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Gruson–Serganova character formulas and the Duflo–Serganova cohomology functor

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Abstract. We establish an explicit formula for the character of an irreducible finite-dimensional representation of $\mathfrak{gl}(m|n)$. The formula is a finite sum with integer coefficients in terms of a basis $\mathcal{E}_\mu$ (Euler characters) of the character ring. We prove a simple formula for the behavior of the "superversion" of $\mathcal{E}_\mu$ in the $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$-case under the map ds on the supercharacter ring induced by the Duflo–Serganova cohomology functor DS. As an application, we get combinatorial formulas for superdimensions, dimensions and $\mathfrak{g}_0$-decompositions for $\mathfrak{gl}(m|n)$ and $\mathfrak{osp}(m|2n)$.

1. Introduction

Let $\mathfrak{g}$ be a finite-dimensional Kac–Moody superalgebra. Denote by W the Weyl group of $\mathfrak{g}$.

1.1. A brief history of character formulas. Let $L(\lambda)$ be a simple finite-dimensional $\mathfrak{g}$-module. In 1977, V. Kac [30] showed that the Weyl character formula

$$Re^{\rho}\operatorname{ch} L(\lambda) = \sum_{w\in W} \operatorname{sgn}(w) w(e^{\lambda+\rho}),$$

where R is the Weyl denominator and ρ is the Weyl vector, holds if $L(\lambda)$ is typical. In 1980, I. Bernstein and D. Leites [1] established for $\mathfrak{g} = \mathfrak{gl}(1|n)$ the character formula

$$Re^{\rho}\operatorname{ch} L(\lambda) = \sum_{w\in W} \operatorname{sgn}(w) w\left(\frac{e^{\lambda+\rho}}{(1+e^{-\beta})}\right), \tag{1.1}$$

where $\beta \in \Delta_1^+$ satisfies $(\beta|\lambda) = 0$. This formula was extended to the $\mathfrak{osp}(2|2n)$-case in [50] and to $\mathfrak{gl}(m|n)$-modules of atypicality one in [51]. In 1998, J. Germoni produced similar character formulas for the two cases $\mathfrak{osp}(3|2)$ and $D(2|1;a)$; except for the case of the standard

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$\mathfrak{osp}(3|2)$-module, Germoni's formula is very similar to (1.1): the only difference is the factor $\frac{1}{2}$ appearing in the right-hand side in certain cases (other formulas were obtained earlier by van der Jeugt in [48,49]). In 1990, J. van der Jeugt, J. Hughes, R. C. King and J. Thierry-Mieg [51] suggested to write the character formula in the general $\mathfrak{gl}(m|n)$-case as a sum of terms

$$\sum_{w\in W}\operatorname{sgn}(w)w\Big(\frac{e^{\lambda}}{\prod_{\beta\in U}(1+e^{-\beta})}\Big)$$

for some $U\subset\Delta_1$ satisfying $(U|\lambda)=0$.

In 1994, V. Kac and M. Wakimoto [32] conjectured that

$$Re^{\rho}\operatorname{ch}L(\lambda)=j^{-1}\sum_{w\in W}\operatorname{sgn}(w)w\Big(\frac{e^{\lambda+\rho}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) \tag{1.2}$$

for some scalar j if the following conditions (which we will call the *KW-conditions*) hold: λ is the highest weight of L with respect to a base Σ containing S, $(S|\lambda+\rho)=(S|S)=0$ and the cardinality of S is equal to the atypicality of L. This conjecture was established in [7,8,23]. For each $S\subset\Delta$ satisfying $(S|\lambda)=(S|S)=0$, we set

$$\mathrm{KW}(\lambda,S):=\sum_{w\in W}\operatorname{sgn}(w)w\Big(\frac{e^{\lambda}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big).$$

(We call the above terms "Kac–Wakimoto terms" since the condition $(S|S)=0$ is crucial for our argument.)

Notice that Germoni's formulas demonstrate that the KW-conditions are not necessary for (1.2) to be valid: the only atypical $\mathfrak{osp}(3|2)$-module satisfying the KW-condition is trivial, whereas (1.1) holds for each $L\not\cong V_{\mathrm{st}}$.

The first general formula for the character $\operatorname{ch}L$ was discovered by V. Serganova [41] in the $\mathfrak{gl}(m|n)$-case by expressing the character as an infinite sum over characters of Kac modules. This algorithmic solution was enhanced by J. Brundan [2] who showed that the values of the coefficients in the infinite sum can be computed in terms of weight diagrams. The equivalence of the two approaches [2,41] was shown in [35]. Another description of the composition factors of Kac modules was obtained in [6]. The description of Brundan [2] was then used by Y. Su and R. Zhang [46] to establish a finite character formula for $\mathfrak{gl}(m|n)$. For the $\mathfrak{osp}$-case, a finite character formula was produced by C. Gruson and V. Serganova in [24]. For the exceptional Lie superalgebras, the character formulas were proven by J. Germoni [18] and L. Martirosyan [34]. For $\mathfrak{q}_n$, an implicit finite character formula was given by I. Penkov and V. Serganova in [37]; Y. Su and R. B. Zhang [47] wrote this formula explicitly using [3]; for the $\mathfrak{p}_n$-case, an infinite Serganova type character formula was recently proven by B.-H. Hwang and J.-H. Kwon [29].

The Kac–Wakimoto character formula suggests the following two refinements of van der Jeugt–Hughes–King–Thierry-Mieg's proposal: to present $Re^{\rho}\operatorname{ch}L$ as a linear combinations of $\mathrm{KW}(\nu,S(\nu))$, where

(I) S_ν is maximal, i.e. $|S_\nu|$ equals the atypicality of $L(\nu)$ or

(II) S_ν can be embedded into a certain base Σ_L.

The Kac–Wakimoto formula is of both types. The Germoni formula and the Su–Zhang formula for $\mathfrak{gl}(m|n)$ are of type I; the Gruson–Serganova formula for $\mathfrak{osp}$ and the Su–Zhang formula for $\mathfrak{q}_n$ are of type II. For the exceptional algebras, the character formulas in [18, 34] can be rewritten as type I or as type II formulas.

1.2. The Gruson–Serganova type character formula. In this paper, we obtain a formula of type II for $\mathfrak{gl}(m|n)$ (by above, such formulas were obtained early for all other cases). Since the $\mathfrak{q}(n)$-case and the exceptional cases can be treated with the same methods, this gives a uniform approach to obtain finite character formulas. The main difference of the $\mathfrak{gl}(m|n)$-case compared to the $\mathfrak{osp}(m|2n)$-case is that Σ_L depends on L.

Let Irr be the set of isomorphism classes of finite-dimensional irreducible $\mathfrak{g}$-modules. The terms $\{\operatorname{ch} L,\ L \in \mathrm{Irr}\}$ form a natural basis of the character ring. We say $\operatorname{ch} L$ is given by a *Gruson–Serganova type character formula* if it can be written as a sum

$$Re^{\rho} \operatorname{ch} L = \sum_{L' \in \mathrm{Irr}} b_{L,L'} \, \mathrm{KW}(L'), \tag{1.3}$$

where the Kac–Wakimoto terms $\mathrm{KW}(L)$ have the following properties.

(i) $\{\mathcal{E}_L := (Re^{\rho})^{-1} \,\mathrm{KW}(L),\ L \in \mathrm{Irr}\}$ form a basis of the character ring (where R is the Weyl denominator). The terms $\mathcal{E}_L$ are equal to the Euler characteristics $\mathcal{E}_\lambda$ (for a suitable choice of parabolic) of Penkov–Serganova, and hence we can equally write the character of $L(\lambda)$ as a finite sum with integral coefficients in the Euler characters).

(ii) The character formula $Re^{\rho} \operatorname{ch} L = \sum_{L' \in \mathrm{Irr}} b_{L,L'} \,\mathrm{KW}(L)$ is finite.

(iii) The matrix $B := (b_{L,L'})$ is a lower triangular matrix with integral entries and 1's on the main diagonal; moreover, there exists a diagonal matrix D with $D^2 = \mathrm{Id}$ such that the entries of DBD^{-1} are non-negative (the entries of DBD^{-1} can be interpreted as a number of certain paths in a directed graph).

(iv) $\mathrm{KW}(L) := j(L)^{-1} \,\mathrm{KW}(\lambda^{\dagger}, S_L)$, where $j(L)$ is a scalar and $\lambda^{\dagger}$ is the ρ-shifted highest weight of L with respect to a certain base Σ_L containing S_L.

The scalar $j(L)$ is an order of the "smallest factor" in $\mathrm{Stab}_W \lambda^{\dagger}$ (for instance, $j(L) = |S|!$ for the $\mathfrak{gl}$-case). The set S_L is a maximal subset of Σ_L satisfying $(\lambda^{\dagger}|S) = (S|S) = 0$.

We call the cardinality of S_L the *tail* of L ($\mathrm{tail}(L)$); this is a non-negative integer which is less than or equal to the atypicality of L. If $b_{L,L'} \neq 0$, then L and L' lie in the same block and $\mathrm{tail}(L') \leq \mathrm{tail}(L)$. We call the highest weight of L a *Kostant weight* if $\mathrm{tail}(L)$ is equal to the atypicality of L; in this case, $b_{L,L'} = \delta_{L,L'}$. From [7, 8], it follows that, for $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$, $L(\lambda)$ satisfies the Kac–Wakimoto character formula if and only if its highest weight is a Kostant weight; this also holds for the $\mathfrak{osp}(2m|2n)$-modules of atypicality greater than one; see Remark 3.5.4.

In the $\mathfrak{osp}$-case, (1.3) was obtained in [24] in a slightly different form (property (iv) is established in Proposition 4.3 below). In this case, $\Sigma_L = \Sigma$ is the usual "mixed" base and $\lambda^{\dagger} = \lambda + \rho$.

In this paper, we will establish (1.3) in the $\mathfrak{gl}$-case. In this case, Σ_L depends on L; in Section 6, we describe the assignment $L \mapsto \lambda^{\dagger}$; this assignment is a one-to-one correspondence between the set of irreducible modules Irr and the set $\Lambda^{\dagger}$ which can be described in terms of weight diagrams as follows: this is the set of the diagrams where at most one position contains more than one of the symbols $\circ, >, <, \times$ and, if such a position exists, it contains $\times^i$ for $i > 1$ with no symbols $\times$ which precede this position. For each diagram $f^{\dagger}$ in $\Lambda^{\dagger}$, we assign S of cardinality $\mathrm{tail}(f^{\dagger})$, where $\mathrm{tail}(f^{\dagger})$ is the maximal number of $\times$'s appearing in the same position in $f^{\dagger}$. For instance, for λ with the diagram $f = {>}{\circ}{\times}{<}{\times}{>}{\circ}{\circ}{\times}$, we have

$$f^{\dagger} = {>}{\circ}{\circ}{<}{\times^2}{>}{\circ}{\circ}{\times} \quad \text{with } \mathrm{tail}(L(\lambda)) := \mathrm{tail}(f^{\dagger}) = 2,$$

and the character formula can be written as

$$\begin{aligned}Re^{\rho}\operatorname{ch} L(\lambda) = \mathrm{KW}(>\circ\times<\times>\circ\circ\times) - \mathrm{KW}(>\circ\times<\times>\circ\times\circ) \\ - \mathrm{KW}(>\times\times<\times>\circ\circ\circ) + 2\,\mathrm{KW}(>\circ\times<\times>\times\circ\circ),\end{aligned}$$

where, for instance,

$$\begin{aligned}\mathrm{KW}(L) =: \mathrm{KW}(>\circ\times<\times>\circ\circ\times) &= \frac{1}{2}\,\mathrm{KW}(>\circ\circ<\times^2>\circ\circ\times; \varepsilon_3-\delta_2, \varepsilon_4-\delta_3),\\ \mathrm{KW}(>\times\times<\times>\circ\circ\circ) &= \frac{1}{6}\,\mathrm{KW}(>\circ\circ<\times^3>\circ\circ\circ, \varepsilon_3-\delta_1, \varepsilon_4-\delta_2, \varepsilon_5-\delta_3).\end{aligned}$$

One has

$$\Sigma_L = \{\delta_1-\varepsilon_1, \varepsilon_1-\varepsilon_2, \varepsilon_2-\varepsilon_3, \varepsilon_3-\delta_2, \delta_2-\varepsilon_4, \varepsilon_4-\delta_3, \delta_3-\delta_4, \delta_4-\varepsilon_5\}$$

which can be naturally encoded as $\delta\varepsilon^2(\varepsilon\delta)^2\delta\varepsilon$.

1.3. Applications and conjectures. For a finite-dimensional module N, define the ξ-character

$$\operatorname{ch}_\xi N := \dim(N_{\bar{0}})_\nu e^\nu + \xi \dim(N_{\bar{1}})_\nu e^\nu$$

(where ξ is a formal variable with $\xi^2 = 1$); clearly, the $\mathbb{Z}$-span of ξ-characters form a ring, which we denote by $\mathrm{Ch}_\xi(\mathfrak{g})$. The character ring $\mathrm{Ch}(\mathfrak{g})$ and the supercharacter ring $\mathrm{Sch}(\mathfrak{g})$ are a factor of $\mathrm{Ch}_\xi(\mathfrak{g})$ by $\xi = 1$ and $\xi = -1$, respectively; these rings were explicitly described by A. N. Sergeev and A. P. Veselov [45]. Several important notions (for instance, $\dim N$ and $\operatorname{sdim} N$) can be viewed as linear maps from $\mathrm{Ch}_\xi(\mathfrak{g})$. The Gruson–Serganova formula gives an expression of $\mathrm{Ch}_\xi L$ see (A.3) and a formula for $\operatorname{sch} L$ in terms of $\mathcal{E}_L^-$ (which are "superanalogues" of $\mathcal{E}_L$).

An important example is the map $\mathrm{mult}_{L'}\colon \mathrm{Ch}(\mathfrak{g}) \to \mathbb{Z}$ which assigns to $\operatorname{ch} N$ the (non-graded) multiplicity $[N : L']$, where L' is a $\mathfrak{g}_0$-module; in Corollary A.5.3, we give formulas for $\mathrm{mult}_{L'}(\mathcal{E}_L)$ and for $\dim(\mathcal{E}_L)$.

Another important linear map is induced by the Duflo–Serganova monoidal functor

$$\mathrm{DS}_x\colon \mathcal{F}\mathrm{in}(\mathfrak{g}) \to \mathcal{F}\mathrm{in}(\mathfrak{g}_x),$$

where $\mathfrak{g}_x$ is a smaller rank Lie superalgebra. In [26, Section 3.8], it was shown that DS_x induces a ring homomorphism

$$\mathrm{ds}_x\colon \mathrm{Sch}(\mathfrak{g}) \to \mathrm{Sch}(\mathfrak{g}_x)$$

given by $\mathrm{ds}_x\colon \operatorname{sch} N \mapsto \operatorname{sch} \mathrm{DS}_x(N)$ (see also [28] for more details); moreover, ds_x coincides with the evaluation of sch to a subalgebra $\mathfrak{h}_x \subset \mathfrak{h}$. In Theorem 7.2, we show that $\mathrm{ds}_x(\mathcal{E}_L^-)$ is given by a simple formula (for the exceptional Lie superalgebras, a similar formula follows from [20]). Since DS_x preserves sdim, this gives a formula for $\operatorname{sdim} \mathcal{E}_L^-$; see Corollary 7.2.8. It turns out that $\operatorname{sdim} \mathcal{E}_L^- = 0$ except for the case when $\operatorname{tail}(L)$ is equal to the defect of $\mathfrak{g}$ (by above, $\operatorname{tail}(L)$ is less than or equal to the defect of $\mathfrak{g}$).

