Multi Dimensional Billiard-like Potentials

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Abstract

Imagine a point particle travelling freely (without friction) on a table, undergoing elastic collisions with the edges of the table. The table is just a bounded region of the plane. This model resembles a game of billiards (or pool) but it looks much simpler - we have only one ball, which is a dimensionless point particle. There is no friction and the table has no pockets. However, the shape of the table can be rather arbitrary. This shape determines the ball dynamics.

The billiard model has numerous applications in physics. First, it works as idealized model for the motion of charged particle in a potential, a model which enables the examination of the relation between classical and quantized systems (see [15],[33] and references therein). Second, this model has been suggested in [30] as a first step for substantiating the basic assumption of statistical mechanics – the ergodic hypothesis of Boltzmann (see especially the discussion and references in [32]). Finally, there exists a direct mechanical realization of this model: The motion of N rigid d-dimensional balls in a d-dimensional box (d=2,3) may be reduced to a billiard problem in n dimensions, where $n = 2N \times d$ This is the main motivation to study higher dimensional billiards.

1 Setup

1.1 The Configuration Space

We consider a billiard flow as the motion of a point mass in a compact domain $D \in \mathbb{R}^d$ or $\mathbb{T}^d$. Assume that the boundary $S = \partial D$ consists of a finite number of submanifolds (C^k, $k \geq 3$ smooth $(d-1)$-dimensional surfaces).

$$S := \partial D = \Gamma_1 \cup \Gamma_2 \cup ... \cup \Gamma_n, \qquad i = 1 \dots n \tag{1.1}$$

The boundaries of these submanifolds, when exist, form *the singular set of* ∂D:

$$\Gamma^* = \partial\Gamma_1 \cup \partial\Gamma_2 \cup ... \cup \partial\Gamma_n, \qquad i = 1 \dots n \tag{1.2}$$

The moving particle has a position $q = (x, y, z^1, ..., z^{d-2}) \in \bar{D}$ and a

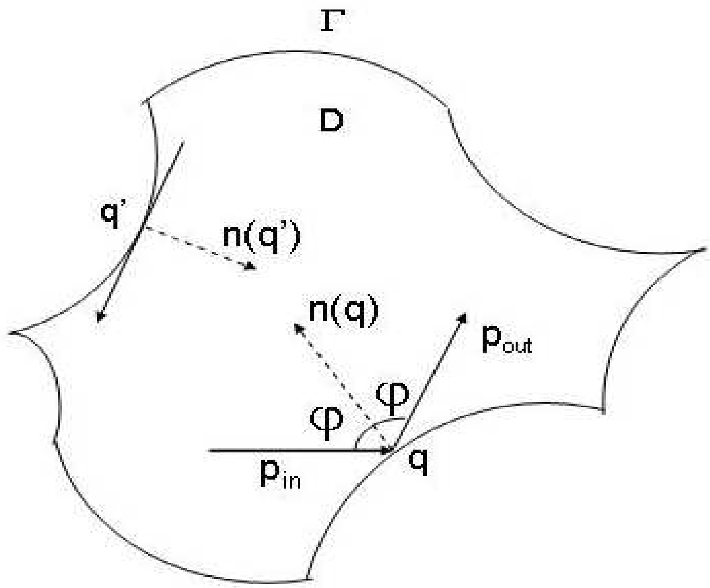

Figure 1: Billiard table, regular reflection and tangent trajectory.

velocity vector $v = (p_x, p_y, p_{z^1}, ..., p_{z^{d-2}}) \in \mathbb{R}^d$ which are functions of time[1]. If $q \in D$, then the particle moves freely with the constant velocity according to the rule:

$$\begin{cases} \dot{q} = v \\ \dot{v} = 0 \end{cases} \tag{1.3}$$

We assume that the particle has mass one, so its momentum is $p = v$ and the equation (1.3) is Hamiltonian with Hamiltonian function

$$H(q,p) = \frac{\|p\|^2}{2} \tag{1.4}$$

So, the point moves at unit speed and bounces off ∂D according to the usual laws of reflection: *the angle of incidence is equal to the angle of reflection.* Formally, the outgoing vector p_{out} is related to the incoming vector p_{in} by

$$p_{out} = p_{in} - 2\langle p_{in}, n(q)\rangle n(q) \tag{1.5}$$

[1]For two dimensional case the position is $q = (x, y)$ and the velocity is $v = (p_x, p_y)$.

where $n(q)$ is the inward unit normal vector to the boundary ∂D at the point q. To use the reflection rule (1.5), we need the normal vector $n(q)$ to be defined, hence the rule cannot be applied at points $q \in \Gamma^*$, where such a vector fails to exist[2].

