Q-type Lie superalgebras

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Abstract The purpose of this paper is to collect some recent results on the representation theory of Lie superalgebras of type Q. Results on the centres, simple weight modules and crystal bases of these superalgebras are included.

1 Introduction

This paper is devoted to the Lie superalgebras of type Q, also known as queer or strange Lie superalgebras. These Lie superalgebras, introduced by V. Kac in [13], have attracted considerable attention of both mathematicians and physicists in the last 40 years. They are especially interesting due to their resemblance to the general linear Lie algebras $\mathfrak{gl}_n$ on the one hand, and because of the unique properties of their structure and representations on the other. By the term "Q-type superalgebras" we mean four series of Lie superalgebras: $\mathfrak{q}(n)$ ($n \geq 2$) and its subquotients $\mathfrak{sq}(n)$, $\mathfrak{pq}(n)$, $\mathfrak{psq}(n)$ (the last one is a simple Lie superalgebra for $n \geq 3$, and in the notation of [13] it is $Q(n)$).

The Q-type Lie superalgebras are rather special in several aspects: their Cartan subalgebras $\mathfrak{h}$ are not abelian and have non-trivial odd part $\mathfrak{h}_{\bar{1}}$; they possess a non-degenerate invariant bilinear form which is *odd*; and they do not have quadratic Casimir elements. Because $\mathfrak{h}_{\bar{1}} \neq 0$, the study of highest weight modules of the Q-type Lie superalgebras requires nonstandard technique, including Clifford algebra methods. The latter are necessary due to the fact that the highest weight space of an irreducible highest weight module $L(\lambda)$ has a Clifford module structure. This

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peculiarity leads to the existence of two different candidates for a role of Verma module of the highest weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$: a module $M(\lambda)$ which is induced from a simple $\mathfrak{h}_{\bar{0}}$-module $\mathbb{C}_\lambda$ and a module $N(\lambda)$ which is induced from a simple $\mathfrak{h}$-module. The character of $M(\lambda)$ nicely depends on λ, and following the kind suggestion of J. Bernstein, we call $M(\lambda)$ a *Verma module* and $N(\lambda)$ a *Weyl module*. Each Verma module $M(\lambda)$ has a finite filtration with the factors isomorphic to $N(\lambda)$ up to a parity change; each Weyl module $N(\lambda)$ has a unique simple quotient, which we denote by $L(\lambda)$. The simple highest weight $\mathfrak{gl}_n$-module of highest weight λ will be denoted by $\dot{L}(\lambda)$.

Note that from categorical point of view it is more natural to call $N(\lambda)$ Verma modules since they are proper standard modules whereas $M(\lambda)$ are standard modules, see [2].

The representation theory of finite dimensional $L(\lambda)$ is well developed. In [24] A. Sergeev established several important results, including a character formula of $L(\lambda)$ for the so called *tensor modules*, i.e. submodules of tensor powers $(\mathbb{C}^{n|n})^{\otimes r}$ of the natural $\mathfrak{q}(n)$-module $\mathbb{C}^{n|n}$. The characters of all simple finite-dimensional $\mathfrak{q}(n)$-modules have been found by I. Penkov and V. Serganova in 1996 (see [21] and [22]) via an algorithm using a supergeometric version of the Borel-Weil-Bott Theorem. This result was reproved by J. Brundan, [1] using a different approach. Very recently, using Brundan's idea and weight diagrams a character formula and a dimension formula for a finite dimensional $L(\lambda)$ were provided by Y. Su and R.B. Zhang in [28]. On the other hand the character formula problem for infinite dimensional $L(\lambda)$ remains largely open, see the conjecture in [1].

The centres of the universal enveloping algebras of the Q-type Lie supealgebras were described by Sergeev and the first author in [5, 26]. An equivalence of categories of strongly typical $\mathfrak{q}(n)$-modules and categories of $\mathfrak{gl}_n$-modules were established recently in [3].

The simple weight modules with finite weight multiplicities of all finite dimensional simple Lie superalgebras were partly classified by Dimitrov, Mathieu, and Penkov in [4]. The most interesting missing case in the classification of [4] is the case of the queer Lie superalgebras $\mathfrak{psq}(n)$. The classification in this case was completed in [6] using a new combinatorial tool - the star action. This action is a mixture of the dot action and the regular action of W depending on the atypicality of the weights.

The combinatorics of the queer Lie superalgebras is also very interesting. One important aspect of the Sergeev duality is the semisimplicity of the category of tensor modules of $\mathfrak{q}(n)$. This naturally raises the question of uniqueness and existence of a crystal bases theory for this category. The crystal bases theory and the combinatorial description of the crystals of the simple tensor modules were obtained in a series of papers of the second author and J. Jung, S.-J. Kang, M. Kashiwara, M. Kim, [7–9].

The goal of the paper is to present a survey on the recent results on the representation theory of the Q-type Lie superalgebras discussed above.

1.1 Content of the paper

The organization of the paper is as follows. In Sect. 2 we include some important definitions and preliminary results. Section 3 is devoted to the description of the centers of the Lie superalgebras of type Q. In Sect. 4 we collect the main results related to the classification of all simple weight $\mathfrak{q}(n)$-modules with finite weight multiplicities. Automorphisms and affine Lie superalgebras of type Q are discussed in Sect. 5. The last section deals with the crystal base theory of the category of tensor representations of $\mathfrak{q}(n)$.

2 Preliminaries

The symbol $\mathbb{Z}_{\geq 0}$ stands for the set of non-negative integers and $\mathbb{Z}_{>0}$ for the set of positive integers.

Let $V = V_{\bar{0}} \oplus V_{\bar{1}}$ be a $\mathbb{Z}_2$-graded vector space. We denote by $\dim V$ the total dimension of V. For a homogeneous element $u \in V$ we denote by $p(u)$ its $\mathbb{Z}_2$-degree; in all formulae where this notation is used, u is assumed to be $\mathbb{Z}_2$-homogeneous. For a subspace $N \subset V$ we set $N_i := N \cap V_i$ for $i = \bar{0}, \bar{1}$. Let Π be the functor which switches parity, i.e. $(\Pi V)_{\bar{0}} = V_{\bar{1}}, (\Pi V)_{\bar{1}} = V_{\bar{0}}$. We denote by $V^{\oplus r}$ the direct sum of r-copies of V.

For a Lie superalgebra $\mathfrak{g}$ we denote by $\mathscr{U}(\mathfrak{g})$ its universal enveloping algebra and by $\mathscr{S}(\mathfrak{g})$ its symmetric algebra.

Throughout the paper the base field is $\mathbb{C}$ and $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ denote one (unless otherwise specified, an arbitrary one) of Q-type Lie superalgebras $\mathfrak{q}(n), \mathfrak{sq}(n)$ for $n \geq 2$, $\mathfrak{pq}(n), \mathfrak{psq}(n)$ for $n \geq 3$.

2.1 Q-type Lie superalgebras

Recall that $\mathfrak{q}(n)$ consists of the matrices with the block form

$$X_{A,B} := \begin{pmatrix} A & B \\ B & A \end{pmatrix}$$

where A, B are arbitrary $n \times n$ matrices; $\mathfrak{q}(n)_{\bar{0}} = \{X_{A,0}\} \cong \mathfrak{gl}_n$, $\mathfrak{q}(n)_{\bar{1}} = \{X_{0,B}\}$ and

$$[X_{A,0}, X_{A',0}] = X_{[A,A'],0}, \quad [X_{A,0}, X_{0,B}] = X_{0,[A,B]}, \quad [X_{0,B}, X_{0,B'}] = X_{0,BB'+B'B}.$$

Define $\mathrm{tr}' : \mathfrak{q}(n) \to \mathbb{C}$ by $\mathrm{tr}'(X_{A,B}) = \mathrm{tr}\, B$. In this notation,

$$\begin{aligned} \mathfrak{sq}(n) &: = \{x \in \mathfrak{q}(n) \mid \mathrm{tr}' x = 0\}, \\ \mathfrak{pq}(n) &: = \mathfrak{q}(n)/(\mathrm{Id}), \\ \mathfrak{psq}(n) &: = \mathfrak{sq}(n)/(\mathrm{Id}), \end{aligned}$$

where Id is the identity matrix.

These definitions are illustrated by the following diagram:

$$\begin{tikzcd} \mathfrak{sq}(n) \arrow[r, hook] \arrow[d, two heads] & \mathfrak{q}(n) \arrow[d, two heads] \\ \mathfrak{psq}(n) \arrow[r, hook] & \mathfrak{pq}(n) \end{tikzcd}$$

Clearly, the category of $\mathfrak{pq}(n)$-modules (resp., $\mathfrak{psq}(n)$-modules) is the subcategory of $\mathfrak{q}(n)$-modules (resp., of $\mathfrak{sq}(n)$-modules) which are killed by the identity matrix Id.

The map $(x,y) \mapsto \mathrm{tr}'(xy)$ gives an odd non-degenerate invariant symmetric bilinear form on $\mathfrak{q}(n)$ and on $\mathfrak{psq}(n)$.

For the quotient algebras $\mathfrak{pq}(n), \mathfrak{psq}(n)$ we denote by $X_{A,B}$ the image of the corresponding element in the appropriate algebra.

