Dynamical systems - Exercise 4

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1. Consider the quadratic map:

\[ F : x_{n+1} = rx_n(1-x_n) \quad x \in I = [0,1] \]  

for large growth rate values:

\[ r > 2 + \sqrt{5}. \]

Prove that the invariant set of \( F \) in \( I \) is a Cantor set \( C \) and that the dynamics on \( C \) is chaotic:

(a) Show that if \( x \in I \) and \( F(x) \notin I \) then \( F^n(x) \to -\infty \).

Solution.

Let us note that \( F(y) < y \) for \( y < 0 \).

Take function \( h(y) = F(y) - y = y[(r-1) - ry] \). Note

- \( h(y) = 0 \) corresponds to the fixed points 0 and \( (r-1)/r \).
- \( h(y) < 0 \) for \( y < 0 \) and \( y > (r-1)/r \).

And we conclude, that \( F(y) < y \) for \( y < 0 \).

Hence, take \( x \in I \) and \( F(x) \notin I \), we have \( F(x) > 1 \), hence

\[ F^2(x) = rF(x)(1-F(x)) < 0, \]

\[ >1 <0 \]

So \( F(x) > F^2(x) > F^3 > \ldots > F^n(x) > \ldots \)

If this sequence would be bounded from below, then there should be a fixed point which is negative, but we know that there are just two fixed points 0 and \( (r-1)/r \) which are non-negative. And hence

\[ \lim_{n \to \infty} F^n(x) = -\infty \]

for \( x \in I \) s.t. \( F(x) \notin I \).
Figure 1: Quadratic map for $r > 2 + \sqrt{5}$

(b) Show that $J_1 = \{x | x \in I \text{ and } F(x) \in I \}$ is composed of two closed intervals: $J_1 = I_0 \cup I_1$ and that for all $x \in J_1$, $|F'(x)| > \lambda > 1$.

Solution.
Have a look at the Figure 1. In order to find $J_1$ we need to find $x \in I$, s.t.

$F(x) = rx(1-x) \leq 1 \leftrightarrow$

$x^2 - x + \frac{1}{r} \geq 0 \leftrightarrow$

$(x - (\frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{r}}))(x - (\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{r}})) \geq 0.$

So we get: $J_1 = I_0 \cup I_1 = [0, \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{r}}] \cup [\frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{r}}, 1]$.

Now let us consider $|F'(x)| = |r - 2rx| = |r(1-2x)|$

For $x \in J_1$ we have:

1. If $x \in I_0$ then:

   $0 \leq x \leq \frac{1}{2} - \frac{1}{2} \sqrt{1 - \frac{4}{r}},$

   $0 \leq 2x \leq 1 - \sqrt{1 - \frac{4}{r}},$
0 \geq -2x \geq -1 + \frac{1}{2} \sqrt{\frac{1}{r} - 4},
\quad 1 \geq 1 - 2x \geq \sqrt{1 - \frac{4}{r}} > 0.

for \ r > 2 + \sqrt{5}.

2. If \ x \in I_1 \ then:
\quad 1 \geq x \leq \frac{1}{2} + \frac{1}{2} \sqrt{1 - \frac{4}{r}},
\quad 2 \geq 2x \leq 1 + \sqrt{1 - \frac{4}{r}},
\quad -2 \leq -2x \leq -1 - \sqrt{1 - \frac{4}{r}},
\quad -1 \leq 1 - 2x \leq -\sqrt{1 - \frac{4}{r}} < 0.

for \ r > 2 + \sqrt{5}.

Hence, form 1. and 2. we get \ |1 - 2x| \geq \sqrt{1 - \frac{4}{r}} \ and
\quad |F'(x)| = |r(1 - 2x)| \geq \lambda' = r \sqrt{1 - \frac{4}{r}} = \sqrt{r^2 - 4r + 4 - 4} =
\quad \sqrt{(r - 2)^2 - 4} > \sqrt{(2 + \sqrt{5} - 2)^2 - 4} = 1

Actually, \lambda' = \{\max \lambda : |F'(x)| \geq \lambda > 1, x \in J_1\}.

(c) Construct \ J_n \ inductively as a union of closed intervals \ I_{0i_1...i_{n-1}} \ and \ prove \ that \ their \ intersection
\quad C = \bigcap_{n=1}^{\infty} J_n = \bigcap_{n=1}^{\infty} \bigcup_{i_j \in \{0,1\}} I_{0i_1...i_{n-1}}

is a Cantor set.