Generally, the incidence angle φ belongs to $[-\frac{\pi}{2}, \frac{\pi}{2}]$, where $\varphi = \pm\frac{\pi}{2}$ correspond to trajectories which are tangent to S (e.g. the point q' in Figure 1; here $\partial D = \Gamma$).

Definition 1.1. The domain D is called *the configuration space* of the billiard system.

1.2 The phase space

Now we construct the phase space of the system. The dynamics (1.3)-(1.5) preserves the Euclidian norm $\|p\|$, so we can set $\|p\| = 1$ (i.e. we restrict the dynamics to the energy surface $H = \frac{1}{2}$). Then, *the phase space* of the system is $\mathcal{P} = \bar{D} \times S^{d-1}$, where S^{d-1} is a $(d-1)$-dimensional unit sphere of velocity vectors. So the elements of $\mathcal{P}$ are

$$\rho \equiv (q, p) = (x, y, z^1, ..., z^{d-2}, p_x, p_y, p_{z^1}, ..., p_{z^{d-2}}).$$

$\Phi^t : \mathcal{P} \to \mathcal{P}$ is *the billiard flow.* $\rho_t = \Phi^t \rho_0$ means that the distance between the trajectory, connecting q_0 with q_t, and the set Γ^*, is bounded away from zero.

It is standard in dynamical system theory to reduce the study of flows to maps by constructing a cross-section. The latter is a hypersurface transversal to the flow. For the flow Φ^t, a hypersurface in $\mathcal{P}$ can be very naturally constructed with the help of the boundary of D. Let

$$P = \{\rho = (q, p) \in \mathcal{P} : q \in \partial D, \langle p, n(q) \rangle \geq 0\} \tag{1.6}$$

This is a $(2d-2)$-dimensional submanifold in $\mathcal{P}$. It consists of all possible outgoing velocity vectors resulting from reflections at ∂D. Clearly, any trajectory of the flow Φ^t crosses the surface P every time it reflects at ∂D.

This defines *the Poincaré return map* by

[2] To be precise, one might define $n(q)$ by continuity at points of Γ^*, but this would give more than one normal vector $n(q)$, hence the dynamics would be multiply defined. This is not good. We adopt a standard convention that the reflection is not defined at any $q \in \Gamma^*$.

$$T : P \to P \text{ such that } Tx = \Phi^{\tau(x)+0}x \tag{1.7}$$

where

$$\tau(x) = \min\{t > 0 : \Phi^{t+0}x \in P\}.$$

Definition 1.2. The map T is *the billiard map.*

If the smooth components of the boundary ∂D are of class C^k, then the map T is C^{k-1}-smooth at all points $\rho \in P \setminus \Sigma$ such that $T\rho \in P \setminus \Sigma$, where

$$\Sigma = \{(q,p) \in P : \langle p, n(q)\rangle = 0\} \cup \{(q,p) \in P : q \in \Gamma^*\}. \tag{1.8}$$

This is the set where the Poincaré map T has singularities.

2 Background

2.1 Boltzmann's hypothesis and Mechanical model

The proper mathematical formulation of Boltzmann's ergodic hypothesis is still not clear. For systems of elastic hard balls on a torus, however Ya. Sinai, in 1963 in [29] gave a stronger, and at the same time mathematically rigorous, version of Boltzmann's Hypothesis:

Boltzmann-Sinai Ergodic Hypothesis: *The system of an arbitrarily fixed number N of identical elastic hard balls moving on T^d torus is ergodic.*

One can reduce the above problem to the billiard problem in higher dimensions by the following procedure.

Let us consider N balls or disks of mass m and radius r enclosed in a bounded domain R, called a container (or reservoir)[3]. The balls (disks) collide elastically with each other and with the walls of the container. Precisely, if a ball with center q hits a wall at a point $w \in \partial R$, then we decompose its velocity vector as $v = v^{||} + v^{\perp}$, where $v^{\perp}$ is the component parallel to

[3]The system of N balls (disks) moving in the open space (or on the plane) without walls is dynamically not very interesting. As it is intuitively clear (and proven mathematically), the total number of collisions between balls is always finite, and after the last collision the balls will fly freely forever. Furthermore, the number of collisions between N balls in the open space is uniformly bounded by a constant that only depends on N. This last fact was proved very recently in 1998 by Burago, Ferleger and Kononenko [8]

the line passing through q and w, and $v^{||}$ is perpendicular to that line (so it is parallel to the line tangent to the boundary at the point w). The new, outgoing, velocity of the ball is $v^{\text{new}} = v^{||} - v^{\perp}$.

