1 Lecture 1 - Isomorphic Theory

We denote by $L_p(\mu) \equiv L_p(\Omega, \Sigma, \mu)$, and unless said otherwise, the spaces will be infinite dimensional and separable, over $\mathbb{R}$. We will also assume that $1 < p < \infty$ and that $p \neq 2$.

The main goal of this lecture is to show that there are not "many" non-isomorphic $L_p(\mu)$ spaces.

Let us consider the two extreme cases:

1. (Ω, Σ, μ) is purely atomic - that is, a countable union $\Omega = \{\omega_1, ...\}$, where $\mu_i = \mu(\{\omega_i\})$ satisfies $0 < \mu_i < \infty$.

2. (Ω, Σ, μ) is purely non-atomic.

Lemma 1.1 *If the measure space is purely atomic then $L_p(\mu)$ is isometric to ℓ_p.*

Proof. The following operator $T : \ell_p \to L_p(\mu)$ by $T((a_i)_{i=1}^{\infty}) = (a_i/\mu_i^{1/p})_{i=1}^{\infty}$ is the desired isometry. ■

Turning to the purely non-atomic case we have the following

Lemma 1.2 *If (Ω, Σ, μ) is purely non-atomic then $L_p(\mu)$ is isometric to $L_p([0,1], \mathcal{B}, \lambda)$, where $\mathcal{B}$ is the Borel σ-algebra and λ is the Lebesgue measure.*

Proof. First, let $(\Omega, \bar{\Sigma}, \mu)$ be the measure space obtained from $(\Omega, \bar{\Sigma}, \mu)$ by identifying sets that differ on sets of measure 0. Recall that if $L_p(\mu)$ is separable then so is $(\Omega, \bar{\Sigma}, \mu)$ - as a quasi-metric space with respect to $d(A, B) = \mu(A \Delta B)$. Also, if $D = (\Omega, \bar{\Sigma}, \mu)$ then D is isomorphic to $C = ([0, a), \mathcal{B}, \lambda)$ for some $0 < a \leq \infty$. In other words, there is a measure preserving map $\phi : [0, a) \to \Omega$ that maps $\mathcal{B}$ onto $\bar{\Sigma}$. Thus, there is an isometry $T : L_p(C) \to L_p(D)$. Indeed, for every measurable $A \in [0, a)$ define $T1_A = 1_{\phi(A)}$, and extend linearly. It is standard to verify that this is an isometry. Thus, it is enough to consider $L_p([0, a])$ for $0 < a \leq \infty$. If $a < \infty$ then clearly $L_p([0, a]) \cong L_p([0, 1])$, as claimed. For $a = \infty$, note that

$$L_p([0, \infty)) \cong L_p([0, 1]) \oplus_p L_p([1, 2]) \oplus_p .....$$

But on the other hand,

$$L_p([0, 1]) \cong L_p((1/2, 1]), \quad L_p([1, 2]) \cong L_p((1/4, 1/2]), ...$$

and thus $L_p([0, \infty)) \cong L_p([0, 1])$. ■

In the general case, Ω can be written as a disjoint union of Ω_1 and Ω_2, the first is purely atomic (that could be finite) and the second is purely non-atomic. Thus, $L_p(\Omega)$ is isometric to $L_p(\Omega_1) \oplus_p L_p(\Omega_2)$; the first is isometric either to ℓ_p or to ℓ_p^m for some m and the other to $L_p([0, 1])$.

Corollary 1.3 *For every* (Ω, Σ, μ) *as above,* $L_p(\Omega, \Sigma, \mu)$ *is isometric to one of the following:*

$$\ell_p,\ L_p, L_p \oplus_p \ell_p^1, L_p \oplus_p \ell_p^2, \ldots, L_p \oplus \ell_p,$$

where by L_p *we denote* $L_p([0,1], \mathcal{B}, \lambda)$.

1.1 Isomorphic equivalence

Recall that $L_p = L_p([0,1], \mathcal{B}, \lambda)$. For two spaces X and Y, the direct sum of X and Y is defined by

$$X \oplus Y = \{(x,y) : x \in X,\ y \in Y, \|(x,y)\| = \|x\|_X + \|y\|_Y\}.$$

Note that for a finite number of components the direct sum can be taken with respect of any p-norm since the resulting spaces will be isomorphic.