Using equation (1.3) and the aforementioned formulas, one obtains the expressions for $\operatorname{sch} \mathrm{DS}_x(L)$, $\operatorname{sdim} L$, $[L : L']$ and $\dim L$ for $L \in \mathrm{Irr}$ (we do not write these long expressions). Note that $\mathrm{DS}_x(L)$ is described in [20, 21, 26]; various formulas for $\dim L$ and $\operatorname{sdim} L$ appeared in [10, 21, 26, 34, 46].

It is well known that the highest weight of L with respect to any base Σ' can be computed by a recursive procedure. In Corollary 4.9.1, we show that, for the $\mathfrak{osp}(2m+1|2n)$-case, the ρ-shifted highest weight of L with respect to any base Σ' is the maximal element in the support of $\mathrm{KW}(L)$ and establish the same formula for certain bases for the $\mathfrak{osp}(2m|2n)$-case; we hope that a similar result holds in the $\mathfrak{gl}$-case as well.

For each base Σ', let ν denote the ρ-shifted highest weight of L with respect to Σ'. Consider the leftmost position in its weight diagram containing the symbol $\times$, and let $\mathrm{tail}'(\nu)$ be the number of $\times$'s in this position. We conjecture that, for the $\mathfrak{gl}$-case, one has

$$\mathrm{tail}(L) = \max_{\Sigma'} \mathrm{tail}'(\nu).$$

1.4. Euler characters. The Euler characters were originally defined as Euler characteristics of the cohomology of vector bundles on a super-flag variety [41]. They were first introduced in [36–38] and play also a crucial role in Brundan's work on characters in the $\mathfrak{q}(n)$-case [3, 4]. We will describe the Euler characters for the "core-free case", i.e. for the principal block of $\mathfrak{gl}(d|d)$ or $\mathfrak{osp}(2d+t|2d)$, where $t = 0, 1, 2$. (A similar description works for all $\mathfrak{osp}$-weights and for the "stable weights" in the $\mathfrak{gl}$-case.)

Fix a flag of parabolic subalgebras

$$\mathfrak{g} = \mathfrak{p}^{(d)} \supset \mathfrak{p}^{(d-1)} \supset \cdots \supset \mathfrak{p}^{(0)} = \mathfrak{b},$$

where d is the defect of $\mathfrak{g}$ and $\mathfrak{l}^{(i)} := [\mathfrak{p}^{(i)}, \mathfrak{p}^{(i)}]$ is of defect i (one has $\mathfrak{l}^{(i)} = \mathfrak{gl}(i|i)$ for $\mathfrak{g} = \mathfrak{gl}(d|d)$, $\mathfrak{l}^{(i)} = \mathfrak{osp}(2i+t|2i)$ for $\mathfrak{g} = \mathfrak{osp}(2d+t|2d)$). For a pair of parabolic subalgebras $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ containing a fixed Borel $\mathfrak{b}$, let $\Gamma_{\mathfrak{p},\mathfrak{q}}(V)$ denote the maximal finite-dimensional quotient of the induced module $\mathcal{U}(\mathfrak{p}) \otimes_{\mathcal{U}(\mathfrak{q})} V$. Then $\Gamma_{\mathfrak{p},\mathfrak{q}}$ defines a functor from the category of finite-dimensional $\mathfrak{q}$-modules $\mathcal{F}\mathrm{in}(\mathfrak{q})$ to $\mathcal{F}\mathrm{in}(\mathfrak{p})$, and we denote by $\Gamma^i_{\mathfrak{p},\mathfrak{q}}$ its derived functors as in [24]. For $\lambda, \mu \in \Lambda^+_{m|n}$, we consider the Poincaré polynomial in the variable z,

$$K^{\lambda,\mu}_{\mathfrak{p},\mathfrak{q}}(z) := \sum_{i=0}^{\infty} [\Gamma^i_{\mathfrak{p},\mathfrak{q}}(L_{\mathfrak{q}}(\lambda)) : L_{\mathfrak{p}}(\mu)] z^i,$$

where $L_{\mathfrak{q}}(\lambda)$ and $L_{\mathfrak{p}}(\mu)$ stand for the corresponding simple $\mathfrak{q}$ and $\mathfrak{p}$-module, respectively. Proposition 1 in [24] expresses the Euler characteristic

$$\mathcal{E}_{\lambda,\mathfrak{p}} := \sum_{\mu \in \Lambda^+_{m|n}} K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}}(-1) \operatorname{ch} L(\mu) \tag{1.4}$$

in terms of $\operatorname{ch} L_{\mathfrak{p}}(\lambda)$. The polynomials $K^{\lambda,\nu}_{\mathfrak{p},\mathfrak{q}}(z)$ for the "neighboring parabolics" were computed in [39] in the $\mathfrak{gl}$-case and in [24] in the $\mathfrak{osp}$-case. These polynomials can be conveniently described in terms of so-called "arc diagrams", Section 5.2.2. The coefficients $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}}(-1)$ can be computed iteratively via the formula

$$K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{q}}(-1) = \sum_{\nu} K^{\lambda,\nu}_{\mathfrak{p},\mathfrak{q}}(-1) K^{\nu,\mu}_{\mathfrak{g},\mathfrak{p}}(-1) \tag{1.5}$$

established in [24, Theorem 1].

In the $\mathfrak{gl}$-case, the matrix $A_{\mathfrak{b}} := (K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{b}}(-1))$ is invertible and the inverse matrix can be explicitly described; this gives the Serganova character formula [39]; this is an "infinite formula": some rows of $A_{\mathfrak{b}}^{-1}$ have infinitely many non-zero entries. For the $\mathfrak{osp}$-case, one has $\mathcal{E}_{\lambda,\mathfrak{b}} = 0$ for some λ's, so the matrix $A_{\mathfrak{b}}$ is not invertible. In order to obtain the Gruson–Serganova character formula, we take for each λ the "maximal suitable parabolic", setting $\mathfrak{p}_\lambda := \mathfrak{p}^{\mathrm{tail}(\lambda)}$, where tail$(\lambda)$ is the maximal i such that $\lambda|_{\mathfrak{h}\cap\mathfrak{l}^{(i)}} = 0$. (For $\mathfrak{osp}$-weights and for the "stable" $\mathfrak{gl}$-weights, one has $\mathrm{tail}(\lambda) = \mathrm{tail}(L(\lambda))$). The iterative formula (1.5) allows to interpret $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}}(-1)$ in terms of "decreasing paths" in a certain directed graph. This graph has several nice properties, which allow to express the inverse matrix $(K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}^\lambda}(-1))^{-1}$ in terms of the paths in this graph; using (1.4), we obtain

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu\in\Lambda^+_{m|n}} (-1)^{\|\lambda\|-\|\mu\|} d^{\lambda,\mu}_{<} \mathcal{E}_{\mu,\mathfrak{p}_\mu} \tag{1.6}$$

for stable weights λ, where $d^{\lambda,\mu}_{<}$ is the number of "increasing paths" from μ to λ in the directed graph and $\|\lambda\|$ is defined in Section 3.5. Since $\dim L_{\mathfrak{p}_\mu}(\mu) = 1$, [24, Proposition 1] gives an explicit formula for $\mathcal{E}_{\mu,\mathfrak{p}_\mu}$; using the denominator identity from [32], this formula can be rewritten as $Re^\rho \mathcal{E}_{\mu,\mathfrak{p}_\mu} = \mathrm{KW}(L)$. This gives (1.3).

For the $\mathfrak{osp}$-case, this program was executed in [24]. In the first part of our paper, we execute a similar program for $\mathfrak{gl}$. The graph in the $\mathfrak{gl}$-case has an easier description than in the $\mathfrak{osp}$-case, but its structure is more complicated: by contrast with the $\mathfrak{osp}$-case, each component has infinitely many sources. We show that, in the $\mathfrak{gl}$-case, each vertex has finitely many predecessors, and this property allows us to obtain (1.6). We will reveal some additional details in Section 1.5 below.

To link (1.6) and (1.3), we show in Proposition 4.3 that the Euler characters are proportional to Kac–Wakimoto terms. This result is also fundamental when we study the effect of ds on Euler characters in the second part of the paper.

A similar approach works for the exceptional Lie superalgebras and for $\mathfrak{q}_n$; in these cases, each component of the graph has a unique source. To the best of our knowledge, the Gruson–Serganova type character formula is not known for the $\mathfrak{p}_n$-case; we expect that, in this case, each component has infinitely many sources as for the $\mathfrak{gl}(m|n)$-case.

1.5. Method of proof. The proof of (1.6) uses iterated parabolic induction. We will outline this proof for the principal block of $\mathfrak{gl}(d|d)$. A similar proof works for all $\mathfrak{osp}$-weights and for the "stable weights" in the $\mathfrak{gl}$-case. On the other hand, the character formula for a simple module of atypicality d can be reduced to this case.

The graphs $\hat{\Gamma}^\chi$ and Γ^χ are directed graphs with the same set of vertices enumerated by the highest weights of the irreducible modules in the principal block. In $\hat{\Gamma}^\chi$, the vertices μ, λ are joined by the edge $\mu \xrightarrow{e} \lambda$ if $K^{\lambda,\mu}_{\mathfrak{p}^{(s)},\mathfrak{p}^{(s-1)}} \neq \delta_{\lambda,\mu}$; for such an edge, we set $b(e) := s$. For Γ^χ, we require that $\mathfrak{p}^{s-1} \supset \mathfrak{p}_\lambda$ (in other words, Γ^χ can be obtained from $\hat{\Gamma}^\chi$ by deleting the edges with $b(e) \leq \mathrm{tail}(\lambda)$). By [35], for the $\mathfrak{gl}(m|n)$-case, $K^{\lambda,\mu}_{\mathfrak{p}^{(s)},\mathfrak{p}^{(s-1)}}$ is either $\delta_{\lambda,\mu}$ or z^i with $i \equiv \|\lambda\| - \|\mu\|$. In particular, if μ and λ are connected by an edge in $\hat{\Gamma}^\chi$, then

$$K^{\lambda,\mu}_{\mathfrak{p}^{(s)},\mathfrak{p}^{(s-1)}}(-1) = (-1)^{\|\lambda\|-\|\mu\|}.$$

We define a "b-decreasing path" in $\hat{\Gamma}^\chi$, Γ^χ as a path with a decreasing function b. Formula (1.5) allows to express $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{b}}(-1)$ as a sum of $(-1)^{\mathrm{length}(P)+\|\lambda\|-\|\mu\|}$, where P runs through the b-decreasing paths from μ to λ in $\hat{\Gamma}^\chi$. We show that $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}_\lambda}(-1)$ has the similar formula in terms of the b-decreasing paths in Γ^χ (the proof uses the fact that a path in $\hat{\Gamma}^\chi$ lies in Γ^χ if the last edge of this path lies in Γ^χ).

The matrix $A_> := (K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}_\lambda}(-1))$ is invertible; by above, its entries can be written in terms of the b-decreasing paths from μ to λ in Γ^χ. We substitute b by another function b' such that b-decreasing paths and b'-decreasing paths are the same. The function b' has the following advantage:

(*) if ν is the start of an edge e_1 and the end of an edge e_2, then $b'(e_1) \neq b'(e_2)$

(this property does not hold for the function b). By above, $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}_\lambda}(-1)$ can be written as a sum of $(-1)^{\mathrm{length}(P)+\|\lambda\|-\|\mu\|}$, where P runs through the b'-decreasing paths from μ to λ in Γ^χ. Property (*) implies that the entries of $A_>^{-1}$ can be written as a sum of $(-1)^{\|\lambda\|-\|\mu\|}$, where P runs through the b'*-increasing* paths; this gives formula (1.6) ($d_<^{\lambda,\mu}$ stands for the number of increasing paths from μ to λ with respect to b'). The graph $\widehat{\Gamma}^\chi$ and the functions $b, b', \deg(e)$ can be naturally described in terms of arc diagrams (see Section 4.5.5).

In order to prove the finiteness of formula (1.6), we show that each vertex λ has a finite set of predecessors in Γ^χ (for the $\mathfrak{gl}$-case, this property does not hold for $\widehat{\Gamma}^\chi$; for the $\mathfrak{osp}$ and $\mathfrak{q}$-cases, the property holds in both cases since $\{\mu \in \Lambda_{m|n} \mid \mu \leq \lambda\}$ is finite).

1.6. Modified superdimensions. By the Kac–Wakimoto conjecture, $\operatorname{sdim} L(\lambda) \neq 0$ if and only if $L(\lambda)$ has a maximal atypicality; this conjecture was proven by V. Serganova in [43]. In the $\mathfrak{gl}(m|n)$-case, $\operatorname{sdim} L(\lambda)$ was computed in [26].

Consider the case $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(M|N)$. Fix a triangular decomposition in the *usual way* (see [24,25], etc.), i.e. a distinguished base for $\mathfrak{gl}(m|n)$ and the mixed base for $\mathfrak{osp}(m|2n)$. Let $L(\lambda)$ be a finite-dimensional simple module of atypicality k. In this case, applying DS_x with $\mathrm{rk}(x) = k$ to $L(\lambda)$ gives by Theorem 7.2 an isotypic representation $L(\lambda')^{\oplus m(\lambda)}$ of

$$\mathfrak{g}_x = \mathfrak{gl}(m-k|n-k)$$

in the $\mathfrak{gl}$-case, and in the $\mathfrak{osp}$-case, either an isotypic representation $L(\lambda')^{\oplus m(\lambda)}$ of

$$\mathfrak{g}_x = \mathfrak{osp}(m-2k|2n-2k)$$

(if $L(\lambda')$ is σ invariant for the involution σ of OSp; see Section 2.2) or $(L(\lambda') \oplus L(\lambda')^\sigma)^{\oplus m(\lambda)}$ otherwise. We put $L^{\mathrm{core}} = L(\lambda')$ in the $\mathfrak{gl}$ and $\mathfrak{osp}(2m+1|2n)$-case and

$$L^{\mathrm{core}} := \begin{cases} L(\lambda') & \text{if } \lambda' \text{ is } \sigma-\text{invariant}, \\ L(\lambda') \oplus L(\lambda')^\sigma & \text{else}, \end{cases}$$

in the $\mathfrak{osp}(2m|2n)$-case. Then L^{core} only depends on the central character of λ. Using this notation, we obtain in case where the atypicality of $L(\lambda)$ equals the rank of x the uniform formula

$$\mathrm{DS}_x(L(\lambda)) \cong \Pi^i (L^{\mathrm{core}})^{\oplus m(\lambda)}$$

for some parity shift Π^i.

Identifying $\mathfrak{g}_x$ with a subalgebra of $\mathfrak{g}$ as in [13, 22], we can interpret the above formula as follows: for a simple $\mathfrak{g}_x$-module L', the "super-multiplicity" of L' in $L(\lambda)$ is zero if $[L^{\mathrm{core}} : L'] = 0$ and is $\pm m(\lambda)$ otherwise; see Remark 2.3.1.