For Q-type Lie superalgebras the set of even roots $(\Delta^+_{\bar 0})$ coincides with the set of odd roots $(\Delta^+_{\bar 1})$. This phenomenon has two obvious consequence. The first one is that all triangular decompositions of a Q-type Lie superalgebra are conjugate with respect to inner automorphisms (this does not hold for other simple Lie superalgebras). The second one is that the Weyl vector $\rho := \frac{1}{2}(\sum_{\alpha\in\Delta^+_{\bar 0}} \alpha - \sum_{\alpha\in\Delta^+_{\bar 1}} \alpha)$ is equal to zero. We set $\rho_0 := \frac{1}{2}\sum_{\alpha\in\Delta^+_{\bar 0}} \alpha$.

We choose the natural triangular decomposition: $\mathfrak{q}(n) = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ where $\mathfrak{h}_{\bar 0}$ consists of the elements $X_{A,0}$ where A is diagonal, $\mathfrak{h}_{\bar 1}$ consists of the elements $X_{0,B}$ where B is diagonal, and $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) consists of the elements $X_{A,B}$ where A, B are strictly upper-triangular (resp., lower-triangular). We consider the induced triangular decompositions of $\mathfrak{sq}(n), \mathfrak{pq}(n), \mathfrak{psq}(n)$.

2.2 Notation

In the standard notation the set of roots of $\mathfrak{gl}_n = \mathfrak{q}(n)_{\bar 0}$ can be written as

$$\Delta^+ = \{\varepsilon_i - \varepsilon_j\}_{1\le i<j\le n}$$

and the set of simple roots as $\pi := \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\}$. Each root space has dimension $(1|1)$.

For $\alpha \in \Delta^+$ let $s_\alpha : \mathfrak{h}^*_{\bar 0} \to \mathfrak{h}^*_{\bar 0}$ be the corresponding reflection: $s_{\varepsilon_i-\varepsilon_j}(\varepsilon_i) = \varepsilon_j$, $s_{\varepsilon_i-\varepsilon_j}(\varepsilon_k) = \varepsilon_k$ for $k \neq i, j$. Denote by W the Weyl group of $\mathfrak{g}_{\bar 0}$ that is the group generated by $s_\alpha : \alpha \in \Delta^+$. Recall that W is generated by $s_\alpha : \alpha \in \pi$.

The space $\mathfrak{h}^*_{\bar 0}$ has the standard non-degenerate W-invariant bilinear form: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

Let E_{rs} be the elementary matrix: $E_{rs} = (\delta_{ir}\delta_{sj})^n_{i,j=1}$.

The elements

$$h_i := X_{E_{ii},0}$$

form the standard basis of $\mathfrak{h}_{\bar 0}$ for $\mathfrak{g} = \mathfrak{q}(n), \mathfrak{sq}(n)$. We use the notation h_i also for the image of h_i in the quotient algebras $\mathfrak{pq}(n), \mathfrak{psq}(n)$.

The elements $H_i := X_{0,E_{ii}}$ $(i = 1,\dots,n)$ form a convenient basis of $\mathfrak{h}_{\bar 1} \subset \mathfrak{q}(n)$; they satisfy the relations $[H_i, H_j] = 2\delta_{ij}h_i$.

For each positive root $\alpha = \varepsilon_i - \varepsilon_j$ we define $\overline{\alpha} = \varepsilon_i + \varepsilon_j$, and

$$\begin{array}{ll} h_\alpha := h_i - h_j,\ h_{\overline{\alpha}} := h_i + h_j, & H_\alpha := H_i - H_j, \\ e_\alpha := X_{E_{ij},0}, & E_\alpha := X_{0,E_{ij}}, \\ f_\alpha := X_{E_{ji},0}, & F_\alpha := X_{0,E_{ji}}. \end{array}$$

All above elements are non-zero in $\mathfrak{sq}(n), \mathfrak{pq}(n), \mathfrak{psq}(n)$ (since we excluded the cases $\mathfrak{pq}(2), \mathfrak{psq}(2)$).

The elements $h_\alpha, e_\alpha, f_\alpha$ $(\alpha \in \Delta^+)$ span $\mathfrak{sl}_n = [\mathfrak{gl}_n, \mathfrak{gl}_n]$; the elements E_α (resp., F_α) form the natural basis of $\mathfrak{n}^+_{\bar 1}$ (resp., of $\mathfrak{n}^-_{\bar 1}$) and the elements H_α span $\mathfrak{h}_{\bar 1} \cap \mathfrak{sq}(n)$.

For each α the elements $h_\alpha, e_\alpha, f_\alpha, h_{\overline{\alpha}}, H_\alpha, E_\alpha, F_\alpha$ span $\mathfrak{sq}(2)$ and one has

$$[e_\alpha, f_\alpha] = h_\alpha,\ \ [E_\alpha, F_\alpha] = h_{\overline{\alpha}},\ \ [H_\alpha, H_\alpha] = 2h_{\overline{\alpha}}$$
$$[E_\alpha, f_\alpha] = [e_\alpha, F_\alpha] = H_\alpha.$$

Set

$$Q(\pi) := \sum_{\alpha\in\Delta^+} \mathbb{Z}\alpha, \quad Q^+(\pi) := \sum_{\alpha\in\Delta^+} \mathbb{Z}_{\geq 0}\alpha.$$

Define a partial order on $\mathfrak{h}^*_{\bar 0}$ by $\nu \geq \mu$ iff $\nu - \mu \in Q^+(\pi)$.

2.3 The algebra $\mathscr{U}(\mathfrak{h})$

Let $\mathfrak{g}$ be a Q-type Lie superalgebra. Denote by HC the Harish-Chandra projection $\mathrm{HC} : \mathscr{U}(\mathfrak{g}) \to \mathscr{U}(\mathfrak{h})$ along the decomposition $\mathscr{U}(\mathfrak{g}) = \mathscr{U}(\mathfrak{h}) \oplus (\mathscr{U}(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^-\mathscr{U}(\mathfrak{g}))$.

The algebra $\mathscr{U}(\mathfrak{h})$ is a Clifford superalgebra over the polynomial algebra $\mathscr{S}(\mathfrak{h}_0)$: $\mathscr{U}(\mathfrak{h})$ is generated by the odd space $\mathfrak{h}_{\bar 1}$ endowed by the $\mathscr{S}(\mathfrak{h}_0)$-valued symmetric bilinear form $b(H, H') = [H, H']$. For each $\lambda \in \mathfrak{h}^*_{\bar 0}$ the evaluation of $\mathscr{U}(\mathfrak{h})$ at λ is a complex Clifford superalgebra. Notice that a non-degenerate complex Clifford superalgebra is either the matrix algebra (if $\dim \mathfrak{h}_{\bar 1}$ is even) or the algebra $Q(n)$ (this is an associative algebra whose Lie algebra is $\mathfrak{q}(n)$), see [5] for details. In particular, it possesses a supertrace which is even if $\dim \mathfrak{h}_{\bar 1}$ is even and odd if $\dim \mathfrak{h}_{\bar 1}$ is odd.

For $\lambda \in \mathfrak{h}^*_{\bar 0}$ let $\mathbb{C}(\lambda)$ be the corresponding one-dimensional $\mathfrak{h}_{\bar 0}$-module. Set

$$\mathscr{C}\ell(\lambda) := \mathscr{U}(\mathfrak{h}) \otimes_{\mathfrak{h}_{\bar 0}} \mathbb{C}_\lambda.$$

Clearly, $\mathscr{C}\ell(\lambda)$ is isomorphic to a complex Clifford algebra generated by $\mathfrak{h}_{\bar 1}$ endowed by the evaluated symmetric bilinear form $b_\lambda(H, H') := [H, H'](\lambda)$. Set

$$c(\lambda) := \dim \operatorname{Ker} b_\lambda.$$

For $\mathfrak{g} = \mathfrak{q}(n)$, $c(\lambda)$ is the number of zeros among $h_1(\lambda), \dots, h_n(\lambda)$. The complex Clifford algebra $\mathscr{C}\ell(\lambda)$ is non-degenerate if and only if $c(\lambda) = 0$.

Denote by $E(\lambda)$ a simple $\mathscr{C}\ell(\lambda)$-module (up to a grading shift, such a module is unique). One has $\dim E(\lambda) = 2^{[\frac{\dim \mathfrak{h}_{\bar 1}+1-c(\lambda)}{2}]}$.

2.4 Highest weight modules

Set $\mathfrak{b} := \mathfrak{h} + \mathfrak{n}^+, \mathfrak{b}^- := \mathfrak{h} + \mathfrak{n}^-$. Endow $\mathscr{C}\ell(\lambda)$ with the $\mathfrak{b}$-module structure via the trivial action of $\mathfrak{n}^+$. Set

$$M(\lambda) := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathscr{C}\ell(\lambda), \quad N(\lambda) := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} E(\lambda).$$

Clearly, $M(\lambda)$ has a finite filtration with the factors isomorphic to $N(\lambda)$ up to parity change. We call $M(\lambda)$ a *Verma module* and $N(\lambda)$ a *Weyl module*.

For a diagonalizable $\mathfrak{h}_{\bar{0}}$-module N and a weight $\mu \in \mathfrak{h}_{\bar{0}}^*$ denote by N^μ the corresponding weight space. Say that a module N has the highest weight λ if $N = \sum_{\mu \leq \lambda} N^\mu$ and $N^\lambda \neq 0$. If all weight spaces N^μ are finite-dimensional we put $\mathrm{ch}\, N := \sum_\mu \dim N^\mu e^\mu$.

