Mathematical Chats Between Two Physicists

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To Martin Gardner — The Master of recreational mathematics

The Luncheon Chat

Joyce is a physicist doing statistical mechanics, and Gill a nuclear physicist specializing in particle interactions. While relaxing with their cups of coffee after a tasty enjoyable light lunch at the TEX (TasteEnjoyrelaχ) — the *Sciences Club* of the University — they began to chat about some common aspects of their specialties.

Gill: The interaction between elements such as particles, nucleons, spins, etc. that are "close" to one another is common to our two disciplines. I wonder whether a lesson can be learned by viewing these phenomena in a unified manner.

Joyce: Hmm...a nice idea. I think that to do this we need some abstract model that reflects the basic common properties of these interactions, and that is amenable to mathematical analysis, such as working with two elements 1 and 0, that form a field called by those pompous mathematicians the *Galois field* of two elements, GF(2).

G: Yes, GF(2) has the advantage that $1 = -1$, so the rule $1 + 1 = 0$ in this field is the same as the annihilation rule of particles and spins: $1 - 1 = 0$. We have of course $0 + 1 = 1 + 0 = 1$ and $0 + 0 = 0$, as well as $1 + 1 = 0$. These addition rules are also known as *Nim sum* or *Xor* — *exclusive or*. Furthermore, to model interactions that are not necessarily neighboring vertices on a grid, it seems best to have a directed graph $G = (V, E)$ — that mathematicians, always tending to succinctness, call *digraph* for short — on whose vertices V

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the "particles" 0 and 1 are initially distributed. Selecting a particle on a vertex u, it is complemented as well as all its neighbors along edges directed away from u.

J: What you describe is a system called *cellular automata* by those inflated logicians, mathematicians and computer scientists, a manifestation of which is the *Merlin Magic Square* game manufactured by Parker Brothers (but Arthur-Merlin games are something else again). Quite a bit is known about such solitaire games. Anyway, a huge literature has been accumulating on cellular automata. A small example, intersecting with solitaire games, is [Gol91], [Pel87], [Sto89], [Sut88], [Sut89], [Sut90], [Sut95]. Incidentally, related but different solitaires are *chip firing games*, see e.g., [BL92], [Lóp97], [Big99].

What seems more attractive and new is to transform these solitaire games into two-player games, where the player first achieving 0s on all the non-leaf vertices wins and the opponent loses. If there is no last move, the outcome is a draw. Moreover, this version will appeal to many of my colleagues who have turned their attention to biology, such as protein folding, where the main aim is to tinker with nature, in order to achieve some doubtful benefits such as designing specialized medicines and genetic engineering (alias tinkering)...For want of a better name, we might call them *Cellata* games, since it reminds me both of the Italian cuisine that I just enjoyed, and of cellular automata.

G (taking a paper napkin and beginning to draw on it): I like your idea, and I share your belief that it appears to be new and interesting. In most of the solitaire games you have mentioned, *any* order of the moves produces the same result. To promote your suggestion of tinkering, I think it's then best to permit the players to select only an *occupied* vertex, i.e., a vertex occupied by a 1. So a move in the game consists of selecting an occupied vertex and *firing* it, i.e., complementing it together with all its directed neighbors. The player making the last move wins. If there is no last move, the outcome is a draw...the order of the moves is then definitely important, unlike in those solitaires. ...Here now is a suggested game on two components with an initial $0,1$-distribution, where 1s are indicated by $\star$s (Figure 1) and vertices occupied by 0s remain unlabeled. As a gentleman, I'm used to "Ladies First" etiquette, so I graciously offer you to move first.

J (pulling a PalmCrash from her handbag and hammering away furiously on its buttons): You propose to play a *sum* of games, i.e., a move consists of selecting a component and firing an occupied vertex on it. The player making the last move in the entire digraph wins, and her opponent loses...it seems to me that your gentlemanly gesture is all but gallant. It is indeed patronizing, since whatever I'll do from this position, you can win. I'll therefore add to your two components two simplified versions, namely deleting vertices 5 and 6 on the two components, with $\star$s as indicated (Figure 3). Under these circumstances I accept your offer to make the first move in the sum consisting of all the 4 components.