If $L(\lambda)$ is maximal atypical, $\mathfrak{g}_x$ is one of the algebras $\mathfrak{gl}_k$, $\mathfrak{o}_k$, $\mathfrak{sp}_k$, $\mathfrak{osp}(1|2k)$.

The numbers $m(\lambda)$ can be computed in the equal rank case. In this case, we have that $m(\lambda) = |\operatorname{sdim} L(\lambda)|$ is equal to the number of increasing paths from the Kostant weights, which are μ's with $\dim L(\mu) = 1$, to λ. (In all cases, it is easy to see that the existence of such paths is equivalent to the maximal atypicality condition). Therefore, we reprove the Kac–Wakimoto conjecture and establish another combinatorial expressions for the superdimensions.

If $L(\lambda)$ is not maximal atypical, one can introduce a modified superdimension sdim^k on the thick tensor ideal spanned by the irreducible representations of atypicality k instead. We show that the modified superdimension is given by $\mathrm{sdim}^k(L(\lambda)) = \pm m(\lambda)\,\mathrm{sdim}^0(L^{\mathrm{core}})$ for $L(\lambda)$ of atypicality k, where sdim^0 is the (unique up to a scalar) modified superdimension on the thick ideal of projective objects, reproving results of [33, 43].

By [21], the isotypic multiplicity in the $\mathfrak{osp}$-case can be expressed in terms of the arc diagram of λ.

1.7. Structure of the article. We recall some background in Sections 2 and 3. In particular, in Section 3, we discuss stability, tail and weight diagrams. We define parabolic induction functors and their derived versions in Section 4. The main results, formula (1.3) for $\mathfrak{gl}(m|n)$ and the behavior of $\mathcal{E}_\mu^-$'s under ds, are proven in Section 5 and Theorem 7.2. For the relationship of (1.3) to other existing $\mathfrak{gl}(m|n)$-character formulas (notably the one from Su–Zhang), see Section 5.4. Section 6 deals with $\mathcal{E}_{\mu,\mathfrak{p}_\mu}$ and $\mathrm{KW}(L)$ in the $\mathfrak{gl}(m|n)$-case; in particular, we describe the assignment $L \mapsto \lambda^\dagger$ and establish property (iv). The results on superdimensions and modified superdimensions are assembled in Section 8. We discuss properties of $\mathrm{KW}(L)$ in Appendix A. In Section A.5, we compute $\dim \mathcal{E}_\lambda$ and obtain a formula for the multiplicity of a $\mathfrak{g}_0$-module in $L(\lambda)$; see Corollary A.5.3, Section A.5.5 and Remark A.5.6 for the graded versions.

1.8. Index of definitions and notation. Throughout the paper, the ground field is $\mathbb{C}$; $\mathbb{N}$ stands for the set of non-negative integers. We will use the standard Kac notation for the root systems.

$\mathrm{KW}(\lambda, S), \mathrm{Ch}(\mathfrak{g}), \mathrm{Sch}(\mathfrak{g})$	Section 1.2
$\mathcal{F}, \Lambda^+_{m\|n}$	Section 2.1
σ	Section 2.2
$\mathfrak{g}_x, \mathrm{rk}(x), \mathrm{ds}_x, \mathrm{ds}_j$	Section 2.3
S_s, Σ, ρ	Section 3.1
iso-set, $\mathrm{at}(\lambda)$, $\mathrm{at}(L)$, stable weight, $\mathrm{tail}(\lambda)$, $\mathfrak{g}_\lambda$	Section 3.2
$\Lambda^{\geq}_{m\|n}$, weight diagrams, $\mathrm{diag}(\lambda)$, $\mathrm{wt}(\lambda)$, stability and tail for the diagrams	Section 3.3
$\mathrm{core}(\lambda)$, χ_λ, Λ^χ, $\mathrm{core}(\chi)$, core-free, $\mathrm{howl}(\lambda)$	Section 3.4
$\\|\lambda\\|_{\mathrm{gr}}$, $\\|\lambda\\|$, τ, Kostant weights	Section 3.5
$K^{\lambda,\mu}_{\mathfrak{p},\mathfrak{q}}(z)$	Section 4.1
$\mathcal{E}_\lambda, \mathfrak{p}_\lambda$	Section 4.2
ρ_L	Proposition 4.3
arc diagrams for $\mathfrak{gl}$, $b(\nu;\lambda)$, $b'(\nu;\lambda)$	Section 4.5
graph $D_\mathfrak{g}$ for $\mathfrak{gl}$, $D^\chi_\mathfrak{g}$	Section 4.6

2. Preliminaries

We denote by Π the parity change functor. Throughout Sections 2–8, $\mathfrak{g}$ stands for one of the Lie superalgebras $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m|2n)$ or $\mathfrak{osp}(2m+1|2n)$.

2.1. Notation. We use the standard notation: the root system Δ lies in the lattice

$$\Lambda_{m|n} \subset \mathfrak{h}^* \quad \text{spanned by} \quad \{\varepsilon_i\}_{i=1}^{m} \cup \{\delta_i\}_{i=1}^{n}.$$

We denote by Λ the lattice spanned by $\{\varepsilon_i\}_{i=1}^{\infty} \cup \{\delta_i\}_{i=1}^{\infty}$ and view $\Lambda_{m|n}$ as a subset of Λ. We define the parity homomorphism $p\colon \Lambda \to \mathbb{Z}_2$ by $p(\varepsilon_i) = \overline{0}$, $p(\delta_j) = \overline{1}$ for all i, j.

For our purposes (character formulas and DS functor), the study of the category $\mathcal{F}$in of finite-dimensional representations of $\mathfrak{g}$ with parity preserving morphisms reduces to study the category $\widetilde{\mathcal{F}}$ with the modules whose weights lie in $\Lambda_{m|n}$.

The category $\widetilde{\mathcal{F}}$ is canonically isomorphic to the category of G-modules, where G is a classical supergroup corresponding to $\mathfrak{g}$,

$$G := \mathrm{GL}(m|n) \text{ for } \mathfrak{gl}(m|n) \quad \text{and} \quad G := \mathrm{SOSp}(m|n) \text{ for } \mathfrak{g} = \mathfrak{osp}(m|n).$$

We fix the same triangular decomposition as in [24, 25, 35]: for $\mathfrak{gl}(m|n)$, we choose the base Σ which contains only one odd root $\varepsilon_m - \delta_1$, and in the $\mathfrak{osp}$-case, we choose a base Σ which contains a maximal possible number of odd roots; we always consider Σ as the ordered set with respect to the usual order (see examples in Section 3.1 below).

We denote by $\Lambda^{+}_{m|n}$ the set of dominant weights in $\Lambda_{m|n}$,

$$\Lambda^{+}_{m|n} := \{\lambda \in \Lambda_{m|n} \mid \dim L(\lambda) < \infty\}.$$

The simple modules in $\widetilde{\mathcal{F}}$ are of the form $L(\lambda), \Pi(L(\lambda))$ for $\lambda \in \Lambda^{+}_{m|n}$.

The category $\widetilde{\mathcal{F}}$ decomposes into a direct sum of two categories,

$$\widetilde{\mathcal{F}} = \mathcal{F} \oplus \Pi\mathcal{F},$$

such that the simple objects in $\mathcal{F}$ are labeled by the dominant integral weights. The category $\mathcal{F}$ is the full subcategory consisting of all modules M with the $\mathbb{Z}_2$-grading induced by the grading on Λ (i.e. $M_i = \sum_{\nu : p(\nu)=i} M_\nu$). Note that $\widetilde{\mathcal{F}}$ and $\mathcal{F}$ are tensor categories.

2.2. OSp, SOSp and σ. One has

$$\mathrm{O}(2r+1) = \mathrm{SO}(2r+1) \times \mathbb{Z}_2 \quad \text{and} \quad \mathrm{O}(2r) = \mathrm{SO}(2r) \rtimes \mathbb{Z}_2;$$

we can choose the subgroup $\mathbb{Z}_2$ in such a way that $\mathbb{Z}_2$ acts on $\mathfrak{osp}(2r|2n)$ by an involutive automorphism σ which stabilizes the Cartan algebra $\mathfrak{h}$. For $r > 1$ (i.e. $\mathfrak{o}_r \neq \mathbb{C}$), σ induces a Dynkin diagram involution given by

$$\begin{aligned} \sigma(\delta_j) &= \delta_j, \qquad j = 1, \ldots, n, \\ \sigma(\varepsilon_i) &= \varepsilon_i, \qquad i = 1, \ldots, r-1, \\ \sigma(\varepsilon_r) &= -\varepsilon_r. \end{aligned}$$

For odd $m = 2r+1$, the orthosymplectic supergroup $\mathrm{OSp}(m|2n)$ is a direct product,

$$\mathrm{OSp}(2r+1|2n) \cong \mathrm{SOSp}(2r+1|2n) \times \mathbb{Z}_2,$$

where the nontrivial element of $\mathbb{Z}_2$ acts as minus the identity. For even $m = 2r$, it is a semidirect product,

$$\mathrm{OSp}(2r|2n) \cong \mathrm{SOSp}(2r|2n) \rtimes \mathbb{Z}_2.$$

The underlying even group of $\mathrm{OSp}(m|2n)$ is $\mathrm{O}(m) \times \mathrm{Sp}(2n)$ and $\mathrm{SO}(m) \times \mathrm{Sp}(2n)$ in the SOSp-case.

The automorphism σ can be extended to $\mathfrak{osp}(2r|2n)$. For $m > 1$, the involution σ on $\mathfrak{h}^*$ is given by

$$\begin{aligned} \sigma(\delta_j) &= \delta_j, \qquad j = 1, \ldots, n, \\ \sigma(\varepsilon_i) &= \varepsilon_i, \qquad i = 1, \ldots, r-1, \\ \sigma(\varepsilon_r) &= -\varepsilon_r. \end{aligned}$$

A finite-dimensional $\mathrm{SO}(2r)$-module N can be extended to $\mathrm{O}(2r)$ if and only if $N^\sigma \cong N$. Similarly, a finite-dimensional $\mathrm{SOSp}(2r|2n)$-module N can be extended to $\mathrm{OSp}(2r|2n)$ if and only if $N^\sigma \cong N$. See also [14] for more details.

2.3. The DS-functor. The DS-functor was introduced in [13]. We recall the definition below.

For a $\mathfrak{g}$-module M and $g \in \mathfrak{g}$, we set

$$M^g := \mathrm{Ker}_M\, g.$$

We fix now an $x \in \mathfrak{g}_1$ with $[x, x] = 0$. We set $\mathfrak{g}_x := \mathfrak{g}^{\mathrm{ad}\, x}/[x, \mathfrak{g}]$; note that $\mathfrak{g}^{\mathrm{ad}\, x}$ and $\mathfrak{g}_x$ are Lie superalgebras. For a $\mathfrak{g}$-module M, we set

$$\mathrm{DS}_x(M) = M^x / xM.$$

Observe that M^x, xM are $\mathfrak{g}^{\mathrm{ad}\, x}$-invariant and $[x, \mathfrak{g}]M^x \subset xM$, so $\mathrm{DS}_x(M)$ is a $\mathfrak{g}^{\mathrm{ad}\, x}$-module and $\mathfrak{g}_x$-module. Thus $\mathrm{DS}_x \colon M \to \mathrm{DS}_x(M)$ is a symmetric monoidal functor from the category of $\mathfrak{g}$-modules to the category of $\mathfrak{g}_x$-modules.

2.3.1. Remark. Notice that the action of x provides a $\mathfrak{g}^{\mathrm{ad}\, x}$-isomorphism

$$M/M^x \xrightarrow{\sim} \Pi(xM).$$

This implies that the "super-multiplicity" of a simple $\mathfrak{g}^{\mathrm{ad}\,x}$-module L' in a $\mathfrak{g}$-module M equals the "super-multiplicity" of L' in the $\mathfrak{g}_x$-module $\mathrm{DS}_x(M)$,

$$[M : L'] - [M : \Pi(L')] = [\mathrm{DS}_x(M) : L'] - [\mathrm{DS}_x(M) : \Pi(L')].$$

In many examples, $\mathfrak{g}_x$ can be identified with a subalgebra of $\mathfrak{g}$; in this case, the same holds for a simple $\mathfrak{g}_x$-module L'. The examples of such situation includes the cases when $\mathfrak{g}$ is a finite-dimensional Kac–Moody algebra (and x is arbitrary); see [13, 22].

2.3.2. Let G_0 denote $\mathrm{GL}(m) \times \mathrm{GL}(n)$ (the $\mathfrak{gl}$-case) or $\mathrm{O}(m) \times \mathrm{Sp}(2n)$ (the $\mathfrak{osp}$-case). There exist $g \in G_0$ and isotropic mutually orthogonal linearly independent roots $\alpha_1, \dots, \alpha_j$ such that

$$\mathrm{Ad}_g(x) = x_1 + \cdots + x_j, \quad \text{where } x_i \in \mathfrak{g}_{\alpha_i}.$$

The number j does not depend on the choice of g and is denoted by rank x (or $\mathrm{rk}(x)$) [13]. The Lie superalgebra $\mathfrak{g}_x$ depends only on the rank of x. For $\mathrm{rk}(x) = k$, we have

$$\mathfrak{g}_x \cong \begin{cases} \mathfrak{gl}(m-k|n-k), & \mathfrak{g} = \mathfrak{gl}(m|n), \\ \mathfrak{osp}(m-2k|2n-2k), & \mathfrak{g} = \mathfrak{osp}(m|2n). \end{cases}$$

Take $x := x_1 + \cdots + x_j$ as above. Then the algebra $\mathfrak{h}_x := \mathfrak{h}^{\mathrm{ad}\,x}/([x, \mathfrak{g}] \cap \mathfrak{h})$ is a Cartan subalgebra of $\mathfrak{g}_x$. The functor DS_x induces a ring homomorphism $\mathrm{ds}_x \colon \mathrm{Sch}(\mathfrak{g}) \to \mathrm{Sch}(\mathfrak{g}_x)$ such that

$$\mathrm{sch}\,\mathrm{DS}_x(N) = \mathrm{ds}_x(\mathrm{sch}\,N) \quad \text{for each } N \in \mathcal{F}\mathrm{in}(\mathfrak{g}).$$

This homomorphism can be described as follows: the restriction $f \mapsto f|_{\mathfrak{h}^{\mathrm{ad}\,x}}$ gives a ring homomorphism $\mathrm{Sch}(\mathfrak{g}) \to \mathrm{Sch}(\mathfrak{g}^{\mathrm{ad}\,x})$; the image of this map lies in $\mathrm{Sch}(\mathfrak{g}_x)$ (which is a subring in $\mathrm{Sch}(\mathfrak{g}^{\mathrm{ad}\,x})$) and $\mathrm{ds}_x \colon \mathrm{Sch}(\mathfrak{g}) \to \mathrm{Sch}(\mathfrak{g}_x)$ is the corresponding map. If we choose

$$\mathfrak{h}_x \subset \mathfrak{h}^{\mathrm{ad}\,x} \quad \text{such that} \quad \mathfrak{h}^{\mathrm{ad}\,x} = \mathfrak{h}_x \oplus ([x, \mathfrak{g}] \cap \mathfrak{h}),$$

then ds_x is given by $f \mapsto f|_{\mathfrak{h}_x}$; see [28, Lemma 4].