If N has a highest weight we denote by $\overline{N}$ the sum of all submodules which do not meet the highest weight space of N. Recall that $L(\lambda) = N(\lambda)/\overline{N}(\lambda)$.

The following conjecture is based on a discussion with V. Mazorchuk.

Conjecture *For any Q-type Lie superalgebra, and any nonzero weight λ,* $\mathrm{ch}\, L(\lambda)_{\bar{0}} = \mathrm{ch}\, L(\lambda)_{\bar{1}}$.

The above conjecture is verified for all but finitely many λ.

2.5 Example: $n = 2$

For $\mathfrak{sq}(2)$ the Cartan algebra is spanned by the even elements $h := h_\alpha, h' := h_{\overline{\alpha}}$ and the odd element $H := H_\alpha$.

The module $N(\lambda)$ is simple if $\lambda(h') \neq 0$ and $\lambda(h) \not\in \mathbb{Z}_{>0}$. If $\lambda(h') = 0$, the simple $\mathfrak{sq}(2)$-module coincides with the simple $\mathfrak{gl}_2$-module $L_{\mathfrak{gl}_2}(\lambda)$; if $\lambda(h') \neq 0, \lambda(h) \in \mathbb{Z}_{>0}$, then $L(\lambda) = L_{\mathfrak{gl}(2)}(\lambda)^{\oplus 2}$ if $\lambda(h) = 1$ and $L(\lambda) = L_{\mathfrak{gl}(2)}(\lambda)^{\oplus 2} \oplus L_{\mathfrak{gl}(2)}(\lambda - \alpha)^{\oplus 2}$ if $\lambda(h) \neq 1$. This can be illustrated by the following diagrams: the module $L(\lambda)$ for $\lambda(h') \neq 0, \lambda(h) = 1$ is of the form

$$\begin{array}{cc} \cdot & \cdot \\ | & | \\ \cdot & \cdot \end{array}$$

and the module $L(\lambda)$ for $\lambda(h') \neq 0, \lambda(h) = 4$ is of the form

$$\begin{array}{cccc} & \cdot & \cdot & \\ & | & | & \\ \cdot & \cdot & \cdot & \cdot \\ | & | & | & | \\ \cdot & \cdot & \cdot & \cdot \\ & | & | & \\ & \cdot & \cdot & \end{array}$$

where the dots on the same level represent the vectors of the same weight and the difference between levels is equal to α; the vertical lines correspond to the action of f_α (so the dots in the same column represent a simple $\mathfrak{gl}_2$-module).

All simple module over $\mathfrak{q}(2), \mathfrak{sq}(2)$ and their quotients $\mathfrak{pq}(2), \mathfrak{psq}(2)$ are classified by V. Mazorchuk in [18].

3 Centres

3.1 Centre of enveloping algebra

A weight λ is called atypical if there exists $\alpha \in \Delta$ such that $h_{\overline{\alpha}}(\lambda) = 0$. The centres of the universal enveloping algebras of Q-type Lie algebras is given by the following theorem.

Theorem *Let $\mathfrak{g}$ be a Q-type Lie superalgebra, $\mathfrak{g} \neq \mathfrak{pq}(2), \mathfrak{psq}(2)$. The restriction of* HC *to $\mathscr{Z}(\mathfrak{g})$ is an algebra isomorphism $\mathscr{Z}(\mathfrak{g}) \xrightarrow{\sim} Z$ where Z is the set of W-invariant polynomial functions on $\mathfrak{h}_{\overline{0}}^*$ which are constant along each straight line parallel to a root α and lying in the hyperplane $h_{\overline{\alpha}}(\lambda) = 0$. In other words,*

$$Z := \mathscr{S}(\mathfrak{h}_0)^W \cap \bigcap_{\alpha \in \Delta} Z_\alpha,$$

where

$$Z_\alpha := \{f \in \mathscr{S}(\mathfrak{h}_0) |\ h_{\overline{\alpha}}(\lambda) = 0 \implies f(\lambda) = f(\lambda - c\alpha)\ \forall c \in \mathbb{C}\}.$$

The theorem is proven in [5, 26]. One has $\mathscr{Z}(U(\mathfrak{q}(n))) = \mathscr{Z}(U(\mathfrak{sq}(n)))$ and $\mathscr{Z}(U(\mathfrak{pq}(n))) = \mathscr{Z}(U(\mathfrak{psq}(n)))$.

3.2 Strongly typical weights

An element a of an associative superalgebra U is called *anticentral* if $ax - (-1)^{p(x)(p(a)+1)}xa = 0$. We denote by $\mathscr{A}(U)$ the set of anticentral elements of U.

Let $\mathfrak{g}$ be a Q-type Lie superalgebra. The anticentre of the Clifford algebra $U(\mathfrak{h})$ is equal to $\mathscr{S}(\mathfrak{h}_0)T_{\mathfrak{h}}$, where the parity of $T_{\mathfrak{h}}$ is equal to the parity of $\dim \mathfrak{h}_{\overline{1}}$ and

$$t_{\mathfrak{h}} := T_{\mathfrak{h}}^2 = \begin{cases} \pm h_1 \dots h_n & \text{for } \mathfrak{g} = \mathfrak{q}(n), \mathfrak{pq}(n) \\ \pm \sum h_1 \dots \hat{h}_i \dots h_n & \text{for } \mathfrak{g} = \mathfrak{sq}(n), \mathfrak{psq}(n). \end{cases}$$

The Harish-Chandra projection provides a linear monomorpism $\mathrm{HC} : \mathscr{A}(U(\mathfrak{g})) \xrightarrow{\sim} \mathscr{A}(U(\mathfrak{h}))$ and the image is equal to $\mathscr{S}(\mathfrak{h}_0)^W T_{\mathfrak{g}}$, where the parity of $T_{\mathfrak{g}}$ is equal to the parity of $\dim \mathfrak{h}_{\overline{1}}$ and

$$\mathrm{HC}(T_{\mathfrak{g}}) = T_{\mathfrak{h}} \prod_{\alpha \in \Delta_{\overline{0}}^+} h_{\overline{\alpha}}.$$

We say that $\lambda \in \mathfrak{h}_{\overline{0}}^*$ is *strongly typical* if $(t_{\mathfrak{h}} \prod_{\alpha \in \Delta_{\overline{0}}^+} h_{\overline{\alpha}})(\lambda) \neq 0$. Note that λ is strongly typical if and only if $T_{\mathfrak{g}} M(\lambda) \neq 0$.

3.3 Equivalence of categories

Let $\mathcal{O}^{\mathfrak{g}}$ (resp., $\mathcal{O}^{\mathfrak{g}_0}$) be the $\mathcal{O}$-category for $\mathfrak{g}$ and $\mathfrak{g}_0$-respectively. We have the natural restriction functor $Res : \mathcal{O}^{\mathfrak{g}} \longrightarrow \mathcal{O}^{\mathfrak{g}_0}$ which sends a $\mathfrak{g}$-module $M = M_{\bar{0}} \oplus M_{\bar{1}}$ to the $\mathfrak{g}_0$-module M_0, and its left adjoint functor $Ind : \mathcal{O}^{\mathfrak{g}_0} \longrightarrow \mathcal{O}^{\mathfrak{g}}$.

The action of the centres of the universal enveloping algebras lead to the block decomposition $\mathcal{O}^{\mathfrak{g}} = \bigoplus \mathcal{O}^{\mathfrak{g}}_{\chi}, \mathcal{O}^{\mathfrak{g}_0} = \bigoplus \mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}}$ indexed by the central characters χ and $\hat{\chi}$ respectively. This gives the projection and inclusions functors $proj_{\chi} : \mathcal{O}^{\mathfrak{g}} \longrightarrow \mathcal{O}^{\mathfrak{g}}_{\chi}, incl_{\chi} : \mathcal{O}^{\mathfrak{g}}_{\chi} \longrightarrow \mathcal{O}^{\mathfrak{g}}$ and $proj_{\hat{\chi}} : \mathcal{O}^{\mathfrak{g}} \longrightarrow \mathcal{O}^{\mathfrak{g}}_{\hat{\chi}}, incl_{\hat{\chi}} : \mathcal{O}^{\mathfrak{g}}_{\hat{\chi}} \longrightarrow \mathcal{O}^{\mathfrak{g}}$.

We say that $\lambda \in \mathfrak{h}_{\bar{0}}^*$ is regular (resp., dominant, integral) if $\lambda(h_\alpha) \neq 0$ (resp., $\lambda(h_\alpha) \notin \mathbb{Z}_{<0}, \lambda(h_\alpha) \in \mathbb{Z}$) for each $\alpha \in \Delta^+$.

If $\mathfrak{g} = \mathfrak{q}(n)$, then a weight $\lambda = (\lambda_1, \dots, \lambda_n)$ ($\lambda_i := \lambda(h_i)$) is a regular dominant strongly typical weight if and only if $\lambda_j - \lambda_i \notin \mathbb{Z}_{\geq 0}$ for $j > i$, $\lambda_i + \lambda_j \neq 0$ for $j > i$, and $\lambda_i \neq 0$ for all i.

