamma(u) < \infty\big\}, \quad V^\infty = V \setminus V^f.$$

We next formulate an algorithm for computing γ. Initially a special symbol ν is attached to the label $\ell(u)$ of every vertex u, where $\ell(u) = \nu$ means that u has no label. We also introduce the notation $V_\nu = \{u \in V: \ell(u) = \nu\}$.

Algorithm GSG for computing the Generalized Sprague-Grundy function for a given finite cyclic digraph $G = (V, E)$.

1. (Initialize labels.) Put $i \leftarrow 0$, $\ell(u) \leftarrow \nu$ for all $u \in V$.
2. (Label.) As long as there exists $u \in V_\nu$ such that no follower of u is labeled i and every follower of u which is either unlabeled or labeled ∞ has a follower labeled i, put $\ell(u) \leftarrow i$.
3. (∞-label.) For every $u \in V_\nu$ which has no follower labeled i, put $\ell(u) \leftarrow \infty$.
4. (Increase label.) If $V_\nu \neq \emptyset$, put $i \leftarrow i + 1$ and return to 2; otherwise end.

We then have $\gamma(u) = \ell(u)$. The γ function has been defined in [22]. See also [4], Ch. 11. The above algorithm is a simplified version of Algorithm G in [16].

Informally, a *P-position* is any position u from which the *P*revious player can force a win, that is, the opponent of the player moving from u. An *N-position* is any position v from which the *N*ext player can force a win, that is, the player who moves from v. The next player can win by moving to a P-position. A *D-position* is any position from which neither player can win, but both have a nonlosing next move, namely, moving to some D-position. Denote the set of all P-positions by $\mathcal{P}$, all N-positions by $\mathcal{N}$ and all D-positions by $\mathcal{D}$.

It turns out that the γ-function on the *sum* of the exponentially large virus graphs enables us to compute a strategy: We first compute the γ-values of each of the components V_i of sizes $O(|V_i|)$. Their *generalized Nim sum* is then the γ-function of the sum [16], [22]. The generalized Nim sum of nonnegative integers is their XOR as for the common Nim sum. For any finite subsets K, L of nonnegative integers and any nonnegative integer a, $\infty(K) \oplus a = a \oplus \infty(K) = \infty(K \oplus a)$, and $\infty(K) \oplus \infty(L) = \infty(\emptyset)$.

The connection between γ on the game graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ of the sum of one or more disjoint games and P, N, D is given by:

$$\mathcal{P} = \{\mathbf{u} \in \mathbf{V}: \gamma(\mathbf{u}) = 0\}, \qquad \mathcal{D} = \{\mathbf{u} \in \mathbf{V}: \gamma(\mathbf{u}) = \infty(L), \ \ 0 \notin L\}, \tag{2}$$

$$\mathcal{N} = \{\mathbf{u} \in \mathbf{V}: 0 < \gamma(\mathbf{u}) < \infty\} \ \cup \ \{\mathbf{u} \in V: \gamma(\mathbf{u}) = \infty(L), \ \ 0 \in L\}, \tag{3}$$

see [16], [22].

The γ-function of the sum of the virus graphs $\mathbf{G}$ induced by the digraphs G on which the game is played, enables us to play optimally. We compute it in the next section.

4 The First Three Examples Revisited

If $\{u_i, u_j, \ldots, u_k\}$ is a collection of vertices of G with $w(u_i) = w(u_j) = \ldots = w(u_k) = 1$ and all other vertices have weight 0, we denote the γ-value of this position of $\mathbf{G}$ by $\gamma(u_i u_j \ldots u_k)$.

Consider G_3 for ≥ 2-regular play. We use Algorithm GSG and inspection to note that $\gamma(e) = 0$: e has only the as yet unlabeled follower cd, which has the follower $\Phi \in \mathbf{V}_0$. Inspection further shows that also $\gamma(ac) = \gamma(bd) = 0$. It turns out that γ is a homomorphism from $\mathbf{V}^f$ onto $\mathrm{GF}(2)^t$ for some nonnegative integer t with kernel $\mathbf{V}_0$ and quotient space $\mathbf{V}^f/V_0 = \{\mathbf{V}_i : 0 \leq i < 2^t\}$ (collection of cosets), and $\dim(\mathbf{V}^f) = t + \dim(\mathbf{V}_0)$.