G (blushing): Well...I really hadn't expected you to find out so soon...I see that on the game consisting of the four components you can win by making an appropriate move... . Since it seems that both of us understand the win/lose

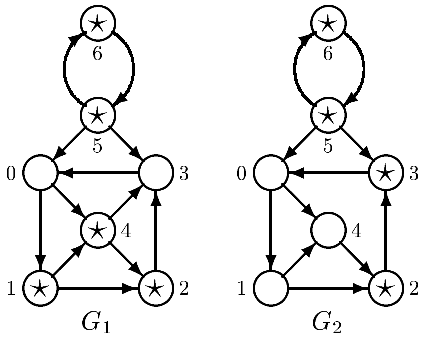

Figure 1: A two-player game $G_1 + G_2$ on cellular automata. A move consists of selecting a vertex v marked with a $\star$ and "firing" it. Once fired, the $\star$ is removed, and $\star$s are placed on every vertex v points to. If two $\star$s appear at a vertex, both are annihilated. Two players play by taking turns firing a vertex. The first player unable to move loses, and the opponent wins. If there is no last move, the outcome is a draw. The result of firing vertex 4 in G_1 is shown in G_1 of Figure 2.

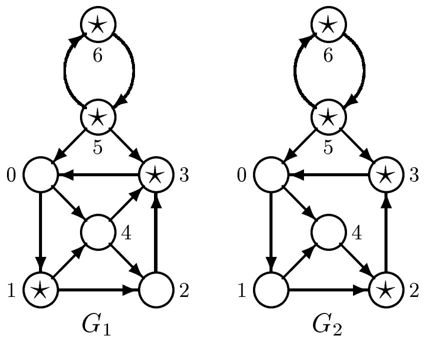

Figure 2: Game $G1 + G2$ from Figure 1 after one move.

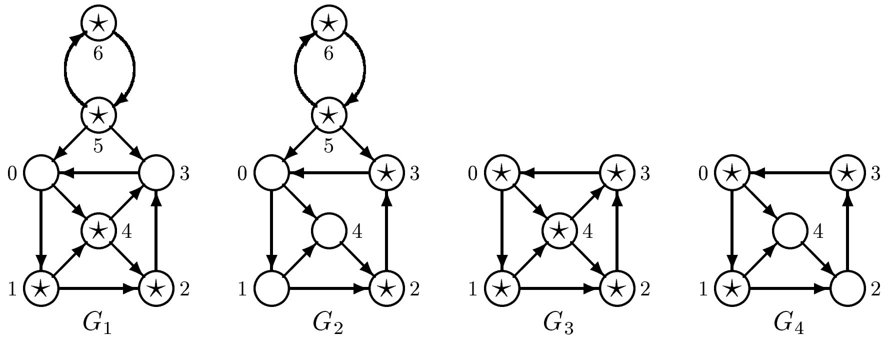

Figure 3: Adding two more components, G_3 and G_4.

positions of this game, I suggest to play the same game with the small change of adjoining a $\star$ on vertex 0 of G_1.

J (consulting her PalmCrash once more and then rising): Alright, the initial position is now a draw. Since we seem to have mastered also the draw positions, it's time to head back to our offices and do some serious physics. . .such as deciding the computational complexity of Cellata games.

A Conversation in Joyce's Office

The next day, Professor Gill Andrin strolled over to Professor Joyce Prato's office.

Gill: Good morning Joyce, I was wondering how you found me out so quickly yesterday when I offered you to play first on Figure 1.

Joyce: Hi Gill, I'm already used to your tricks. When I saw that you proposed to play on two components of a game that obviously has cycles, I assumed that you had computed the generalized Sprague-Grundy function γ for the game [Smi66], [Con76, Ch. 11], [FY86]; otherwise you would hardly be able to beat a sharp opponent and be so smug about it. (Walking over to the whiteboard.) I suspected that γ is *additive* (also called *linear*) on the digraph $G = (\mathbf{V}, \mathbf{E})$, induced by the given groundgraph $G = (V, E)$, where $\mathbf{V}$ is the collection of all subsets of vertices from $V = (z_1, \dots, z_n)$. That is, $\gamma(\mathbf{u}) \oplus \gamma(\mathbf{v}) = \gamma(\mathbf{u} \oplus \mathbf{v})$ whenever either $\gamma(\mathbf{u}) < \infty$ or $\gamma(\mathbf{v}) < \infty$. The $\oplus$ denotes Nim sum, and every $\mathbf{w} \in \mathbf{V}$ is an n-dimensional binary vector with 1s precisely in locations i where z_i is an occupied vertex in G. I proved linearity with the aid of my PalmCrash. This enabled me to compute γ very easily.