Let χ (resp., $\hat{\chi}$) be the $\mathfrak{g}$ (resp., $\mathfrak{g}_0$) central character which corresponds to a strongly typical weight λ (so $L(\lambda) \in \mathcal{O}^{\mathfrak{g}}_{\chi}, \dot{L}(\lambda) \in \mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}}$). We set $\tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi} := \mathcal{O}^{\mathfrak{g}}_{\chi}$ if $\dim \mathfrak{h}_{\bar{1}}$ is odd; if $\dim \mathfrak{h}_{\bar{1}}$ is even one has a decomposition $\mathcal{O}^{\mathfrak{g}}_{\chi} = \tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi} \oplus \Pi(\tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi})$, where Π is the parity change functor. Note that for an integral weight λ the blocks $\mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}}, \tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi}$ are indecomposable.

The functors $F := proj_{\chi} \circ Ind \circ incl_{\hat{\chi}} : \mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}} \longrightarrow \tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi}$ and $G := proj_{\hat{\chi}} \circ Res \circ incl_{\chi} : \tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi} \longrightarrow \mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}}$ are adjoint. The main result of [3] is that for a regular dominant strongly typical weight λ both functors F and G decompose in direct sums of k copies of some functors $F_1 : \mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}} \longrightarrow \tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi}$ and $G_1 : \tilde{\mathcal{O}}^{\mathfrak{g}}_{\chi} \longrightarrow \mathcal{O}^{\mathfrak{g}_0}_{\hat{\chi}}$ respectively and the functors F_1, G_1 are mutually inverse equivalences of categories.

4 Bounded, cuspidal and weight modules of $\mathfrak{q}(n)$

4.1 Bounded weights

We call a weight $\lambda \in \mathfrak{h}_{\bar{0}}^*$ *bounded* if the set of weight multiplicities of $L(\lambda)$ is uniformly bounded, i.e. there exists a constant C such that the $\dim L(\lambda)^{\nu} < C$ for all ν. Conditions when λ is bounded are obtained in [6]. These conditions are formulated in terms of the $*$-action, see below.

4.1.1 Definition

For $\lambda \in \mathfrak{h}_{\bar{0}}^*$ and $\alpha \in \pi$ we set $s_\alpha \cdot \lambda := s_\alpha(\lambda + \rho_0) - \rho_0$ and

$$s_\alpha * \lambda = \begin{cases} s_\alpha \lambda & \text{if } \lambda(h_{\overline{\alpha}}) \neq 0, \\ s_\alpha \cdot \lambda & \text{if } \lambda(h_{\overline{\alpha}}) = 0. \end{cases}$$

For $i = 1, \dots, n-1$ we set $s_i * \lambda := s_{\alpha_i} * \lambda$.

Note that $s_\alpha * s_\alpha * \lambda = \lambda$ and $s_\alpha * s_\beta * \lambda = s_\beta * s_\alpha * \lambda$ if $(\alpha, \beta) = 0$. Therefore the group $\widetilde{W}$ generated by the symbols $s_1, \dots, s_{n-1}$ subject to the relations $s_i^2 = 1$, $s_i s_j = s_j s_i$ for $i - j > 1$ acts on $\mathfrak{h}_{\bar{0}}^*$ via $*$-action. Note that $\widetilde{W}$ is an infinite Coxeter group.

4.1.2 Description of bounded weights

Recall that $\mathfrak{gl}_n$-module $\dot{L}(\lambda)$ is finite-dimensional if and only if $s_i \cdot \lambda < \lambda$ for each $i = 1, \dots, n-1$ (the partial order was introduced in §2.2).

The $\mathfrak{q}(n)$-module $L(\lambda)$ is finite-dimensional if and only if for each $i = 1, \dots, n-1$ one has $(\lambda, \varepsilon_i - \varepsilon_{i+1}) \in \mathbb{Z}_{>0}$ or $(\lambda, \varepsilon_i) = (\lambda, \varepsilon_{i+1}) = 0$, see [20], which can be rewritten as $s_i * \lambda < \lambda$ for each $i = 1, \dots, n-1$.

For each weight μ there exists a sequence $\mu = \mu_0 < \mu_1 < \mu_2 < \dots < \mu_s$ such that $\mu_{i+1} = s_{k_i} * \mu_i$ for some $k_i \in \{1, 2, \dots, n-1\}$ and μ_s is $W*$-maximal (i.e., $s_i * \mu \not< \mu$ for each i). We call such sequence a *$W*$-increasing string* starting at μ.

Bounded weights for $\mathfrak{gl}_n$ were described in [17]. For an integral weight μ the conditions on μ being bounded can be reformulated as follows: μ is bounded if and only if

(i) there exists a unique increasing W-string $\mu = \mu_0 < \mu_1 < \mu_2 < \dots < \mu_s$;
(ii) the set $\{i : s_i \cdot \mu_j = \mu_j\}$ is empty for $j < s$ and has cardinality at most one for $j = s$.

In [6] we proved that the same description for bounded weights is valid for $\mathfrak{q}(n)$ if we change the dot action by the $*$-action. The non-integral bounded weights can be also described in terms of the $*$-action.

4.2 Example: the case $n = 3$

4.2.1 The case $\mathfrak{gl}_3$

Consider first the case $\mathfrak{gl}_3$. There are three types of W-orbits $W \cdot \lambda$ for integral λ: the trivial orbit for $\lambda + \rho_{\bar{0}} = (a, a, a)$ (these weights are not bounded), the regular orbits which contain six elements (each element has a trivial stabilizer) and singular orbits which contain three elements (the stabilizer of each element is $\mathbb{Z}_2$). The regular orbits are of the form:

$$\begin{tikzcd}
 & \lambda \arrow[dl, dash] \arrow[dr, dash] & \\
s_1 \cdot \lambda \arrow[d, dash] & & s_2 \cdot \lambda \arrow[d, dash] \\
(s_2 s_1) \cdot \lambda \arrow[dr, dash] & & (s_1 s_2) \cdot \lambda \arrow[dl, dash] \\
 & (s_2 s_1 s_2) \cdot \lambda = (s_1 s_2 s_1) \cdot \lambda &
\end{tikzcd}$$

The non-trivial singular orbits are of the form:

$$\begin{tikzcd}
\lambda = s_1 \cdot \lambda \arrow[d, dash] \\
s_2 \cdot \lambda \arrow[d, dash] \\
(s_1 s_2) \cdot \lambda = s_2 s_1 s_2 \cdot \lambda
\end{tikzcd}$$

(or the same with interchanged s_1, s_2).

The edges of the diagrams correspond to simple reflections s_1, s_2 and the upper vertex in a given edge is bigger with respect to the partial order. We say that a vertex is a *top* (resp., *bottom*) vertex if there is no edge ascending (resp., descending) from this vertex.

Note that $\dot{L}(\lambda)$ is finite-dimensional if and only if λ is represented by a top vertex which belongs to $n-1$ edges. Hence $\dot{L}(\lambda)$ is finite-dimensional if and only if λ is the top vertex in the regular orbit (the diagram above).

The increasing strings are represented by the paths going in upward direction, for instance $s_1 s_2 s_1 \cdot \lambda < s_2 s_1 \cdot \lambda < s_1 \lambda < \lambda$. The condition (i) for a given vertex means that there exists a unique ascending path; the condition (ii) means that each vertex in this path, except the top one, belongs to $n-1$ edges and the top one belongs to at least $n-2$ edges.

We see that all non-bottom vertices represent bounded weights for $\mathfrak{gl}_3$.

4.2.2 The case $\mathfrak{q}(3)$

We now look at the $\widetilde{W}$-orbits in the case $\mathfrak{q}(3)$. There are 6 types of $\widetilde{W}$-orbits, which we describe below (up to the interchange s_1 and s_2).

(1) The trivial orbit corresponds to the case $\lambda = (a,a,a), a \neq 0$; these weights are not bounded.

(2) The orbits of the form

$$\begin{tikzcd}
 & \lambda \arrow[dl, dash] \arrow[dr, dash] & \\
s_1 * \lambda \arrow[d, dash] & & s_2 * \lambda \arrow[d, dash] \\
(s_2 s_1) * \lambda \arrow[dr, dash] & & (s_1 s_2) * \lambda \arrow[dl, dash] \\
 & (s_2 s_1 s_2) * \lambda = (s_1 s_2 s_1) * \lambda &
\end{tikzcd}$$

(3) The orbit

$$\begin{tikzcd}
& 0 \arrow[dl, dash] \arrow[dr, dash] & \\
s_1 * 0 = -\alpha_1 \arrow[dr, dash] & & s_2 * 0 = -\alpha_2 \arrow[dl, dash] \\
& (s_2 s_1) * 0 = (s_1 s_2) * 0 = -\alpha_1 - \alpha_2 &
\end{tikzcd}$$

(4) The orbits of the form

$$\begin{tikzcd}
\lambda = s_1 * \lambda \arrow[d, dash] \\
s_2 * \lambda \arrow[d, dash] \\
(s_1 s_2) * \lambda = (s_2 s_1 s_2) * \lambda
\end{tikzcd}$$

(5) The orbits of the form

$$\begin{tikzcd}
& & & & \lambda \arrow[dl, dash] \arrow[drr, dash] & & \\
& \lambda' = s_2 * \lambda' \arrow[dl, dash] & & s_1 * \lambda \arrow[dr, dash] & & & s_2 * \lambda \arrow[dl, dash] \\
s_1 * \lambda' \arrow[drr, dash] & & & & s_2 s_1 * \lambda \arrow[dll, dash] & s_1 s_2 * \lambda \arrow[dr, dash] & \\
& & s_2 s_1 * \lambda' = s_1 s_2 s_1 * \lambda & & & & s_2 s_1 s_2 * \lambda = s_1 s_2 s_1 s_2 * \lambda
\end{tikzcd}$$

with $\lambda' = \lambda + \alpha_1$.