2.3.3. In this paper, we will describe the action of ds_x on a certain basis of $\mathrm{Sch}(\mathfrak{g})$. We do not use DS, but ds only, and while DS_x depends on x (even for fixed rank [26]), ds_x depends only on the rank of x, and we simply write ds_k for ds_x with $\mathrm{rk}(x) = k$. Then

$$\mathrm{ds}_j = (\mathrm{ds}_1)^j. \tag{2.1}$$

3. Weights, roots and diagrams

We use the standard notation for the roots of $\mathfrak{g}_0$ and denote by Π_0 a standard set of simple roots. In what follows, we consider only bases Σ of Δ which are compatible with Π_0, that is $\Delta^+(\Sigma)_0 = \Delta^+(\Pi_0)$. By [40], all such bases are connected by chains of odd reflections. These bases can be encoded by words consisting of m letters ε and n letters δ (see examples below).

We fix a standard bilinear form on $\mathfrak{h}^*$ as follows:

$$(\varepsilon_i|\varepsilon_j) = \delta_{ij} = -(\delta_i|\delta_j), \quad (\varepsilon_i|\delta_j) = 0.$$

3.1. The base Σ and the sets S_s. We set $S_0 := \emptyset$ and, for $s = 1, \dots, \min(m,n)$, introduce the sets S_s as follows:

$$S_s := \begin{cases} \{\varepsilon_{m+1-i} - \delta_i\}_{i=1}^{s} & \text{for } \mathfrak{gl}(m|n), \\ \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=0}^{s-1} & \text{for } \mathfrak{osp}(2m|2n), \\ \{\varepsilon_{m-i} - \delta_{n-i}\}_{i=0}^{s-1} & \text{for } \mathfrak{osp}(2m+1|2n). \end{cases}$$

Notice that $S_{\min(m,n)}$ is a basis of a maximal isotropic subspace of $\mathfrak{h}^*$.

For $\mathfrak{gl}(m|n)$, we take the base Σ corresponding to the word $\varepsilon^m\delta^n$,

$$\Sigma := \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-1} - \varepsilon_m, \varepsilon_m - \delta_1, \delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\}.$$

For the $\mathfrak{osp}$-case, we denote by Σ a base containing $S_{\min(m,n)}$ (such a base is unique): this are the bases $\delta^{n-m}(\delta\varepsilon)^m$ and $\varepsilon^{m-n}(\delta\varepsilon)^n$ for $\mathfrak{osp}(2m|2n)$ with $n \geq m$ and $n \leq m$, respectively, and $\delta^{n-m}(\varepsilon\delta)^m$ and $\varepsilon^{m-n}(\varepsilon\delta)^n$ for $\mathfrak{osp}(2m+1|2n)$ with $n \geq m$ and $n \leq m$, respectively. For instance,

$$\Sigma = \{\delta_1 - \delta_2, \dots, \delta_{n-m+1} - \varepsilon_1, \varepsilon_1 - \delta_{n-m+2}, \delta_{n-m+2} - \varepsilon_2, \dots, \varepsilon_{m-1} - \delta_n, \delta_n \pm \varepsilon_m\}$$

for $\mathfrak{osp}(2m|2n)$ with $n \geq m$.

3.1.1. Remark. In [21], we used the same bases Σ, but in the $\mathfrak{osp}(2m|2n)$-case, we chose different S for different blocks. The formulas in Proposition 4.3 hold for both choices of S.

3.1.2. We denote by ρ the Weyl vector of $\mathfrak{g}$ (see Section A.1.1). Note that ρ is unique for $\mathfrak{osp}(M|2n)$ with $M \neq 2$; we take

$$\rho = \begin{cases} \sum_{i=1}^{n} (n-i)\delta_i & \text{for } \mathfrak{osp}(2|2n), \\ \sum_{i=1}^{m} (1-i)\varepsilon_i + \sum_{i=1}^{n} (m-i)\delta_i & \text{for } \mathfrak{gl}(m|n). \end{cases}$$

Notice that $(\rho|S_{\min(m,n)}) = 0$; in the $\mathfrak{osp}(2m|2n)$-case, one has $\sigma(\rho) = \rho$.

3.1.3. Remark. For our choice of Σ, there are several natural choices of x of each rank s: this can be $x \in \sum_{\beta \in S_s} \mathfrak{g}_\beta$ or $x \in \sum_{\beta \in S_s} \mathfrak{g}_{-\beta}$. For these choices, we have a natural embedding $\mathrm{DS}_x(\mathfrak{g}) = \mathfrak{g}_x \subset \mathfrak{g}$ which is compatible with the triangular decomposition with the set of positive roots

$$\Delta_x^+ := \{\alpha \in \Delta^+ \mid (\alpha|S_s) = 0\} \setminus S.$$

Our choice of ρ is compatible with the embedding $\mathfrak{g}_x \subset \mathfrak{g}$, i.e. $\rho|_{\mathfrak{h}_x}$ is the Weyl vector of $\mathfrak{g}_x$. For instance, for $\mathfrak{gl}(m|n)$ with $x \in \mathfrak{g}_{\varepsilon_m - \delta_1}$, we have

$$\rho|_{\mathfrak{h}_x} = \sum_{i=1}^{m-1} (1-i)\varepsilon_i + \sum_{i=2}^{n} (m-i)\delta_i$$

which is the Weyl vector for $\mathfrak{g}_x \cong \mathfrak{gl}(m-1|n-1)$.

3.2. Atypicality, stability and tails. We call $S \subset \Delta_1$ an *iso-set* if S forms a basis of an isotropic subspace in $\mathfrak{h}^*$, i.e. S is linearly independent and $(S|S) = 0$. For instance, S_r is an iso-set.

For $\lambda \in \mathfrak{h}^*$, we denote by $\operatorname{at}(\lambda)$ the atypicality of λ (i.e. the maximal cardinality of an iso-set orthogonal to λ). The atypicality of $L(\lambda)$ is equal to $\operatorname{at}(\lambda + \rho)$.

The stability is usually introduced for a weight diagram. Below, we will introduce this notion for a weight (and a fixed base Σ).

We say that $\mathfrak{g}_s \subset \mathfrak{g}$ is an *equal rank subalgebra* if $\mathfrak{g}_s$ is of the following form:

$$\begin{array}{ll} \mathfrak{g}_s = \mathfrak{gl}(s|s) & \text{for the } \mathfrak{gl}\text{-case}, \\ \mathfrak{g}_s = \mathfrak{osp}(2s+1|2s) & \text{for } \mathfrak{g} = \mathfrak{osp}(2m+1|2n), \\ \mathfrak{g}_s = \mathfrak{osp}(2s|2s) \text{ or } \mathfrak{osp}(2s+2|2s) & \text{for } \mathfrak{g} = \mathfrak{osp}(2m|2n), \end{array}$$

and, in addition, $\mathfrak{g}_s$ has a base $\Sigma_s \subset \Sigma$. Note that $\rho'_s := \rho|_{\mathfrak{g}_s \cap \mathfrak{h}}$ satisfies $(\rho'_s|\alpha) = 2(\alpha|\alpha)$ for each $\alpha \in \Sigma_s$, so ρ'_s is "a Weyl vector" for $\mathfrak{g}_s$ (ρ'_s is the usual Weyl vector except for the $\mathfrak{gl}$-case). Observe that $\mathfrak{g}$ contains a unique copy of $\mathfrak{g}_s$ for each s with $0 < s \leq \min(m, n)$.

3.2.1. Definition. In the $\mathfrak{gl}$-case, we say that $\lambda \in \Lambda^+_{m|n}$ is a *stable* weight if there exists an equal rank subalgebra $\mathfrak{g}_s \subset \mathfrak{g}$ such that

$$\operatorname{at}(\lambda + \rho)|_{\mathfrak{h} \cap \mathfrak{g}_s} = \operatorname{at}(\lambda + \rho) = s.$$

3.2.2. Definition. Take $\lambda \in \Lambda^+_{m|n}$, which is assumed to be stable for the $\mathfrak{gl}(m|n)$-case. We denote by $\mathfrak{g}_\lambda$ the maximal equal rank subalgebra of $\mathfrak{g}$ satisfying $\lambda|_{\mathfrak{h} \cap [\mathfrak{g}_\lambda, \mathfrak{g}_\lambda]} = 0$; we call $\mathfrak{g}_\lambda$ the *tail subalgebra* of λ and denote by $\operatorname{tail}(\lambda)$ the defect of $\mathfrak{g}_\lambda$.

3.2.3. Examples. The $\mathfrak{gl}(3|3)$-weight λ with $\lambda + \rho = 3\varepsilon_1 + 2\varepsilon_2 - 2\delta_2 - 5\delta_3$ is stable (with $\mathfrak{g}_s = \mathfrak{gl}(2|2)$), and one has $\mathfrak{g}_\lambda = \mathfrak{gl}(1|1)$ and $\operatorname{tail}(\lambda) = 1$; the $\mathfrak{gl}(3|3)$-weight ν with $\nu + \rho = 3\varepsilon_1 + \varepsilon_2 - \delta_2 - 5\delta_3$ is stable (with $\mathfrak{g}_s = \mathfrak{gl}(2|2)$), and one has $\mathfrak{g}_\nu = \mathfrak{gl}(2|2)$ and $\operatorname{tail}(\nu) = 2$.

3.3. Weight diagrams. Many properties of a finite-dimensional representation $L(\lambda)$ can be better understood by assigning a *weight diagram* to the weight λ (see e.g. [6,15,26,43]). These were first defined in [6] for $\mathfrak{gl}(m|n)$ and then for $\mathfrak{osp}(m|2n)$ in [24, 43] and for OSp in [14]. Note that the conventions how to draw these weight diagrams differ. The original weight diagrams of [6] use a different labeling of the vertices: our $>$ is a $\times$, our $<$ a $\circ$ and our $\times$ a $\vee$. For the difference between the weight diagrams of [24] and [14] in the $\mathfrak{osp}$-case, see [14, Proposition 6.1]. We follow essentially [24].

We denote by $\Lambda^{\geq}_{m|n}$ the following subgroup of $\mathfrak{h}^*$:

$$\Lambda^{\geq}_{m|n} := \left\{ \sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_i \delta_j \;\middle|\; a_1 \in \frac{1}{2}\mathbb{Z},\ a_1 - b_1 \in \mathbb{Z},\ a_i - a_j, b_i - b_j \in \mathbb{N} \text{ for } i < j \right\}.$$

The set $\Lambda^{\geq}_{m|n}$ contains ρ, $\Lambda^+_{m|n}$ and $\Lambda^+_{m|n} + \rho$.

3.3.1. We assign the weight diagram (a labeling of the real line $\mathbb{R}$ by certain symbols) to each weight $\sum_{i=1}^{m} a_i \varepsilon_i + \sum_{j=1}^{n} b_i \delta_j \in \Lambda^{\geq}_{m|n}$ using the following rules:

- for $\mathfrak{gl}(m|n)$, we put $>$ and $<$ at the position with the coordinate j if $a_i = j$ and $b_i = -j$ for some i, respectively;
- for $\mathfrak{osp}(2m|2n)$, we put $>$ and $<$ at the position with the coordinate t if $|a_i| = j$ and $|b_i| = j$ for some i, respectively; if $a_m \neq 0$, we add the signs $+$ and $-$ if $a_m > 0$ and $a_m < 0$, respectively;
- for $\mathfrak{osp}(2m+1|2n)$, we put $>$ and $<$ at the position with the coordinate $j - \frac{1}{2}$ if $|a_i| = j$ and $|b_i| = j$ for some i, respectively; we add the signs $+$ and $-$ if the zero position is occupied by $\times^p$ for $p > 0$, $(\lambda + \rho|\varepsilon_i) = \frac{1}{2}$ for some i and $(\lambda + \rho|\varepsilon_i) \neq \frac{1}{2}$ for each i, respectively.

If $>, <$ occupy the same position, we write these symbols as $\times$ ($\times^s$ stands for s symbols $<$ and s symbols $>$; $\overset{>}{\times^s}$ stands for s symbols $<$ and $s+1$ symbols $>$). We put an "empty symbol" $\circ$ at the non-occupied positions with the coordinates in $a_1 + \mathbb{Z}$; sometimes, instead of $\circ$, we put its coordinate (for instance, $0\circ\times$ means that $\times$ has the coordinate 2). For a diagram f, we denote by $f(a)$ the symbols at the a-th position.

Note that, for the $\mathfrak{osp}(2m+1|2n)$-case, our diagram is obtained from the diagram used in [24] by the shift by $-\frac{1}{2}$.

3.3.2. Examples. The diagram of ρ for $\mathfrak{osp}(2n+1|2n)$ has the sign $-$ and contains n symbols $\times$ in the zero position; we write this as $(-)\times^n$; similarly, for $\mathfrak{osp}(2n|2n)$, the diagram of ρ is $\times^n$, and for $\mathfrak{gl}(3|3)$, the diagram of ρ is $\times\times\times$, where the rightmost symbol $\times$ appears in the position 0. For $\mathfrak{gl}(m|n)$, we sometimes add a coordinate of $\circ$ instead one empty symbol; for instance, for $\mathfrak{gl}(4|3)$, the diagram of ρ can be written as

$$\times\times\times{>}1 \quad \text{or} \quad -4\times\times\times{>}.$$

3.3.3. We assign to each $\lambda \in \Lambda^{\geq}_{m|n}$ the diagram of $\lambda + \rho$ constructed as above; this diagram will be denoted by $\operatorname{diag}(\lambda)$. For a diagram f, the corresponding weight in $\Lambda^{\geq}_{m|n}$ will be denoted by $\operatorname{wt}(f)$; notice that

$$\operatorname{wt}(\operatorname{diag}(\lambda)) = \lambda + \rho.$$

The map $\lambda \mapsto \operatorname{diag}(\lambda)$ gives a one-to-one correspondence between $\Lambda^{+}_{m|n}$ and the diagrams containing k symbols $\times$, $m-k$ symbols $>$ and $n-k$ symbols $<$ (where $k \leq \min(m,n)$) with the following additional properties:

- the atypicality of $L(\lambda)$ is equal to the number of symbols $\times$ in the diagram of $\lambda + \rho$;
- in the $\mathfrak{gl}$-case, the coordinates of the occupied positions lie in $\mathbb{Z}$, and each occupied position contains exactly one of the signs $\{>, <, \times\}$;
- in the $\mathfrak{osp}$-case, the coordinates of the occupied positions lie in $\mathbb{N}$, and each non-zero occupied position contains exactly one of the signs $\{>, <, \times\}$;
- in the $\mathfrak{osp}(2m|2n)$-case, the zero position does not contain $<$, contains at most one symbol $>$ and an arbitrary number of $\times$; a diagram f has a sign if and only if $f(0) = \circ$; see [24, Section 6] for details;

- in the $\mathfrak{osp}(2m+1|2n)$-case, the zero position contains at most one of the symbols $>, <$ and an arbitrary number of $\times$; a diagram f has a sign if and only if $f(0) = \times^i$ for $i > 0$; see [24, Section 6] for details.

3.3.4. Remark (OSp$(2m|2n)$-modules). By Section 2.2, simple OSp$(2m|2n)$-modules are in one-to-one correspondence with the unsigned $\mathfrak{osp}(2m|2n)$-diagrams.

