Curriculum Vitae

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1. Studies

2008- : Post-doc at the Weizmann Institute of Science, Israel.

2007-2008 : Attaché Temporaire d'Enseignement et de Recherche (temporary teaching and research position) at the University Lyon 1.

2003-2007 : Ph.D. in Mathematics, under the supervision of Prof. Olivier Mathieu, within the algebra team, at the University Lyon 1.
Thesis : *Une étude combinatoire de la géométrie des fibres de Springer de type A* (A combinatorial study of the geometry of Springer fibers in type A), defended on December 12th, 2007.

2002-2003 : 1) DEA in Mathematics (Master degree) at the University Lyon 1, Dissertation : *Représentations du groupe spécial linéaire en caractéristique naturelle* (Representations of the special linear group in positive characteristic). Supervisor : Prof. Olivier Mathieu.
2) Agrégation of Mathematics (national competition).

2001-2002 : Maîtrise de Mathématiques at the École Normale Supérieure de Lyon. Report : *Classification des groupes de réflexions* (Classification of reflection groups). Supervisor : Prof. David Bessis, at the University Lyon 1.

2000-2001 : Licence de Mathématiques at the École Normale Supérieure de Lyon. Report : *Représentations linéaires des groupes finis* (Linear representations of finite groups). Supervisor : Prof. Cédric Bonnafé, at the University of Besançon, France.

2000-2004 : Scholarship at the École Normale Supérieure de Lyon.

1996-1997 : Baccalauréat.

2. Research

Research subject : Geometry of Springer fibers.

Domains : Algebraic geometry, combinatorics and representation theory.

Keywords : Flag manifolds, Springer fibers, nilpotent orbits, orbital varieties, Young diagrams and tableaux, Springer correspondence, Kazhdan-Lusztig theory.

Works

1. *Une étude combinatoire de la géométrie des fibres de Springer de type A*, Ph.D. thesis, University Lyon 1, 2007.
2. *Nombres de Betti des fibres de Springer de type A*, Comptes-Rendus de l'Académie des Sciences, Paris, Série I, volume 347 (2009) pages 283–287.
3. *Composantes singulières des fibres de Springer dans le cas deux-colonnes*, Comptes-Rendus de l'Académie des Sciences, Paris, Série I, volume 347 (2009) pages 631–636.
4. *Betti numbers of Springer fibers in type A*, Journal of Algebra, volume 322 (2009) pages 2566–2579.
5. *Singular components of Springer fibers in the two-column case*, Annales de l'Institut Fourier (Grenoble), volume 59 (2009) pages 2429–2444.
6. *On the singularity of the irreducible components of Springer fibers in* $sl(n)$ (with A. Melnikov), preprint `arXiv:0905.1617` (2009). Accepted for publication in Selecta Mathematica.
7. *A unified approach on Springer fibers in the hook, two-row and two-column cases*, preprint `arXiv:0803.2183` (2009), 42 pages. Accepted for publication in Transformation Groups.
8. *Some characterizations of the singular components of Springer fibers in the two-column case* (with A. Melnikov), preprint `arXiv:0909.4008` (2009). Accepted for publication in Algebras and Representation Theory.
9. *On the singularity of some special families of components of Springer fibers*, preprint (2010), 33 pages, submitted.

3. Talks in seminars

- Algebra seminar of the University Lyon 1 (June 19, 2007).
- Seminar on enveloping algebras of the University Paris 7 (February 15, 2008).
- Algebraic geometry seminar of the University Grenoble 1 (March 31, 2008, June 15, 2009).
- Workshop on Lie theory and symplectic geometry, University Lyon 1 (July 1, 2008).

– Representation theory and algebraic geometry seminar, Weizmann Institute (September 24, 2008, November 5, 2008, February 5, 2009, December 4, 2009).
– Algebra seminar of the University of Haifa (November 13, 2008, November 19, 2009).
– Algebra seminar of the Hebrew University of Jerusalem (November 20, 2008).
– Seminar on algebraic combinatorics, University of Bar-Ilan (November 25, 2008).
– Seminar on Lie groups and module spaces of the University of Genève (February 25, 2009).
– Algebra seminar of the Israel Institute of Technology (Technion) (March 16, 2009).
– Seminar on mathematical physics of the University of Dijon (April 2, 2009).
– Algebra and topology seminar of the University ETH Zürich (April 8, 2009).
– Algebra seminar of the University of Besançon (April 16, 2009).
– Seminar on Lie groups and harmonic amalysis of the University Nancy 1 (June 4, 2009).
– Workshop on representation theory, University of Caen (June 9, 2009).
– Algebra seminar of the University Ben Gurion of Beer Sheva (November 4, 2009).

