Instructions: Please write your solutions in \LaTeX{} / Word or exquisite handwriting. Submission can be done individually or in pairs.

In this exercise we will study the field of constants, defined as follows.

**Definition 1** Let $F/K$ be a function field. The field of constants of $F/K$ is defined by

\[ \tilde{K} = \{ z \in F \mid z \text{ is algebraic over } K \} \]  

**Why care about the field of constants?** The motivation for this definition is the following. It could be the case that in the field extension $F/F_q$, where $F_q$ is the field of $q$ elements, the elements of $F_{q^2}$, say, are contained in $F$. Thus, even though we “started” from $F_q$, we got the elements of $F_{q^2}$ inside $F$.

1. Give an example of an algebraic function field $F/F_2$ such that $F_4 \subseteq F$.

Let $F/K$ be a field extension. Recall that the extension is called finite if $[F : K] < \infty$. The extension $F/K$ is called algebraic if any $x \in F$ is algebraic over $K$. That is, there exists a polynomial $f$ with coefficients in $K$, such that $f(x) = 0$.

2. Prove that any finite extension is algebraic.

3. For an element $a \in F$, consider the field $K(a)$ obtained by adjoining $a$ to $K$. Prove that $K(a)/K$ is a finite extension.

We are now ready to prove that $\tilde{K}$ is a field. This is not obvious – if $a, b$ are algebraic, namely, there exist $f_a, f_b$ polynomials over $K$, such that $f_a(a) = f_b(b) = 0$, what should be the polynomial over $K$ having root $a + b$?

4. Prove that $\tilde{K}$ is a field. *Guidance: given $a, b \in \tilde{K}$, use the previous two items to show that $K(a, b)$ is an algebraic extension of $K$.*

Informally speaking, in the rest of the exercise you will be asked to show that $K$ can be “replaced” by $\tilde{K}$ in the results we have seen so far during the course. From here on, $F/K$ is an algebraic function field, $\mathcal{O}$ is a valuation ring of $F/K$, with the corresponding place $P$, and discrete valuation $v$.

5. Show that $\tilde{K} \subseteq \mathcal{O}$, and that $\tilde{K} \cap P = \{0\}$.

6. Let $x \in \tilde{K}$. Prove that $v(x) = 0$. 

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7. In class we showed that $K$ is embedded in $F_P$, the residue class field of $P$. Extend this and show that $\tilde{K}$ is embedded in $F_P$.

8. Why does $\tilde{K}$ called the field of constants of $F/K$?