3.3.5. Tail in the diagrammatic language. It is easy to see that $\lambda \in \Lambda^{\geq}_{m|n}$ is stable in the $\mathfrak{gl}$-case if and only if all symbols $\times$ in $\mathrm{diag}(\lambda)$ precede all symbols $<, >$.

For $\lambda \in \Lambda^{+}_{m|n}$, we can easily express $\mathrm{tail}(\lambda)$ in terms of $f := \mathrm{diag}(\lambda)$ as follows:

- for the $\mathfrak{osp}(2m|2n)$-case, $\mathrm{tail}(\lambda)$ is equal to the number of symbols $\times$ in the zero position of f;
- for the $\mathfrak{osp}(2m+1|2n)$-case, $\mathrm{tail}(\lambda)$ is the number of symbols $\times$ in the zero position of f if f does not have $(+)$ sign and is less by 1 if f has the sign $(+)$;
- for a stable weight λ in the $\mathfrak{gl}$-case, $\mathrm{tail}(\lambda)$ is equal to the maximal length of the subdiagram $\times\times\ldots\times$ which starts from the first symbol $\times$ in $\mathrm{diag}(\lambda)$.

For instance, in the $\mathfrak{osp}$-case, $\mathrm{tail}(\circ\times\times) = 0$ and $\mathrm{tail}((+)\times^3\times) = 2$; in the $\mathfrak{gl}$-case, one has $\mathrm{tail}(\circ\times\times\circ\times\times\times) = 2$.

Note that, in the $\mathfrak{gl}$-case, $\mathrm{tail}(\lambda) \neq 0$ if λ is an atypical stable weight.

3.4. Cores and howls. We call the symbols $>, <$ the *core symbols*. A *core diagram* is a weight diagram which does not contain symbols $\times$ and does not have a sign.

For a weight diagram f, we denote by $\mathrm{core}(f)$ the core diagram which is obtained from the diagram of f by replacing all symbols $\times$ by $\circ$ and deleting the sign. For instance, $\mathrm{core}(<\circ\times>) = <\circ\circ>$. For a weight λ, we set

$$\mathrm{core}(\lambda) := \mathrm{core}(\mathrm{diag}(\lambda)).$$

3.4.1. We say that a $\mathfrak{g}$-central character is *dominant* if $\mathcal{F}(\mathfrak{g})$ contains modules with this central character. We denote by χ_λ the central character of $L(\lambda)$. For a dominant central character χ, we set

$$\Lambda^{\chi} := \{\lambda \in \Lambda^{+}_{m|n} \mid \chi_\lambda = \chi\}.$$

For the $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(2m+1|2n)$-cases, the dominant central characters are parametrized by the core diagrams, i.e. for $\lambda, \nu \in \Lambda^{+}_{m|n}$,

$$\chi_\lambda = \chi_\nu \implies \mathrm{core}(\lambda+\rho) = \mathrm{core}(\nu+\rho);$$

for $\mathfrak{osp}(2m|2n)$, the same holds for the atypical dominant central characters, and one has

$$\chi_\lambda \in \{\chi_\nu, \chi_{\nu^\sigma}\} \implies \mathrm{core}(\lambda) = \mathrm{core}(\nu).$$

For a dominant central character $\chi = \chi_\lambda$, we set $\mathrm{core}(\chi) := \mathrm{core}(\lambda)$. By above, a dominant central character is determined by its core for $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2m+1|2n)$, and for $\mathfrak{osp}(2m|2n)$, this holds for atypical dominant central characters.

For the $\mathfrak{osp}(2m|2n)$-case, we introduce $t \in \{0,2\}$ for each dominant central character χ and for each $\lambda \in \Lambda^+_{m|n}$ in the following way: $t = 0$ if core(χ) and core(λ), respectively, have an empty zero position and $t = 2$ otherwise (i.e. the zero position is occupied by $>$); for $\mathfrak{osp}(2m+1|2n)$, we set $t = 1$, and for $\mathfrak{gl}(m|n)$, we set $t := 0$. We will sometimes use the notation $t(\chi)$ or $t(\lambda)$; one has $t(\lambda) := t(\chi_\lambda)$.

3.4.2. We say that a diagram f is *core-free* if core$(f) = \emptyset$ or $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ and core$(f) = >$ ($>$ occupies the zero position).

3.4.3. By [24, Theorem 2], the blocks in $\mathcal{F}(\mathfrak{g})$ are parametrized by the dominant central characters; for $\mathfrak{gl}(m|n)$, the block of atypicality s is equivalent to the block χ_0 in $\mathfrak{gl}(s|s)$; for the $\mathfrak{osp}(M|2n)$-case, the block of atypicality s is equivalent to the block χ_0 in $\mathfrak{osp}(2s+t|2s)$. The equivalences are described in [24, Section 6]. For $\lambda \in \Lambda^+_{m|n}$, let howl$(\lambda)$ be the corresponding weight in χ_0. Diagrammatically, the passage from λ to howl(λ) essentially amounts to removing the core symbols $<, >$ from diag(λ) except for the symbol $>$ at the zero position in the $\mathfrak{osp}(2m|2n)$-case (see [24, Section 6] and [21, Section 3.7] for details). (In particular, howl(λ) has a core-free diagram.) If tail(λ) is defined, then tail$(\lambda) =$ tail(howl(λ)).

For example, if diag$(\lambda) = {>}{\times}{<}{\times}{\circ}{<}{\circ}{\times}$, then

$$\operatorname{diag}(\operatorname{howl}(\lambda)) = \begin{cases} \times\times\circ\circ\times & \text{for } \mathfrak{g} = \mathfrak{gl}(4|5), \\ {>}\times\times\circ\circ\times & \text{for } \mathfrak{g} = \mathfrak{osp}(8|10), \\ (+)\times\times\circ\circ\times & \text{for } \mathfrak{g} = \mathfrak{osp}(9|10). \end{cases}$$

3.5. The functions $\|\lambda\|$ and $\|\lambda\|_{\mathrm{gr}}$. Let $\lambda \in \Lambda^+_{m|n}$ be such that $\lambda + \rho$ is atypical. Set $f :=$ diag(howl(λ)).

Definition. Let $a_1 \le \cdots \le a_j$ be the coordinates of the symbols $\times$ in f ($j = \operatorname{at}(\lambda+\rho)$). Then

$$\|\lambda\| := \begin{cases} \displaystyle\sum_{i=1}^{j} a_i & \text{for } t(\lambda) \neq 2, \\ \displaystyle\operatorname{tail}(\lambda) - j + \sum_{i=1}^{j} a_i & \text{for } t(\lambda) = 2, \end{cases}$$

$$\|\lambda\|_{\mathrm{gr}} := \begin{cases} \displaystyle\sum_{i=1}^{j}(a_i - a_1) - \frac{j(j-1)}{2} & \text{for } \mathfrak{gl}(m|n), \\ \displaystyle\sum_{i=1}^{j} a_i & \text{for } \mathfrak{osp}(2m|2n), \\ \displaystyle j - \operatorname{tail}(\lambda) + \sum_{i=1}^{j} a_i & \text{for } \mathfrak{osp}(2m+1|2n). \end{cases}$$

Notice that $\|\lambda\|_{\mathrm{gr}} \in \mathbb{N}$ and that $\|\lambda\| \in \mathbb{N}$ for $\mathfrak{osp}(M|N)$ and $\|\lambda\| \in \mathbb{Z}$ for $\mathfrak{gl}(m|n)$. Moreover, $\|\lambda\|_{\mathrm{gr}} = 0$ if and only if howl$(\lambda) = 0$ in the $\mathfrak{osp}$-case and howl$(\lambda) \in \mathbb{Z}\sum_{i=1}^{j}(\varepsilon_i - \delta_i)$ for the $\mathfrak{gl}$-case (i.e. $\dim L(\operatorname{howl}(\lambda)) = 1$). The condition $\|\lambda\| = 0$ is equivalent to howl$(\lambda) = 0$ and howl$(\lambda) = 0, \varepsilon_1$ for $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ and for $\mathfrak{g} = \mathfrak{osp}(2m+1|2n)$, respectively. Note that, in the $\mathfrak{gl}(m|n)$-case, $\|\lambda\|_{\mathrm{gr}}$ is invariant under the shift of the diagram.

If $\lambda+\rho$ is typical, we set $\|\lambda\|_{\mathrm{gr}}=0$ (we do not define $\|\lambda\|$ in this case).

3.5.1. Remark. For the $\mathfrak{osp}$-case with $t=0,1$, one has $(-1)^{p(\mathrm{howl}(\lambda))}=(-1)^{\|\lambda\|}$. If λ,ν are stable $\mathfrak{gl}$-weights with $\chi_\lambda=\chi_\nu$, then $(-1)^{p(\lambda)-p(\nu)}=(-1)^{\|\lambda\|-\|\nu\|}$.

3.5.2. Remark. The odd-looking formulas for $\|\lambda\|$ with $t(\lambda)=2$ and for $\|\lambda\|_{\mathrm{gr}}$ with $t(\lambda)=1$ can be interpreted as follows. Consider f', which is obtained from f by removing $>$ from the zero position and then shifting all entries at the non-zero positions of f by one position to the left; then $\|\lambda\|=\sum_{i=1}^{j}a_i'$, where a_i' are the coordinates of $\times$ in f'. The above operation induces a bijection τ between the core-free $\mathfrak{osp}(2m+2|2m)$-weights and the core-free $\mathfrak{osp}(2m+1|2m)$-weights: this bijection, introduced in [24], assigns to f the diagram f' with the sign chosen in such a way that $\mathrm{tail}(f)=\mathrm{tail}(\tau(f))$. For instance,

$$\tau(\overset{\times}{>}\circ\times)=-\times\times,\quad \tau(\overset{\times}{>})=-\times,\quad \tau(>\times)=+\times,\quad \tau(>\circ\times)=\circ\times.$$

One has

$$\|\lambda\|=\|\tau(\lambda)\|,\quad \|\lambda\|_{\mathrm{gr}}=\|\tau(\lambda)\|_{\mathrm{gr}}.$$

3.5.3. Definition. We call $\lambda\in\Lambda^+_{m|n}$ *a Kostant weight* if $\dim L(\mathrm{howl}(\lambda))=1$.

Note that $\dim L(\mathrm{howl}(\lambda))=1$ means that $\mathrm{howl}(\lambda)=0$ and $\mathrm{howl}(\lambda)\in\mathbb{Z}\,\mathrm{str}$ for the $\mathfrak{osp}$-case and for the $\mathfrak{gl}$-case, respectively.

Observe that $\|\lambda\|_{\mathrm{gr}}=0$ if and only if λ is a Kostant weight ($\|\lambda\|_{\mathrm{gr}}$ can be seen as the "distance" to the nearest Kostant weight). For instance, for $\mathfrak{gl}(3|3)$ and $\mathrm{diag}(\lambda)=\times\times\circ\times$, one has $\|\lambda\|_{\mathrm{gr}}=1$.

For the $\mathfrak{gl}$-case, this term was used in [5]; in [8], these weights are called *totally connected*. If in addition λ is stable, such weight is called a *ground state* in [26, 52]. The Kostant weights are precisely the weights where all $\times$ are adjacent to each other discounting possible core symbols.

3.5.4. Remark. For the $\mathfrak{gl}(m|n)$-case, the modules satisfying the KW-conditions (see Section 1.1) were classified in [8]; for the $\mathfrak{osp}(M|N)$-case, this was done in [7]. The results of these classification can be formulated as follows. Except for the case $\mathfrak{g}=\mathfrak{osp}(2m|2n)$ with $t=0$ and atypicality 1, $L(\lambda)$ satisfies the KW-conditions if and only if λ is a Kostant weight. For the case $\mathfrak{g}=\mathfrak{osp}(2m|2n)$ with $t=0$, all simple finite-dimensional modules of atypicality 1 satisfy the KW-conditions. The latter case has the following interpretation. Let $\mathcal{F}(\mathfrak{osp}(2m|2n))^\chi$ be a block of atypicality 1 with $t=0$. Since $\mathfrak{osp}(2|2)=\mathfrak{sl}(1|2)$, we have

$$\mathcal{F}(\mathfrak{osp}(2m|2n))^\chi\xrightarrow{\sim}\mathcal{F}(\mathfrak{osp}(2|2))^{\chi_0}=\mathcal{F}(\mathfrak{sl}(1|2))^{\chi_0}\xrightarrow{\sim}\mathcal{F}(\mathfrak{sl}(1|1))^{\chi_0},$$

so the image of each simple module $L\in\mathcal{F}(\mathfrak{osp}(2m|2n))^\chi$ is the trivial $\mathfrak{sl}(1|1)$-module.

From the above description, it follows that the KW-conditions are compatible with the equivalence of categories given by the translation functors $T_{a,a+1}$ described in Section 4.7. This is not true in general: the switch functor

$$\mathcal{F}^{\chi_0}(\mathfrak{osp}(2m+1|2n))\xrightarrow{\sim}\mathcal{F}^{\chi_0}(\mathfrak{osp}(2m+1|2n))$$

given by $N\mapsto(N\otimes V_{\mathrm{st}})^{\chi_0}$ maps the trivial module (satisfying the KW-conditions) to the standard module, which does not satisfy these conditions.

4. Parabolic induction, Euler characters and character formulas

We define parabolic induction functors $\Gamma^i_{\mathfrak{p},\mathfrak{q}}$ and the Poincaré polynomials $K^{\lambda,\mu}_{\mathfrak{p},\mathfrak{q}}(z)$ in Section 4.1 and Euler characters $\mathcal{E}_\lambda$ in Section 4.2. We give a diagrammatic description of the Poincaré polynomials in the $\mathfrak{gl}(m|n)$-case. This leads to a character formula for $\operatorname{ch} L(\lambda)$.

4.1. The functors $\Gamma^i_{\mathfrak{p},\mathfrak{q}}$. Let $\mathfrak{q} \subset \mathfrak{p} \subset \mathfrak{g}$ be a pair of parabolic subalgebras containing $\mathfrak{b}$, and let V be a finite-dimensional $\mathfrak{q}$-module. We denote by $\Gamma_{\mathfrak{p},\mathfrak{q}}(V)$ the maximal finite-dimensional quotient of the induced module $\mathcal{U}(\mathfrak{p}) \otimes_{\mathcal{U}(\mathfrak{q})} V$. View $\Gamma_{\mathfrak{p},\mathfrak{q}}$ as a functor from the category of finite-dimensional $\mathfrak{q}$-modules to the category of finite-dimensional $\mathfrak{p}$-modules and define the derived functors $\Gamma^i_{\mathfrak{p},\mathfrak{q}}$ as in [24] ($\Gamma^i_{\mathfrak{p},\mathfrak{q}} := \Gamma_i(P/Q, \bullet)$ in the notation of [24]). By [25], for $\mathfrak{gl}(m|n)$, these functors coincide with the functors $\Gamma^i_{\mathfrak{p},\mathfrak{q}}$ defined in [35].

For $\lambda, \mu \in \Lambda^+_{m|n}$, we consider the Poincaré polynomial in the variable z as

$$K^{\lambda,\mu}_{\mathfrak{p},\mathfrak{q}}(z) := \sum_{i=0}^{\infty} [\Gamma^i_{\mathfrak{p},\mathfrak{q}}(L_{\mathfrak{q}}(\lambda)) : L_{\mathfrak{p}}(\mu)] z^i,$$

where $L_{\mathfrak{q}}(\lambda)$ and $L_{\mathfrak{p}}(\mu)$ stand for the corresponding simple $\mathfrak{q}$ and $\mathfrak{p}$-module, respectively.