In particular, $\mathbf{V}_0$ is a linear subspace of $\mathbf{V}$, so it contains all Nim sums of Φ, e, ac, bd. Inspection shows that it contains nothing else. Hence

$$\mathbf{V}_0 = \{\Phi, e, ac, bd, ace, bde, abcd, abcde\}. \tag{4}$$

Inspection further shows that $\gamma(cd) = 1$. Thus we get the coset

$$\mathbf{V}_1 = cd \oplus \mathbf{V}_0 = \{cd, cde, ad, bc, ade, bce, ab, abe\}.$$

We see that $ade \in \mathbf{V}_1$, as claimed in Example 2. Also $\dim(\mathbf{V}_0) = 3$, $\dim(\mathbf{V}^f) = 4$, $t = 1$. (We also have $\gamma(a) = \infty$, but $\mathbf{V}^\infty$ is clearly not a subspace. However, $\mathbf{W} = \{a, \Phi\}$ is a subspace such that every vector in $\mathbf{V}$ can be written uniquely as a sum of two vectors: one in $\mathbf{V}^f$ and one in $\mathbf{W}$.)

The same considerations show that for G_4,

$$\mathbf{V}_0 = \{\Phi, b, ac, de, abc, bde, acde, abcde\}, \tag{5}$$

$$\mathbf{V}_1 = cd \oplus \mathbf{V}_0 = \{cd, bcd, ad, ce, abd, bce, ae, abe\}.$$

The γ-function of the sum of G_3 and G_4 is thus $1 \oplus 1 = 0$. By (2) this is a P-position, substantiating the claim of Example 2.

Note that the lower part of G_1 is congruent to G_3, and the lower part of G_2 is congruent to G_4. Inspection shows that for both G_1 and G_2, $\gamma(gh) = 2$. Therefore $t = 2$ for both components. Further, $\mathbf{V}_0$ and $\mathbf{V}_1$ of G_1 (G_2) are the same as $\mathbf{V}_0$ and $\mathbf{V}_1$ of G_3 (G_4) respectively. For G_1 we thus get the cosets $\mathbf{V}_2 = gh \oplus \mathbf{V}_0$, $\mathbf{V}_3 = adgh \oplus \mathbf{V}_0$ where $\mathbf{V}_0$ is given by (4), $\dim \mathbf{V}_0 = 3$, $\dim \mathbf{V}^f = 5$. The same holds for G_2, but with $\mathbf{V}_0$ given by (5).

We thus conclude that for G_1, $\gamma(bcegh) = 3$, and for G_2, $\gamma(abcdegh) = 2$, as claimed in Example 1. The winning moves suggested in Examples 1–3 all yield positions with γ-value 0, so they are P-positions by (2).

5 Further Examples

Example 4 We now wish to play a 2-regular game on the sum of the two components of Figure 1, rather than a 3-regular game. This affects only vertex g, whose outdegree is 3 in both components. The outcome is now a draw by (2), since $\gamma(gh) = \infty(0)$ for both components. Note that not every sequence of moves will maintain a draw: thoughtless moves might give the opponent the opportunity to win.

Example 5 Play a 2-regular game on the sum of the two components G_2 with position g, and G_4 with the position shown in Figure 2. We have $\gamma(g) = \infty(1)$, so the sum has γ-value $\infty(1) \oplus 1 = \infty(1 \oplus 1) = \infty(0)$. By (3) this is an N-position. The first player has a unique winning move by firing g and complementing a and d. Any other move leads to either a draw; or to an N-position, from which the opponent can win.