The first result of this section is the following:

Theorem 1.4 L_p *is isomorphic to* $\ell_p \oplus L_p$ *and to* $\ell_p^m \oplus L_p$ *for any* m.

Before proving Theorem 1.4 we need a few preliminary results on complemented subspaces of a Banach space. For the sake of simplicity we shall denote the fact that X is isomorphic to Y by $X \approx Y$.

Definition 1.5 *Let* X *be a Banach space. Two closed subspaces* Y *and* Z *of* X *are called complemented if every* $x \in X$ *has a unique presentation* $x = y + z$*, where* $y \in Y$ *and* $z \in Z$.

Lemma 1.6 *If* Y *and* Z *are complemented then* $Y \cap Z = \{0\}$ *and there are bounded projections* $P_Y : X \to Y$ *and* $P_Z : X \to Z$.

Proof. Note that the mapping $x \to y = P_Y x$ (where y is the unique point in Y that satisfies $x = y + z$) is well defined and linear. Indeed, to prove linearity, let $x = y + z$ and $u = v + w$. Then $x + u = (y+v) + (z+w)$, and clearly the first summand is in Y while the second is in Z. By the uniqueness of the presentation, $P_Y(x+u) = y + v = P_Y x + P_Y u$.

To prove that P_Y is bounded we use the Closed Graph Theorem. Since Y is a closed subspace of a Banach space, it is a Banach space and thus P_Y is a linear operator between Banach spaces. Consider the graph $\{(x, P_Y x) : x \in X\} \subset X \times Y$ and assume that $(x_n, P_Y x_n) \to (x, y)$. Clearly $x_n = P_Y x_n + (Id - P_Y)x_n$. From the definition of P_Y it is clear that $(Id - P_Y)x_n \in z$, and since $x_n \to x$ and $P_Y x_n \to y$, then $(Id - P_Y)x_n \to x - y$. Moreover, since Z is closed, $x - y \in Z$, and from the uniqueness of the presentation of x, $y = P_Y x$. Thus, the graph is closed, proving that P_Y is bounded.

Finally, to show that $P = P_Y$ is a projection,

$$P^2 x = P(Px) = P(Px + 0) = Px,$$

because the unique presentation of $Px \in Y$ is $Px + 0$. ∎

Remark 1.7 *Note that if P is a bounded projection $P : X \to X$ then PX and $(Id - P)(X)$ are complemented subspaces.*

Definition 1.8 *If Y is an image of a bounded projection of X then Y is called a complemented subspace of X.*

Lemma 1.9 *ℓ_p is isometric to a complemented subspace of L_p and the projection is of norm 1.*

Proof. Let $(A_i)_{i=1}^{\infty}$ be a sequence of disjoint subsets of $[0, 1]$ of positive Lebesgue measure. The argument of Lemma 1.2 shows that ℓ_p is isometric to the subspace Y of functions that take a constant value on each of the A_i. Define $P : L_p \to Y$ by

$$(Pf)(t) = \frac{1}{\lambda(A_i)} \int_{A_i} f d\lambda$$

for $t \in A_i$, and 0 if t does not belong to any of the sets A_i. This map is clearly linear and onto Y and $P^2 = P$, i.e., P is a projection. To prove that it is bounded, observe that if $1/p + 1/q = 1$ then

$$\left| \int_{A_i} f d\lambda \right|^p \leq \lambda^{p/q}(A_i) \int_{A_i} |f|^p d\lambda.$$

Thus,

$$\|Pf\|_{L_p} = \left(\sum_i \frac{1}{\lambda^p(A_i)} \left| \int_{A_i} f d\lambda \right|^p \lambda(A_i) \right)^{1/p} \leq \left(\sum_i \lambda^{1-p}(A_i) \lambda^{p/q}(A_i) \int_{A_i} |f|^p d\lambda \right)^{1/p}$$
$$\leq \|f\|_{L_p}$$

■

Remark 1.10 *Observe that if Y and Z are complemented subspaces of X then $X \approx Y \oplus Z$. Indeed, consider the map $x \to y + z$. Then,*

$$\|x\| = \|y + z\| \leq \|y\| + \|z\| = \|y \oplus z\|.$$

To prove the reverse direction, if P is the bounded projection onto Y and $K = \|P\|$ then $\|y\| = \|Px\| \leq K\|x\|$ and $\|z\| = \|(Id - P)x\| \leq (K + 1)\|x\|$. Hence, $\|y\| + \|z\| \leq K'\|x\|$.