Now we reduce the system of N hard balls in a container R to a billiard. We denote by $q_i = (q_i^1, q_i^2, q_i^3)$ the center of the ith ball[4] and by $v_i = (v_i^1, v_i^2, v_i^3)$ its velocity vector, $1 \leq i \leq N$. Now the entire system can be described by a configuration point

$$q = (q_1^1, q_1^2, q_1^3, q_2^1, \ldots, q_N^2, q_N^3) \in \mathbb{R}^{3N}$$

and its velocity vector

$$v = (v_1^1, v_1^2, v_1^3, v_2^1, \ldots, v_N^2, v_N^3) \in \mathbb{R}^{3N}$$

Since the balls cannot overlap, one needs to exclude configurations that satisfy

$$(q_i^1 - q_j^1)^2 + (q_i^2 - q_j^2)^2 + (q_i^3 - q_j^3)^2 < (2r)^2 \tag{2.1}$$

for $1 \leq i < j \leq N$. The inequality (2.1) specifies a spherical cylinder in $\mathbb{R}^3$, which we denote by C_{ij}. After removing all forbidden configurations we get a configuration space

$$D = R^N \backslash \bigcup_{i \neq j} C_{ij}$$

One can check by direct inspection that the trajectory of the configuration point q in D is governed by the billiard rules. Specular reflections at the surface of a cylinder C_{ij} correspond to collisions between the balls i and j. Thus, the study of the mechanical model of N balls or disks is reduced to the study of billiard dynamics in the domain D.

We note that the conservation of the total kinetic energy $\frac{1}{2}\sum_i m\|v_i\|^2$ is equivalent to the preservation of the norm $\|v\|$ of the velocity vector. If the container R is a torus $\mathbb{T}^d$, then there are no walls ($\partial R = 0$) and the total momentum $P = \sum_{i=1}^N mv_i$ is also a first integral of motion.

The singularity set Γ^* contains all intersection of the cylindrical surfaces C_{ij} with each other. Such intersections correspond to simultaneous collisions of three or more balls. The outcome of such multiple collisions is not defined.

[4] For disks on a plane, we have two coordinates instead of three.

Indeed it is clear that different order of 2-body collisions which are arbitrary close to the multiple solutions may have different outcome.

2.2 Basic types of billiards

Definition 2.1. A smooth component of the boundary $\Gamma_i \in \partial D$ is said to be *dispersing*, if its curvature is positive, *focusing* if its curvature is negative and *neutral* if its curvature is zero.

2.2.1 Completely integrable billiards

It was proved by Birkhoff that *every classical ellipsoidal billiard is integrable*, however the opposite question, known as Birkoff-Poritski conjecture is still open.

Conjecture 2.2. (Birkoff-Poritski)*: Any 2-dimensional integrable smooth, convex billiard is an ellipse.*

In [37] this conjecture was generalized to the higher dimensions. Delshams et al ([11]) showed that the Birkhoff conjecture is *locally* true for symmetric entire perturbations, i.e. any non-trivial symmetric perturbation of an ellipse is non-integrable.

For higher dimensions they established the splitting of separatrices under *very general* perturbation of a *prolate*[5] ellipsoid.

2.2.2 Smooth convex tables

Billiards in general oval-shaped tables, even if not completely integrable, have many common features with billiards in ellipses. In 1973 Lazutkin [19] proved: if D is a strictly convex domain (the curvature of the boundary never vanishes) with sufficiently smooth boundary, then there exists a positive measure set $N \subset P$ that is foliated by invariant curves. The set N accumulates near the boundary ∂P. Almost all trajectories starting in the set N have caustics, which are convex closed curves lying inside D. Of course, the billiard cannot be ergodic since N has a positive measure. The Lyapunov exponents for points $x \in N$ are zero. However, away from N the dynamics might be quite different.

[5]Prolate ellipsoid is the one with all its axis of equal length except one, which is larger.

Originally Lazutkin demanded 553 continuous derivatives in his theorem, but later R. Douady proved that 6 are enough and conjectured that the curve may be C^4.