G: Congratulations. But how could you possibly prove linearity with the aid of a computer?

J: I took lots of examples, and it always confirmed linearity. There was no counterexample at all.

G: Hmm...Is this a standard method of proof in statistical mechanics?

J: Well, I don't need the formal proofs of those highbrow mathematicians. I perceive truth when I meet it.

G: It appears that you have been a little hard on mathematicians, especially yesterday. Many phenomena are counterintuitive. I concur with the mathematicians that proofs of claims are necessary, though the precise notion of "proof" might be debatable. Of course one might formulate a *conjecture*, and base further results on it.

To come back to our Cellata game, $\gamma(u)$, when finite, is the smallest nonnegative integer not appearing among the *options* (direct followers) of vertex $u \ldots$instead of using n-dimensional vectors to denote vertices of $\mathbf{V}$, it will now be more convenient to denote them by $n_1 \ldots n_k$, where $z_{n_1}, \ldots, z_{n_k}$ are the occupied vertices of V. Thus you presumably noticed that on G_1, $\gamma(4) = 0$, since it has the as yet unlabeled option 23, that has the option Φ, the configuration with no $\star$s, for which obviously $\gamma(\Phi) = 0$. Similarly, $\gamma(02) = \gamma(13) = 0$. Using linearity, we then get

$$\mathbf{V}_0 = \{\Phi, 4, 02, 13, 024, 134, 0123, 01234\},$$

where $\mathbf{V}_i$ is the subset of $\mathbf{V}$ on which γ assumes the value i ($i < \infty$). In fact, γ is a homomorphism from $\mathbf{V}^f$ (the linear subspace of the vector space $\mathbf{V}$ on which γ is finite) onto $\mathrm{GF}(2)^t$ for some nonnegative integer t with kernel $\mathbf{V}_0$ and quotient space $\mathbf{V}^f/V_0 = \{\mathbf{V}_i : 0 \le i < 2^t\}$, and $\dim(\mathbf{V}^f) = t + \dim(\mathbf{V}_0)$. We have $\mathbf{V}^\infty = \mathbf{V} \setminus \mathbf{V}^f$, where $\mathbf{V}^\infty$ is the subset on which $\gamma = \infty$. For G_1, $\gamma(23) = 1$, since its only options are $\{\Phi, 02\} \subseteq \mathbf{V}_0$. Also $\gamma(56) = 2$. We thus get the cosets

$$\mathbf{V}_1 = 23 \oplus \mathbf{V}_0 = \{23, 234, 03, 12, 034, 124, 01, 014\},$$

$\mathbf{V}_2 = 56 \oplus \mathbf{V}_0$, $\mathbf{V}_3 = 0356 \oplus \mathbf{V}_0$, $\dim \mathbf{V}_0 = 3$, $\dim \mathbf{V}^f = 5$, $t = 2$.

For G_2 we get

$$\mathbf{V}_0 = \{\Phi, 1, 02, 34, 012, 134, 0234, 01234\},$$

$\mathbf{V}_1 = 23 \oplus \mathbf{V}_0$, $\mathbf{V}_2 = 56 \oplus \mathbf{V}_0$, $\mathbf{V}_3 = 0356 \oplus \mathbf{V}_0$, $\dim \mathbf{V}_0 = 3$, $\dim \mathbf{V}^f = 5$, $t = 2$.

It follows that the γ-value on G_1 is $\gamma(56) \oplus (124) = 2 \oplus 1 = 3$, and also on G_2 we have a γ-value of 3. Their Nim sum is thus 0, which means that whoever moves from this position loses. Is this how you figured things out?