Proof of Theorem 1.4. We will prove that $L_p \approx L_p \oplus \ell_p$. The other claims follow the same path and their proof is omitted. First, by Lemma 1.9, there are complemented subspaces X and Y of L_p such that X is isometric to ℓ_p. Thus, by Remark 1.10,

$$L_p \approx X \oplus Y \approx \ell_p \oplus Y \approx \ell_p \oplus_p Y.$$

It is standard to verify that $\ell_p \approx \ell_p \oplus_p \ell_p$ (for example, by decomposing $\mathbb{N}$ to two sequences and considering the subspaces spanned by the corresponding coordinate vectors). Hence,

$$\ell_p \oplus_p Y \approx \ell_p \oplus_p (\ell_p \oplus_p Y) \approx \ell_p \oplus L_p.$$

■

1.2 ℓ_p and L_p are not isomorphic

Next, we will show that for $1 \le p < 2$ and for $2 < p < \infty$, ℓ_p and L_p are not isomorphic.

Theorem 1.11 *For $1 \le p < \infty$, ℓ_2 is isometric to a subspace of L_p, and for $1 < p < \infty$ this subspaces is a complemented subspace. On the other hand, for $1 \le p < \infty$ and $p \neq 2$, ℓ_2 is not isomorphic to a subspace of ℓ_p.*

Since L_p contains ℓ_2 as a subspace but ℓ_p does not, the two spaces are not isomorphic.
Proof. To prove the first part, let $(g_i)_{i=1}^{\infty}$ be independent, standard Gaussian variables defined on a separable probability space that is purely non-atomic (e.g., generated by the Gaussian measure on $\mathbb{R}^{\aleph_0}$). Note that each one of these Gaussians is in L_p for any $1 \le p < \infty$ and let X be the closure in L_p of the span of $\{g_1, ..., \}$. Since the Gaussian measure is rotation invariant then for every $(a_i)_{i=1}^{\infty} \in \ell_2$, $\sum a_i g_i$ has the same distribution as $g\|a\|_2$. Thus,

$$\left(\mathbb{E}|\sum a_i g_i|^p\right)^{1/p} = \|a\|_2 \|g\|_{L_p}.$$

The desired isometry is $(a_i) \to C_p^{-1} \sum a_i g_i$, where C_p is the L_p norm of a standard Gaussian.

Next, we will show that if $p > 1$ then X is a complemented subspace. To that end, we define the following projection by

$$Pf = \sum_{i=1}^{\infty} [\mathbb{E}(f \cdot g_i)] g_i.$$

It is straightforward to verify that P is a projection onto X. It remains to show that P is bounded. To that end, note that for every n,

$$\mathbb{E}\left(\sum_{i=1}^{n}[\mathbb{E}(f\cdot g_i)]g_i\right)^p = C_p^p\left(\sum_{i=1}^{n}(\mathbb{E}(f\cdot g_i))^2\right)^{p/2}$$
$$= C_p^p \sup_{\|a\|_2\le 1}|\sum_{i=1}^{n} a_i\mathbb{E}(f\cdot g_i)|^p = C_p^p \sup_{\|a\|_2\le 1}|\mathbb{E}(f\cdot\sum_{i=1}^{n} a_i g_i)|^p$$
$$\le C_p^p\mathbb{E}|f|^p \sup_{\|a\|_2\le 1}\left(\mathbb{E}\left(\sum_{i=1}^{n} a_i g_i\right)^q\right)^{p/q} \le C_p^p C_q^p \mathbb{E}|f|^p \sup_{\|a\|_2\le 1}(\sum_{i=1}^{n} a_i^2)^{p/2}$$
$$= (C_p C_q)^p\mathbb{E}|f|^p,$$

where $1/p + 1/q = 1$. Thus, by taking n to infinity it follows that $\|P\| \le C_p C_q$.