(6) The orbits of the form

$$\begin{tikzcd}
& \lambda' = s_2 * \lambda' \arrow[dl, dash] & & & & \lambda = s_1 * \lambda \arrow[dr, dash] & \\
s_1 * \lambda' \arrow[drrr, dash] & & & & & & s_2 * \lambda \arrow[dlll, dash] \\
& & & s_2 s_1 * \lambda' = s_1 s_2 * \lambda & & &
\end{tikzcd}$$

with $\lambda' = \lambda + \alpha_1$.

We see that $L(\lambda)$ is finite-dimensional if and only if λ is a top vertex in one of the orbits (2), (3) or the right top (the highest) vertex in (5); the bounded weights correspond to the non-bottom vertices in the orbits (2)–(6).

4.3 Cuspidal and weight modules

Definition Let M be a $\mathfrak{q}(n)$-module.

(i) We call M a *weight module* if $M = \bigoplus_{\lambda \in \mathfrak{h}_{\bar{0}}^*} M^\lambda$ and $\dim M^\lambda < \infty$ for every $\lambda \in \mathfrak{h}_{\bar{0}}^*$.

(ii) We call M a *cuspidal module* if M is a weight module and every nonzero even root vector $e_\alpha \in \mathfrak{q}(n)^\alpha$ acts injectively on M for every $\alpha \in \Delta$.

Remark In many cases the condition $\dim M^\lambda < \infty$ in (i) is not included in the definition of a weight module. We include this condition for convenience.

The following theorem is proved in [4] and reduces the classification of all simple weight modules of $\mathfrak{q}(n)$ to all simple cuspidal modules of $\mathfrak{q}(n)$. For the definition of "parabolically induced" we refer the reader to [4].

Theorem *Every simple weight $\mathfrak{q}(n)$-module is parabolically induced from a cuspidal module over $\mathfrak{q}(n_1) \oplus \ldots \oplus \mathfrak{q}(n_k)$, for some positive integers $n_1, \ldots, n_k$ with $n_1 + \ldots + n_k = n$.*

To classify all simple cuspidal modules we used the so called "twisted localization" technique - we present every simple cuspidal as a twisted localization of a highest weight module. Some details are listed below.

4.3.1 Twisted localization

Set $U := U(\mathfrak{q}(n))$. Then $\mathbf{F}_\alpha := \{f_\alpha^n \mid n \in \mathbb{Z}_{\geq 0}\} \subset U$ satisfies Ore's localization conditions because $\operatorname{ad} f_\alpha$ acts locally finitely on U. Let $\mathscr{D}_\alpha U$ be the localization of U relative to $\mathbf{F}_\alpha$, and for a $\mathfrak{q}(n)$-module M, set $\mathscr{D}_\alpha M = \mathscr{D}_\alpha U \otimes_U M$. For $x \in \mathbb{C}$ and $u \in \mathscr{D}_\alpha U$ we set

$$\Theta_x(u) := \sum_{i \geq 0} \binom{x}{i} (\operatorname{ad} f_\alpha)^i (u) f_\alpha^{-i},$$

where $\binom{x}{i} = \frac{x(x-1)\ldots(x-i+1)}{i!}$. Since $\operatorname{ad} f_\alpha$ is locally nilpotent on $\mathscr{D}_\alpha U$, the sum above is actually finite. Note that for $x \in \mathbb{Z}$ we have $\Theta_x(u) = f_\alpha^x u f_\alpha^{-x}$. For a $\mathscr{D}_\alpha U$-module M by $\Phi_\alpha^x M$ we denote the $\mathscr{D}_\alpha U$-module M twisted by the action

$$u \cdot v^x := (\Theta_x(u) \cdot v)^x,$$

where $u \in \mathscr{D}_\alpha U$, $v \in M$, and v^x stands for the element v considered as an element of $\Phi_\alpha^x M$. In particular, $v^x \in M^{\lambda + x\alpha}$ whenever $v \in M^\lambda$. Set $\mathscr{D}_\alpha^x M := \Phi_\alpha^x(\mathscr{D}_\alpha M)$.

The classification of the simple cuspidal $\mathfrak{q}(n)$-modules is obtained in the following theorem proved in [6]. This together with the description of the bounded weights completes the classification of all simple weight $\mathfrak{q}(n)$-modules. The uniqueness part of the theorem involves the definition of a bounded weight type and it is skipped here.

4.3.2 Theorem

Let M be a simple cuspidal $\mathfrak{q}(n)$–module. Then there is a bounded weight λ, and a tuple $(x_1,\ldots,x_{n-1})$ of $n-1$ complex nonintegral numbers such that $M \simeq \mathscr{D}_{\alpha_1}^{x_1}\mathscr{D}_{\alpha_1+\alpha_2}^{x_2}\cdots\mathscr{D}_{\alpha_1+\ldots+\alpha_{n-1}}^{x_{n-1}}L(\lambda)$.

5 Automorphisms of $\mathfrak{q}(n)$ and affine Lie superalgebra $\mathfrak{q}(n)^{(2)}$

5.1 Automorphisms of Q-type Lie superalgebra

Although the Q-type superalgebras are not invariant with respect to the supertransposition, they are invariant with respect to the *q-supertransposition* $\sigma_q : \begin{pmatrix} A & B \\ B & A\end{pmatrix} \mapsto \begin{pmatrix} A^t & \zeta B^t \\ \zeta B^t & A^t\end{pmatrix}$, where $\zeta = \zeta_4 \in \mathbb{C}$ is a fixed primitive 4th root of unity.

Let $\mathfrak{g}$ be a Q-type Lie superalgebra. The natural homomorphisms $\mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{Aut}\,\mathfrak{g}$ given by $X \mapsto \mathrm{Ad}_R(X,X)$, where

$$\mathrm{Ad}_R(X,X)\begin{pmatrix} A & B \\ B & A\end{pmatrix} = \begin{pmatrix} XAX^{-1} & XBX^{-1} \\ XBX^{-1} & XAX^{-1}\end{pmatrix}$$

induces an embedding $\mathrm{PSL}_n(\mathbb{C}) \longrightarrow \mathrm{Aut}\,\mathfrak{g}$. In the light of [10, 23] one has $\mathrm{Aut}\,\mathfrak{g} = \mathrm{PSL}_n(\mathbb{C}) \times \mathbb{Z}_4$, where $\mathbb{Z}_4$ is generated by $-\sigma_q$.

5.2 Affine Lie superalgebra $\mathfrak{q}(n)^{(2)}$

Recall that semisimple Lie algebras are finite-dimensional Kac-Moody algebras and affine Lie algebras are Kac-Moody algebras of finite growth. Each affine Lie algebra is a (twisted or non-twisted) affinization of a finite-dimensional Kac-Moody algebra; this means that an affine Lie algebra can be described in terms of a finite-dimensional Kac-Moody algebra and its finite order automorphism, see [14, Chap. VI–VIII]. The Cartan matrices of finite-dimensional and affine Lie algebras are symmetrizable [14, Chap. IV].

The superalgebra generalization of Kac-Moody algebras was introduced in [13]; see [12,30] for details. Call a Kac-Moody superalgebra *affine* if it has a finite growth and *symmetrizable* if it has a symmetrizable Cartan matrix. The affine symmetrizable Lie superalgebras are classified in [23, 29], and, as in Lie algebra case, they are (twisted or non-twisted) affinizations of finite-dimensional Kac-Moody superalgebras. Non-symmetrizable affine Lie superalgebras were described in [12]. The classification includes two degenerate superalgebras, one family of constant rank and one series $\mathfrak{q}(n)^{(2)}$. The superalgebras of the series are twisted affinizations of $\mathfrak{sq}(n)$ corresponding to the automorphism $\sigma_q^2 : a \mapsto (-1)^{p(a)}a$.

A symmetrizable affine Lie superalgebra has an even non-degenerate invariant bilinear form and a Casimir element. The Q-type Lie superalgebras and the affine

Kac-Moody algebra $\mathfrak{q}(n)^{(2)}$ do not have even invariant bilinear forms, but have odd ones.

An interesting feature of $\mathfrak{q}(n)^{(2)}$ is that any Verma module is reducible; more precisely if $M(\lambda)$ is a Verma module over $\mathfrak{q}(n)^{(2)}$ and $\overline{M}(\lambda)$ is its maximal submodule (i.e., $L(\lambda) = M(\lambda)/\overline{M}(\lambda)$), then $\overline{M}(\lambda)^{\lambda-2\delta} \neq 0$, where δ is the minimal imaginary root of $\mathfrak{q}(n)^{(2)}$. The proof is given in [11].

5.2.1 Description of $\mathfrak{q}(n)^{(2)}$

We introduce $\mathfrak{q}(n)^{(1)} := \mathscr{L}(\mathfrak{sq}(n)) \oplus \mathbb{C}D$, where $\mathscr{L}(\mathfrak{sq}(n)) := \mathfrak{sq}(n) \otimes \mathbb{C}[t^{\pm 1}]$ is the loop superalgebra, D acts on $\mathscr{L}(\mathfrak{sq}(n))$ by $[D, x \otimes t^k] = kx \otimes t^k$. Note that $\mathfrak{sq}(n)$, $\mathfrak{q}(n)^{(1)}$ are not Kac-Moody superalgebras since their Cartan subalgebras contain odd elements.