J: Precisely. For G_3 and G_4 that I adjoined to the game, we have $\mathbf{V}_0$, $\mathbf{V}_1$ as for G_1 and G_2 respectively, but $\dim \mathbf{V}_0 = 3$, $\dim \mathbf{V}^f = 4$, $t = 1$. Therefore on G_3, $\gamma(01234) = 0$ and on G_4, $\gamma(013) = \gamma\big((23) \oplus (012)\big) = 1$. Thus firing vertex 0 on G_4, results in 34, with $\gamma(34) = 0$. This is a winning move, since γ now vanishes on the entire digraph.

G: Yes. By adjoining a $\star$ at vertex 0 in G_1, we get $\gamma(012456) = \infty$, so the sum of the four components has also γ-value infinity, and the outcome is now a draw, as you said. I better leave now, as I got to teach my Graduate Mesoscopic Physics course.

J: Enjoy — bye.

The Truncated Chat in the Faculty Room

Joyce and Gill met again next day in the Faculty room where doughnuts, cookies, coffee and tea were served in anticipation of an important gathering.

Joyce (moving to the whiteboard): I thought it would be interesting to change the rules, a particular case of which would be to fire the selected vertex u and complement precisely any *two* of its options in the groundgraph if $d_{\rm out}(u) \ge 2$; and complement all the options of u if $d_{\rm out}(u) \le 2$. (I'm now using the terminology "firing" in a new sense: complementing the selected vertex and some subset of its options.) I conjecture that additivity holds also for this game. The digraph $G(s)^2$ I would like to play this game on depends on a parameter $s \in \mathbb{Z}^+$. It has vertex set $\{x_1,\ldots,x_s,y_1,\ldots,y_s\}$, and edges:

$$\begin{aligned} F(x_i) &= y_i \qquad \text{for } i = 1,\ldots,s,\\ F(y_k) &= \{y_i\colon 1 \le i < k\} \cup \{x_j\colon 1 \le j \le s \text{ and } j \ne k\} \qquad \text{for } k = 1,\ldots,s. \end{aligned}$$

As an example, I'm drawing $G(4) = G(4)^2$ on the board (Figure 4). Suppose we play on $G(7)$, and place 1s precisely on the 8 vertices $x_7, y_1, \ldots, y_7$. Can you figure out the nature of this position?

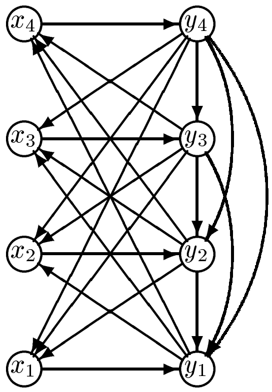

Figure 4: Playing on a parametrized digraph.

Gill: (fingering the knobs of the WallComp next to the whiteboard): Before doing that, why didn't you consider the G^1 version, i.e., firing an occupied vertex (in your new sense of "firing"), and complementing precisely *one* of its options?

J: Well, this would be a pure particle physics game without much appeal to statistical mechanics, and it was *you* who had suggested to consider a unified approach. Besides, this special case was solved in [Fra74], [FY76], [FY82], where a polynomial strategy was formulated. The misère version was analyzed in [Fer84].

G: I expected you to say this, but it gave me time to think about the question you asked me…I concur with your conjecture about additivity. It seems that though the groundgraph $G(s)$ has no leaf, the game-graph $G(s)$ has no γ-value ∞. It also appears that any collection of x_i is in $\mathbf{V}_0$. The value of the y_i seem to be more tricky...I think that $\gamma(y_i)$ =the ith *odious* number, where the odious numbers are those positive integers whose binary representations have an odd number of 1-bits. Incidentally, odious numbers arise in the analysis of other games, such as Grundy's game, Kayles, Mock Turtles, Turnips. See [BCG82]. They arose earlier in a certain two-way splitting of the nonnegative integers [LM59] (but without this odious terminology...). More information about this and over 54,000 other integer sequences is available on-line from

`http://www.research.att.com/~njas/sequences/`

thanks to the mathematician Neil Sloane, who probably contributed more to a larger number of mathematicians than any other mathematician!