4.1.1. Fix a central character χ and a flag of parabolic subalgebras

$$\mathfrak{g} = \mathfrak{p}^{(d)} \supset \mathfrak{p}^{(d-1)} \supset \cdots \supset \mathfrak{p}^{(0)} = \mathfrak{b},$$

where d is the defect of $\mathfrak{g}$ and $\mathfrak{l}^{(i)} := [\mathfrak{p}^{(i)}, \mathfrak{p}^i]$ is given by $\mathfrak{l}^{(i)} = \mathfrak{gl}(i|i)$ for $\mathfrak{g} = \mathfrak{gl}(d|d)$, $\mathfrak{l}^{(i)} = \mathfrak{osp}(2i+t|2i)$ for $\mathfrak{g} = \mathfrak{osp}(2d+t|2d)$.

The polynomials $K^{\lambda,\nu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i+1)}}(z)$ for the "neighboring parabolics" were given in [39] in the $\mathfrak{gl}$-case and in [24] in the $\mathfrak{osp}$-case. In the $\mathfrak{gl}$-case, we will describe these polynomials in terms of so-called "arc diagrams" in Section 4.5 below. Using these polynomials, the values $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}_\lambda}(-1)$ can be computed iteratively using the formula (established in [24, Theorem 1])

$$K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{q}}(-1) = \sum_{\nu} K^{\lambda,\nu}_{\mathfrak{p},\mathfrak{q}}(-1) K^{\nu,\mu}_{\mathfrak{g},\mathfrak{p}}(-1), \tag{4.1}$$

where the summation is taken over $\nu \in \mathfrak{h}^*$ with $\dim L_{\mathfrak{p}}(\nu) < \infty$.

4.2. The terms $\mathcal{E}_\lambda$. Take $\lambda \in \Lambda^+_{m|n}$, which is assumed to be stable for the $\mathfrak{gl}$-case. Let $\mathfrak{g}_\lambda$ be the tail subalgebra of λ (see Definition 3.2.2). As in [24], we introduce

$$\mathcal{E}_\lambda := R^{-1} e^{-\rho} \operatorname{J}_W \left(\frac{e^{\lambda+\rho}}{\prod_{\alpha \in \Delta(\mathfrak{g}_\lambda)^+_1} (1 + e^{-\alpha})} \right);$$

see Section A.1 for notation. Clearly, $\mathcal{E}_\lambda \in \mathcal{R}$; see Section A.2 for notation. By [24, Proposition 1] (Euler characteristic formula), one has

$$\mathcal{E}_\lambda = \sum_{\mu \in \Lambda^+_{m|n}} K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}_\lambda}(-1) \operatorname{ch} L(\mu), \quad \text{where } \mathfrak{p}_\lambda := \mathfrak{b} + \mathfrak{g}_\lambda. \tag{4.2}$$

The sum in the right-hand side of the formula is finite (see, for example, [24, Lemma 3]).

4.2.1. Remark. The perspective of [24, 41] is a bit different. The $\mathcal{E}_\lambda$'s are defined as actual Euler characters. It is important not to confuse the Euler character $\mathcal{E}_\lambda$ of [24] with the Euler character $\mathcal{E}_\lambda$ of [25]. In the latter case, $\mathcal{E}_\lambda$ simply equals for $\mathfrak{gl}(m|n)$ the character of the Kac module $K(\lambda)$.

4.2.2. In the case when $\mathfrak{g}_\lambda = \mathfrak{g}$, formula (A.2.2) gives $\mathcal{E}_\lambda = e^\lambda = \operatorname{ch} L(\lambda)$.

4.3. Proposition. *Take $\lambda \in \Lambda^+_{m|n}$, which is stable in the $\mathfrak{gl}(m|n)$-case. Set $s := \operatorname{tail}(\lambda)$.*

(i) *In the $\mathfrak{osp}$-case, we have*

$$j_s Re^\rho \mathcal{E}_\lambda = \mathrm{KW}(\lambda + \rho, S_s),$$

where $j_s = \max(1, 2^{s-1}s!)$ for $t = 0$ and $j_s = 2^s s!$ for $t = 1, 2$.

(ii) *If $\mathfrak{g} = \mathfrak{gl}(m|n)$ and λ is stable, then*

$$j_s Re^\rho \mathcal{E}_\lambda = \mathrm{KW}(\lambda + \rho_L, S_s),$$

where $j_s = (-1)^{[\frac{s}{2}]}s!$ and ρ_L is the Weyl vector for the base $\varepsilon^{m-s}(\varepsilon\delta)^s\delta^{n-s}$.

Proof. In the $\mathfrak{osp}$-case, set $\Sigma_L := \Sigma$; in the $\mathfrak{gl}$-case, let Σ_L be the base $\varepsilon^{m-s}(\varepsilon\delta)^s\delta^{n-s}$. Denote by ρ_λ and ρ'_λ the Weyl vectors for $\Delta(\mathfrak{g}_\lambda)$ with respect to the bases $\Sigma \cap \Delta(\mathfrak{g}_\lambda)$ and $\Sigma_L \cap \Delta(\mathfrak{g}_\lambda)$, respectively. Let $W_\lambda \subset W$ be the Weyl group of $\mathfrak{g}_\lambda$. Note that S_s is the maximal iso-set in $\Delta(\mathfrak{g}_\lambda)$. Combining (A.2.2) and (A.4.5), we obtain

$$\mathrm{J}_{W_\lambda}\left(\frac{e^{\rho_\lambda}}{\prod_{\alpha\in\Delta(\mathfrak{g}_\lambda)^+_1}(1+e^{-\alpha})}\right) = j_s^{-1}\,\mathrm{J}_{W_\lambda}\left(\frac{e^{\rho'_\lambda}}{\prod_{\alpha\in S_s}(1+e^{-\alpha})}\right).$$

One has $\mathrm{J}_W = \mathrm{J}_{W/W_\lambda}\cdot \mathrm{J}_{W_\lambda}$, where W/W_λ stands for any set of representatives. Using W_λ-invariance of λ and $\rho - \rho_\lambda$, we obtain

$$\begin{aligned}
Re^\rho\mathcal{E}_\lambda &= \mathrm{J}_W\left(\frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta(\mathfrak{g}_\lambda)^+_1}(1+e^{-\alpha})}\right) = \mathrm{J}_{W/W_\lambda}\left(\mathrm{J}_{W_\lambda}\left(\frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta(\mathfrak{g}_\lambda)^+_1}(1+e^{-\alpha})}\right)\right)\\
&= j_s^{-1}\,\mathrm{J}_{W/W_\lambda}\left(\mathrm{J}_{W_\lambda}\left(\frac{e^{\lambda+\rho-\rho_\lambda+\rho'_\lambda}}{\prod_{\alpha\in S_s}(1+e^{-\alpha})}\right)\right) = j_s^{-1}\,\mathrm{J}_W\left(\frac{e^{\lambda+\rho-\rho_\lambda+\rho'_\lambda}}{\prod_{\alpha\in S_s}(1+e^{-\alpha})}\right)\\
&= \mathrm{KW}(\lambda+\rho-\rho_\lambda+\rho'_\lambda, S_s).
\end{aligned}$$

For the $\mathfrak{osp}$-case, one has $\Sigma = \Sigma_L$, so $\rho_\lambda = \rho'_\lambda$; this gives (i). For the $\mathfrak{gl}$-case, notice that Σ_L is obtained from Σ by the chain of odd reflections with respect to the roots in $\Delta(\mathfrak{g}_\lambda)$; this gives $\rho_L - \rho = \rho'_\lambda - \rho_\lambda$ and establishes (ii). □

4.4. The $\mathfrak{osp}$-case. Consider the case $\mathfrak{g} = \mathfrak{osp}(M|2n)$ ($M = 2m$ or $M = 2m+1$). Theorems 3, 4 and the remark after Theorem 3 of [24] imply that, for $\lambda \in \Lambda^+_{m|n}$, one has

$$\operatorname{ch} L(\lambda) = \sum_{\mu\in\Lambda^+_{m|n}} (-1)^{\|\lambda\|-\|\mu\|} d_<^{\lambda,\mu}\mathcal{E}_\mu,$$

where $d_<^{\lambda,\mu}$ is the number of "increasing paths" from $\operatorname{diag}(\mu)$ to $\operatorname{diag}(\lambda)$ in the graph $D_\mathfrak{g}$ described in [24, Section 11]; we will recall some properties of this graph below.

4.4.1. Properties of $D_{\mathfrak{g}}$. The connected components of $D_{\mathfrak{g}}$ correspond to the dominant central characters, so for each component $D_{\mathfrak{g}}^{\chi}$, we can define $t \in \{0,1,2\}$ via the corresponding central character. The map $\lambda \to \mathrm{howl}(\lambda)$ gives an isomorphism $D_{\mathfrak{g}}^{\chi} \xrightarrow{\sim} D^{\chi_0}_{\mathfrak{osp}(2k+t|2k)}$ for $k := \mathrm{at}(\chi)$, $t := t(\chi)$; the map τ induces an isomorphism $D^{\chi_0}_{\mathfrak{osp}(2k+1|2k)} \xrightarrow{\sim} D^{\chi_0}_{\mathfrak{osp}(2k+t|2k)}$.

Assume that $\mathrm{diag}(\mu)$ is a predecessor of $\mathrm{diag}(\lambda)$ in $D_{\mathfrak{g}}$. From [24, Section 11], we conclude that, for the cases $t = 0, 2$, $\mathrm{diag}(\lambda)$ is obtained from $\mathrm{diag}(\mu)$ by moving several symbols $\times$ to the right; moreover, if $\mathrm{diag}(\lambda)$ has a sign, then $\mathrm{diag}(\mu)$ has the same sign. Using the isomorphism induced by τ, we conclude that, for $t = 1$, $\mathrm{diag}(\lambda)$ is obtained from $\mathrm{diag}(\mu)$ by moving several symbols $\times$ to the right or by changing the sign $-$ to the sign $+$. This implies

$$\lambda > \mu, \quad \mathrm{howl}(\lambda) > \mathrm{howl}(\mu), \quad \|\lambda\| \geq \|\mu\|, \quad \|\lambda\|_{\mathrm{gr}} > \|\mu\|_{\mathrm{gr}}, \quad \mathrm{tail}(\mu) \geq \mathrm{tail}(\lambda)$$

and that if λ is stable, then μ is stable. Moreover,

$$\lambda - \mu \in \begin{cases} \frac{1}{2}\mathbb{N}\Pi_0 + \mathbb{N}(\delta_n - \varepsilon_m) + \mathbb{N}(\delta_n + \varepsilon_m) & \text{for } t = 0, \\ \frac{1}{2}\mathbb{N}\Pi_0 & \text{for } t = 1, 2. \end{cases} \tag{4.3}$$

By above, $D_{\mathfrak{g}}$ is $\mathbb{N}$-graded with respect to $\|\ \|_{\mathrm{gr}}$ (if $\mathrm{diag}(\mu)$ is a predecessor of $\mathrm{diag}(\lambda)$, then $\|\mu\|_{\mathrm{gr}} < \|\lambda\|_{\mathrm{gr}}$). In particular, each vertex in $D_{\mathfrak{g}}$ has finitely many predecessors.

The map τ described in Remark 3.5.2 gives an isomorphism of the graph $D_{\mathfrak{osp}(2m+1|2n)}$ and the subgraph of $D_{\mathfrak{osp}(2m+2|2n)}$ which correspond to the union of connected components with $t = 2$.

4.4.2. We conclude that, for $\mathfrak{osp}(M|N)$, we have

$$d_<^{\lambda,\lambda} = 1, \quad d_<^{\lambda,\mu} = d_<^{\mathrm{howl}\,\lambda, \mathrm{howl}\,\mu} \geq 0,$$
$$d_<^{\lambda,\mu} \neq 0 \implies \chi_\lambda = \chi_\mu, \quad \|\mu\| \leq \|\lambda\|, \quad \mathrm{tail}(\mu) \geq \mathrm{tail}(\lambda), \quad \|\lambda\|_{\mathrm{gr}} > \|\mu\|_{\mathrm{gr}}.$$

Moreover, the sum in the right-hand side of the character formula is finite and the terms $\{\mathcal{E}_\lambda\}_{\lambda \in \Lambda^+_{m|n}}$ form a basis in the character ring of $\mathcal{F}$.

Using Remark 3.5.1, we obtain

$$(-1)^{p(\lambda)} \operatorname{ch} L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} (-1)^{p(\mu)} d_<^{\lambda,\mu} \mathcal{E}_\mu \quad \text{if } \lambda \text{ is stable and } t(\lambda) \neq 2.$$

4.4.3. Remark. The $\mathfrak{q}(n)$-case can be treated with the same methods. The character of $L(\lambda)$ can be written as a finite sum in the Euler characters where the coefficients are again given by the number of increasing paths in a certain bimarked graph. As for the $\mathfrak{osp}$-case, the finiteness is automatic since each vertex λ in this graph has a finite number of predecessors. However, Y. Su and R. B. Zhang already obtained in [47] a similar character formula based on earlier work of J. Brundan [3] so that we have refrained from including this case.

4.5. The Poincaré polynomials in the $\mathfrak{gl}$-case. Let k be the degree of atypicality of $\lambda + \rho$. For $i = 0, \dots, k-1$, the polynomials $K^{\lambda,\mu}_{\mathfrak{p}^{(i+1)},\mathfrak{p}^{(i)}}(z)$ were computed in [39] (see also [35, Corollary 3.8]). We will recall the diagrammatic interpretation (which was provided by Serganova in one of her wonderful talks in Rehovot) in Definition 4.5.3.

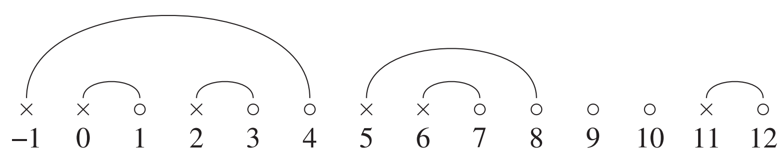
Let $\mathfrak{g} := \mathfrak{gl}(m|n)$. We identify a weight $\lambda \in \Lambda^+_{m|n}$ with $\mathrm{diag}(\lambda)$.

4.5.1. Arc diagrams. Take a weight $\nu \in \Lambda^+_{m|n}$. Denote by $\mathrm{diag}(\nu)$ the weight diagram of $\nu + \rho$. The arc diagram $\mathrm{Arc}(\nu)$ consists of the arcs $\mathrm{arc}(a; a')$, where $a < a'$ and $\mathrm{diag}(\nu)$ has $\times$ and $\circ$ at the position a and a', respectively. These arcs satisfy the following properties:

- each symbol $\times$ is connected by an arc to exactly one symbol $\circ$;
- each symbol $\circ$ is connected to at most one symbol $\times$;
- the arcs do not intersect;
- each symbol $\circ$ situated under an arc is connected to a symbol $\times$.

The arc diagram $\mathrm{Arc}(\nu)$ is unique and can be constructed in the following way: we pass from right to left through the weight diagram and connect each of the finitely many crosses $\times$ with the next empty symbol to the right by an arc (ignoring core symbols).

