February 18, 2004

Estimates for the Number of Collisions in the Boltzmann’s Gas and Billiards

(based on the article of C. A. Galperin “Billiards and other elastic collisions of particles and balls”)

by Anna Rapoport
Around 1960, when Sinai tried to prove the Boltzmann Hypothesis, he considered equal balls which move inside empty $\mathbb{R}^3$ and collide elastically. Actually, he considered gas molecules as equal balls which move inside empty $\mathbb{R}^3$ and collide elastically.

What is the maximum number of collisions if the initial position and velocity could be chosen arbitrarily (under the condition that the answers to 1) and 2) are negative)?

1. Could the collisions in the system occur infinitely long?
2. Could the number of collisions be infinite at finite time?
3. What is the maximum number of collisions if the initial position and velocity could be chosen arbitrarily (under the condition that the answers to 1) and 2) are negative)?

Which arose in the prove of ergodicity was the following: ergodicity of the gas inside the reservoir, the important questions which arise in the prove of ergodicity was the following.
Motivation

The gas could be reduced to the billiard. Consider the dispersion.

What happens to $F$ in backward time?

The dispersion gives us

$$
\frac{1}{2} \cos \theta \cdot \frac{1}{R^+} + \frac{1}{R^-} = \frac{-1}{R}
$$

Here $R^+$ is the radius of curvature of the boundary at the point of collision and $\theta$ is the angle of reflection. $R^-$ is the radius of curvature at the point of collision (after collision).

What happens to $F$ in backward time?

The evolution of the wave front (particularity its curvature) let $x \in \mathcal{F}$. In order to obtain the sign of Lyapunov exponents we look at the boundary. Consider the dispersion.

Motivation
the curvatures are finite. Be finite.

this it is sufficient that the number of collisions on each finite interval to

and for

because \( 2 \phi' \cos' \phi / i^2 \phi') < 0 \), \( \forall i \). And for \( \infty = i \cdot 2^i = \infty \int_0^1 \), \( \forall i \) it is sufficient that the criterion of Seidel-Stern is finite.

By the criterion of Seidel-Stern

\[
\frac{\ldots + \frac{t_2}{1} + \frac{\cos \phi}{2\phi} + \frac{t_1}{1}}{1} = (x) \chi
\]

In such a way we can set a continued fraction:

\[
\frac{+\chi + \frac{\cos \phi}{2\phi}}{1} + \frac{t_1}{1} = \frac{-\chi}{1} + \frac{t_1}{1} = \frac{-\gamma + \frac{t_1}{1}}{1} = \frac{(x) \gamma}{1} = (x) \chi
\]

After one collision:

\[
\frac{t_1 \phi \cos}{z} \cdot \frac{+\chi}{1} = -\chi
\]

The curvatures are finite, and we set \( H/I = H', Y/I = \pm \chi \).
Consider 3 equal balls in \( \mathbb{R}^3 \).

In 1964, American physicists Sandri, Sullivan, and Norem proved that in the system of 3 equal balls 1, 2, 3:

Only in 1993, chemist T. Murphy and physicist E. Cohen proved that in such a system the number of collisions not larger than 4 is independent of the same year it was independently numerically calculated that in such a system the number of collisions not larger than 4 is independent of the system of 3 equal balls 1, 2, 3.

<table>
<thead>
<tr>
<th>Possible</th>
<th>Impossible</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>a</td>
</tr>
<tr>
<td>((12)(13))((12)(13))</td>
<td>((12)(13))((12)(13))</td>
</tr>
<tr>
<td>e1</td>
<td>e1</td>
</tr>
<tr>
<td>((12)(13)(31))</td>
<td>((12)(13)(31))</td>
</tr>
<tr>
<td>c1</td>
<td>c1</td>
</tr>
<tr>
<td>((12)(23)(12))</td>
<td>((12)(23)(12))</td>
</tr>
<tr>
<td>f1</td>
<td>f1</td>
</tr>
<tr>
<td>d1</td>
<td>d1</td>
</tr>
<tr>
<td>a</td>
<td>a</td>
</tr>
</tbody>
</table>