4. Conferences attended

– *Workshop on Problems and Progress in Lie Algebraic Theory* at the Weizmann Institute of Science, Israel (July 7–8, 2010), talk.
– *Algebraic Groups and Invariant Theory* at Ascona, Switzerland (August 30 – September 4, 2009), short talk.
– *Structures in Lie Representation Theory* at the University of Brême, Germany (August 9–22, 2009), plenary talk.
– *The Israel Mathematical Union annual meeting* at the Weizmann Institute of Science, Israel (April 30 – May 1, 2009), talk in the topology-geometry session.
– *Algebraic Lie Structures with Origins in Physics* at the Newton Institute, England (March 2009), poster presented.
– *Workshop on Enveloping Algebras and Related Topics* at the Weizmann Institute of Science, Israel (January – February 2008).
– *Groupes and geometry* at Luminy, France (December 2006).
– *Semaine dérivée* (week conference on derived categories) at the University Paris 7 (January 2005).

5. Activities

Referee for the following journals :

- Annales de l'Institut Fourier,
- Bulletin de la Société Mathématique de France,
- Journal of Algebra,
- Proceedings of the American Mathematical Society.

7. Teaching

Tutorials in mathematics for undergraduate students, at the University Lyon 1. I taught the following topics :

- "Basic mathematical techniques" (real analysis, limits, continuity, derivation, integration, trigonometry, first order differential equations).
- "Math 2" (multivariable calculus, differentials, multiple integration).
- "Math 3" (mathematical analysis, series, Fourier series, power series, partial differential equations).
- "Math 5" (differential calculus, curves, surfaces, volumes).
- "Math II algebra" (basic linear algebra, polynoms).
- "Math IV algebra" (linear algebra, reduction of endomorphisms).
- "Math IV analysis" (topology, metric spaces, differential calculus).

For each topic, within the teaching team, I took part to the elaboration of exercice slips, subjects for exams, marking.

Research statement

Research subject : Geometry of Springer fibers.

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Keywords : Flag manifolds, Springer fibers, nilpotent orbits, orbital varieties, Young diagrams and tableaux, Springer correspondence, Kazhdan-Lusztig theory.

Introduction

Let G be a reductive algebraic group over $\mathbb{C}$ et let $\mathfrak{g}$ denote its Lie algebra. The set $\mathcal{B}$ of the Borel subgroups $B \subset G$ is an algebraic projective variety, called the *flag variety*. For a nilpotent element $x \in \mathfrak{g}$, the set

$$\mathcal{B}_x = \{B \in \mathcal{B} : x \in Lie(B)\}$$

is a closed subvariety of $\mathcal{B}$. The variety $\mathcal{B}_x$ is called a *Springer fiber*, since this is the fiber over x for the Springer resolution $\mathcal{X} \to \mathcal{N}$, $(B, x) \mapsto x$, where $\mathcal{N} \subset \mathfrak{g}$ is the subset of nilpotent elements and $\mathcal{X} = \{(B, x) \in \mathcal{B} \times \mathcal{N} : x \in Lie(B)\}$.

The variety $\mathcal{B}_x$ is an algebraic projective variety, equidimensional, but in general it is not irreducible.

The study of Springer fibers involves different domains such as algebraic geometry, representation theory and combinatorics. It takes its origin in the works of T.A. Springer who obtained a geometric realization of the irreducible representations of Weyl groups in the cohomology of Springer fibers (cf. [Spr]). D. Kazhdan et G. Lusztig gave a topological construction of the Springer representations (cf. [KL2]) and they conjectured a link between the configuration of the irreducible components in Springer fibers and the construction of bases for representations of the Hecke algebras (cf. [KL1, §6.3]). The geometry of Springer fibers and their irreducible components has become an important topic of study for thirty years.

However, up to now, few questions have been solved, and the Springer fibers remain misterious objectifs. Even for the type A, there have been few advances, and mainly for few particular cases.