Let ε be an automorphism of $\mathfrak{sq}(n)$ given by $\varepsilon(x) := (-1)^{p(x)}x$, i.e. $\varepsilon = \sigma_q^2$. We extend ε to $\mathfrak{q}(n)^{(1)}$ by $\varepsilon(t) = -t$, $\varepsilon(D) = D$. Then $\mathfrak{q}(n)^{(2)}$ is the quotient of the subalgebra $(\mathfrak{q}(n)^{(1)})^{\varepsilon}$ of elements fixed by ε by the abelian ideal $\sum_{i\neq 0} \mathbb{C}X_{I,0} \otimes t^{2i}$, where I stands for the $n \times n$ identity matrix (see Sect. 2.1 for notation). We may identify $\mathfrak{q}(n)^{(2)}$ with the vector space

$$\mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 1}] \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

where $K := X_{I,0}$ and

$$\begin{aligned} \mathfrak{q}(n)^{(2)}_{\bar{0}} &= \mathfrak{gl}_n \oplus \mathbb{C}D \oplus \Big(\sum_{k \in \mathbb{Z}\setminus\{0\}} \mathfrak{sl}_n \times t^{2k} \Big) \\ &= \mathfrak{sl}_n \otimes \mathbb{C}[t^{\pm 2}] \oplus \mathbb{C}K \oplus \mathbb{C}D, \quad \mathfrak{q}(n)^{(2)}_{\bar{1}} = \mathfrak{sl}_n \otimes t\mathbb{C}[t^{\pm 2}]; \end{aligned}$$

the commutator is given by

$$[x \otimes t^k, y \otimes t^m] = \begin{cases} (xy - yx) \otimes t^{k+m}, \\ \qquad \text{if } km \text{ is even}, \\ [x \otimes t^k, y \otimes t^m] = \iota(xy + yx) \otimes t^{k+m} + 2\delta_{-k,m} \operatorname{tr}(xy) K, \\ \qquad \text{if } km \text{ is odd}, \end{cases}$$

where $\iota : \mathfrak{gl}_n \to \mathfrak{sl}_n$ is the natural map $\iota(x) := x - \operatorname{tr}(x) I/n$.

6 Crystal bases of $\mathfrak{q}(n)$

6.1 The quantum queer superalgebra

Let $\mathbf{F} = \mathbb{C}((q))$ be the field of formal Laurent series in an indeterminate q and let $\mathbf{A} = \mathbb{C}[[q]]$ be the subring of $\mathbf{F}$ consisting of formal power series in q. For $k \in \mathbb{Z}_{\geq 0}$, we define

$$[k] = \frac{q^k - q^{-k}}{q - q^{-1}}, \quad [0]! = 1, \quad [k]! = [k][k-1] \cdots [2][1].$$

For an integer $n \geq 2$, let $P = \mathbb{Z}\varepsilon_1 \oplus \cdots \oplus \mathbb{Z}\varepsilon_n$ and $P^\vee = \mathbb{Z}h_1 \oplus \cdots \oplus \mathbb{Z}h_n$. Then $\mathfrak{h}_{\bar{0}} = \mathbb{C} \otimes_{\mathbb{Z}} P^\vee$.

Definition The *quantum queer superalgebra* $U_q(\mathfrak{q}(n))$ is the unital superalgebra over $\mathbf{F}$ generated by the symbols e_i, f_i, E_i, F_i $(i = 1, \ldots, n-1)$, q^h $(h \in P^\vee)$, H_j $(j = 1, \ldots, n)$ with the following defining relations.

$$
\begin{aligned}
& q^0 = 1, \;\; q^h q^{h'} = q^{h+h'} \;\; (h, h' \in P^\vee), \\
& q^h e_i q^{-h} = q^{\alpha_i(h)} e_i \;\; (h \in P^\vee), \\
& q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i \;\; (h \in P^\vee), \\
& q^h H_j = H_j q^h, \\
& e_i f_j - f_j e_i = \delta_{ij} \frac{q^{h_i - h_{i+1}} - q^{-h_i + h_{i+1}}}{q - q^{-1}}, \\
& e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0 \quad \text{if } |i-j| > 1, \\
& e_i^2 e_j - (q + q^{-1}) e_i e_j e_i + e_j e_i^2 = 0 \quad \text{if } |i-j| = 1, \\
& f_i^2 f_j - (q + q^{-1}) f_i f_j f_i + f_j f_i^2 = 0 \quad \text{if } |i-j| = 1, \\
& H_i^2 = \frac{q^{2h_i} - q^{-2h_i}}{q^2 - q^{-2}}, \\
& H_i H_j + H_j H_i = 0 \;\; (i \neq j), \\
& H_i e_i - q e_i H_i = E_i q^{-h_i}, \\
& H_i f_i - q f_i H_i = -F_i q^{h_i}, \\
& e_i F_j - F_j e_i = \delta_{ij} (H_i q^{-h_{i+1}} - H_{i+1} q^{-h_i}), \\
& E_i f_j - f_j E_i = \delta_{ij} (H_i q^{h_{i+1}} - H_{i+1} q^{h_i}), \\
& e_i E_i - E_i e_i = f_i F_i - F_i f_i = 0, \\
& e_i e_{i+1} - q e_{i+1} e_i = E_i E_{i+1} + q E_{i+1} E_i, \\
& q f_{i+1} f_i - f_i f_{i+1} = F_i F_{i+1} + q F_{i+1} F_i, \\
& e_i^2 E_j - (q + q^{-1}) e_i E_j e_i + E_j e_i^2 = 0 \quad \text{if } |i-j| = 1, \\
& f_i^2 F_j - (q + q^{-1}) f_i F_j f_i + F_j f_i^2 = 0 \quad \text{if } |i-j| = 1.
\end{aligned}
$$

The generators e_i, f_i $(i = 1, \ldots, n-1)$, q^h $(h \in P^\vee)$ are regarded as *even* and E_i, F_i $(i = 1, \ldots, n-1)$, H_j $(j = 1, \ldots, n)$ as *odd*. From the defining relations, it is easy to see that the even generators together with H_1 generate the whole algebra $U_q(\mathfrak{q}(n))$.

The superalgebra $U_q(\mathfrak{q}(n))$ is a bialgebra with the comultiplication $\Delta \colon U_q(\mathfrak{q}(n)) \to U_q(\mathfrak{q}(n)) \otimes U_q(\mathfrak{q}(n))$ defined by

$$
\begin{aligned}
& \Delta(q^h) = q^h \otimes q^h \quad \text{for } h \in P^\vee, \\
& \Delta(e_i) = e_i \otimes q^{-h_i + h_{i+1}} + 1 \otimes e_i,
\end{aligned}
$$

$$\Delta(f_i) = f_i \otimes 1 + q^{h_i - h_{i+1}} \otimes f_i,$$
$$\Delta(H_1) = H_1 \otimes q^{h_1} + q^{-h_1} \otimes H_1.$$

6.2 The category $\mathscr{O}^{\geq 0}_{\text{int}}$.

A $U_q(\mathfrak{q}(n))$-module M is called a *weight module* if M has a weight space decomposition $M = \bigoplus_{\mu \in P} M^\mu$, where

$$M^\mu = \left\{ m \in M \,; q^h m = q^{\mu(h)} m \text{ for all } h \in P^\vee \right\},$$

and $\dim M^\mu < \infty$ for every μ. The *set of weights of* M is defined to be

$$\text{wt}(M) = \{\mu \in P \,; M^\mu \neq 0\}.$$

6.2.1 Definition

A weight $U_q(\mathfrak{q}(n))$-module V is called a *highest weight module with highest weight* $\lambda \in P$ if V^λ is finite-dimensional and satisfies the following conditions:

(i) V is generated by V^λ,
(ii) $e_i v = E_i v = 0$ for all $v \in V^\lambda$. $i = 1, \ldots, n-1$,

6.2.2 Strict partitions

Set

$$P^{\geq 0} = \{\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \in P;\, \lambda_j \in \mathbb{Z}_{\geq 0} \text{ for all } j = 1, \ldots, n\},$$
$$\Lambda^+ = \{\lambda = \lambda_1 \varepsilon_1 + \cdots + \lambda_n \varepsilon_n \in P^{\geq 0};\, \lambda_i \geq \lambda_{i+1} \text{ and } \lambda_i = \lambda_{i+1} \text{ implies}$$
$$\lambda_i = \lambda_{i+1} = 0 \text{ for all } i = 1, \ldots, n-1\}.$$

Note that each element $\lambda \in \Lambda^+$ corresponds to a *strict partition* $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_r > 0)$. Thus we will often call $\lambda \in \Lambda^+$ a strict partition.