Three-particle collisions
Configuration, "velocity" and phase spaces

Take $N$ particles which are moving in $\mathbb{R}^3$. Each of them has

\[
\begin{aligned}
\text{Coordinates: } & (x_i(t), y_i(t), z_i(t)), \\
\text{Velocities: } & (v_{ix}(t), v_{iy}(t), v_{iz}(t)), \\
\text{Form a configuration space: } & \bigoplus_N \mathbb{R}^3 \cong \mathbb{R}^6. \\
\text{Usually, the phase space is not the whole } \mathbb{R}^6, \text{ but some part of it:}
\end{aligned}
\]

\[
\begin{aligned}
\mathbb{R}^N \oplus \mathbb{R}^3 \cong \mathbb{R}^6 \\
\text{Phase space: } & \mathbb{R}^6 \ni (t^\uparrow, x(t)) = (t, x), \\
\text{\"velocity\" space: } & \mathbb{R}^3 \ni (z(t), v(t)), \\
\text{For high-dimensional points:}
\end{aligned}
\]

\[
\begin{aligned}
\forall \mathbb{N}^3 = \{ ((t), (z), (v)) \in \mathbb{R}^6 \} \\
\text{Velocities: } & (v_{iz}(t), v_{iy}(t), v_{ix}(t)), \\
\text{Coordinates: } & (x(t), y(t), z(t))
\end{aligned}
\]

Take $N$ particles which are moving in $\mathbb{R}^3$. Each of them has

Configuration, "velocity" and phase spaces
Examples

1. We have particles in $\mathbb{R}^3$ with equal masses and the energy in the system is conserved, then

$$v_2^2 + \cdots + v_n^2 = \text{const},$$

the distance between centers is $\geq 2R$.

2. Two equal balls in $\mathbb{R}^3$ with radius $r$. If they were transparent, then their configuration space would be $\mathbb{R}^6$, but they are not and then the configuration space of radius $r$ will be $\mathbb{R}^6$. If they were transparent, this corresponds to the outer part of some cylinder $C \in \mathbb{R}^6$ which forms angle $\pi/4$ with horizontal. So the point moves out of this cylinder. The point hits cylinder when the balls collide.

The projection of $z(\tilde{z}) = z - \tilde{z}$ is the curve lying on the projection of $\mathbb{R}^6$ into $\mathbb{R}^3$.

$$\text{const} = zN \omega + \dot{z}N \omega + xN \omega + \cdots + zI \omega + \dot{z}I \omega + xI \omega \Rightarrow \text{const} = vN + \cdots + vI \omega$$

system is conserved, then

$\mathbb{R}^3$ with equal masses and the energy in the system.
Reduction to the billiard.

We exclude the cases of collisions of more than 2 particles and

restrictions could be found only after deep research as for 3 balls.

In dimensions higher than 2 there are no such restrictions (some

\[ r_i = 0, \quad i = 1, N. \]

So it could be reduced to the system of elastic points in \( \mathbb{R} \).

And it could collide only with balls \#(1 - i)\# and \#(i + 1)\#. And it

multi-dimensional (\( d \geq 2 \)) gas. In one-dimensional system ball

there is a big difference between one-dimensional gas and

Take \( N \) balls with masses \( m_1, \ldots, m_N \) and radiiueses \( r_1, \ldots, r_N \).

