Dynamical systems - Exercise 6

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1. Give an example of a polynomial with real coefficients which does not have negative Schwarzian derivative on $[0,1]$. 

**Solution.** Consider $P(x) = x^3 + 6x + 1$. Then:

$$P'(x) = 3x^2 + 6 
eq 0,$$

$$P''(x) = 6x,$$

$$P'''(x) = 6.$$

Now let us calculate a Schwarzian derivative on $[0, 1]$:

$$SP(x) = \frac{P'''(x)}{P'(x)} \left( \frac{P''(x)}{P'(x)} \right)^2 - \frac{3}{2} \left( \frac{P'(x)}{P'(x)} \right)^2 = \frac{6}{3x^2 + 6} - \frac{3}{2} \left( \frac{6x}{3x^2 + 6} \right)^2 = \begin{cases} 0 & \text{for } x \in [0, 1] \\ \geq 0 & \text{otherwise} \end{cases}.$$

2. Consider a piecewise linear map $f : I \to I$, such that $f'(x) = \lambda_i$ for $x \in \Delta_i$ and $I = \bigcup_{i=1}^{n} \Delta_i$, and the intervals $\Delta_i$ have pairwise disjoint intervals. Assume that the image of the end points of the interval $\Delta_i$ belong to the set of the end points $\Lambda$ of all the intervals. 

(a) Construct a topological Markov chain describing the dynamics of $f$. 

**Solution.** Because the image of the end points of the interval $\Delta_i$ belong to the set of the end points $\Lambda$, we have that each interval $\Delta_i$ covers the integer number of the intervals $\Delta_j$, $j = 1, n$.

In this case the topological Markov chain corresponds to the graph $G$ which has $n$ vertices $v_j$, corresponding to each $\Delta_i$ and directed edges $u_{ij}$ if $\Delta_i \to \Delta_j$. Let $A$ be a corresponding transition matrix with the elements

$$A_{ij} = \begin{cases} 1, & \text{if } \Delta_i \to \Delta_j; \\ 0, & \text{otherwise}. \end{cases}$$
(b) Let $B = \text{diag}(\frac{1}{\lambda_1}, \frac{1}{\lambda_2}, \ldots, \frac{1}{\lambda_n}) A$, where $A$ is the transition matrix associated with this topological Markov chain. Show that if $p(x) = a_i, x \in \Delta_i$ is a positive invariant density for $f$ then the vector $a = (a_1, \ldots, a_n)$ is a left eigenvector of $B$ with eigenvalue $1$ and $a$ satisfies the normalization $\sum a_i \text{length}(\Delta_i) = 1$.

Solution. First, since $p(x)$ is a positive invariant density it satisfies Perron-Frobenious equation. Also mention that $\{y : y \in f^{-1}(x), x \in \Delta_i\}$ belong to those $\Delta_j$ which cover $\Delta_i$, i.e. $\Delta_j \rightarrow \Delta_i$. We get:

$$a_i = \sum_{y \in f^{-1}(x), x \in \Delta_i} \frac{p(y)}{|f'(y)|} = \sum_{j=1}^n \frac{a_j}{|\lambda_j|} A_{ji}. \quad (1)$$

Second, $a$ satisfies normalization condition since $p(x)$ is a positive invariant density:

$$1 = \int \rho(x) dx = \sum_{i=1}^n \int_{\Delta_i} a_i dx = \sum_{i=1}^n a_i |\Delta_i|$$

Third, we want to prove that

$$aB = \lambda a = a. \quad (2)$$

Left hand side:

$$aB = (a_1, a_2, \ldots, a_n) \left( \begin{array}{ccc} 1 \frac{1}{|\lambda_1|} & 0 & \cdots & 0 \\ 0 & \frac{1}{|\lambda_2|} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{|\lambda_n|} \end{array} \right) \left( \begin{array}{cccc} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{array} \right) =$$

by (1)

$$= \left( \sum_{j=1}^n \frac{a_j}{|\lambda_j|} A_{j1}, \sum_{j=1}^n \frac{a_j}{|\lambda_j|} A_{j2}, \ldots, \sum_{j=1}^n \frac{a_j}{|\lambda_j|} A_{jn} \right) = (a_1, a_2, \ldots, a_n) = a$$

And (2) is proved. ■

(c) Bonus (non-trivial): show that if the transition matrix $A$ associated with this topological Markov chain is transitive then such an eigenvector must exist.
(d) Construct a specific (numerical) example for such a map, present its graph and find the invariant measure for your map.

Solution.

Let us consider the following example:

\[ f(x) = \begin{cases} 
3x, & \Delta_1 = [0, \frac{1}{4}); \\
\frac{x + \frac{1}{2}}{2}, & \Delta_2 = [\frac{1}{4}, \frac{1}{2}); \\
-x + \frac{3}{2}, & \Delta_3 = [\frac{1}{2}, \frac{3}{4}); \\
-3x + 3, & \Delta_4 = [\frac{3}{4}, 1]. 
\end{cases} \]

The graph is presented at Figure 1.