For the position concocted by you, $\gamma(x_7 y_1 \dots y_7) = 0 \oplus 1 \oplus 2 \oplus 4 \oplus 7 \oplus 8 \oplus 11 \oplus 13 = 14$. Thus the player to move can win: either by firing y_7 and complementing y_1, y_2, or by firing y_6 and complementing y_1, y_3, or by firing y_5 and complementing y_2 and y_3.

J: Very nice...suppose we take an identical clone of $G(s)^2$, and begin with precisely the same initial configuration, but change the rule for the clone: complement the selected vertex u together with any *three* of its options if $d_{\text{out}}(u) \geq 3$; and all the options of u if $d_{\text{out}}(u) \leq 3$. We better call this new clone $G(s)^3$, to distinguish it from $G(s)^2$. Can the first player win also here?

G (moving to within reach of both the whiteboard and the WallComp): Let's see...on the clone, all collections of an even number of x_i are in $\mathbf{V}_0$; and $\mathbf{V}^f$ consists precisely of all collections of an *even* number of 1*s*...we seem to have $\gamma(x_j y_j)$ = smallest nonnegative integer not the Nim sum of at most three $\gamma(x_i y_i)$ for $i < j$. Thus $\gamma(x_7 y_1 \dots y_7) = 0 \oplus 1 \oplus 2 \oplus 4 \oplus 8 \oplus 15 \oplus 16 \oplus 32 = 48$. So firing y_7 and complementing y_6 and any two of the x_i ($i < 7$) is a winning move. ...Incidentally, the sequence $\{1, 2, 4, 8, 15, 16, 32, 51, \dots\}$ appears also in Neil's Encyclopædia, and has been used in [BCG82] for a special case of the game "Turning Turtles".

J: How about playing the sum of $G(s)^2$ and $G(s)^3$ with the same given initial position on both clones?

G: That's easy. The value of the sum is simply the Nim sum of their γ-values which is $14 \oplus 48$. To win we have to move in $G(s)^3$ to a position with γ-value 14. There is a unique winning move of changing the γ-value 32 to 30. This is affected by firing y_7 and complementing y_6, y_5 and y_1...I hear in the corridor the President talking with the Cabinet Minister of Science approaching...we better adjourn before we'll have to explain to the minister that we are playing a game.

J (moving to the WallComp): Not before we briefly summarize where we stand...We still should address the question of the computational complexity of Cellata games...and yes, I concede that it would be nice to prove additivity formally for the family of all Cellata games...In these games, is every draw position necessarily such that *every* move from it leads to another draw? This is the case for all the games we considered, but it would be nice to provide a case where this doesn't hold...We played *impartial* games. How about playing a sort of *partizan* game on, say, $G(s)^3$ and $G(s)^2$ simultaneously, i.e., one player follows the $G(s)^3$ rules and her opponent the $G(s)^2$ rules?...I think I can see some interesting applications in fields other than physics. Incidentally, the case of $G(s)^4$, where a vertex on G_s is fired and any *four* of its options are complemented, seems to give rise to the sequence $1, 2, 4, 8, 16, 31, 32, 64, 103, \ldots$. It is the sequence $\gamma(x_j y_j)$ defined as the smallest nonnegative integer not the Nim sum of at most four earlier terms. This sequence was not in the Encyclopædia of integers, so I just sent a message, via the WallComp, to your latest mathematics hero Neil Sloane, together with the fact that it appears in Table 3, Chapter 14 of [BCG82]. Note that the strategy of our Cellata games on just Figure 4 alone subsumes and unifies that of a battery of games there...I just noted that Sloane has added the new sequence into his Encyclopædia.

If it wouldn't be for our own University President who seeks to elicit more money from this narrow-minded minister, I'd proudly tell the latter that we are playing a game, followed by a quote from the founder of our *Sciences Club*:

"*...A third purpose of this book is to have fun. Indeed, pleasure has probably been the main goal all along. But I hesitate to admit it, because computer scientists want to maintain their image as hard-working individuals who deserve high salaries. Sooner or later society will realise that certain kinds of hard work are in fact admirable even though they are more fun than just about anything else.*" ([Knu93b, p. iii], see also [Knu77].)

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