2 Lecture 2

Thus far, we have seen that there are at most two non-isomorphic spaces - ℓ_p and L_p. Now we will show that there are exactly two. Recall that we proved that ℓ_2 isometrically embeds in L_p for $1 \le p < \infty$. We also proved that ℓ_2 is isometric to a complemented subspace of L_p for $1 < p < \infty$ but this will not be relevant here. To complete the proof it is enough to show that ℓ_2 does not embed in ℓ_p.

Theorem 2.1 *Let $1 \le p < \infty$, $p \neq 2$. Then ℓ_2 does not isomorphically embed in ℓ_p.*

We will actually prove a stronger claim than Theorem 2.1.

Definition 2.2 *We say that $(y_n) \subset \ell_p$ is a block basis if there is an increasing sequence $0 = p_0 < p_1 < ....$ such that $y_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i$.*

Theorem 2.3 *Let $1 \le p < \infty$ and consider a sequence $(x_n) \subset \ell_p$ that converges weakly to 0 such that for every n, $0 < a \le \|x_n\| \le b < \infty$. Then, for every $\varepsilon_n > 0$ there is a block basis (y_n) and a subsequence (m_n) such that*

$$\left\|\frac{x_{m_n}}{\|x_{m_n}\|} - y_n\right\| \le \varepsilon_n.$$

Also, for every $\varepsilon > 0$ there is a subsequence (x_{m_n}) such that $x_{m_n}/\|x_{m_n}\|$ is $1+\varepsilon$ equivalent to the standard unit vector basis of ℓ_p.

Proof. The first part of the claim follows from a "gliding hump" argument, which will be sketched here. Fix $m_1 = 1$ and $p_0 = 0$. We shall denote $x = \sum_{i=1}^{\infty} e_i^*(x)e_i$ and $\bar{x} = x/\|x\|$. Fix $\varepsilon_1 > 0$ and note that there there is some p_1 such that

$$\left(\sum_{i=p_1+1}^{\infty} |e_i^*(\bar{x}_1)|^p\right)^{1/p} \leq \varepsilon_1,$$

as the tail of a converging series. Set $y_1 = \sum_{i=1}^{p_1} e_i^*(\bar{x}_1)e_i$. To construct y_2 and m_2, observe that there is some $m_2 > m_1$ such that

$$\left(\sum_{i=1}^{p_1} |e_i^*(\bar{x}_{m_2})|^p\right)^{1/p} \leq \frac{\varepsilon_2}{2}.$$

Indeed, this is the case because for every n, $e_i^*(\bar{x}_n) \to 0$ for every i - by the weak convergence of (x_n) allowing one to control a finite number of indices i uniformly. In addition, there is some $p_2 > p_1$ for which

$$\left(\sum_{i=p_2+1}^{\infty} |e_i^*(\bar{x}_{m_2})|^p\right)^{1/p} \leq \frac{\varepsilon_2}{2}.$$

Define $y_2 = \sum_{i=p_1+1}^{p_2} e_i^*(\bar{x}_{m_2})e_i$ and it is easy to verify that $\|y_2 - \bar{x}_{m_2}\| \leq \varepsilon_2$. The rest of the construction follows a similar path.

Turning to the second part, let $\varepsilon > 0$, fix $\varepsilon_n > 0$ to be named later and consider (y_n) and $(\bar{x}_{m_n})$ as in the first part. In particular,

$$y_n = \sum_{i=p_{n-1}+1}^{p_n} a_i e_i, \quad 1 - \varepsilon_n \leq \|y_n\| \leq 1 + \varepsilon_n.$$