The next examples involve play on the digraph $G(p)$ (Figure 3) which depends on a parameter $p \in \mathbb{Z}^+$. It has vertex set $\{x_1, \ldots, x_s, y_1, \ldots, y_s\}$, and edges:

$$F(x_i) = y_i \quad \text{for} \quad i = 1, \ldots, s,$$
$$F(y_k) = \{y_i : 1 \leq i < k\} \cup \{x_j : 1 \leq j \leq s \text{ and } j \neq k\} \text{ for } k = 1, \ldots, s.$$

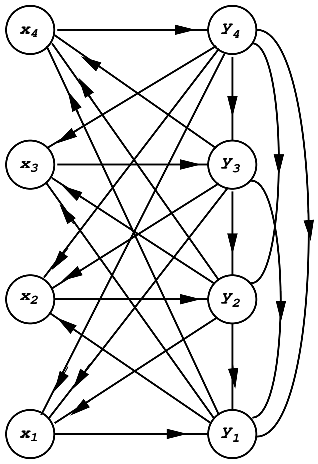

Fig. 3. Virus games on a parametrized digraph. © 2000

Example 6 Play a 3-regular game on $G(7)$, where $w(x_7) = w(y_1) = \ldots = w(y_7) = 1$, and all the other vertices have weight 0. Inspection shows that all collections of an even number of x_i are in $\mathbf{V}_0$ (i.e., an even number of x_i having weight 1, all other vertices weight 0); and $\mathbf{V}^f$ consists precisely of all collections of an even number of 1s. Further, $\gamma(x_j y_j)$ = smallest nonnegative integer not the Nim sum of at most three $\gamma(x_i y_i)$ for $i < j$. Thus $\gamma(x_7 y_1 \dots y_7) = 0 \oplus 1 \oplus 2 \oplus 4 \oplus 8 \oplus 15 \oplus 16 \oplus 32 = 48$. So firing y_7 and complementing y_6 and any two of the x_i $(i < 7)$ is a winning move. Incidentally, the sequence $\{1, 2, 4, 8, 15, 16, 32, 51, \dots\}$ has been used in [1] for a special case of the game "Turning Turtles".

Example 7 Play a 4-regular game on $G(7)$, on the same initial position as in the previous example. Here any collection of x_i is in $\mathbf{V}_0$. (More generally, in an s-regular game on $G(p)$ any collection of x_i is in $\mathbf{V}_0$ if s is even, otherwise any collection of an even number of x_i is in $\mathbf{V}_0$.) Moreover, $\big(\gamma(y_i)\big)_{i \ge 1} = (1, 2, 4, 8, 16, 31, 32, 64, 103, \dots)$. It is the sequence defined as the smallest nonnegative integer not the Nim sum of at most four earlier terms. This sequence appears in Table 3, Ch. 14 of [1]. Note that the strategy of our two-player virus games on just Fig. 3 alone subsumes and unifies that of a battery of games there. Thus $\gamma(x_7 y_1 \dots y_7) = 0 \oplus 1 \oplus 2 \oplus 4 \oplus 8 \oplus 16 \oplus 31 \oplus 32 = 32$. Firing y_7 and complementing any four x_i $(i < 7)$ results in a P-position.

Example 8 Play the sum of the games in the two previous examples. In the 3-regular games, fire y_6 and complement any two of the x_i $(i \ne 6)$. This results in γ-value 32 for this game, so its sum with the 4-regular game constitutes a P-position.

Example 9 Now consider a play on $G(7)$ which is not regular: play a 2-regular game on the subset of all the y_{2i-1} and on all the x_i, and a 3-regular game on the subset of all the y_{2i} $(i \ge 1)$. Clearly any collection of x_{2i-1} is in $\mathbf{V}_0$, and any collection of even size of x_{2i} is in $\mathbf{V}_0$. We can now use induction on the number of even indexed x_i to show that actually any collection of x_i is in $\mathbf{V}_0$. Inspection shows that $\big(\gamma(y_i)\big)_{i \ge 1} = (1, 2, 4, 8, 7, 16, 11, 29, 13 \dots)$. This strange looking sequence is not (yet) in the useful encyclopædia of integers of Neil Sloane available on-line at http://www.research.att.com/˜njas/sequences/

Example 10 Play a 1-regular game on the subset of all the y_{2i-1} and on all the x_i, and a 2-regular game on the subset of all the y_{2i} $(i \ge 1)$. We can show again that any collection of x_i is in $\mathbf{V}_0$. Inspection shows further that $\gamma(y_i)_{i \ge 1} = \{1, 2, 3, 4, 5, 8, 6, 15, 7, 16, 9, 26, 10, 32, 11, 44, 12, 49, 13 \dots\}$. Also this sequence is not yet in the encyclopædia of integers.

6 Homework

Consider the digraph depicted in Fig. 4. It is suggested to play three types of games on it.