Using the (a) and (b) let us find the invariant density \( \rho(x) \) in the form \( \rho(x) = a_i \) for each of the intervals \( \Delta_i \). Also calculate \( \lambda_1 = 3, \lambda_2 = 1, \lambda_3 = -1, \lambda_4 = -3 \).

We have:
- \( \Delta_1 \to \Delta_1, \Delta_2 \to \Delta_2, \Delta_3 \to \Delta_3 \);
Figure 2: The Markov chain

- $\Delta_2 \rightarrow \Delta_4$;
- $\Delta_3 \rightarrow \Delta_4$;
- $\Delta_4 \rightarrow \Delta_1, \Delta_4 \rightarrow \Delta_2, \Delta_4 \rightarrow \Delta_3$.

The transition matrix in this case is

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

Now using equation (1) we get:

$$a_1 = \sum_{j=1}^{4} \frac{a_j}{\lambda_j} A_{j1} = \frac{a_1}{3} + \frac{a_4}{3}$$

$$a_2 = \sum_{j=1}^{4} \frac{a_j}{\lambda_j} A_{j2} = \frac{a_1}{3} + \frac{a_4}{3}$$

$$a_3 = \sum_{j=1}^{4} \frac{a_j}{\lambda_j} A_{j3} = \frac{a_1}{3} + \frac{a_4}{3}$$
\[ a_4 = \sum_{j=1}^{4} \frac{a_j}{|\lambda_j|} A_{ja} = a_2 + a_3 \]

Also \( \rho(x) \) should satisfy \( \int_0^1 \rho(x) \, dx = 1 \), hence \( a_1 + a_2 + a_3 + a_4 = 4 \) and we find:

\[ a_1 = a_2 = a_3 = \frac{4}{5}, \quad a_4 = \frac{8}{5}. \]

And

\[ \rho(x) = \begin{cases} 4/5, & x \in [0, 3/4); \\ 8/5, & x \in [3/4, 1]. \end{cases} \]

Then the invariant measure is

\[ \mu(A) = \int_A \rho(x) \, dx \]

for any measurable \( A \).

E.g. \( \mu(f^{-1}(\Delta_4)) = \mu(\Delta_2) + \mu(\Delta_3) = 1/5 + 1/5 = 2/5 = \mu(\Delta_4) \)

\[ \blacksquare \]

(e) Demonstrate numerically (for your example) the convergence to the invariant measure.

Solution.

Choose \( N \) to be the number of iterates and \( Q \) - the number of "bins".

Take some typical \( x_0 \). Calculate the sequence \( x_0, f(x_0), \ldots, x_{N-1}(x_0) \).

Obtain the histogram: find vector of frequencies \( F = (F_1, \ldots, F_Q) \), \( F_i \) gives the percent \( f^i(x_0) \) which fall into \([i - 1/Q, i/Q]\). Multiply now by \( Q \). The result for \( x_0 = 0.2789, N = 6000, Q = 200 \) is presented at Figure 3. \[ \blacksquare \]

3. Let \( f : M \to M, g : N \to N \) be smoothly conjugate maps:

\[ f(x) = h^{-1} \circ g \circ h(x), \]

where \( h : M \to N \) is a \( C^r, r > 1 \) diffeomorphism. Assume \( \rho_g(u) \) is a positive invariant density for \( g \).

(a) Show that

\[ \rho_f(x) = |h'(x)| \rho_g(h(x)) \]

is a positive invariant density for \( f \) on \( M \) (i.e. that \( \rho_f(x) \) satisfies the Perron-Frobenious equation of \( f \) and that its corresponding absolutely continuous invariant measure is a probability measure on \( M \)).

Solution.
Figure 3: Numerical demonstration for $x_0 = 0.2789$, $N = 6000$, $Q = 200$. Averages are depicted by lines, note that they coincide with the results from part (d).
i. First, let us show that $\rho_f(x)$ satisfies the Perron-Frobenious equation of $f$.