The smoothness here is essential. *The Bunimovich stadium* is a convex domain with C^1 (not C^2!) boundary, consisting of two identical half-circles connected by two parallel line segments. The stadium was the first billiard with convex boundary proved to be ergodic and hyperbolic (see [3]).

Since [3] the principal question remained open: Does the mechanism of defocusing also generate chaotic billiards in higher dimensions? Recently L. Bunimovich and J. Rehacek found that under some restrictions on the shape of the billiard one can produce convex higher-dimensional billiards which are ergodic ([5]).

2.2.3 Dispersing and semi-dispersing billiards

Definition 2.3. If all the components of the boundary ∂D are dispersing, the billiard is said to be *dispersing.* If ∂D consists of dispersing and neutral components, the billiard is said to be *semi-dispersing.*

Dispersing billiards are also known as *Sinai billiards*, since Ya. Sinai introduced them in 1970 in his classical paper [30]. Based on this work and [6] it was understood that any 2-dimensional dispersing billiard is ergodic (actually, they proved the Boltzmann-Sinai hypothesis for two hard disks on the two-dimensional unit torus $\mathbb{T}^2$).

The generalization of this result to higher dimensions $d > 2$ took fourteen years, and was done by Chernov and Sinai in [32].

Although the model of two hard balls on $\mathbb{T}^d$ is already rather involved technically, it is still a so called *strictly dispersive* billiard system, i. e. such that the smooth components of the boundary ∂D of the configuration space are strictly concave from outside D. The billiard systems of more than two hard balls on $\mathbb{T}^d$ are no longer strictly dispersive, but just *semi-dispersive* (strict concavity of the smooth components of ∂D is lost, merely concavity persists), and this circumstance causes a lot of additional technical troubles in their study.

After many years of hard work, just in 1999, Simányi and Szász in [28] established the complete hyperbolicity (nonzero Lyapunov exponents almost everywhere) for N balls in T^d. Full hyperbolicity together with Katok-Strelcyn theory provides that the ergodic components of these system are of positive measure. So there are countably many of them and each of them is K-mixing. In 2003 the same authors managed to prove the ergodicity (actually, the B-mixing property) of the above system.

2.2.4 Billiards with a mixed phase space

It is well known that a typical (generic) Hamiltonian system is neither chaotic nor integrable [22] but rather demonstrates a mixed behavior when islands of stability (KAM islands) are situated in a "chaotic sea" formed by one or several ergodic (chaotic) components each of which occupies a subset of a positive measure (volume) in the phase space. Such systems are called "Hamiltonian systems with *divided (or mixed) phase space*".

There are many examples of such billiards, e.g. *the cardioid*, non elliptical convex billiards etc. One of the interesting examples of such a mixed phase space is *the mushroom billiard* recently suggested in [4]. It provides continuous transition from completely integrable circle billiard to completely chaotic stadium billiard. The system also exhibit easily localized chaotic sea and island of stability. In [4] Bunimovich makes the following remark:

Remark 2.4. *For mushroom the basic mechanism of chaos (of hyperbolicity) is the mechanism of defocusing as in stadium and other chaotic billiards with focusing components of the boundary rather than the mechanism of dispersing as in Sinai (or dispersing billiards). It is quite natural because systems with the defocusing mechanism occupy an intermediate position between systems with the dispersing mechanism and integrable systems. Indeed, dispersing means that neighboring orbits just diverge, integrability means that their divergence and convergence are balanced while defocusing means that divergence (on average) prevails over convergence.*

2.3 Statistical properties

2.3.1 Invariant measure

The Hamiltonian equations (1.3) preserve the Liouville measure $dqdp$ on the phase space $\mathcal{P}$, where dq and dp are the Lebesgue measures on D and S^{d-1}, respectively. This measure is also invariant under reflections (1.5). We can get a probability measure

$$d\mu = c_\mu dqdp \quad \text{where} \quad c_\mu = (|D||S^{d-1}|)^{-1}$$

$|\cdot|$ stands for d-dimensional volume. The invariant measure for the map T is

$$d\nu = c_\nu |\langle p, n(q)\rangle| dq dp$$

2.3.2 CLT and Decay of Correlations

When the system behavior is chaotic one would like to characterize it by the trajectory's statistical properties. Two such properties which play a central role in physics are *the rate of the decay of correlations* and the *central limit theorem (CLT)*.