Moreover, for every (α_n),

$$\begin{aligned} \|\sum_n \alpha_n y_n\| = \|\sum_n \alpha_n \sum_{i=p_{n-1}+1}^{p_n} a_i e_i\| &= \left(\sum_n |\alpha_n|^p \sum_{i=p_{n-1}+1}^{p_n} |a_i|^p\right)^{1/p} \\ &= \left(\sum_n |\alpha_n|^p \|y_n\|^p\right)^{1/p} \equiv \Delta. \end{aligned}$$

Hence, setting $\delta = \sup_n \varepsilon_n$ it is evident that

$$(1-\delta)\left(\sum_{i=1}^{\infty} |\alpha_n|^p\right)^{1/p} \leq \Delta \leq (1+\delta)\left(\sum_{i=1}^{\infty} |\alpha_n|^p\right)^{1/p}.$$

Next, by the triangle inequality, followed by Hölder's inequality,

$$\|\sum \alpha_n(\bar{x}_{m_n} - y_n)\| \leq \sum |\alpha_n| \|\bar{x}_{m_n} - y_n\| \leq \left(\sum_n |\alpha_n|^p\right)^{1/p} \cdot \left(\sum_n \varepsilon_n^q\right)^{1/q},$$

where $1/p + 1/q = 1$. Thus, if we set $\eta = (\sum_n \varepsilon_n^q)^{1/q}$ then for every (α_n),

$$\|\sum_n \alpha_n \bar{x}_{m_n}\| \leq (1 + \delta + \eta) \left(\sum_n |\alpha_n|^p\right)^{1/p},$$

and

$$\|\sum_n \alpha_n \bar{x}_{m_n}\| \geq (1 - \delta - \eta) \left(\sum_n |\alpha_n|^p\right)^{1/p}.$$

To conclude the proof one has to assume that $(1 + \delta + \eta)/(1 - \delta - \eta) \leq 1 + \varepsilon$. ■

Proof of Theorem 2.1. Assume that ℓ_2 embeds in ℓ_p. Thus, there is an operator $T : \ell_2 \to \ell_p$ with $\|T\| \cdot \|T^{-1}\| < \infty$. Set $x_n = Te_n$ and recall that a bounded operator maps a weakly converging sequence to a weakly converging sequence. Hence, (x_n) converges weakly to 0 and by Theorem 2.3, there is a subsequence $(\bar{x}_{m_n})$ which is $1 + \varepsilon$ equivalent to the unit vector basis of ℓ_p. On the other hand, it is equivalent to the unit vector basis of ℓ_2. This implies that the unit vector basis in ℓ_2 is equivalent to the one in ℓ_p and that is clearly not true. ■

Remark 2.4 *In a similar way to the proof of Theorem 2.3 one can find a subsequence m_n as above with the additional property that there is a bounded projection on $\overline{\text{span}}(x_{m_1}, x_{m_2}, ...)$. [hint: show that this is true for (y_n) and therefore the same holds for a small perturbation].*

Corollary 2.5 *If X is an infinite dimensional complemented subspace of ℓ_p for $1 < p < \infty$ then it is isomorphic to ℓ_p.*

Proof. First, observe that every infinite dimensional subspace of ℓ_p has a normalized sequence that converges weakly to 0. Indeed, since $H_n = \{x : x_1 = 0, ..., x_n = 0, x_{n+1} \neq 0\}$ intersects X for every n, one can select $x_n \in H_n$ and normalize. The resulting sequence converges weakly to 0. By Remark 2.4, there is a subsequence (x_{m_n}) which is equivalent to the unit vector basis in ℓ_p and the closure of its span is complemented in X. Since X is complemented in ℓ_p, the closure of the span of (x_{m_n}), denoted by Y, is complemented in ℓ_p and isomorphic to it. Moreover, since X is complemented in ℓ_p there is a subspace Z such that $\ell_p \approx X \oplus_p Z$. Therefore,

$$\begin{aligned}\ell_p &\cong \ell_p \oplus_p \ell_p \oplus_p .... \approx (X \oplus_p Z) \oplus_p (X \oplus_p Z) \oplus_p .... \\ &\approx X \oplus_p [(Z \oplus_p X) \oplus_p (Z \oplus_p X) \oplus_p ...] \cong X \oplus_p \ell_p,\end{aligned}$$

and since Y is complemented in X and isomorphic to ℓ_p

$$X \approx Y \oplus W \approx \ell_p \oplus W \approx \ell_p \oplus_p \ell_p \oplus_p W \approx \ell_p \oplus_p (\ell_p \oplus_p W) \approx \ell_p \oplus_p X.$$