Let us remind two equalities for homeomorphism: if $y := h(x)$ then

$$f'(x) = (h^{-1})'g'h' = \frac{h'(x)}{h'(h^{-1}(g[h(x)]))}g'(y) = \frac{h'(x)}{h'(f(x))}g'(y); \quad (4)$$

$$f^{-1} = h^{-1} \circ g^{-1} \circ h \Rightarrow g^{-1} = h \circ f^{-1} \circ h^{-1}. \quad (5)$$

Want to prove

$$\rho_f(x) = \sum_{u \in f^{-1}(x)} \frac{\rho_f(u)}{|f'(u)|}. \quad (6)$$

Let $y = h(x), \; v = h(u)$, then (6) is equivalent to

$$\rho_f(h^{-1}(y)) = \sum_{h^{-1}(v) \in f^{-1}(h^{-1}(y))} \frac{\rho_f(h^{-1}(v))}{|f'(h^{-1}(v))|}$$

Now using (3), (4), we obtain:

$$|h'(h^{-1}(v))|\rho_g(h(h^{-1}(y))) = \sum_{v \in h(f^{-1}(h^{-1}(y)))} \frac{|h'(h^{-1}(v))|\rho_g(h(h^{-1}(v)))}{|h'(h^{-1}(v))| |g'(h(h^{-1}(v)))|}$$

$$|h'(h^{-1}(y))|\rho_g(y) = \sum_{v \in g^{-1}(y)} \frac{|h'(f(h^{-1}(v)))|\rho_g(v)}{|g'(v)|}$$

Note $|h'(h^{-1}(y))| = |h'(h^{-1}(g(v)))|$, but also $h^{-1}(g(v)) = f(h^{-1}(v))$, hence $|h'(h^{-1}(y))| = |h'(f(h^{-1}(v)))|$ and we get:

$$\rho_g(y) = \sum_{v \in g^{-1}(y)} \frac{|h'(f(h^{-1}(v)))|\rho_g(v)}{|h'(f(h^{-1}(v)))| |g'(v)|}$$

And hence:

$$\rho_g(y) = \sum_{v \in g^{-1}(y)} \frac{\rho_g(v)}{|g'(v)|}$$

This one is true, because $\rho_g(y)$ is a positive invariant density for $g$ and, hence satisfies the Perron-Frobenious equation (which is exactly the above one) for $g$. So we can conclude, going backwards that $\rho_f(x)$ satisfies the Perron-Frobenious equation for $f$.

ii. Left to prove that its corresponding absolutely continuous invariant measure is a probability measure on $M$. Need to show that
\[
\mu_f(M) = \int_M \rho_f(x) \, dx = 1
\]

Again \( y = h(x) \) and \( dy = h'(x) \, dx \), then

\[
\int_M \rho_f(x) \, dx = \int_M |h'(x)| \rho_g(h(x)) \, dx = \int_{h(M)} \rho_g(y) \, \text{sign}(h'(x)) \, dy = \int_N \rho_g(y) \, dy = 1
\]

(b) Consider the case \( M = N = [0, 1] \). Show that the Lyapunov exponent of \( f \) for \( \rho_f \)– typical point is equal to the Lyapunov exponent of \( g \) for \( \rho_g \)– typical point.

Solution.

Lyapunov exponent for \( \rho_f \)– typical point is equal to

\[
\lambda_f(x_0) = \int_0^1 \ln |f'| \, d\mu_f(x).
\]

Denote \( y := h(x) \), then \( \frac{dy}{dx} = h'(x) \). By part (a):

\[
d\mu_g(y) = \rho_g(y) \, dy,
\]

\[
d\mu_f(x) = \rho_f(x) \, dx = |h'(x)| \rho_g(h(x)) \, dx = |h'(x)| \frac{d\mu_g(y)}{dy} \, dx = |h'(x)| \rho_g(y) \frac{1}{h'(x)}.
\]

So we get: \( d\mu_f(x) = \text{sign}(h'(x)) \, d\mu_g(y) \).

Using (4) and that \( h \) is a homeomorphism (hence takes the boundary to the boundary) we get:

\[
\lambda_f(x_0) = \int_0^1 \ln |f'| \, d\mu_f(x) = \int_0^1 \ln \frac{h'(x)}{|h'(f(x))|} \, g'(h(x)) \, d\mu_f(x) = \int_{h(0)}^{h(1)} \ln |g'(y)| \, \text{sign}(h'(x)) \, d\mu_g(y) + \int_0^1 \ln |h'(x)| \, d\mu_f(x) - \int_0^1 \ln |h'(f(x))| \, d\mu_f(x)
\]

Note that by Birkhoff Ergodic Theorem (take \( \phi(x) = \ln |h'(x)| : [0, 1] \to \mathbb{R} \) and \( \phi \in L^1([0, 1], \mu_f) \)) the following two expressions are the same:

\[
\int_0^1 \ln |h'(x)| \, d\mu_f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \ln |h'(f^i(x))|,
\]

\[
\int_0^1 \ln |h'(f(x))| \, d\mu_f(x) = \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n-1} \ln |h'(f^i(x))|.
\]
And hence,

\[
\lambda_f(x_0) = \int_{h(0)}^{h(1)} \ln |g'(y)| \text{sign}(h') \, d\mu_g(y) = \int_0^1 \ln |g'(y)| \, d\mu_g(y) = \lambda_{g}(y_0)
\]