Results achieved

I have undertaken the study of the geometry of Springer fibers in my Ph.D. thesis (reference [1]). In my Ph.D. thesis, as well as in subsequent works, I have obtained a set of new results, which I am going to describe. All along this presentation (and from now on) we consider Springer fibers for $G = GL(\mathbb{C}^n)$, so that $x : \mathbb{C}^n \to \mathbb{C}^n$ is a nilpotent endomorphism. We denote by $\lambda(x) = (\lambda_1 \geq \ldots \geq \lambda_r)$ the Jordan block sizes of x.

Results obtained in my Ph.D. thesis "Une étude combinatoire de la géométrie des fibres de Springer de type A" (A combinatorial study of Springer fibers in type A)

First, I obtained a calculation of the Poincaré polynomial of the variety $\mathcal{B}_x$. This calculation is based on the construction of a cell decomposition of $\mathcal{B}_x$. If $x = 0$, then $\mathcal{B}_x$ is the flag variety, which admits a decomposition into Schubert cells $S(\sigma)$ parameterized by the elements of the symmetric group $\sigma \in \mathbf{S}_n$, and the codimension of $S(\sigma)$ is the inversion number of σ. For x general, I constructed a cell decomposition of $\mathcal{B}_x$ parameterized by a set of row-standard tableaux, and whose codimension of the cells can be interpreted as a number of inversions (cf. [1, §II]).

I studied the components of Springer fibers by considering the natural action of the centralizer of x and certain special orbits parameterized again by row-standard tableaux. In the case of a nilpotent endomorphism of hook, two-row or two-column type, I obtained necessary and sufficient criteria for such an orbit be contained in a given component, these criteria involve dominance relations between Young diagrams and combinatorial algorithms (cf. [1, §III]). I derived some topological properties of the intersections of components in the same three cases, in particular I determined the dimension of a general intersection of components (cf. [1, §IV-V]).

Finally, I established a sufficient condition of singularity for components of Springer fibers in the case where x is of nilpotent order 2 (cf. [1, §V]).

Newer results

After my Ph.D., I obtained several new results on questions related to the singularity of Springer fiber components. First, I established a necessary and sufficient criterion of singularity for the components in the case where x is of nilpotent order 2 (cf. [5]). In collaboration with A. Melnikov, I derived from this first criterion some other characterizations of the singular components in the case $x^2 = 0$, which show in particular analogy between components and Schubert varieties, and a relation with Kazhdan-Lusztig theory (cf. [8]). For x of general Jordan form, I studied the singularity for special families of components and especially I began to study the relation between the singularity of components and the question of existence of dense orbit for the action of the stabilizer of x. (cf. [9]).

In collaboration with A. Melnikov, I also obtained the complete description of the Jordan forms $\lambda(x)$ such that all the components of $\mathcal{B}_x$ are nonsingular (cf. [6]) :

Theorem 1 *All the irreducible components of $\mathcal{B}_x$ are nonsingular exactly in the following cases :*

(i) $\lambda(x) = (\lambda_1, 1, 1, \ldots)$ *(hook case);*
(ii) $\lambda(x) = (\lambda_1, \lambda_2)$ *(two-row case);*
(iii) $\lambda(x) = (\lambda_1, \lambda_2, 1)$ *("two-row-plus-one-box" case);*
(iv) $\lambda(x) = (2, 2, 2)$.

In what follows, I describe the above results in more details.

0. Preliminaries

Let $V = \mathbb{C}^n$, and let $x \in \mathrm{End}(V)$ be a nilpotent element. The Springer fiber $\mathcal{B}_x$ interprets as the set of the complete flags $(0 = V_0 \subset V_1 \subset \ldots \subset V_n = V)$ which are stable by x, i.e. $x(V_i) \subset V_i$ for all $i = 1, \ldots, n$. We let $\lambda(x) = (\lambda_1 \geq \ldots \geq \lambda_r)$ denote the sizes of the Jordan blocks of x, and we let $Y(x)$ be the Young diagram of rows of lengths $\lambda_1, \ldots, \lambda_r$.