Solution.
\quad J_0 = [0, 1] = I,
\quad J_1 = \{x | F(x) \in [0, 1]\} = F^{-1}(I) = I_0 \cup I_1,

See Figure 1:
\quad J_2 = \{x | F^2(x) \in [0, 1]\} = F^{-2}(I) = F^{-1}(J_1) = F^{-1}(I_0 \cup I_1) = I_{00} \cup I_{01} \cup I_{10} \cup I_{11}
\[ J_n = \{ x | F^n(x) \in [0, 1] \} = F^{-n}(I) = F^{-1}(J_{n-1}) = \bigcup_{i_j \in 0, 1} I_{i_0i_1...i_{n-1}}. \]

Now let us take their intersection \( C = \bigcap_{n=1}^{\infty} J_n \). Note that \( C \) is the set of those points in \([0, 1]\) which never escape under iterations of \( F \).

**Lemma 1** (Properties of \( J_n \):)

1. They are nested, i.e. \( J_n \subset J_{n-1} \).
2. Each \( J_n \) consists of \( 2^n \) disjoined closed intervals.
3. The length of each interval in \( J_n \) is less than \( \lambda^{-n} \).

**Proof.**

1. Follows from the construction.
2. By induction \( J_1 \) consists of \( 2^1 \) intervals, let \( J_{n-1} \) consists of \( 2^{n-1} \) intervals, and each of them mapped under \( F^{-1} \) into two subintervals one contained in \( I_0 \) and one in \( I_1 \). So there are \( 2^n \) intervals.
3. Take some interval \( I_{i_0i_1...i_{n-1}} \in J_n \), let \( x, y \) to be its endpoints. Note that all endpoints of \( J_n \) belong to \( C \). Then by the Mean Value Theorem applied to \( F^n \) on the interval \([x, y]\), there exists a point \( c \in (x, y) \), s.t.

\[ F^n(y) - F^n(x) = (F^n)'(c)(y - x) \]

We showed in (b) that \( |F'(x)| > \lambda > 1 \) for all \( x \in J_1 \). Since \( c \in J_n \), we have \( F^i(c) \in J_1 \) for all \( 0 \leq i \leq n - 1 \), hence by the chain rule \( |(F^n)'(c)| > \lambda^n > 1 \).

\[ |F^n(y) - F^n(x)| > \lambda^n |y - x| \]

Note that \( \{y, x\} \in C \) hence \( |F^n(y) - F^n(x)| < 1 \) and we get

\[ |y - x| < \lambda^{-n} \]

,** Lemma 2** \( C \) is a Cantor set, i.e. \( C \) is

1. Non-empty.
2. Compact.
3. Perfect.
4. Totally disconnected.

**Proof.**

1. As the intersection of closed nested sets.
2. Since each $J_n$ is a finite union of closed intervals, hence $J_n$ are closed for any $n$. Thus $C$ is an infinite intersection of closed intervals, and hence close. Moreover, $C$ is bounded ($C \subseteq [0, 1]$). Hence $C$ is compact.

3. All the endpoints of each $J_n$ are contained in $C$. Let $x \in C$, then denote for each $n$ by $J_{n,i}$ the component of $C$ which contains $x$. Since $|J_{n,i}| \to 0^1$ as $n \to \infty$, then there are endpoints from $J_{n,i}$ which are arbitrarily close to $x$. Hence $x$ is in the closure of $C \setminus \{x\}$.

4. $J_n = F^{-1}(J_{n-1})$, from (b) $F^{-1}$ shrinks the length of each component of $J_{n-1}$ by at least the amount $\lambda^{-1} < 1$, so the length of the components tends to zero as $n \to \infty$.

Or, more formally one can repeat the argument as in the proof of the length of the intervals. Just suppose that exists some fixed interval $[a, b] \subset C$. And get

$$|F^n(b) - F^n(a)| > \lambda^n D > 1$$

Where $D = |b - a|$. For all $n$ sufficiently large. But $C$ is invariant, thus $\{F^n(b), F^n(a)\} \subset C \in [0, 1]$. We have a contradiction.

(d) Prove that there is a homeomorphism between points in $C$ and $\Omega_+$, the set of infinite sequences of $\{0, 1\}$.

Solution.

$J_1 = I_0 \cup I_1$. Since $C \in J_1$, hence $C \subset I_0 \cup I_1$.

For each $x \in C$ define $h : C \to \Omega_+$ by

$$h(x) = (i_0 i_1 i_2 \ldots)$$

where $i_n = \begin{cases} 0, & \text{if } F^n(x) \in I_0; \\ 1, & \text{if } F^n(x) \in I_1. \end{cases}$

Clearly $h$ is well defined.

Note that $I_{i_0 i_1 \ldots i_{n-1}}$ is one of the intervals in $J^n$.