Let $f : M \to M$ be a dynamical system, μ an invariant probability measure, and $\varphi : M \to \mathbb{R}$ a function which we think of as a quantity that can be measured or observed (for example, temperature in an experiment). Consider the sequence of observables

$$\varphi, \quad \varphi \circ f, \quad \varphi \circ f^2, \ldots, \varphi \circ f^n, \ldots$$

and ask how they compare qualitatively with genuinely random stochastic processes on the probability space $(M; \mu)$.

For example, one could ask if the *Central Limit Theorem* holds: that is to say, for φ with zero average ($\int \varphi d\mu = 0$) we may ask if

$$\frac{1}{\sqrt{n}} \sum_{i=0}^{n-1} \varphi \circ f^i \longrightarrow_{\mathcal{D}} \mathcal{N}(0, \sigma)$$

for some $\sigma > 0$ where $\mathcal{N}(0; \sigma)$ is the normal distribution with variance σ^2 and $\mathcal{D}$ means convergence in distribution[6].

Another standard question concerns the decay of correlation between φ and $\varphi \circ f^n$ for large n; Defining

$$\Phi(n) := \left| \int (\varphi \circ f^n) \varphi d\mu - \left(\int \varphi d\mu \right)^2 \right|,$$

[6]Let $X_1, X2, \ldots$ and X be real random variables with respective distribution functions $F_1, F_2, \ldots$ and F. One says X_n *converges* to X *in distribution* as $n \to \infty$, and writes $X_n \to_{\mathcal{D}} X$, if

$$\lim_{n\to\infty} F_n(x) = F(x) \text{ for all } x \in C_F.$$

Here $C_F := \{x \in \mathbb{R} : F \text{ is continuous at } x\} = \{x \in \mathbb{R} : P[X = x] = 0\}$ is the set of continuity points of F.

one could ask whether $\Phi(n)$ tends to zero as n tends to infinity and at what speed. For example, if $\Phi(n) \sim e^{-\alpha n}$ for some $\alpha > 0$ for all φ in some class of test functions, then this is a property of the dynamical system $(f;\mu)$ and we say $(f;\mu)$ has an *exponential decay of correlations*. Similarly, if $\Phi(n) \sim n^{-\alpha}$ for some $\alpha > 0$, then we say $(f;\mu)$ has a *polynomial decay*, and so on.

For Anosov diffeomorphisms, Sinai, Ruelle and Bowen ([31], [24], [25] and [1]) proved the CLT in the early seventies, and at the same time it was established that the correlations decay exponentially, which is accepted as the fastest possible rate of decay.

For billiards the question is much more difficult. Since 1980, a major project was to extend the Sinai-Ruelle-Bowen techniques to dispersing billiards. The work has already spanned 20 years and is still in progress. First, in [6] and [7] the CLT was proved. Only in the late nineties, it was shown that correlations decayed exponentially by a different approach ([39], [10]). It finally becomes clear that for the purpose of physical applications, dispersing billiards behave just like Anosov diffeomorphisms. This was summarized by Gallavotti and Cohen in their "Axiom C" paper [14][7].

2.4 Smooth approximation of scattering billiards

From statistical mechanics, one knows that under certain situations, such as high pressure and low temperature, the hard sphere model is a poor predictor of gas properties. A more accurate model is obtained by replacing the elastic collision between particles by an interaction caused by a smooth potential field V.

In [12] the case of two particles with a finite range potential moving on a two-dimensional torus was examined. They showed that this system can contain a stable elliptic periodic orbit and hence be non-ergodic. A system of two hard spheres interacting by elastic collision is always ergodic as was shown by Sinai in [30].

V.Rom-Kedar and D.Turaev in [35],[34] and [36] considered the effect of smoothing of the potential on scattering billiards, and in particular, its effect on the ergodic properties.

More precisely, they considered of the form

[7] These questions are still open for continuous time (i.e. for flows) and for billiards with infinite horizon.

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x, y; \epsilon). \quad (2.2)$$

which limit, as $\epsilon \to 0$, to a singular Hamiltonian with a potential which vanishes in the interior of the billiard domain D and is strictly positive on its boundaries.

It was proved that smooth scattering two-dimensional billiard-like potentials may give rise to elliptic islands.

Thus, it was demonstrated that *proving Boltzmann hypothesis for the hard sphere model is a-priori insufficient for proving the achievement of thermodynamical equilibrium for a gas.*

In [34] they derived a rigorous estimate of the size of islands appearing in such flows. It was found that for all scattering geometries and for many types of natural potentials which limit to the billiard flow as a parameter $\epsilon \to 0$, islands of a *polynomial* size in ϵ appear.

