Dynamical systems - Exercise 1

Answers

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Problem 1:
Consider the space of all infinite sequences of 0’s and 1’s:
\[ B = \{ \bar{a} = (a_0, a_1, a_2, \ldots) | a_i \in \{0, 1\} \}, \]
with the distance function:
\[ \text{dist}(\bar{a}, \bar{b}) = \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^{|i|}}. \]
Consider the shift map:
\[ f(\bar{a}) = (a_1, a_2, \ldots). \]

(a) Prove that the shift map is continuous.

Solution.
\[ f : B \to B, \]
We need to prove that
\[ \forall \epsilon > 0 \ \exists \delta > 0 \ s.t. \text{ if } \text{dist}(\bar{a}, \bar{b}) < \delta \text{ then } \text{dist}(f(\bar{a}), f(\bar{b})) < \epsilon. \]

Way 1 First note the following fact. For \( \{\bar{a}, \bar{b}\} \in B \)
\[ a_i = b_i \text{ for } i = 0, n \iff \text{dist}(\bar{a}, \bar{b}) < \frac{1}{2^n}. \]
Now take \( \delta < 1/2^{n+1} \), then \( a_i = b_i \) for \( i = 0, n + 1 \). Applying shift map \( f \), we get \( a_i = b_i \) for \( i = 1, n + 1 \) and hence, \( \text{dist}(f(\bar{a}), f(\bar{b})) < 1/2^n \). Now take \( n \) s.t. \( 1/2^n < \epsilon \) and this finishes the proof.

Way 2 For any \( \epsilon > 0 \) take \( \delta = \frac{\epsilon}{2} \), then
\[ \text{dist}(f(\bar{a}), f(\bar{b})) = \sum_{i=0}^{\infty} \frac{|a_{i+1} - b_{i+1}|}{2^{|i|}} = 2 \sum_{i=0}^{\infty} \frac{|a_{i+1} - b_{i+1}|}{2^{|i|+1}} = 2(\text{dist}(\bar{a}, \bar{b}) - \frac{|a_0 - b_0|}{2}) = 2 \text{dist}(\bar{a}, \bar{b}) - |a_0 - b_0| \leq 2 \text{dist}(\bar{a}, \bar{b}) < \epsilon \]
(b) Find the fixed points of the shift map.

Solution. Fixed points \( \bar{a} = (a_0, a_1, a_2, \ldots) \) satisfy \( f(\bar{a}) = \bar{a} \), hence:
\[
f(\bar{a}) = (a_1, a_2, \ldots) = (a_0, a_1, a_2, \ldots) = \bar{a}.
\]
So \( a_0 = a_1 = a_2 = \ldots \) and fixed points are \((1, 1, 1, \ldots)\) and \((0, 0, 0, \ldots)\). ■

(c) Find the periodic orbits of the shift map of period 2 and 3.

Solution. Periodic points of period 2 correspond to \( f^2(\bar{a}) = \bar{a} \) (and \( f(\bar{a}) \neq \bar{a} \)), hence:
\[
f^2(\bar{a}) = (a_2, a_3, \ldots) = (a_0, a_1, a_2, \ldots) = \bar{a}.
\]
So \( a_0 = a_2 = \ldots = a_{2n} = \ldots \) and \( a_1 = a_3 = \ldots = a_{2n-1} = \ldots \) and periodic points of period 2 are
\[
(1, 0, 1, 0 \ldots),
(0, 1, 0, 1 \ldots).
\]
And the periodic orbit is \( \{(1, 0, 1, 0 \ldots), (0, 1, 0, 1 \ldots)\} \).

Periodic points of period 3 correspond to \( f^3(\bar{a}) = \bar{a} \) (and \( f(\bar{a}) \neq \bar{a}, f^2(\bar{a}) = \bar{a} \)), hence:
\[
f^3(\bar{a}) = (a_3, a_4, a_5, \ldots) = (a_0, a_1, a_2, \ldots) = \bar{a}.
\]
So \( a_0 = a_3 = \ldots = a_{3n} = \ldots \) and \( a_1 = a_4 = \ldots = a_{3n-1} = \ldots \) and \( a_2 = a_5 = \ldots = a_{3n-2} = \ldots \) periodic points of period 3 are
\[
(1, 0, 0, 1, 0 \ldots),
(0, 1, 0, 0, 1 \ldots),
(0, 0, 1, 0, 0, 1 \ldots),
(1, 1, 0, 1, 1, 0 \ldots),
(1, 0, 1, 1, 0, 1 \ldots),
(0, 1, 1, 0, 1, 1 \ldots).
\]
And the periodic orbits are:
\[
\{(1, 0, 1, 0, 0 \ldots), (0, 0, 1, 0, 0, 1 \ldots), (0, 1, 0, 0, 1, 0 \ldots)\};
\{(1, 1, 0, 1, 1, 0 \ldots), (1, 0, 1, 1, 0, 1 \ldots), (0, 1, 1, 0, 1, 1 \ldots)\}
\]
■
(d) Explain the relations of the above to binary representation of numbers in $[0, 1]$.