The variety $\mathcal{B}_x$ is almost all the time reducible (unless x is zero or regular) and its irreducible components are parameterized by the standard Young tableaux of shape $Y(x)$. Indeed, following N. Spaltenstein [Spa], there is a natural partition $\mathcal{B}_x = \bigsqcup_T \mathcal{B}_x^T$ parameterized by the standard tableaux of shape $Y(x)$. The subsets $\mathcal{B}_x^T$ in the partition are locally closed, irreducible and all $\dim \mathcal{B}_x^T = \dim \mathcal{B}_x$ (cf. [Spa, §II.5]). It follows that $\mathcal{B}_x = \bigcup_T \overline{\mathcal{B}_x^T}$ is the decomposition of $\mathcal{B}_x$ into irreducible components, and thus all the components $\mathcal{K}^T := \overline{\mathcal{B}_x^T}$ have the same dimension.

1. Calculation of the Betti numbers and the Poincaré polynomial

The Springer representations consist of a linear action of the symmetric group $\mathbf{S}_n$ on the cohomology spaces $H^m(\mathcal{B}_x, \mathbb{Q})$. Moreover, the representation attached to the space of maximal degree $H^{\max}(\mathcal{B}_x, \mathbb{Q})$ is irreducible and isomorphic to the Specht module corresponding to the Young diagram $Y(x)$. In particular, we obtain that $\dim H^{\max}(\mathcal{B}_x, \mathbb{Q})$ is equal to the number of standard tableaux of shape $Y(x)$, which in fact is also implied by the construction of Spaltenstein presented above.

I have shown that, more generally, each Betti number $b_m := \dim\, H^m(\mathcal{B}_x, \mathbb{Q})$ can be described as the cardinal of a set of tableaux. For this calculation of the Betti numbers, I have constructed a cell decomposition of $\mathcal{B}_x$. The cells are parameterized by the set of row-standard tableaux of shape $Y(x)$, and the codimension of the cells is interpreted as an inversion number.

Définition 1 We call *row-standard tableau* a tableau of shape $Y(x)$, numbered from 1 to n, such that the entries in the rows are increasing from left to right.

Note that, if τ is a row-standard tableau, and if we put the numbers in each column in increasing order from top to bottom, then we construct a standard tableau $\mathrm{st}(\tau)$. Thus, a row-standard tableau can be interpreted as a permutation of a standard tableau.

$$\text{Example :} \qquad \tau = \begin{array}{|c|c|c|} \hline 2 & 3 & 8 \\ \hline 4 & 6 & 7 \\ \hline 1 & 5 \\ \cline{1-2} \end{array} \qquad \mathrm{st}(\tau) = \begin{array}{|c|c|c|} \hline 1 & 3 & 7 \\ \hline 2 & 5 & 8 \\ \hline 4 & 6 \\ \cline{1-2} \end{array}$$

Définition 2 Let τ be a row-standard tableau. We call *inversion* a pair of entries (i, j), $i < j$ in a same column of τ and such that one of the following conditions is satisfied :

– i has no neighbor entry on its right and i is situated under j,

– i, j have respective right-neighbor entries i', j', and $i' > j'$.

We denote by $n_{\mathrm{inv}}(\tau)$ the number of inversions of τ.

For example, the previous tableau τ has four inversions : the pairs $(1, 2)$, $(3, 6)$, $(5, 6)$, $(7, 8)$. The inversion number measures the difference between τ and its "standardization" $\mathrm{st}(\tau)$.

Let $d = \dim \mathcal{B}_x$. We have the following result (cf. [1, §II], [2], [4]).

Theorem 2 *The variety $\mathcal{B}_x$ admits a cell decomposition $\mathcal{B}_x = \bigsqcup_\tau C(\tau)$ parameterized by the row-standard tableaux of shape $Y(x)$, with the following properties*

(a) $C(\tau) \subset \mathcal{B}_x^T$ for $T = \mathrm{st}(\tau)$ the "standardization" of τ.

(b) $\dim C(\tau) = d - n_{\mathrm{inv}}(\tau)$.

From the theorem, it follows that $b_m = 0$ for m odd, and the Betti number b_{2m} is equal to the number of row-standard tableaux τ of inversion number $n_{\mathrm{inv}}(\tau) = d - m$.

By studying the combinatorics of the number of inversions, I have derived an explicit formula for the Poincaré polynomial $P_x(t) := \sum_{m=0}^{d} b_m t^m$. Moreover, I have established an inductive formula for the polynomial $P_x(t)$ (cf. [4]).