The appearance of these islands in a large "chaotic sea" is very important, because there may occur "stickiness" regions [40], which influence the temporal correlation function of initial conditions from the "chaotic see".

3 Research plan

There are two main directions I am going to pursue in my Ph.D. thesis. There is a strong causal relationship between them. One is more theoretical – the extension of the above works of V.Rom-Kedar and D. Turaev to higher dimensional billiards. Another is more practical – the investigation of the dynamical properties of particular billiards in three dimensions. Precisely, the following goals are set:

3.1 Objectives

1. Provide the conditions for a smooth Hamiltonian flow, converging to a singular billiard flow, to exist. Prove that this convergence is smooth, where the billiard flow is smooth and continuous where the billiard flow loses its smoothness. Consider the case of curved trajectories.

2. Study the behavior of a particle in the three-dimensional billiards which are constructed experimentally by *The Laser Cooling and Trapping Group* of Dr. Nir Davidson.

3. Find a mechanism which by a perturbation of a scattering billiard to a smooth Hamiltonian flow may create stability islands and find their sizes. Such islands are expected to appear near singular periodic orbits and corner polygons.

3.2 Methodology

1. This part of the work is mostly done. The singular set of the billiard flow (where it looses smoothness) is defined and the following theorem is proved.

   Theorem 3.1. *Consider a Hamiltonian system defined by*

   $$H = \frac{\|p\|^2}{2} + V(q;\epsilon) + H_\epsilon(q,p;\epsilon). \tag{3.1}$$

   *If the potential $V(q;\epsilon)$ satisfies conditions **I-IV** (see Appendix), and*

   $$H_\epsilon(q,p;\epsilon) \rightrightarrows 0$$

   uniformly along with all its derivatives as $\epsilon \to 0$ in some neighborhood of $\bar{D}$, then the Hamiltonian flow (3.1) r-converges to the billiard flow in D.

   This theorem could be directly applied to the works of Delshams group. We can extend their results obtained for ellipsoidal billiards to the corresponding smooth Hamiltonian system. We also intend to extend the above theorem to billiards with curved trajectories. Several mechanisms of forming the later are possible, e.g. gravitation and magnetic field. They influence the kinetic energy in the Hamiltonian function, but locally the reflection rule stays the same.

2. We plan to work with a 3-dimensional billiard which was built by Nir Davidson et al. Its boundary is formed by two three-dimensional cones, glued together by their basis (here, circle; but other shapes such as ellipse, stadium and dispersing boundaries may be considered). We will investigate ergodic and hyperbolic properties of this system and the influence of the smooth potential on them.

3. There are several ideas for obtaining non-linear elliptic periodic points of the Poincaré map for a smooth system (i.e. elliptic orbits for the Hamiltonian flow corresponding to the original billiard flow):

- First is the mechanism of multiple tangency. As for 3-dimensional space the orbit with only one tangency gives one pair of elliptic eigenvalues and another is hyperbolic. We hope that by arranging two tangencies in a proper way it would be possible to obtain two pairs of elliptic eigenvalues. If this idea works we will generalize it for higher dimensions.
- Second idea is a smooth approximation of polygonal trajectory which starts and ends in the same ”corner”, i.e. the point which belongs to Γ^*. It is not clear yet which kind of ”corner” should be considered e.g. for 3-dimensional case there are two possibilities: an edge (curve of intersection of two boundary surfaces) or a corner (the point of intersection of 3 surfaces).

If we succeed to find an elliptic periodic point the next step will be to find whether the return map is nonlinearly stable so that KAM theory applies as in [13] and [34]. Here additional issues like resonances will naturally arise.

Appendix

Following the definitions of the 2-dimensional case of [35], we consider the family of Hamiltonian systems associated with

$$H = \frac{\|p\|^2}{2} + V(q;\epsilon) \tag{3.2}$$

where the potential $V(q;\epsilon) \to 0$ inside the region D as $\epsilon \to 0$ and it tends to infinity outside. The potential V is called *a billiard-like potential.* It satisfies the following conditions:

Condition I. *For any compact region $K \subset D$ the potential $V(q;\epsilon)$ diminishes along with all its derivatives as $\epsilon \to 0$:*

$$\lim_{\epsilon\to 0} \|V(q;\epsilon)|_{q\in K}\|_{C^{r+1}} = 0 \tag{3.3}$$

The growth of the potential to infinity across the boundary needs to be treated more carefully. We assume that V is evaluated along the level sets of some *finite* function near the boundary. In other words, suppose, that in a neighborhood of $\bar{D}\backslash\Gamma^*$ there exists *a pattern function* $Q(q;\epsilon)$ which is C^{r+1} with respect to q and it depends continuously on ϵ (in C^{r+1}-topology) at $\epsilon \geq 0$ (it has, along with all derivatives, a proper limit as $\epsilon \to 0$). Assume that away from Γ^*:

Condition IIa. *The billiard boundary is composed of level surfaces of $Q(q;0)$:*

$$Q(q;\epsilon=0)|_{q\in S_i} \equiv Q_i = constant \tag{3.4}$$

For each boundary component S_i, for Q close to Q_i, let us define *a barrier function* $W_i(Q;\epsilon)$ which does not depend explicitly on q and assume that:

Condition IIb. *There exists a small neighborhood N_i of the surface S_i in which:*

$$V(q;\epsilon)|_{q\in N_i} \equiv W_i(Q(q;\epsilon);\epsilon) \tag{3.5}$$

and

Condition IIc. ∇V *does not vanish in a finite neighborhood of the boundary surfaces, thus:*

$$\nabla Q|_{q\in N_i} \neq 0 \tag{3.6}$$

and

$$\frac{d}{dQ}W_i(Q;\epsilon) \neq 0. \tag{3.7}$$

A rapid growth of the potential across the boundary may be described in terms of the barrier function along. Now we can choose one of surfaces Γ_i and call it just S. Without loss of generality we can assume $Q = 0$ on S. By (3.6), the pattern function Q is monotonic function across S. Assume that Q is positive inside D near S and negative outside (otherwise consider $-Q$ so that (3.8) is always true).

Condition III. *As* $\epsilon \to +0$ *the barrier function increases from zero to infinity across the boundary* S:

$$\lim_{\epsilon \to +0} W(Q;\epsilon) = \begin{cases} 0 & Q > 0 \\ +\infty & Q < 0 \end{cases} \tag{3.8}$$

Condition IV. *As* $\epsilon \to +0$, *for any finite strictly positive* W_1 *and* W_2, *the function* $Q(W;\epsilon)$ *tends to zero uniformly on the interval* $[W_1, W_2]$ *along with all its* $(r+1)$ *derivatives.*

The condition **IV** makes sense since by (3.7) Q could be considered as a function of W and ϵ near the boundary. For small ϵ a finite change in W corresponds to a small change in Q.

References

[1] R. Bowen *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, 1975, Lect. Notes Math. 470, Springer, Berlin.

[2] L. A. Bunimovich, Ya. G. Sinai *The fundamental theorem of the theory of scattering billiards*,1973, Math. USSR-Sb. 19, 407-423.

[3] L.A, Bunimovich *On the ergodic properties of certain billiards*, 1974, Functional Analysis and Applications 8, 254-255.

[4] L.A. Bunimovich *Mushrooms and other billiards with divided phase space*, 2001, Chaos 11, 802.

[5] L. A. Bunimovich and J. Rehacek *On the ergodicity of high-dimensional focusing billiards*, 1999, Ann. Inst. Henri Poincaré 68, 421448.

[6] L. A. Bunimovich and Ya. G. Sinai *Statistical properties of Lorentz gas with periodic configuration of scatterers*, 1981, Commun. Math. Phys. 78 , 479-497.

[7] L. A. Bunimovich, Ya. G. Sinai and N. I. Chernov *Markov partitions for two-dimensional billiards*, 1990, Russ. Math. Surv. 45, 105152.

[8] D. Burago, S. Ferleger and A. Kononenko *Uniform estimates on the number of collisions in semi-dispersing billiards*, 1998, Ann. of Math. 147, 695-708.

[9] N.I. Chernov and R.Markarian *Introduction to the Ergodic Theory of chaotic Billiards.*

[10] N.I. Chernov *Decay of correlations and dispersing billiards*, 1999, J. Statist. Phys. 94, 513-556.

[11] A. Delshams and R.Ramírez-Ros *Homoclinic orbits of twist maps and billiards*, 1998, Symmetry and Perturbation Theory ed D Bambusi and G Gaeta (Florence: Quaderni GNFM), 46-86.

[12] V.J. Donnay *Non-ergodicity of Two Particles Interecting via a Smooth Potential*, 1999, Journal of Statistical Physics, 96, Nos. 5/6.