Since both spaces are isomorphic to $\ell_p \oplus X$ they are isomorphic to each other. ■

3 Lecture 3: Rosenthal's Inequality

So far, we have seen some of the five classical spaces that are complemented in L_p: L_p itself, ℓ_p, ℓ_2, $\ell_2 \oplus \ell_p$ and $\sum \oplus_p \ell_2$. At a later date we will show that all of them are not isomorphic. This leads to a natural question - are these all the non-isomorphic, complemented subspaces of L_p? Our next goal is to show that the answer to this question is negative.

Definition 3.1 *Let $(w_i) \subset \mathbb{R}^+$, $2 < p < \infty$ and define the $X_{p,w}$ norms:*

$$\|(a_i)\|_{X_{p,w}} = \max\left\{ \left(\sum_{i=1}^{\infty} |a_i|^p\right)^{1/p}, \left(\sum_{i=1}^{\infty} a_i^2 w_i^2\right)^{1/2} \right\}.$$

We will show the following facts:

Theorem 3.2 *1. The sequence space $X_{p,w}$ is isomorphic to a complemented subspace of L_p, $2 < p < \infty$, and the norm of the isomorphism and of the projection do not depend on the sequence (w_i).*

2. Under certain assumptions on the sequence (w_i), $X_{p,w}$ is not isomorphic to one of the other "classical" spaces.

3. Under the same assumption as in (2), all the spaces $X_{p,w}$ are isomorphic.

The space in (3) is denoted by X_p.

The first step in the proof of Theorem 3.2 is Khintchine's inequality.

Theorem 3.3 *For every $1 \le p < \infty$ there are constants c_p and C_p such that for every n and every $a_1, ..., a_n \in \mathbb{R}$,*

$$c_p \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} \le \left(\mathbb{E}|\sum_{i=1}^{n} \varepsilon_i a_i|^p\right)^{1/p} \le C_p \left(\sum_{i=1}^{n} a_i^2\right)^{1/2},$$

where $(\varepsilon_i)_{i=1}^n$ are independent, Bernoulli random variables.

Lemma 3.4 *For every* $(a_i)_{i=1}^n$,

$$Pr\left(|\sum_{i=1}^n a_i\varepsilon_i| > t\right) \le 2\exp\left(-\frac{t^2}{2\sum_{i=1}^n a_i^2}\right).$$

Proof. The proof is based on a standard estimate on the moment generating function. It is evident that for every $t \in \mathbb{R}$, $\exp(t)+\exp(-t) \le 2\exp(-t^2/2)$. Fix $\lambda > 0$ and observe that by independence,

$$\begin{aligned}\mathbb{E}\exp(\lambda\sum_{i=1}^n \varepsilon_i a_i) &= \mathbb{E}\prod_{i=1}^n \exp(\lambda\varepsilon_i a_i) = \prod_{i=1}^n \mathbb{E}\exp(\lambda\varepsilon_i a_i) \\ &= \prod \frac{\exp(\lambda a_i)+\exp(-\lambda a_i)}{2} \le \prod \exp(\lambda^2 a_i^2/2) \le \exp((\lambda^2/2)\sum_{i=1}^n a_i^2).\end{aligned}$$

Hence, by Chebyshev's inequality,

$$\begin{aligned}Pr\left(\sum_{i=1}^n a_i\varepsilon_i > t\right) &= Pr\left(\exp(\lambda\sum_{i=1}^n a_i\varepsilon_i) > \exp(\lambda t)\right) \le \mathbb{E}\exp(\lambda\sum_{i=1}^n \varepsilon_i a_i - \lambda t) \\ &\le \exp(-\lambda t + (\lambda^2/2)\sum_{i=1}^n a_i^2).\end{aligned}$$

The claim now follows by minimizing with respect to λ. The other tail estimate follows in a similar fashion. ■