Perspectives

Theorem 2 is somehow related to the theory of symmetric functions, an important domain in algebraic combinatorics, (cf. [McD]). Specifically, A. Garsia and C. Procesi have described a link between the geometry of Springer fibers and the notion of Kostka polynomials (cf. [GP]). This link had be established before by G. Lusztig (cf. [L]). It would be interesting to understand a possible connected between the combinatorics of number of inversions of row-standard tableaux and the Kostka polynomials.

Moreover it would be interesting to study if the row-standard tableaux could form a basis for a combinatorial description of the Springer representations of the symmetric group, generalizing the Specht modules.

2. Computation of irreducible components and their intersections

Following N. Spaltenstein, the irreducible components of $\mathcal{B}_x$ are parameterized by the standard tableaux of shape $Y(x)$, and more precisely each component $\mathcal{K}^T$ is obtained as the closure of a subset $\mathcal{B}_x^T \subset \mathcal{B}_x$. However, though the subsets $\mathcal{B}_x^T$ are well understood, determining their closures is a difficult problem. The difficulty is that the study strongly depends on the Jordan form of x. Until now, the components have been explicitely described (by a system of equations) only in three particular cases : the cases where the diagram $Y(x)$ is of hook type (i.e. only one row of length ≥ 2), has two rows (cf. [Fu]) or has two columns (cf. [MP]). In the general case, it is even not possible to say whether two given components do have a nonempty intersection. Each case is very specific.

I have studied the components under the point of view of a family of special elements : the elements of $\mathcal{B}_x$ fixed under the action of the standard torus relative to a fixed Jordan basis of x. These flags are parameterized by the row-standard tableaux of shape $Y(x)$, and we denote by F_τ the flag corresponding to τ.

These elements are not generic in the components of $\mathcal{B}_x$ in general. In [9], I have described the components containing a generic element of the form F_τ.

In addition, we can show that two components $\mathcal{K}^T, \mathcal{K}^S \subset \mathcal{B}_x$ have a nonempty intersection if and only if they contain a common element of the form F_τ.

We write $\tau \in T$ if $F_\tau \in \mathcal{K}^T$. In [1, §III], [7], I have studied the relation $\tau \in T$, combinatorially. I have obtained necessary or sufficient conditions in the general case, and a complete, common description in the hook, two-row and two-column cases.

For these three cases, I propose two criteria. The first criterion relies on the lower semicontinuity of the rank. It involves dominance relations between two sequences of Young diagrams attached respectively to the tableaux τ, T.

The second criterion relies on a combinatorial algorithm. This algorithm aims to reconstruct the tableau τ by successive insersions of its entries $1, 2, \ldots$, according to certain rules imposed by T. A failure can occur, and the success of the algorithm means exactly that $\tau \in T$.

The fact to have criteria which are common to the three cases can be underlined. The proof of the criteria for each case is very different.

Intersections of components

Let us consider the graph whose vertices are the irreducible components $\mathcal{K} \subset \mathcal{B}_x$, and with an edge between $\mathcal{K}, \mathcal{K}'$ if $\operatorname{codim} \mathcal{K} \cap \mathcal{K}' = 1$. The conjecture [KL1, §6.3] says that this graph has the structure of a W-graph. This conjecture has motivated the research on the question of the intersections of components of Springer fibers. Until today, it has been checked only in the hook, two-row and two-column cases.

For these three cases, I have shown a link between the algorithm mentioned above and the combinatorics involved in the description of the pairs of components of $\mathcal{B}_x$ which intersect in codimension 1, in [Fu] and [MP] (in particular the parentheses diagrames). By application, I obtain for instance that, for $\mathcal{K}^T, \mathcal{K}^S \subset \mathcal{B}_x$ any two components, we have

$$\operatorname{codim} \mathcal{K}^T \cap \mathcal{K}^S = 1 \;\Rightarrow\; (S \in T \text{ or } T \in S).$$

It follows several topological properties of the components of $\mathcal{B}_x$ (cf. [7, §7–8]).

Through a computation relying on the algorithm presented above and on the decomposition of the flag variety into Schubert cells, I have determined the dimension of any finite intersection of irreducible components of the Springer fiber $\mathcal{B}_x$, for x of hook, two-row or two-column type (cf. [1, §IV–V]).