6.2.3 Example

Let

$$\mathbf{V} = \bigoplus_{j=1}^{n} \mathbf{F} v_j \oplus \bigoplus_{j=1}^{n} \mathbf{F} v_{\bar{j}}$$

be the vector representation of $U_q(\mathfrak{q}(n))$. The action of $U_q(\mathfrak{q}(n))$ on $\mathbf{V}$ is given as follows:

$$e_i v_j = \delta_{j,i+1} v_i,\ e_i v_{\bar{j}} = \delta_{j,i+1} v_{\bar{i}},\ f_i v_j = \delta_{j,i} v_{i+1},\ f_i v_{\bar{j}} = \delta_{j,i} v_{\overline{i+1}},$$
$$E_i v_j = \delta_{j,i+1} v_{\bar{i}},\ E_i v_{\bar{j}} = \delta_{j,i+1} v_i,\ F_i v_j = \delta_{j,i} v_{\overline{i+1}},\ F_i v_{\bar{j}} = \delta_{j,i} v_{i+1},$$
$$q^h v_j = q^{\varepsilon_j(h)} v_j,\ q^h v_{\bar{j}} = q^{\varepsilon_j(h)} v_{\bar{j}},\ H_i v_j = \delta_{j,i} v_{\bar{j}},\quad H_i v_{\bar{j}} = \delta_{j,i} v_j.$$

Note that $\mathbf{V}$ is an irreducible highest weight module with highest weight ε_1 and $\mathrm{wt}(\mathbf{V}) = \{\varepsilon_1, \ldots, \varepsilon_n\}$.

6.2.4 Definition

We define $\mathscr{O}_{\mathrm{int}}^{\geq 0}$ to be the category of finite-dimensional weight modules M satisfying the following conditions:

(i) $\mathrm{wt}(M) \subset P^{\geq 0}$,
(ii) for any $\mu \in P^{\geq 0}$ and $i \in \{1, \ldots, n\}$ such that $\mu(h_i) = 0$, we have $H_i|_{M^\mu} = 0$.

The following proposition is proved in [7].

6.2.5 Proposition

Any irreducible $U_q(\mathfrak{q}(n))$-module in $\mathscr{O}_{\mathrm{int}}^{\geq 0}$ appears as a direct summand of a tensor power of $\mathbf{V}$.

6.3 *Crystal bases in* $\mathscr{O}_{\mathrm{int}}^{\geq 0}$

Let M be a $U_q(\mathfrak{q}(n))$-module in $\mathscr{O}_{\mathrm{int}}^{\geq 0}$. For $i = 1, 2, \ldots, n-1$, we define the *even Kashiwara operators* on M in the usual way. That is, for a weight vector $u \in M_\lambda$, consider the *i-string decomposition* of u:

$$u = \sum_{k \geq 0} f_i^{(k)} u_k,$$

where $e_i u_k = 0$ for all $k \geq 0$, $f_i^{(k)} = f_i^k / [k]!$, and define the even Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i = 1, \ldots, n-1)$ by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k,$$
$$\tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

On the other hand, we define the *odd Kashiwara operators* $\widetilde{H}_1$, $\widetilde{E}_1$, $\widetilde{F}_1$ by

$$\begin{aligned}
\widetilde{H}_1 &= q^{h_1 - 1} H_1, \\
\widetilde{E}_1 &= -(e_1 H_1 - q H_1 e_1) q^{h_1 - 1}, \\
\widetilde{F}_1 &= -(H_1 f_1 - q f_1 H_1) q^{h_2 - 1}.
\end{aligned}$$

For convenience we will use the following notation $e_{\bar{1}} = E_1, f_{\bar{1}} = F_1, \tilde{e}_{\bar{1}} = \widetilde{E}_1, \tilde{f}_{\bar{1}} = \widetilde{F}_1$.

Recall that an abstract $\mathfrak{gl}(n)$-crystal is a set B together with the maps $\tilde{e}_i, \tilde{f}_i \colon B \to B \sqcup \{0\}$, $\varphi_i, \varepsilon_i \colon B \to \mathbb{Z} \sqcup \{-\infty\}$ $(i = 1, \ldots, n-1)$, and $\mathrm{wt} \colon B \to P$ satisfying the conditions given in [16]. For an abstract $\mathfrak{gl}(n)$-crystal B and $\lambda \in P$, we set $B^\lambda = \{b \in$

$B \mid \mathrm{wt}(b) = \lambda\}$. We say that an abstract $\mathfrak{gl}(n)$-crystal is a $\mathfrak{gl}(n)$-*crystal* if it is realized as a crystal basis of a finite-dimensional integrable $U_q(\mathfrak{gl}(n))$-module. In particular, we have $\varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geq 0}\,;\, \tilde{e}_i^n b \neq 0\}$ and $\varphi_i(b) = \max\{n \in \mathbb{Z}_{\geq 0}\,;\, \tilde{f}_i^n b \neq 0\}$ for any b in a $\mathfrak{gl}(n)$-crystal B.

6.3.1 Crystal basis

Definition Let $M = \bigoplus_{\mu \in P^{\geq 0}} M^\mu$ be a $U_q(\mathfrak{q}(n))$-module in the category $\mathscr{O}_{\mathrm{int}}^{\geq 0}$. A *crystal basis* of M is a triple $(L, B, l_B = (l_b)_{b \in B})$, where

(i) L is a free $\mathbf{A}$-submodule of M such that

  (1) $\mathbf{F} \otimes_{\mathbf{A}} L \xrightarrow{\sim} M$,
  (2) $L = \bigoplus_{\mu \in P^{\geq 0}} L^\mu$, where $L^\mu = L \cap M^\mu$,
  (3) L is stable under the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i = 1, \ldots, n-1)$, $\widetilde{H}_{\bar{1}}$, $\widetilde{E}_{\bar{1}}$, $\widetilde{F}_{\bar{1}}$.

(ii) B is a finite $\mathfrak{gl}(n)$-crystal together with the maps $\widetilde{E}_{\bar{1}}, \widetilde{F}_{\bar{1}} : B \to B \sqcup \{0\}$ such that

  (1) $\mathrm{wt}(\widetilde{E}_{\bar{1}} b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\widetilde{F}_{\bar{1}} b) = \mathrm{wt}(b) - \alpha_1$,
  (2) for all $b, b' \in B$, $\widetilde{F}_{\bar{1}} b = b'$ if and only if $b = \widetilde{E}_{\bar{1}} b'$.

(iii) $l_B = (l_b)_{b \in B}$ is a family of $\mathbb{C}$-vector spaces such that

  (1) $l_b \subset (L/qL)^\mu$ for $b \in B^\mu$,
  (2) $L/qL = \bigoplus_{b \in B} l_b$,
  (3) $\widetilde{H}_{\bar{1}} l_b \subset l_b$,
  (4) for $i = 1, \ldots, n-1, \bar{1}$, we have
    (1) if $\tilde{e}_i b = 0$ then $\tilde{e}_i l_b = 0$, and otherwise $\tilde{e}_i$ induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{e}_i b}$.
    (2) if $\tilde{f}_i b = 0$ then $\tilde{f}_i l_b = 0$, and otherwise $\tilde{f}_i$ induces an isomorphism $l_b \xrightarrow{\sim} l_{\tilde{f}_i b}$.

As proved in [8], for every crystal basis (L, B, l_B) of a $U_q(\mathfrak{q}(n))$-module M we have $\widetilde{E}_{\bar{1}}^2 = \widetilde{F}_{\bar{1}}^2 = 0$ as endomorphisms on L/qL.

6.3.2 Example

Let

$$\mathbf{V} = \bigoplus_{j=1}^{n} \mathbf{F} v_j \oplus \bigoplus_{j=1}^{n} \mathbf{F} v_{\bar{j}}$$

be the vector representation of $U_q(\mathfrak{q}(n))$. The action of $U_q(\mathfrak{q}(n))$ on $\mathbf{V}$ is given as follows:

$e_i v_j = \delta_{j,i+1} v_i$, $e_i v_{\bar{j}} = \delta_{j,i+1} v_{\bar{i}}$, $f_i v_j = \delta_{j,i} v_{i+1}$, $f_i v_{\bar{j}} = \delta_{j,i} v_{\overline{i+1}}$, $E_i v_j = \delta_{j,i+1} v_{\bar{i}}$, $E_i v_{\bar{j}} = \delta_{j,i+1} v_i$, $F_i v_j = \delta_{j,i} v_{\overline{i+1}}$, $F_i v_{\bar{j}} = \delta_{j,i} v_{i+1}$, $q^h v_j = q^{\varepsilon_j(h)} v_j$, $q^h v_{\bar{j}} = q^{\varepsilon_j(h)} v_{\bar{j}}$, $H_i v_j = \delta_{j,i} v_{\bar{j}}$, $H_i v_{\bar{j}} = \delta_{j,i} v_j$.

Set

$$\mathbf{L} = \bigoplus_{j=1}^{n} \mathbf{A} v_j \oplus \bigoplus_{j=1}^{n} \mathbf{A} v_{\bar{j}},$$

$l_j = \mathbb{C} v_j \oplus \mathbb{C} v_{\bar{j}}$, and let $\mathbf{B}$ be the crystal graph given below:

$$\begin{tikzcd} \boxed{1} \arrow[r, shift left, "1"] \arrow[r, shift right, dashed, "\bar{1}"'] & \boxed{2} \arrow[r, "2"] & \boxed{3} \arrow[r, "3"] & \cdots \arrow[r, "n-1"] & \boxed{n} \end{tikzcd}$$

Here, the actions of $\tilde{f}_i$ $(i = 1, \dots, n-1)$ are expressed by i-arrows and of $\widetilde{F}_{\bar{1}}$ by an $\bar{1}$-arrow. Then $(\mathbf{L}, \mathbf{B}, l_{\mathbf{B}} = (l_j)_{j=1}^{n})$ is a crystal basis of $\mathbf{V}$.