4.5.2. Example. We have the following arc diagram for $\times\times\circ\times\circ\circ\times\times\circ\circ\circ\circ\times\circ$:



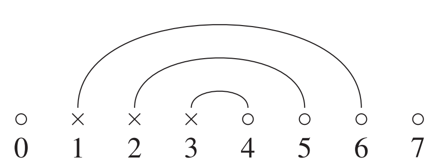
4.5.3. Definition. For a weight diagram f, we denote by f^u_a the weight diagram obtained from f by interchanging the symbols at the positions u and a.

Let $\lambda, \nu \in \Lambda^+_{m|n}$ be such that $\mathrm{diag}(\lambda) = \mathrm{diag}(\nu)^u_a$. We say that λ is obtained from ν by a *move* if $\mathrm{diag}(\nu)$ has $\times$ at the position a, $\circ$ at the position u and u lies under the arc originated at a, that is $\mathrm{Arc}(\nu)$ contains $\mathrm{arc}(a; a')$ with $a < u \leq a'$. For such a move, we define the weight as the number of arcs in $\mathrm{Arc}(\nu)$ which are "strictly above" u (for instance, if $u = a'$, then the move has zero weight).

Observe that if λ can be obtained from ν by a move as above, then such a move is unique. In this case, we set $b'(\nu; \lambda) := u$. Note that $\mathrm{diag}(\lambda)$ has $\times$ at the u-th position; we set $b(\nu; \lambda) := i + 1$, where i is the number of the symbols $\times$ with the coordinates less than u in $\mathrm{diag}(\lambda)$.

We will consider only the case of stable λ, so the symbols $\times$ in $\mathrm{diag}(\lambda)$ precede the core symbols. If λ is obtained from ν by a move, then ν is stable. A move is called a *non-tail move* if $b(\nu; \lambda) > \mathrm{tail}(\lambda)$.

4.5.4. Example. Take ν with $\mathrm{diag}(\nu) = 0\times\times\times$ and with the following arc diagram for $0\times\times\times\circ$:



There are 6 weights $\lambda_1, \ldots, \lambda_6$ which can be obtained from ν; in all cases, $b(\nu; \lambda) = 3$. For instance, λ_1 with $\mathrm{diag}(\lambda_1) = 1\times\times\times$ can be obtained from ν by a move of weight 2 with $b'(\nu; \lambda_1) = 4$. Similarly, λ_2 with $\mathrm{diag}(\lambda_2) = 1\times\times\circ\times$ can be obtained from ν by a move of weight 1 with $b'(\nu;\lambda_1)=5$. Another example is λ_3 with $\mathrm{diag}(\lambda_3)=0\times\circ\times\times$ can be obtained from ν by a move of weight 1 with $b'(\nu;\lambda_1)=4$.

From the weight λ_2, we can obtain μ with $\mathrm{diag}(\mu)=1\times\circ\times\times$ by a move of weight 0 with $b'(\lambda_2;\mu)=4$ and $b(\lambda_2;\mu)=2$.

Among the above examples, only the first move is a tail move.

4.5.5. Let λ be a stable weight and $k:=\mathrm{at}(\lambda+\rho)$. For $i=1,\dots,k$, the results of [39] (see also [35, Corollary 3.10 (b)]) give

$$K^{\lambda,\lambda}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}}(z)=1,$$
$$K^{\lambda,\mu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}}(z)=z^s \quad \text{for } \mu\neq\lambda,$$

if λ is obtained from μ by a move of weight s and $b(\mu;\lambda)=i$. In all other cases,

$$K^{\lambda,\mu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}}(z)=0.$$

Moreover, $K^{\lambda,\mu}_{\mathfrak{g},\mathfrak{p}^{(k)}}(z)=\delta_{\lambda,\mu}$ (see [24, Lemma 5], a “Typical Lemma”).

4.5.6. Lemma. *Take a stable weight $\lambda\in\Lambda^+_{m|n}$. Assume that λ is obtained from ν by a move of weight w.*

(i) *Then ν is stable,* $\mathrm{core}(\lambda)=\mathrm{core}(\nu)$ *and*

$$\lambda>\nu,\quad \|\lambda\|-\|\nu\|-(w+1)\in 2\mathbb{Z},\quad \mathrm{tail}(\nu)\leq b(\nu;\lambda).$$

(ii) *If the move is a non-tail move, then*

$$\|\lambda\|_{\mathrm{gr}}>\|\nu\|_{\mathrm{gr}},\quad \mathrm{tail}(\lambda)\leq\mathrm{tail}(\nu).$$

Proof. The first assertion immediately follows from the formula $\mathrm{diag}(\lambda)=\mathrm{diag}(\nu)^u_a$ for $a<u$. Consider the case of a non-tail move, i.e. $b(\nu;\lambda)>\mathrm{tail}(\lambda)$.

Since λ,ν are stable, their diagrams start from the subdiagrams $\times\dots\times$ containing, respectively, $\mathrm{tail}(\lambda)$ and $\mathrm{tail}(\nu)$ symbols $\times$. Let A_λ and A_ν be the coordinates of symbols $\times$ in these subdiagrams (one has $A(\lambda)=\{u_\lambda+i\}_{i=0}^{\mathrm{tail}(\lambda)-1}$, where u_λ is the minimal coordinate of the non-empty symbol in $\mathrm{diag}(\lambda)$). The inequality $b(\nu;\lambda)>\mathrm{tail}(\lambda)$ means that $u>\max A_\lambda$. This gives $A_\lambda\subset A_\nu$ and implies (ii). □

4.6. Graph $D_{\mathfrak{g}}$. Let $D_{\mathfrak{g}}$ be a graph with the set of vertices enumerated by $\Lambda^+_{m|n}$. We identify the weight λ with $\mathrm{diag}(\lambda)$. We join f,g by the edge $f\to g$ if $\mathrm{core}(f)=\mathrm{core}(g)$ and $\mathrm{howl}(g)$ is obtained from $\mathrm{howl}(f)$ by a non-tail move described in Section 4.5.

Recall that $f\to g$ implies that $\mathrm{howl}(g)$ is obtained from $\mathrm{howl}(f)$ by moving $\times$ from a position a to an empty position $u>a$. We mark each edge by the corresponding u.

4.6.1. Subgraphs $D^\chi_{\mathfrak{g}}$. Clearly, if λ and ν lie in the same connected component of $D_{\mathfrak{g}}$, then $\chi_\lambda=\chi_\nu$. Denote by $D^\chi_{\mathfrak{g}}$ the full subgraph with the vertices λ such that $\chi_\lambda=\chi$. If χ has atypicality s, then the map $f\mapsto\mathrm{howl}(f)$ gives an isomorphism of $D^\chi_{\mathfrak{g}}$ and $D^{\chi_0}_{\mathfrak{gl}(s|s)}$.

If $\mathrm{at}\,\lambda\leq 1$, then the corresponding vertex is isolated. It is not hard to see that $D^\chi_{\mathfrak{g}}$ is connected if χ has atypicality greater than 1.

4.6.2. Corollary. *The following statements hold.*

(i) *Let ν be a predecessor of λ. Then* $\operatorname{diag}(\lambda)$ *is obtained from* $\operatorname{diag}(\nu)$ *by moving some symbols $\times$ to the right. In particular,* $\operatorname{core}(\lambda)=\operatorname{core}(\nu)$, *$\lambda>\nu$ and*

$$\operatorname{howl}(\lambda)>\operatorname{howl}(\nu),\quad \|\lambda\|>\|\nu\|,\quad \|\lambda\|_{\mathrm{gr}}>\|\nu\|_{\mathrm{gr}},\quad \mathfrak{g}_{\operatorname{howl}(\lambda)}\subset\mathfrak{g}_{\operatorname{howl}(\nu)}.$$

(ii) *Any vertex in $D_{\mathfrak{g}}$ has a finite number of predecessors.*

Proof. Let ν be a predecessor of λ. Then $\operatorname{diag}(\lambda)$ is obtained from $\operatorname{diag}(\nu)$ by moving several symbols $\times$ to the right; this gives $\lambda>\nu$ and $\|\lambda\|>\|\nu\|$; the rest of the formulas in (i) follow from Lemma 4.5.6. By above, for (ii), it is enough to consider the case $\mathfrak{g}=\mathfrak{gl}(s|s)$ and $\operatorname{core}(\lambda)=\emptyset$. By Lemma 4.5.6, the predecessors of λ in $D_{\mathfrak{g}}$ lie in the following set:

$$\{\nu\in\Lambda^+_{s|s}\mid \operatorname{core}(\nu)=\emptyset,\ \|\nu\|_{\mathrm{gr}}<\|\lambda\|_{\mathrm{gr}},\ A_\lambda\subset A_\nu\},$$

where A_λ, A_ν are as in the proof of Lemma 4.5.6. In particular, the coordinates of all non-empty symbols in $\operatorname{diag}(\nu)$ lie between $u_\lambda-s$ and $u_\lambda-s+\|\lambda\|_{\mathrm{gr}}$, where u_λ is the minimal coordinate of the non-empty symbol in $\operatorname{diag}(\lambda)$. This gives (ii). □

4.6.3. We call a path in $D_{\mathfrak{g}}$ *increasing* and *decreasing* if the marks strictly increase and decrease, respectively, along the path.

4.6.4. Example. If λ is a Kostant weight, then $\mathcal{E}_\lambda=e^\lambda$. For $\mathfrak{gl}(n|n)$, the adjoint representation **Ad** has a three step Loewy filtration

$$\mathbf{Ad}=\begin{pmatrix}\mathbb{C}\\ \Pi(L(\varepsilon_1-\delta_n))\\ \mathbb{C}\end{pmatrix}$$

The middle term with highest weight $\lambda=\varepsilon_1-\delta_n$ corresponds to the diagram

$$-n\,\underbrace{\times\times\ldots\times}_{n-1\text{ times}}\circ\times;$$

this diagram is connected to the Kostant weights

$$-n\,\underbrace{\times\times\ldots\times}_{n\text{ times}},\qquad -n-1\,\underbrace{\times\times\ldots\times}_{n\text{ times}},$$

the corresponding weights 0 and $\mu=\sum_{i=1}^n(\delta_i-\varepsilon_i)$. This gives

$$\operatorname{ch}L(\varepsilon_1-\delta_n)=\mathcal{E}_{\varepsilon_1-\delta_n}-1-e^\mu,\quad \operatorname{sch}L(\varepsilon_1-\delta_n)=\mathcal{E}^-_{\varepsilon_1-\delta_n}+1+e^\mu.$$

Notice that $\operatorname{sdim}(\mathbf{Ad})=0$; hence $\operatorname{sdim}L(\varepsilon_1-\delta_n)=2$.

4.6.5. Examples. For $\mathfrak{gl}(1|1)$, the graph $D_{\mathfrak{g}}$ does not have edges. For $\mathfrak{gl}(2|2)$, the vertices λ with $\operatorname{core}(\lambda)\neq\emptyset$ are isolated; the vertices with $\operatorname{core}(\lambda)=\emptyset$ form a connected component $D_{\mathfrak{g}}^{\chi_0}$ of the following form:

$$\begin{array}{l}\searrow 3\\ 0\times\times\xrightarrow{3}0\times\circ\times\xrightarrow{4}0\times\circ\circ\times\xrightarrow{5}0\times\circ\circ\circ\times\ldots\\ \searrow 4\\ 1\times\times\xrightarrow{4}1\times\circ\times\xrightarrow{5}1\times\circ\circ\times\xrightarrow{6}1\times\circ\circ\circ\times\ldots\end{array}$$

The left column corresponds to the Kostant weights ($\|\lambda\|_{\text{gr}}=0$); the next column to the λ's with $\|\lambda\|_{\text{gr}}=1$ and so on.

4.6.6. Proposition. *The following statements hold.*

(i) *Each vertex is connected to a Kostant weight by an increasing path.*

(ii) *The Kostant weights are the sources of the graph $D_{\mathfrak{g}}$.*

Proof. For (i), take f_0 with $\|f_0\|_{\text{gr}}\neq 0$, and let u be the coordinate of the rightmost symbol $\times$ (which is not in the tail since f is not a Kostant weight) and let a be maximal such that $a<u$ and $f(a)=\circ$. Set $f_1=(f)^u_a$. Then $f=(f_1)^u_a$, and f is obtained from f_1 by a non-tail move with $b'(f_1,f)=u$. If $\|f_1\|_{\text{gr}}\neq 0$, we construct f_2 by the same rule. Continuing this process, we obtain an increasing path

$$f_r\to f_{r-1}\to\cdots\to f_1$$

with $\|f_{i+1}\|_{\text{gr}}<\|f_i\|_{\text{gr}}$; thus, for some r, one has $\|f_r\|_{\text{gr}}=0$. This gives (i). Now (ii) follows from (i) and the inequality $\|\lambda\|_{\text{gr}}>\|\nu\|_{\text{gr}}$ if ν is a predecessor of λ. □

4.7. Character formula for $\mathfrak{gl}(m|n)$. Take $\mathfrak{g}=\mathfrak{gl}(m|n)$ with a distinguished base Σ. For $\mu,\lambda\in\Lambda^+_{m|n}$, denote by $\mathcal{P}^>(\mu,\lambda)$ the set of decreasing paths from μ to λ and by $d_<^{\lambda,\mu}$ the number of increasing paths from μ to λ in the graph $D_{\mathfrak{g}}$. Set

$$d'_{\lambda,\mu}:=(-1)^{\|\lambda\|-\|\mu\|}\sum_{P\in\mathcal{P}^>(\mu,\lambda)}(-1)^{\text{length}\,P}.$$

By above, $d_<^{\lambda,\mu}=d_<^{\text{howl}(\lambda),\text{howl}(\mu)}$ and $d'_{\lambda,\mu}=d'_{\text{howl}(\lambda),\text{howl}(\mu)}$.

4.7.1. Let Λ^χ_{st} be the set of stable weights in Λ^χ. In the next section, we will prove the following formulas for $\lambda\in\Lambda^\chi_{\text{st}}$:

$$\begin{aligned}\mathcal{E}_\lambda&=\sum_{\mu\in\Lambda^\chi}d'_{\lambda,\mu}\,\text{ch}\,L(\mu),\\ \text{ch}\,L(\lambda)&=\sum_{\mu\in\Lambda^\chi_{\text{st}}}(-1)^{\|\lambda\|-\|\mu\|}d_<^{\lambda,\mu}\mathcal{E}_\mu.\end{aligned}\tag{4.4}$$

Notice that, by Corollary 4.6.2, the right-hand sides of the above formulas have finite number of non-zero summands.

4.7.2. For a non-stable weight λ, we introduce $\mathcal{E}_\lambda$ by the first formula in (4.4), i.e.

$$\mathcal{E}_\lambda:=\sum_{\mu\in\Lambda^\chi}d'_{\lambda,\mu}\,\text{ch}\,L(\mu).\tag{4.5}$$

If λ is stable and μ is not stable, then $d_<^{\lambda,\mu}=0$ (by Corollary 4.6.2). Therefore, the second formula in (4.4) can be rewritten as

$$\text{ch}\,L(\lambda)=\sum_{\mu\in\Lambda^\chi}(-1)^{\|\lambda\|-\|\mu\|}d_<^{\lambda,\mu}\mathcal{E}_\mu=\sum_{\mu\in\Lambda^+_{m|n}}(-1)^{\|\lambda\|-\|\mu\|}d_<^{\lambda,\mu}\mathcal{E}_\mu\tag{4.6}$$

if λ is stable. By (4.4), the matrices

$$(d'_{\lambda,\nu}) = (d'_{\mathrm{howl}(\lambda),\mathrm{howl}(\nu)}) \quad \text{and} \quad ((-1)^{\|\lambda\|-\|\mu\|} d_{<}^{\lambda,\mu})$$

are mutually inverse. Using (4.5), we deduce (4.6) for each $\lambda \in \Lambda^+_{m|n}$.