**Solution.** A lot of answers were considered correct in this part (actually, you should give 1-1 correspondence).

E.g.:

Let $\bar{a}$ be a binary representation of some number $R \in [0, 1)$ then

$$R = ||\bar{a}|| = \sum_{i=0}^{\infty} \frac{|a_i|}{2^{|i|+1}}.$$

More properties:

Shift map gives a multiplication by 2 and taking a fractional part.

Periodic orbits correspond to rational numbers. ■

**Problem 2:**

Prove that the Lorenz system:

$$\begin{align*}
\frac{dx}{dt} &= -\sigma(x - y) \\
\frac{dy}{dt} &= -xz + rx - y, \quad \sigma, r, b > 0 \\
\frac{dz}{dt} &= xy - bz.
\end{align*}$$

is dissipative.

**Solution.**

Take as a Lyapunov function

$$\varphi(x, y, z) = \frac{1}{2}(x^2 + \sigma y^2 + \sigma z^2) - r\sigma z$$

(0.0.1)

First let’s show that the level sets of $\varphi \{ \varphi(x, y, z) = c \}$ are homeomorphic to a circle for $c$ sufficiently large.

Rewrite level sets of $\varphi$:

$$\frac{1}{2}x^2 + \frac{1}{2}\sigma y^2 + \frac{1}{2}\sigma(z - r)^2 = c + \frac{1}{2}\sigma r^2$$

For $c$ large they are ellipsoids with center in $(0, 0, r)$, and, hence, are homeomorphic to sphere.

Another option: consider the map $h$:

$$\begin{align*}
x' &= \frac{1}{\sqrt{2}}x \\
y' &= \sqrt{2}y \\
z' &= \frac{1}{\sqrt{2}}(z - r)
\end{align*}$$

And show that $h$ is homeomorphism which takes $\varphi(x, y, z) = c$ to a sphere.

Now let’s show that $\dot{\varphi} < 0$
\( \dot{\varphi} = \frac{\partial \varphi}{\partial x} \dot{x} + \frac{\partial \varphi}{\partial y} \dot{y} + \frac{\partial \varphi}{\partial z} \dot{z} = -\sigma x(x - y) + \sigma y(-xz + rx - y) + (\sigma z - \sigma r)(xy - bz) = -\sigma x^2 - xy + y^2 + bz^2 - rbz = G \)

now,

\[
G = -\sigma \left( \frac{x - y}{2} \right)^2 + \frac{3}{4} y^2 + b \left( z - \frac{1}{2} r \right)^2 \right) + \frac{1}{4} b \sigma r^2. \]

\( \downarrow \infty \) as \( c \to \infty \)

hence for sufficiently large \( c \), \( G \) is indeed negative. \( \blacksquare \)

**Remark 0.1** Some people wrote **INCORRECTLY** that:

\[
G = -\sigma \left( \frac{x - y}{2} \right)^2 + \frac{3}{4} y^2 + b \left( z - \frac{1}{2} r \right)^2 \right) + \sigma rbz \leq 0
\]

and then ”showed” that \( \sigma rbz < 0 \) as from (0.0.1):

\[
\frac{1}{2} (x^2 + \sigma y^2 + \sigma z^2) - c = \sigma rz,
\]

\[
b \frac{1}{2} (x^2 + \sigma y^2 + \sigma z^2) - cb = \sigma rbz,
\]

So, they argued, for \( c \) sufficiently large it will be negative. However, this is **incorrect**, because \( z \sim c \), yet \( \sigma z^2 \sim c^2 \gg c \)

**Problem 3:**

Consider a mechanical Hamiltonian system with dissipation

\[
\begin{cases}
\frac{dx}{dt} = p \\
\frac{dp}{dt} = -V'(x) - \delta p.
\end{cases}
\]

Under what conditions on the potential \( V(x) \) does the system have a ball of dissipation? Give examples and explain.

**Solution.** Take as a Lyapunov function, the Hamiltonian function:

\[
H(x, p) = \frac{p^2}{2} + V(x) + \alpha p
\]

\( V(x) \) should be such a function that, \( \frac{p^2}{2} + V(x) + \alpha p = c \) is a compact, simply connected and grows with \( c \).

E.g. if \( V(x) \) is a polynomial of even degree with a **positive** leading coefficient.

More general condition: \( V(x) \) is continuous and

\[
\lim_{x \to \infty} V(x) = \lim_{x \to -\infty} V(x) = +\infty
\]
Let’s calculate

$$\dot{H} = \frac{\partial H}{\partial x} \dot{x} + \frac{\partial H}{\partial p} \dot{p} = (V'(x) + \alpha p)p + (p + \alpha x)(-V'(x) - \delta p)$$

$$= -[(\delta - \alpha)p^2 + \alpha x(V'(x) + \delta p)]$$

Hence, in order to have $\dot{H} < 0$ we need

$$(\delta - \alpha)p^2 + \alpha x(V'(x) + \delta p) > 0.$$ 

Examples:

1. Take as $V(x) = x^2/2$, then the level sets of $H$ are circles and there is a ball of dissipation.

2. Take $V(x) = -x^2/2$, then the level sets are hyperbolas which are not homeomorphic to a circle.