[13] V.J. Donnay *Elliptic Islands in generalized Sinai billiards*, 1996, Ergod. Th. Dynam. Sys., 16, 975–1010.

[14] G. Gallavotti and E. D. G. Cohen *Dynamical ensembles in stationary states*, 1995, J. Stat. Phys. 80, 931-970.

[15] M.Gutzwiller *Chaos in Classical and Quantum Mechanic*, 1990, New York: Springer

[16] A. Kaplan, N. Friedman, M. Andersen, and N. Davidson *Observation of Islands of Stability in Soft Wall Atom-Optics Billiards*, 2001, Phys. Rev. Lett. 87, 274101.

[17] A.Katok and B. Hasselblatt *Introduction to the Modern Theory of Dynamical systems*

[18] A. Krámli, N. Simányi, D. Szász *The K-Property of Four Billiard Balls*, 1992, Commun. Math. Phys. 144, 107-148.

[19] V.F. Lazutkin *The existence of caustics for a billiard problemin a convex domain*, 1973, Izv. Akad. Nauk SSSR, Ser. Math. 37, 186-216. English translation: Math. USSR Izv. 7, 185-214 (1974).

[20] V.F. Lazutkin *Convex billiards and eigenfunctions of the Laplace operator*, 1981, (in Russian) Leningrad. Univ. Publ.

[21] C. Liverani and M. Wojtkowski *Ergodicity in Hamiltonian Systems Dynamics reported*, 1995, Dynam. Report. Expositions Dynam. Systems (N.S.) 4, Springer, Berlin, 130-202.

[22] L. Markus and K. R. Meyer *Generic Hamiltonian dynamical systems are neither integrable nor ergodic*, 1978, Mem. Am. Math. Soc. 144, 152.

[23] R. Ramírez-Ros *Singular phenomena for convex billiard tables*

[24] D. Ruelle *A measure associated with Axiom-A attractors*,1976, Amer. J. Math. 98, 619-654.

[25] D. Ruelle *Thermodynamic Formalism*, 1978, Addison-Wesley, Reading, Mass.

[26] N. Simányi *The K-property of N billiard balls I*, 1992, Invent. Math. 108, 521-548.

[27] N. Simányi *The K-property of N billiard balls II*, 1992, Computation of the Neutral Linear Space, Invent. Math. 110, 151-172.

[28] N. Simányi, D. Szász *Hard ball systems are completely hyperbolic*, 1999, Annals of Mathematics 149, 35–96.

[29] Ya. G. Sinai *On the Foundation of the Ergodic Hypothesis for a Dynamical System of Statistical Mechanics*, 1963 Soviet Math. Dokl. 4, 1818–1822.

[30] Ya. G. Sinai *Dynamical systems with elastic reflections: Ergodic properties of scattering billiards*,1970, Russ. Math. Surv. 25, 137-189

[31] Ya. G. Sinai *Gibbs measures in ergodic theory*, 1972,Russ. Math. Surveys 27, 21-69.

[32] Y.Sinai and N.Chernov *Ergodic properties of some systems of two-dimensional disks and threedimensional balls*, 1987, Usp. Mat. Nauk 42, 153-174, 256.

[33] U.Smilansky *Semiclassical quantization of chaotic billiards a scattering approach*, 1995, Proc. 1994 Les-Houches Summer School on 'Mesoscopic Quantum Physics' ed A Akkermans et al.

[34] V.Rom-Kedar and D.Turaev *Big Islands in dispersing billiard-like potentials*, 1999, Phys. D 130, no. 3–4, 187–210.

[35] D.Turaev and V.Rom-Kedar *Elliptic Islands appearing in near-ergodic flows*, 1998, Nonlinearity, 11(3), 575–600

[36] D.Turaev and V. Rom-Kedar *Soft Billiards with corners*, 2003, Journal of Statistical Physics 112 (3–4), 765–813

[37] A.P. Veselov *Intagrable maps*, 1991, Russian Math. Surveys, 46(5), 1–51.

[38] L.-S. Young *Developments in Chaotic Dynamics*, 1998, Notices of the AMS, Volume 45, N. 10

[39] L.-S. Young *Statistical properties of dynamical systems with some hyperbolicity*, 1998, Annals of Math. 147, 585-650.

[40] G.M. Zaslavsky *Chaos, fractional kinetics, and anomalous transport*, 2002, Physics Reports 371, 461–580.