Reduction to the billiard.
The configuration space of $N$ particles is a subset of $\mathbb{R}^N$ corresponding to the coordinates $x_1, x_2, \ldots, x_N$ of the particles with masses $m_1, m_2, \ldots, m_N$. Configuration points $x = (x_1, \ldots, x_N)$ move uniformly between collisions and change their trajectory when $m_i$ and $m_{i+1}$ collide. If $m_i = m_{i+1}$, then the collision is a billiard (incidence angle = reflection angle), otherwise it is not so. If $m_i < m_{i+1}$ then the point $x_i$ reflects from $x_{i+1}$. Hyperplanes $\Pi_1, \ldots, \Pi_N$ form the walls of the $(N-1)$-sided angle. Somehow the $x_i$'s reflect from $\Pi_i$. This defines a hyperplane $\Pi_i \in \mathbb{R}^N$. So the point different. The configuration $x \in \mathbb{R}^N$ moves uniformly between collisions and changes its trajectory when $m_1, m_2, \ldots, m_N$ collide. Therefore, configuration points $x \in \mathbb{R}^N$ are the coordinates of the particles with mass $m_1, m_2, \ldots, m_N$. Let $x_1 \geq x_2 \geq \ldots \geq x_N$ be the coordinates of the particles in $\mathbb{R}$. The part of $\mathbb{R}^N$ corresponding to $x_1 \geq x_2 \geq \ldots \geq x_N$ is a configuration space for the particles.
Let two balls of radii $u$ and $v$ have collided in $\mathbb{R}^n$.

The configurations space for the balls in $\mathbb{R}^n$
Note that the reflection will be of billiard type only if $m = \ell$. 

$$\left( \bigcap_{i \neq j}^{N_u} (\text{int} C) \right) - N_uR = \emptyset$$

The space in this case is

So $x$ lies outside of $\ell$, because the balls are not transparent.

The point $x$ cannot be inside $C$, because the balls are not transparent.

Finally we get a $u$-dimensional configuration point $x$.

For all other pairs.

Then

Now let us consider $N$ balls in $\mathbb{R}^n$ and let $\ell \# \ell$ and let $j \# j$ and let
There's a linear transformation:

$$e^{ix}=\sum_{m} p_{m} e^{ix_{m}}.$$ 

Then the matrix of this transformation is:

$$A=\text{diag}(p_{m_{1}},\ldots,p_{m_{N}}).$$

The conservation of momentum gives us:

$$I=\mathcal{H}=\sum_{i=1}^{N} (\mathfrak{a}|\mathfrak{a}|)_{i,\mathfrak{i}}.$$ 

Now if one takes the entire kinetic energy to be $I=\mathcal{H}$, then

$$\sum_{i=1}^{N} m_{i} v_{i} = \sum_{i=1}^{N} p_{m_{i}} e^{ix_{i}} = j! m_{j}.$$ 

Then the matrix of this transformation is:

$$A = \text{diag}(p_{m_{1}},\ldots,p_{m_{N}}).$$ 

In linear transformation:

Take the configuration space $\mathcal{C} \subset \mathfrak{M}$ or $u < \mathfrak{i}$ (or $1 \neq u \geq 2$). Consider the changes in the sign after a reflection:

$$\phi = \phi \cos \theta.$$ 

Hence $\cos \theta = \phi$ and

$$\phi \cos \theta = \phi /|\mathfrak{a}|.$$ 

Therefore

$$\phi \cos \theta = \phi /|\mathfrak{a}| = \phi /|\mathfrak{a}|.$$ 

How to obtain the correct reflection?
Sinai's assumptions

In the problem for $N$ particles in $\mathbb{R}^1$ (let us call this problem (a)), Sinai decided to consider not a polyhedral angle of special boundaries. So in (q) we have a polyhedral angle with boundaries. So in (q) we have a polyhedral angle with boundaries.

3. The configuration point reflects elastically from the boundaries.

2. Their intersection is nonempty.

1. All the bodies are smooth.

Following requirements:

- Convex bodies in $\mathbb{R}^n$ (let us call this problem (b))

In the problem for $N$ balls in $\mathbb{R}^n$ (let us call this problem $\mathcal{H}$) convex bodies in $\mathbb{R}^n$ is not necessary equal to the finite number of arbitrary cylinders.

Sinai changed cylinders to the finite number of arbitrary cylinders in $\mathbb{R}^n$. (In the problem for $N$ particles in $\mathbb{R}^1$ (let us call this problem $\mathcal{H}$) convex bodies in $\mathbb{R}^n$ is not necessary equal to the finite number of arbitrary cylinders.)

Sinai's assumptions
Theorem 1. The number of reflections of billiard trajectory in polyhedral angle $\Omega$ is finite. The number of collisions of $x$ inside $\Omega$ is finite. Hence the number of collisions of $x$ inside $\Omega$ is finite. Hence the point $x$ could visit each of $\Omega$, $\Omega^\circ$, and its neighborhoods at most $\lceil C/\nu \rceil$ times.