- A weight distribution over $\mathbb{Z}^0$ on the vertices is given initially. A move consists of selecting a vertex u with weight > 0 and shifting a weight of 1 to any follower v. After the move, $w(u)$ has diminished by 1, and $w(v)$ has increased by 1.

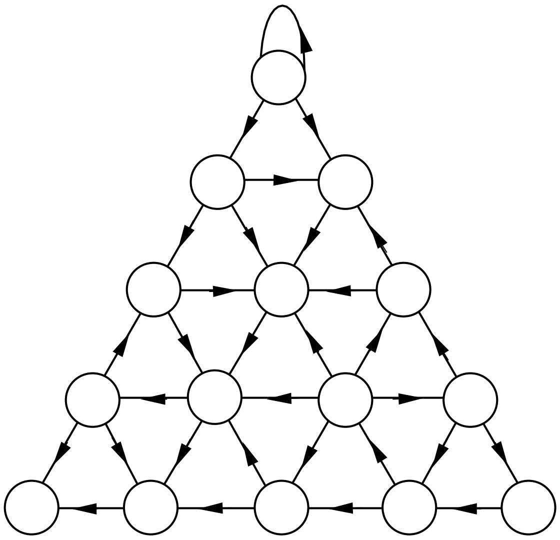

Fig. 4. Triple Strike game. © 2000

- Play a 2-regular game on the digraph.
- Play a 3-regular game on the digraph.
- Play a sum of three instances of the previous games.

Compute an optimal strategy for each of the four games.

7 Epilogue

The special case of 1-regular games are the so-called *annihilation games*, analyzed in [8], [13], [15]. For annihilation games it is convenient to consider a vertex with weight 1 to be occupied by a token, and one with weight 0 to be unoccupied. Two tokens are then mutually annihilated on impact, hence the name. Misère play (in which the player making the last move loses, and the opponent wins) of annihilation games was investigated in [7]. Our motivation for examining annihilation games, suggested to us by John Conway, was to create games which exhibit some interaction between tokens and yet still have a polynomial strategy.

Annihilation games are "barely" polynomial, in several senses. Their complexity is $O(|V|^6)$, the polynomial computation of a winning move may require a "strategy in the broad sense" (see [10] §4) and just about any perturbation of them yields Pspace-hard games (see [14], [12], [17]). They also lead to linear error correcting codes [9], but their Hamming distance is ≤ 4.

The current work is a generalization of annihilation games. It was motivated by the desire to create games which naturally induce codes of Hamming distance > 4. Virus games may provide such codes, a topic to be taken up elsewhere. In practice, $\mathbf{V}_0$, which is all that's needed for the codes, can often be computed by inspection. The best codes may be derived from a digraph which is a simplification of that of Figure 3, where $\mathbf{V}_0$ can also be computed by direct inspection. The lexicodes method [6], [5], [21], related to games, is clearly exponential.

Another motivation was to further explore polynomial games with "token interactions". Last but not least was the desire to create two-player virus games, which traditionally have been solitaire only.

Cellular Automata Games have traditionally either been 0-player games such as Conway's "Life", or solitaire games played on a grid or digraph $G = (V, E)$. Two commercial manifestations are *Lights Out* manufactured by Tiger Electronics, and *Merlin Magic Square* by Parker Brothers (but Arthur-Merlin games are something else again). Quite a bit is known about such solitaire games. Background and theory can be found e.g., in [18], [20], [23], [24], [25], [26], [27]. Incidentally, related but different solitaires are *chip firing games*, see e.g., [3], [19], [2]. In the present paper we have extended cellular automata games to two-player games in a natural way, and indicated a strategy for them, which is based on three key ideas: I. Showing that γ is essentially additive: $\gamma(\mathbf{u}_1 \oplus \mathbf{u}_2) = \gamma(\mathbf{u}_1) \oplus \gamma(\mathbf{u}_2)$ if $\mathbf{u}_1 \in \mathbf{V}^f$ or $\mathbf{u}_2 \in \mathbf{V}^f$. II. Showing that $\mathbf{V}_0$, $\mathbf{V}^f$ and γ can be computed by restricting attention to the linear span of "sparse" vectors. III. Computing a winning move.

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