6.3.3 Tensor product rule

The *tensor product rule* for the crystal bases in the category $\mathscr{O}_{\mathrm{int}}^{\geq 0}$ is given by the following theorem (Theorem 2.7 in [8]).

Theorem *Let M_j be a $U_q(\mathfrak{g})$-module in $\mathscr{O}_{\mathrm{int}}^{\geq 0}$ with crystal basis (L_j, B_j, l_{B_j}) $(j = 1, 2)$. Set $B_1 \otimes B_2 = B_1 \times B_2$ and*

$$l_{B_1 \otimes B_2} = (l_{b_1} \otimes l_{b_2})_{b_1 \in B_1, b_2 \in B_2}.$$

Then

$$(L_1 \otimes_{\mathbf{A}} L_2, B_1 \otimes B_2, l_{B_1 \otimes B_2})$$

is a crystal basis of $M_1 \otimes_{\mathbf{F}} M_2$, where the action of the Kashiwara operators on $B_1 \otimes B_2$ are given as follows.

$$\begin{aligned} \tilde{e}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{e}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2), \\ b_1 \otimes \tilde{e}_i b_2 & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2), \end{cases} \\ \tilde{f}_i(b_1 \otimes b_2) &= \begin{cases} \tilde{f}_i b_1 \otimes b_2 & \text{if } \varphi_i(b_1) > \varepsilon_i(b_2), \\ b_1 \otimes \tilde{f}_i b_2 & \text{if } \varphi_i(b_1) \leq \varepsilon_i(b_2), \end{cases} \end{aligned} \tag{1}$$

$$\begin{aligned} \widetilde{E}_{\bar{1}}(b_1 \otimes b_2) &= \begin{cases} \widetilde{E}_{\bar{1}} b_1 \otimes b_2 & \text{if } \langle h_1, \mathrm{wt}\, b_2 \rangle = 0, \\ & \quad \langle h_2, \mathrm{wt}\, b_2 \rangle = 0, \\ b_1 \otimes \widetilde{E}_{\bar{1}} b_2 & \text{otherwise}, \end{cases} \\ \widetilde{F}_{\bar{1}}(b_1 \otimes b_2) &= \begin{cases} \widetilde{F}_{\bar{1}} b_1 \otimes b_2 & \text{if } \langle h_1, \mathrm{wt}\, b_2 \rangle = 0, \\ & \quad \langle h_2, \mathrm{wt}\, b_2 \rangle = 0, \\ b_1 \otimes \widetilde{F}_{\bar{1}} b_2 & \text{otherwise}. \end{cases} \end{aligned} \tag{2}$$

6.3.4 Abstract crystal

Definition An *abstract* $\mathfrak{q}(n)$*-crystal* is a $\mathfrak{gl}(n)$-crystal together with the maps $\widetilde{E}_1, \widetilde{F}_1 : B \to B \sqcup \{0\}$ satisfying the following conditions:

(a) $\mathrm{wt}(B) \subset P^{\geq 0}$,
(b) $\mathrm{wt}(\widetilde{E}_1 b) = \mathrm{wt}(b) + \alpha_1$, $\mathrm{wt}(\widetilde{F}_1 b) = \mathrm{wt}(b) - \alpha_1$,
(c) for all $b, b' \in B$, $\widetilde{F}_1 b = b'$ if and only if $b = \widetilde{E}_1 b'$.
(d) if $3 \leq i \leq n-1$, we have

  (i) the operators $\widetilde{E}_1$ and $\widetilde{F}_1$ commute with $\tilde{e}_i$ and $\tilde{f}_i$.
  (ii) if $\widetilde{E}_1 b \in B$, then $\varepsilon_i(\widetilde{E}_1 b) = \varepsilon_i(b)$ and $\varphi_i(\widetilde{E}_1 b) = \varphi_i(b)$.

Recall that W is the Weyl group of $\mathfrak{gl}_n$. The *Weyl group action* on an abstract $\mathfrak{q}(n)$-crystal B is the action of W on $\mathfrak{gl}_n$-crystal B, which is given in [15].

Let B_1 and B_2 be abstract $\mathfrak{q}(n)$-crystals. The *tensor product* $B_1 \otimes B_2$ of B_1 and B_2 is defined to be the $\mathfrak{gl}(n)$-crystal $B_1 \otimes B_2$ together with the maps $\widetilde{E}_1$, $\widetilde{F}_1$ defined by (2). Then it is an abstract $\mathfrak{q}(n)$-crystal. Note that $\otimes$ satisfies the associative axiom on the set of abstract $\mathfrak{q}(n)$-crystals.

6.3.5 Example

(a) If (L, B, l_B) is a crystal basis of a $U_q(\mathfrak{q}(n))$-module M in the category $\mathcal{O}^{\geq 0}_{\mathrm{int}}$, then B is an abstract $\mathfrak{q}(n)$-crystal.
(b) The crystal graph $\mathbf{B}$ is an abstract $\mathfrak{q}(n)$-crystal.
(c) By the tensor product rule, $\mathbf{B}^{\otimes N}$ is an abstract $\mathfrak{q}(n)$-crystal. When $n = 3$, the $\mathfrak{q}(n)$-crystal structure of $\mathbf{B} \otimes \mathbf{B}$ is given below:

$$\begin{tikzcd}
\boxed{1}\otimes\boxed{1} \arrow[r, "1"] \arrow[d, dashed, "\bar{1}"] & \boxed{2}\otimes\boxed{1} \arrow[r, "2"] \arrow[d, shift right, "1"'] \arrow[d, shift left, dashed, "\bar{1}"] & \boxed{3}\otimes\boxed{1} \arrow[d, shift right, "1"'] \arrow[d, shift left, dashed, "\bar{1}"] \\
\boxed{1}\otimes\boxed{2} \arrow[d, "2"] & \boxed{2}\otimes\boxed{2} \arrow[r, "2"'] & \boxed{3}\otimes\boxed{2} \arrow[d, "2"] \\
\boxed{1}\otimes\boxed{3} \arrow[r, shift left, dashed, "\bar{1}"] \arrow[r, shift right, "1"'] & \boxed{2}\otimes\boxed{3} & \boxed{3}\otimes\boxed{3}
\end{tikzcd}$$

Let B be an abstract $\mathfrak{q}(n)$-crystal. For $i = 1, \ldots, n-1$, we set

$$w_i = s_2 \cdots s_i s_1 \cdots s_{i-1}.$$

Then w_i is the shortest element in W such that $w_i(\alpha_i) = \alpha_1$. We define the *odd Kashiwara operators* $\widetilde{E}_i$, $\widetilde{F}_i$ $(i = 2, \ldots, n-1)$ by

$$\widetilde{E}_i = S_{w_i^{-1}} \widetilde{E}_1 S_{w_i}, \quad \widetilde{F}_i = S_{w_i^{-1}} \widetilde{F}_1 S_{w_i}.$$

6.3.6 Definition

Let B be an abstract $\mathfrak{q}(n)$-crystal.

(a) An element $b \in B$ is called a $\mathfrak{gl}(n)$*-highest weight vector* if $\tilde{e}_i b = 0$ for $1 \le i \le n$.
(b) An element $b \in B$ is called a *highest weight vector* if $\tilde{e}_i b = \widetilde{E}_i b = 0$ for $1 \le i \le n$.
(c) An element $b \in B$ is called a *lowest weight vector* if $w_0 b$ is a $\mathfrak{q}(n)$-highest weight vector, where w_0 is the element of W of longest length.

The description of the set of the highest (and hence of the lowest) weight vectors in $\mathbf{B}^{\otimes N}$ for $N > 0$ is given by the following proposition (see Theorem 4.6 (c) in [8]).

6.3.7 Proposition

An element b_0 in $\mathbf{B}^{\otimes N}$ is a highest weight vector if and only if $b_0 = 1 \otimes \tilde{f}_1 \cdots \tilde{f}_{j-1} b$ for some j, and some highest weight vector b in $\mathbf{B}^{\otimes(N-1)}$ such that $\mathrm{wt}(b_0) = \mathrm{wt}(b) + \varepsilon_j$ is a strict partition.

The following uniqueness and existence theorem is one of the main results in [8].

6.3.8 Theorem

(a) Let $\lambda \in \Lambda^+$ be a strict partition and let M be a highest weight $U_q(\mathfrak{q}(n))$-module with highest weight λ in the category $\mathcal{O}_{\mathrm{int}}^{\ge 0}$. If (L, B, l_B) is a crystal basis of M, then L^λ is invariant under $\widetilde{K}_i := q^{h_i - 1} H_i$ for all $i = 1, \ldots, n$. Conversely, if M^λ is generated by a free $\mathbf{A}$-submodule L_λ^0 invariant under $\widetilde{K}_i$ $(i = 1, \ldots, n)$, then there exists a unique crystal basis (L, B, l_B) of M such that

(i) $L^\lambda = L_\lambda^0$,
(ii) $B^\lambda = \{b_\lambda\}$,
(iii) $L_\lambda^0 / q L_\lambda^0 = l_{b_\lambda}$,
(iv) B is connected.

Moreover, B, as an abstract $\mathfrak{q}(n)$-crystal depends only on λ. Hence we may write $B = B(\lambda)$.

(b) The $\mathfrak{q}(n)$-crystal $B(\lambda)$ has a unique highest weight vector b_λ and unique lowest weight vector l_λ.

A combinatorial description of the crystal bases in terms of semistandard decomposition tableaux has been obtained in [9].

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