Dynamical systems - Exercise 3

Anna Rapoport

November 29, 2003

1. Let $\gamma(t)$ be a periodic solution of the system $\dot{x} = f(x)$. In studying the stability of this solution we are led to the linearized system

$\dot{\xi} = Df(\gamma(t))\xi$

To determine the stability of the periodic orbit, it would be tempting to compute "time dependent" eigenvalues of $Df(\gamma(t))$ and claim that $\gamma(t)$ is stable if their real part is negative for all time. The following example demonstrates that this is false.

Let $\gamma(t) = \gamma(t + 2\pi)$ and

$Df(\gamma(t)) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2 t & 1 - \frac{3}{2}\cos t\sin t \\ -1 + \frac{3}{2}\sin t\cos t & -1 + \frac{3}{2}\sin^2 t \end{pmatrix}$.

(a) Compute the eigen-values of $Df(\gamma(t))$.

Solution. Let us calculate the eigenvalues of $A = Df(\gamma(t))$. They satisfy the characteristic equation $\det(A - \lambda I) = 0$:

$$\det(A - \lambda I) = (-1 + \frac{3}{2}\cos^2 t - \lambda)(-1 + \frac{3}{2}\sin^2 t - \lambda) - (\frac{9}{4}\cos^2 t\sin^2 t - 1) = 0,$$

$$\lambda^2 + \frac{1}{2}\lambda + \frac{1}{2} = 0,$$

$$\lambda_{1,2} = -\frac{1}{4} \pm \frac{i\sqrt{7}}{4}.$$

One can see that they do not depend on $t$, and $\text{Re}[\lambda_{1,2}] < 0$. ❑

(b) Compute a Poincare map in a neighborhood of $\gamma(t)$ and find the stability of $\gamma(t)$.

[Hint: one solution of (1) is given by:

$\xi(t) = (-\exp(t/2)\cos t, \exp(t/2)\sin t) \equiv (\phi_1, \phi_2)$]
The substitution
\[ \xi = \begin{pmatrix} 1 & \phi_1 \\ 0 & \phi_2 \end{pmatrix} y \]
will reduce the time dependent system to a system of the form:
\[ \begin{pmatrix} 1 & \phi_1 \\ 0 & \phi_2 \end{pmatrix} y = \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} y \]
which can be easily solved for \( y \) \((a_{11}, a_{21}) \) is the first column of \( D f(\gamma(t)) \))

**Solution.**

\[
\begin{align*}
\dot{y} &= \begin{pmatrix} 1 & \phi_1 \\ 0 & \frac{\phi_1}{\phi_2} \end{pmatrix} \begin{pmatrix} a_{11} & 0 \\ a_{21} & 0 \end{pmatrix} y = \begin{pmatrix} a_{11} - \frac{\phi_1}{\phi_2} a_{22} & 0 \\ a_{21} & 0 \end{pmatrix} y \\
\end{align*}
\]

\[
\begin{align*}
\dot{y} &= \left( -1 + \frac{3}{2} \cos^2 t + \cot t(-1 - \frac{3}{2} \sin t \cos t) \right) y = \left( \frac{\exp(-t/2)}{\sin t} (-1 - \frac{3}{2} \sin t \cos t) \right) y = \begin{pmatrix} \exp(-t/2) & -1 - \cot t \\ \sin t & 0 \end{pmatrix} y \\
\end{align*}
\]

\[
\begin{align*}
\left\{ \begin{array}{l}
\dot{y}_1 = (-1 - \cot t)y_1 \\
\dot{y}_2 = \frac{\exp(-t/2)}{\sin t} (-1 - \frac{3}{2} \sin t \cos t)y_1 \\
\end{array} \right.
\end{align*}
\]

\[
\ln |y_1| = -t - \ln |\sin t| + C_1
\]

\[
y_1 = \frac{C \exp(-t)}{\sin t}
\]

\[
y_2 = \int \frac{C \exp(-3t/2)}{\sin^2 t} (-1 - \frac{3}{2} \sin t \cos t) dt = C \left[ \int \frac{\exp(-3t/2)}{\sin^2 t} dt - \int \frac{3}{2} \exp(-3t/2) \cot t dt \right] = C \left[ \frac{\exp(-3t/2)}{\sin^2 t} \right] - \int \frac{3}{2} \exp(-3t/2) \cot t dt
\]

First integral by parts: \( u = \exp(-3t/2), \ dv = -\frac{1}{\sin^2 t}, \) hence \( du = -\frac{3}{2} \exp(-3t/2), \ v = \cot t.\)

\[
y_2 = C \exp(-3t/2) \cot t + \int \frac{3}{2} \exp(-3t/2) \cot t dt - \int \frac{3}{2} \exp(-3t/2) \cot t dt
\]

\[
y_2 = C \exp(-3t/2) \cot t
\]

Hence

\[
\xi(t) = \begin{pmatrix} 1 & -\exp(t/2) \cos t \\ 0 & \exp(t/2) \sin t \end{pmatrix} \begin{pmatrix} C \frac{\exp(-t)}{\sin t} \\ C \exp(-3t/2) \cot t \end{pmatrix} = \begin{pmatrix} C \exp(-t) \sin t \\ C \exp(-t) \cos t \end{pmatrix}
\]
Hence the fundamental matrix of solutions is:

\[
X(t) = \begin{pmatrix}
-\exp(t/2) \cos t & C \exp(-t) \sin t \\
\exp(t/2) \sin t & C \exp(-t) \cos t
\end{pmatrix}
\]

\[
X(t + 2\pi) = \begin{pmatrix}
-\exp(t/2 + \pi) \cos t & C \exp(-t - 2\pi) \sin t \\
\exp(t/2 + \pi) \sin t & C \exp(-t - 2\pi) \cos t
\end{pmatrix}
\]

Want to find monodromy matrix \( M \), s.t.:

\[
X(t + 2\pi) = X(t)M
\]

\[
M = X^{-1}(t)X(t + 2\pi) = \begin{pmatrix}
-\exp(-t/2) \cos t & \exp(-t/2) \sin t \\
\frac{1}{C} \exp(t) \sin t & \frac{1}{C} \exp(t) \cos t
\end{pmatrix} \begin{pmatrix}
-\exp(t/2 + \pi) \cos t & C \exp(-t - 2\pi) \sin t \\
\exp(t/2 + \pi) \sin t & C \exp(-t - 2\pi) \cos t
\end{pmatrix} = \begin{pmatrix}
\exp(\pi) & 0 \\
0 & \exp(-2\pi)
\end{pmatrix}
\]

We get the characteristic multipliers \( \rho_1 = \exp(\pi) \) and \( \rho_2 = \exp(-2\pi) \).

And we see that \( |\rho_1| > 1 \), so the periodic orbit is not stable.