Perspectives

The concrete topological study of Springer fiber components in the general case looks difficult, it could give at least certain partial answers (for instance, necessary or sufficient criteria to say whether two given components intersect). On the other hand, point (iii) of Theorem 1 in the introduction shows that the case $\lambda(x) = (\lambda_1, \lambda_2, 1)$ is also specific. It would be therefore natural to study the topology of Springer fiber components for this particular case of the Jordan form, relying on the techniques presented above, and using possibly in addition other methods which would be better suited for this case.

3. On the singularity of components

First, let us briefly sum up the results already achieved about the singularity of the irreducible components of the Springer fiber $\mathcal{B}_x$.

1) In the cases where x is of hook or two-row type, all the irreducible components of $\mathcal{B}_x$ are smooth (cf. [Va] for the hook case, [Fu] for the two-row case).
2) Singular components exist. J.A. Vargas [Va] has proved that, for x with Jordan blocks of sizes $\lambda(x) = (2, 2, 1, 1)$, the variety $\mathcal{B}_x$ admits a singular component (and moreover it is the unique singular component for $n \leq 6$).
3) $\mathcal{B}_x$ always admits smooth components, for instance the so-called Richardson components, defined to be the components which are homogeneous under the action of a parabolic subgroup. J. Pagnon and N. Ressayre [PR] have constructed another special family of smooth components.

I have continued the study in the three directions, obtaining :

a) a classification of the diagrams $Y(x)$ such that all the components of $\mathcal{B}_u$ are smooth,
b) several characterizations of the singular components in the two-column case,
c) a study of the singularity of certain special components, for $Y(x)$ general.

Concerning a), see Theorem 1 stated in the introduction.

Characterization of the singular components in the two-column case

We suppose that the diagram $Y(x)$ has two columns, equivalently x is of nilpotent order two. This case presents two specific properties : indeed, we can show that in the two-column case, the group $Z_x = \{g \in GL(\mathbb{C}^n) : gxg^{-1} = x\}$ acts on $\mathcal{B}_x$ with a finite number of orbits, and that there is a unique closed orbit $\mathcal{Z}_0 \subset \mathcal{B}_x$ for this action.

I have obtained a first necessary and sufficient criterion of singularity for the components of $\mathcal{B}_x$, which relies on these properties. First, fixing an element $F_0 \in \mathcal{Z}_0$, we have that F_0 belongs to any component $\mathcal{K} \subset \mathcal{B}_x$, and moreover $\mathcal{K}$ is singular if and only if F_0 is a singular point of $\mathcal{K}$. We introduce a finite set $X_0 \subset \mathcal{B}_x$ of elements which are connected to F_0 by a projective curve (induced by the action of a one-parameter subgroup of Z_x). Then, for each $F \in X_0 \cap \mathcal{K}$ we obtain that the curve connecting F to F_0 lies in the component $\mathcal{K}$, thus we get an element in the tangent space $\mathcal{T}_{F_0}\mathcal{K}$. In fact, we prove that the vectors so-obtained form a basis of $\mathcal{T}_{F_0}\mathcal{K}$, whence $\dim \mathcal{T}_{F_0}\mathcal{K} = |X_0 \cap \mathcal{K}|$. The following singularity criterion is therefore obtained (cf. [3], [5]).

$$\mathcal{K} \textit{ is singular } \Leftrightarrow |X_0 \cap \mathcal{K}| > \dim \mathcal{K}.$$

Note that this criterion is analogous to criteria of the same kind for the singularity of Schubert varieties.

In collaboration with A. Melnikov, relying on the previous criterion, I have obtained three other criteria, more explicit, for the singularity of the irreducible components of $\mathcal{B}_x$ in the two-column case (cf. [8]). The first criterion describes precisely the form of the standard tableaux T which correspond to the singular components $\mathcal{K}^T \subset \mathcal{B}_x$. The second criterion is based on the construction of a cell decomposition of each component (which allows in addition to compute the Poincaré polynomials of the components) and it is stated as follows :

$$\mathcal{K} \subset \mathcal{B}_x \textit{ is smooth if and only if its Poincaré polynomial is palindromic.}$$

It results in particular that $\mathcal{K} \subset \mathcal{B}_x$ is smooth if and only if it is rationally smooth. Again the property is the same as for Schubert varieties (for simply laced root systems).