In $\Omega_+$ take dist as in Exercise 1:

$$\text{dist}(\bar{a}, \bar{b}) = \sum_{i=0}^{\infty} \frac{|a_i - b_i|}{2^i}.$$ 

Lemma 3 $h$ is homeomorphism, i.e.

1. $h$ is one-to one and onto.
2. $h$ is continuous.
3. $h^{-1}$ is continuous.

Proof. 

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1Here I denote by $|K|$ the length of $K$. 

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1. Let \( i_0i_1i_2 \ldots \) be an element of \( \Omega^+ \). To show that \( h \) is one-to-one and onto we need to show that \( h^{-1}(i_0i_1i_2 \ldots) \) contains exactly one point. If \( x \in h^{-1}(i_0i_1i_2 \ldots) \), hence \( x \in I_{i_0i_1 \ldots i_n} \) for all \( n \). Thus

\[
h^{-1}(i_0i_1i_2 \ldots) = \bigcap_{n=0}^{\infty} I_{i_0i_1 \ldots i_n}.
\]

We need to show that this intersection is non-empty and contains exactly one point. It is non-empty as the intersection of closed nested intervals.

To show that there is only one point, suppose that \( h^{-1}(i_0i_1i_2 \ldots) \) contains two points \( x \) and \( y \). Let \( I_{i_0i_1 \ldots i_n} = [a_n, b_n] \). Then \( |x - y| \leq |b_n - a_n| \) for all \( n \). From part 4 of Lemma 1 we have that \( |b_n - a_n| < \lambda^{-n-1} \to 0 \) as \( n \to \infty \). Hence \( |x - y| = 0 \) and \( x = y \).

2. Take \( \{x, y\} \in C \). We need to show

\[
\forall \varepsilon \exists \delta : |x - y| < \delta \Rightarrow \text{dist}(h(x), h(y)) < \varepsilon.
\]

If we choose \( n \) so that \( \frac{1}{n} < \varepsilon \), so (from ex 1) it is sufficient to show that \( \exists \delta > 0 \) such that the sequences \( h(x) \) and \( h(y) \) agree on the first \( n + 1 \) digits whenever \( |x - y| < \delta \), i.e. both \( x \) and \( y \) are in \( I_{i_0i_1 \ldots i_n} \) whenever \( |x - y| < \delta \).

Now, let \( [a_1, b_1], [a_2, b_2], \ldots, [a_{n+1}, b_{n+1}] \) be the \( n + 1 \) intervals in \( J_{n+1} \), such that \( b_{i-1} < a_i \). Set

\[
\delta = \frac{1}{2} \min\{|a_i - b_{i-1}|\}
\]

Since the intervals do not overlap and there are \( 2^{n+1} \) of them, \( \delta > 0 \). Hence if \( x \) and \( y \) both in \( I_n \) they should be in the same interval, i.e. if \( x \in I_{i_0i_1 \ldots i_n} \) so does \( y \), and their first \( n + 1 \) entries really coincide.

3. Take \( \{u, v\} \in \Omega^+ \). We need to show

\[
\forall \varepsilon \exists \delta : \text{dist}(u, v) < \delta \Rightarrow |h^{-1}(u) - h^{-1}(v)| < \varepsilon.
\]

Denote \( x := h^{-1}(u) \), \( y := h^{-1}(v) \), \( \{x, y\} \in C \). For any \( \varepsilon \) find \( n \) s.t. \( \lambda^{-n-1} < \varepsilon \). Then choose \( \delta = \frac{1}{2^n} \), \( \text{dist}(u, v) < \frac{1}{2^n} \) means that first \( n + 1 \) digits of \( u \) and \( v \) coincide, hence \( x \) and \( y \) belong to the same \( I_{i_0i_1 \ldots i_n} \) and \( |x - y| = |h^{-1}(u) - h^{-1}(v)| < \lambda^{-n-1} < \varepsilon \).

(e) Prove that the shift map on \( \Omega^+ \), and the map \( F \mid_C \) are topologically conjugate.

**Proof.** We need to show that \( h \circ F = \sigma \circ h \). Let \( x \in C \) and \( h(x) = i_0i_1i_2 \ldots \)

\[
h \circ F(x) = h(F(x)) = (i_1i_2 \ldots)
\]

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\[ \sigma \circ h = \sigma(h(x)) = (i_1 i_2 \ldots). \]

(f) Conclude that the map \( F|_C \) is chaotic - include a few properties which demonstrate the chaotic behavior.

**Solution.** As \( F|_C \) is topologically conjugate to the shift map, hence

1. The periodic points of \( F|_C \) are dense in \( C \);
2. \( F|_C \) is topologically transitive on \( C \);
3. \( F|_C \) is expansive.
4. \( F|_C \) is chaotic.

\[ \blacksquare \]