Hence the theorem $C$, $\nu$. Then we are left with disconnected sides of $\text{codim} = 1$. The distance between each two of them is bounded from above with some constant $C$. Then, we prove that the number of reflections is uniformly bounded by induction. Remove each of $\nu$, $\Omega$. Hence the number of reflections is finite by induction. Consider the union of $\nu$, $\Omega$.

Let us prove that the number of reflections is finite by induction. Consider the union of $\nu$, $\Omega$.

Let us prove that the number of reflections is finite by induction. Consider the union of $\nu$, $\Omega$.

Proof (by Galperin). Let us consider the sphere $S^{n-1}$ with the center in the center of the angle. Project this angle in the center in the center of the angle. Project this angle in the center in the center of the angle. Project this angle in the center in the center of the angle.
By Lemma 2, the number of reflections is finite.

\((f) := \langle VO, \alpha O \rangle = BO \cos |VO| \cdot |\alpha O| < BO \cos |BO| \cdot |\alpha O| = \langle BO, \alpha O \rangle =: (B)f\)

\(BO \angle BOF < BO \angle BOII\), hence some fixed point inside \(O\). Because \(AO \angle BOII = BO \angle BOII\), hence \(POO \angle BO = \angle BO II\) and \(PO \angle BO = N\). Let \(f\) be \(O\). Move every vector to \(O\). So that \(\langle PO, \alpha O \rangle = \langle VO, \alpha O \rangle\) and \(\langle PO, \alpha O \rangle = \langle VO, \alpha O \rangle\) is the vertex of the angle.

Proof (of "Small" Sinai's Problem) Let \(O\) to be the vertex of the angle location of hyperplanes but not on the initial conditions. Then the number of collisions is finite and depends on the each collision. And some linear function \(f(x)\) strictly increases after each collision. And some linear function \(f(x)\) strictly increases after each collision. And some linear function \(f(x)\) strictly increases after each collision. And some linear function \(f(x)\) strictly increases after each collision.

Another proof by Sevryuk M.B.
Theorem 3.
1. The maximum number of collisions in the system of 3 particles is (precise estimate)

$$\arccos \frac{\epsilon w/\lambda + \tau w/\lambda}{\epsilon w/\lambda} = \arccos \frac{\epsilon w/\lambda + \tau w/\lambda}{\epsilon w/\lambda}$$

2. The number of collisions in the system of N particles is finite and uniformly bounded with the constant \(C\).

$$\left( \frac{\epsilon w/\lambda + \tau w/\lambda}{\epsilon w/\lambda} \right)^{-1} \cdot 2 = (Nw, \ldots, w)' \cdot C$$

Proof. 1. Note that the maximum number \(\lambda_{\alpha}(\alpha)\) of reflections of billiard particle in the linear angle for \(Q\) equals \(\lambda/\alpha \). We then have a billiard motion inside the dihedral angle \(\epsilon w/\lambda, \tau w/\lambda\) if \(\{ \epsilon x \leq \tau x \leq \lambda x | \epsilon \in (\epsilon w, \tau w, \lambda w) \} \). The configuration space for the system of 3 particles is the dihedral angle \(\epsilon w/\lambda, \tau w/\lambda\) for the system of 2 particles, or after the transformation \(A = \text{diag}(\sqrt{m_1}, \sqrt{m_2}, \sqrt{m_3})\), we have a billiard motion inside the dihedral angle \(\epsilon w/\lambda, \tau w/\lambda\).

Then

\[
\text{accos} = \arccos \frac{\epsilon w/\lambda + \tau w/\lambda}{\epsilon w/\lambda} = \arccos \frac{\epsilon w/\lambda + \tau w/\lambda}{\epsilon w/\lambda}
\]
Sinai's problem

Theorem 4.

1. (a) Thenumber of reflections of the billiard trajectory in the polyhedral "curvilinear" angle $\mathcal{Q}$ (with some restrictions on the geometry of the angle) is finite.