The third criterion involves the intersections of components in codimension 1 :

$$\mathcal{K} \textit{ is smooth } \Leftrightarrow \#\{\mathcal{K}' \subset \mathcal{B}_x \text{ component} : \operatorname{codim} \mathcal{K} \cap \mathcal{K}' = 1\} > \min(r, s+1),$$

where r and s, $r \geq s$, denote the lengths of the columns of $Y(x)$. This criterion shows that the singular components correspond to the vertices with a great number of edges in the Kazhdan-Lusztig graph (cf. section 2).

On the singularity of certain special components, for $Y(x)$ general

In [9], I have studied the singularity of components for a Jordan form $Y(x)$ general, but by considering certain special families of components. The first two special families are the following ones.

1) A component $\mathcal{K} \subset \mathcal{B}_x$ is said to be a Richardson component if it is homogeneous for the action of a parabolic subgroup $P \subset GL(\mathbb{C}^n)$. Equivalently, $\mathcal{K}$ is isomorphic to a product of flag varieties.

2) A component $\mathcal{K} \subset \mathcal{B}_x$ is said to be a Bala-Carter component if is contains a dense orbit of a special type for the action of Z_x, the stabilizer of x : the orbit of a flag adapted to a Jordan basis, i.e. of the form $F = (\langle e_1, \ldots, e_i \rangle)_{i=0}^n$ where $x(e_i) \in \{0, e_{i-1}\}$.

These two families are in duality. For a standard tableau T, we denote by T^* the transposed tableau (i.e. th i-th row of T corresponds to the i-th column of T^*). We have :

$$\mathcal{K}^T \text{ is a Bala-Carter component} \Leftrightarrow \mathcal{K}^{T^*} \text{ is a Richardson component.}$$

Whereas the Richardson components are always smooth, the Bala-Carter components are often singular. I have given a characterization of the singular Bala-Carter components. From this criterion, we get in addition that the Bala-Carter components are, among the components of Springer fiber, those the most susceptible to be singular, in the sense that if $\mathcal{B}_x$ admits a singular component, then it admits one of Bala-Carter type.

The third family of components we consider enlarges the family of Bala-Carter components :

3) We consider the family of components which contain a dense Z_x-orbit of the form $Z_x F$, where F is the flag adapted to a permutation of a Jordan basis, i.e. of the form $F = (\langle e_{\sigma_1}, \ldots, e_{\sigma_i} \rangle)_{i=0}^n$, $\sigma \in \mathbf{S}_n$.

Equivalently, the components in this family are those which contain a dense orbit of the form Z_xF_τ, where F_τ is the flag like in section 2. I have given a parameterization of the components of this type by a family of graphs which generalize the parentheses diagrams.

The fourth family of components we consider enlarges the family of Richardson components :

4) The family of components which are iterated fiber bundles of projective spaces.

In particular, all the components in this family are smooth. I have shown the following relation between the families 3) and 4) :

Theorem 3 *If $\mathcal{K}^T$ contains a dense Z_x-orbit of the form Z_xF, where F is the flag adapted to a permutation of a Jordan basis, then $\mathcal{K}^{T^*}$ is an iterated fiber bundle of projective spaces.*

In the two-column case, every component is of type 3) and in the two-row case, every component is of type 4). We retrieve in particular the description due to F. Fung [Fu] of the structure of iterated bundle of the components in the two-row case.

Perspectives

The study of questions related to the singularity of Springer fiber components can be continued in the following directions. First, it is natural to make deeper the study of the singularity in the two-column case (type, singular locus, resolution of singularities). We could also try to adapt ideas from the two-column case in order to obtain necessary or sufficient singularity criteria in other cases.

It would be interesting to study whether a version of Theorem 1 can be found for Springer fibers of other types than type A. Finally, it seems profitable to continue the study of the link mentioned above between the singularity of components and the intersections in codimension one. As for the link between the singularity of components and the existence of dense orbits for the action of the stabilizer of x.

References to my works

1. *Une étude combinatoire de la géométrie des fibres de Springer de type A*, Ph.D. thesis, Université Lyon 1, 2007.

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