(b) This number is less than some constant $C_2(m_1, \ldots, m_N)$ which depends only on the geometry of the angle $\mathcal{Q}$.

2. (a) The number of collisions of $N$ balls in $\mathbb{R}^n$ with masses $m_1, \ldots, m_N$ and radiiuses $r_1, \ldots, r_N$ is finite.

(b) This number is less than some constant $C_2(m_1, \ldots, m_N)$ which depends only on the geometry of the angle $\mathcal{Q}$ and is independent of the initial conditions.

Theorem 4. I. (a) The number of reflections of the billiard "large" Sinai's problem...
Restrictions on the geometry of the "curvilinear" angle.

Let us consider a "curvilinear" angle on the plane, such that its two sides

$s_1$ and $s_2$ has the common tangent $t$. If the billiard particle moves along

the trajectory which is parallel to $t$, then after finite number of

reflections it will exit the angle. But if it becomes closer to $t$ the number

of reflections goes to infinity.

Let us consider "curvilinear" angle on the plane, such that its two sides

inside the "curvilinear" angle is bounded by some constant $C_1$, then

the number of collisions inside the "curvilinear" angle is bounded by

the number of reflections.

The idea of the proof of the Theorem 4: If the number of reflections

inside the "polyhedral" angle is bounded by some constant $C_1$, then

the number of collisions inside the "curvilinear" angle is bounded by

$C_1$. But this approach suffers from technical problems, related to the

almost tangent reflections from the "curvilinear" sides.

In $n$-dimensional space such a condition could be replaced by the

following condition: Let $q$ be the corner point. Then the angle $\theta$

having

$q$ as its vertex is of "correct geometry" if in the small neighborhood of $\theta$

we can insert a nonzero "that" polyhedral angle.

For the fact angle it is sufficient that the angle between the tangent lines

of reflections goes to infinity.
Nondegeneracy condition: $$d = C$$

More easy and effective condition was introduced by Burstingen, Kanonenko.

\[
\begin{align*}
\left( \frac{\min \theta}{\max \theta} \right)^N & = (N, 1, \ldots, N, 1) = (N, 1, \ldots, N, 1, \ldots, N) = \cfrac{1}{16} \left( \frac{\min \theta}{\max \theta} \right)^N \\
\end{align*}
\]

Later they proved that

\[
\begin{align*}
\left( \frac{\min \theta}{\max \theta} \right)^N & = (N, 1, \ldots, N, 1) = (N, 1, \ldots, N, 1, \ldots, N) = \cfrac{1}{16} \left( \frac{\min \theta}{\max \theta} \right)^N \\
\end{align*}
\]

authors first proved the estimate uniform in initial conditions.

Roughly speaking, it means that if a point is$$ \frac{d}{C} $$-close to their intersection, using this notion of nondegeneracy the $ C $$/p $$C $$ is called C -nondegenerate if

Definition 5. The "curvilinear" angle $ \theta $ is called C -nondegenerate if

\[ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} \]

Then and all $ y \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$. And.

and Kanonenko. Let us fix some constant $ C < 0 $. And for arbitrary point

More easy and effective condition was introduced by Burstingen, Perel'ger.

Nondegeneracy condition: $ C/p $$C $$/p $$C $$ is called C -nondegenerate if

\[ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} \]

Then and all $ y \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$ \gamma $$ \frac{\text{dist}(b, f)}{\text{dist}(\gamma, W, f)} = \frac{1}{\gamma} $$. And
Actually, we can say that the number of collisions also finite in just corners. But in the corner we have proved that the number of collisions is finite. Since the source of problem could be the distance between the boundaries always bounded from zero to infinity $t = t_0$. In order to have an infinite number of reflections, we have $P_i = 1$. In finite time $t$, in finite time $t_1$ the length of the billiard trajectory also finite. Since $\|\alpha\|$ is the length of the billiard some compact domain with dispersing and semi-dispersing boundaries. The billiard which corresponds to such a gas is the billiard inside finite time for the gas in some reservoir.

Concluding remarks