4.7.3. We retain notation of Section A.3.1. Fix a central character χ and denote by $\mathcal{F}\mathrm{in}^\chi$ the full subcategory of $\mathcal{F}\mathrm{in}$ of the modules with the central character χ. We will consider translation functors $T^V_{\chi,\chi'}$ for special cases when these functors are equivalence of categories and V is either the standard representation or its dual. These functors can be described as follows.

Recall that, for a weight diagram f, we denote by $(f)_a^{a+1}$ the diagram f' obtained from f by interchanging the symbols in the positions a and $a+1$. We denote by $T_{a,a+1}$ the corresponding operations on $\Lambda^{\geq}_{m|n}$ and on the central characters: $T_{a,a+1}(\nu) = \nu'$ such that $\mathrm{diag}(\nu') = T_{a,a+1}(\mathrm{diag}(\nu))$ and $T_{a,a+1}(\chi) = \chi'$ such that $\mathrm{core}(\chi') = T_{a,a+1}(\mathrm{core}(\chi))$.

For $V = V_{\mathrm{st}}, V^*_{\mathrm{st}}$, the translation functor

$$T^V_{\chi,\chi'} \colon \mathcal{F}\mathrm{in}^\chi \xrightarrow{\sim} \mathcal{F}\mathrm{in}^{\chi'}$$

is an equivalence of categories if $\chi' = T_{a,a+1}(\chi)$ for some a and exactly one of the positions $a, a+1$ in $\mathrm{core}(\chi)$ is empty (so for $\lambda \in \Lambda^\chi$, exactly one of the positions $a, a+1$ in $\mathrm{diag}(\lambda)$ is occupied by a core symbol and $T_{a,a+1}$ interchanges this core symbol with $\circ$ or $\times$, respectively).

One has

$$T^V_{\chi,\chi'}(L(\lambda)) = L(T_{a,a+1}(\lambda)).$$

Note that $\mathrm{howl}(\lambda) = \mathrm{howl}(T_{a,a+1}(\lambda))$. By a slight abuse of notation, we denote such functor by $T_{a,a+1}$.

4.7.4. Lemma. *For $\lambda' := T_{a,a+1}(\lambda)$, one has*

$$Re^\rho \mathcal{E}_\lambda = \Theta_{\chi,\chi'}(Re^\rho \mathcal{E}_{\lambda'}),$$

where $\Theta_{\chi,\chi'} \colon \mathcal{R}_\Sigma \to \mathcal{R}_\Sigma$ is the ring homomorphism corresponding to $T_{a,a+1}$ (Section A.3.1*).*

Proof. For each $\mu \in \Lambda^\chi$, set $\mu' := T_{a,a+1}(\mu)$. By (4.5),

$$Re^\rho \mathcal{E}_\lambda = \sum_{\mu' \in \Lambda^{\chi'}} d'_{\lambda,\mu} Re^\rho \,\mathrm{ch}\, L(\mu).$$

Using Section A.3.1, we get

$$\Theta_{\chi,\chi'}(Re^\rho \mathcal{E}_\lambda) = \sum_{\mu \in \Lambda^\chi} d'_{\lambda,\mu} Re^\rho \,\mathrm{ch}\, L(\mu').$$

Since $\mathrm{howl}(\mu') = \mathrm{howl}(\mu)$, one has $d'_{\lambda,\mu} = d'_{\lambda',\mu'}$, so

$$Re^\rho \mathcal{E}_{\lambda'} = \sum_{\mu' \in \Lambda^{\chi'}} d'_{\lambda',\mu'} Re^\rho \,\mathrm{ch}\, L(\mu') = \sum_{\mu' \in \Lambda^{\chi'}} d'_{\lambda,\mu} Re^\rho \,\mathrm{ch}\, L(\mu')$$

as required. □

4.8. Another form of the character formula. In the $\mathfrak{osp}$-case, we retain the notation of Proposition 4.3 and set

$$\mathrm{KW}(\nu) := \mathrm{KW}(\nu+\rho, S_{\mathrm{tail}(\nu)}), \quad j(\nu) := j_{\mathrm{tail}(\nu)}.$$

For the $\mathfrak{gl}$-case, we will introduce $\mathrm{KW}(\nu)$ in Section 6.1.4 and set $j(\nu) := \mathrm{tail}(\nu)!$.

4.8.1. Corollary. *We have*

$$Re^{\rho}\,\mathrm{ch}\,L(\lambda) = \sum_{\mu\in\Lambda^+_{m|n}} (-1)^{\|\lambda\|-\|\mu\|}\frac{d_<^{\lambda,\mu}}{j(\mu)}\,\mathrm{KW}(\mu).$$

Proof. Combining Proposition 4.3 and Section 4.4, we obtain the assertion for the $\mathfrak{osp}$-case. For the $\mathfrak{gl}$-case, we combine (4.6), Lemma 4.7.4 and Corollary 6.5 (ii). □

4.8.2. Remark. Setting $\mathrm{KW}(L(\nu)) := j(\nu)^{-1}\,\mathrm{KW}(\nu)$, we obtain formula (1.3).

4.8.3. Remark. The graph $D_{\mathfrak{g}}$ is an oriented graph; this graph does not have multi-edges for the $\mathfrak{gl}$-case and for the $\mathfrak{osp}$-case with $t=1,2$; for $t=0$, the graph has double edges.

4.9. Highest weights of L with respect to different bases. Fix any base $\tilde{\Sigma}$ compatible with Π_0 (i.e. $\Delta^+(\tilde{\Sigma})\cap\Delta_0 = \Delta^+(\Pi_0)$) and denote the Weyl vector by $\tilde{\rho}$. For a simple finite-dimensional module L, denote by $\mathrm{hwt}\,L$ the "ρ-twisted highest weight of L" i.e.

$$\mathrm{hwt}_{\tilde{\Sigma}}\,L = \nu+\tilde{\rho},$$

where ν is the highest weight of L with respect to $\tilde{\Sigma}$. If $\beta\in\tilde{\Sigma}$ is isotropic and r_β is the corresponding odd reflection, then

$$\mathrm{hwt}_{r_\beta\tilde{\Sigma}}\,L = \begin{cases}\mathrm{hwt}_{\tilde{\Sigma}}\,L & \text{if } (\mathrm{hwt}_{\tilde{\Sigma}}\,L|\beta)\neq 0,\\ \mathrm{hwt}_{\tilde{\Sigma}}\,L+\beta & \text{otherwise.}\end{cases}$$

Using this procedure, one can compute $\mathrm{hwt}_{\tilde{\Sigma}}\,L(\lambda)$ recursively. The character formula in Corollary 4.8.1 allows to give the following formula for $\mathrm{hwt}_{\tilde{\Sigma}}\,L(\lambda)$ for $t(\lambda)=1,2$.

4.9.1. Corollary. *Consider the partial order $\tilde{>}$ on $\mathfrak{h}^*$ given by $\nu\,\tilde{>}\,\mu$ if $\nu-\mu\in\mathbb{N}\tilde{\Sigma}$. View* $\mathrm{KW}(\lambda)$ *as an element of $\mathcal{R}_{\tilde{\Sigma}}$.*

(i) *In the $\mathfrak{osp}$-case with $t(\lambda)=1,2$, the weight* $\mathrm{hwt}_{\tilde{\Sigma}}\,L(\lambda)$ *is a unique maximal element in* $\mathrm{supp}\,\mathrm{KW}(\lambda)$ *with respect to the partial order $\tilde{>}$.*

(ii) *In the $\mathfrak{osp}$-case with $t(\lambda)=0$, the same holds if $\delta_n\pm\varepsilon_m\in\tilde{\Sigma}$.*

(iii) *In the $\mathfrak{gl}$-case, $\lambda+\rho$ is a unique maximal element in* $\mathrm{supp}\,\mathrm{KW}(\lambda)$ *with respect to the partial order $>$.*

Proof. Assertions (i), (ii) follow by induction on $\|\lambda\|_{\mathrm{gr}}$ if we combine Corollary 4.8.1 with (4.3). Similarly, (iii) follows by induction on $\|\lambda\|_{\mathrm{gr}}$ from Corollary 4.8.1 and the fact that $d_<^{\lambda,\mu}\neq 0$ implies $\mu<\lambda$. □

5. Proof of formulas (4.4)

The proof of (4.4) follows the plan explained in [20, Section 3]. In Sections 5.1, 5.2 below, we recall the main constructions of [20].

5.1. Marked graphs. Consider a directed graph (V, E), where V and E are at most countable, where the set of edges E is equipped by two functions: $b\colon E \to \mathbb{Z}$ and a function κ from E to a commutative ring.

We say that $\iota\colon V \to \mathbb{N}$ and $\iota\colon V \to \mathbb{Z}$ define an $\mathbb{N}$-*grading* and a $\mathbb{Z}$-grading, respectively, on this graph if, for each edge $\nu \xrightarrow{e} \lambda$, one has $\iota(\nu) < \iota(\lambda)$.

5.1.1. For a path

$$P := \nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3 \longrightarrow \cdots \xrightarrow{e_s} \nu_{s+1},$$

we define

$$\text{length}(P) := s, \quad \kappa(P) := \prod_{i=1}^{s} \kappa(e_i).$$

We call the path P *decreasing* and *increasing* if

$$b(e_1) > b(e_2) > \cdots > b(e_s) \quad \text{and} \quad b(e_1) < \cdots < b(e_s),$$

respectively. We consider a path $P = \nu$ (with one vertex and zero edges) as a decreasing/increasing path of zero length with $\kappa(P) = 1$.

5.1.2. Definition. We call two functions $b, b'\colon E \to \mathbb{Z}$ *decreasingly equivalent* if, for each path $\nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3$, one has

$$b(e_1) > b(e_2) \iff b'(e_1) > b'(e_2).$$

5.1.3. Observe that two decreasingly equivalent graphs have the same set of decreasing paths.

5.1.4. We denote the set of decreasing and increasing paths from ν to λ by $\mathcal{P}^{>}(\nu, \lambda)$ and $\mathcal{P}^{<}(\nu, \lambda)$, respectively.

Let (V, E) be a $\mathbb{Z}$-graded graph with a finite number of edges between any two vertices. Notice that, in this case, the number of paths between any two vertices is finite.

We introduce the square matrices $A^{<}(\kappa) = (a^{<}_{\lambda,\nu})_{\lambda,\nu \in V}$ and $A^{>}(\kappa) = (a^{>}_{\lambda,\nu})_{\lambda,\nu \in V}$ by

$$a^{>}_{\lambda,\nu} := \sum_{P \in \mathcal{P}^{>}(\nu,\lambda)} \kappa(P), \qquad a^{<}_{\lambda,\nu} := \sum_{P \in \mathcal{P}^{<}(\nu,\lambda)} (-1)^{\text{length}(P)} \kappa(P).$$

Since the graph is $\mathbb{Z}$-graded, these matrices are lower-triangular with $a^{>}_{\lambda,\lambda} = a^{<}_{\lambda,\lambda} = 1$. The following lemma is proven in [20, Section 3.4] (the proof is similar to one in [24, Theorem 4]).

5.1.5. Lemma. *Let (V, E) be a $\mathbb{Z}$-graded graph with a finite number of edges between any two vertices. Assume that $b\colon E \to \mathbb{Z}$ satisfies the following property:*

(BB) for each path $\nu_1 \xrightarrow{e_1} \nu_2 \xrightarrow{e_2} \nu_3$, one has $b(e_1) \neq b(e_2)$.

Then $A^{>}(\kappa) \cdot A^{<}(\kappa) = A^{<}(\kappa) \cdot A^{>}(\kappa) = \text{Id}$.

5.2. Graphs $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ and $\Gamma^{\chi}_{\mathrm{st}}$. We take $\mathfrak{g} := \mathfrak{gl}(m|n)$ and fix a central character χ. We define $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ and its subgraph $\Gamma^{\chi}_{\mathrm{st}}$ similarly to [20].

5.2.1. Graph $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$. Let $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ be a graph with the set of vertices $V := \Lambda^{\chi}_{\mathrm{st}}$ and the following edges: if

$$K^{\lambda,\nu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}} \neq \delta_{\nu,\lambda}$$

(where $\delta_{\nu,\lambda}$ is the Kronecker symbol), we join ν, λ by the edge of the form $\nu \xrightarrow{e} \lambda$ with $b(e) = i$. The graph $\Gamma^{\chi}_{\mathrm{st}}$ is obtained from $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ by removing the edges of the form $\nu \xrightarrow{e} \lambda$ with $b(e) \leq \mathrm{tail}(\lambda)$. For the core-free case, $\Lambda^{\chi}_{\mathrm{st}} = \Lambda^{\chi}$, and we denote the resulting graphs by $\widehat{\Gamma}^{\chi}$ and Γ^{χ}, respectively.

We denote by $P^{>}(\nu,\lambda)$ the set of decreasing paths from ν to λ in the graph $\Gamma^{\chi}_{\mathrm{st}}$.

By Section 4.5.5, if $\nu \xrightarrow{e} \lambda$ is an edge in $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ and in $\Gamma^{\chi}_{\mathrm{st}}$, then λ is obtained from ν by a move and a non-tail move, respectively, of weight s, and for $i := b(\lambda;\nu) = b(e)$, one has

$$K^{\lambda,\nu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}} = z^{s}.$$

In particular, $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ does not have multi-edges and is $\mathbb{Z}$-graded with respect to $\|\lambda\|$.

5.2.2. Take $\lambda \in \Lambda^{\chi}$ with $\nu \neq \lambda$. By Lemma 4.5.6 (i) and Section 4.5.5,

$$K^{\lambda,\nu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}}(-1) = (-1)^{\|\lambda\|-\|\nu\|+1} \tag{5.1}$$

if $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ contains an edge $\nu \xrightarrow{e} \lambda$ with $b(e) = i$ and

$$K^{\lambda,\nu}_{\mathfrak{p}^{(i)},\mathfrak{p}^{(i-1)}} = 0$$

otherwise (in particular, if ν is not dominant). By Lemma 4.5.6 (ii), the graph $\Gamma^{\chi}_{\mathrm{st}}$ is $\mathbb{N}$-graded with respect to $\|\lambda\|_{\mathrm{gr}}$ and satisfies the following condition:

(Tail) for each edge $\nu \xrightarrow{e} \lambda$ in $\Gamma^{\chi}_{\mathrm{st}}$, one has $\mathrm{tail}(\nu) \leq b(e)$.

This condition implies the following important property: a decreasing path P in $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$ lies in $\Gamma^{\chi}_{\mathrm{st}}$ if and only if the last edge of P lies in this graph. Using this property, (4.1) and (5.1), we obtain, for $\lambda \in \Lambda^{\chi}_{\mathrm{st}}$ and $\mu \in \Lambda^{+}_{m|n}$,

$$K^{\lambda,\nu}_{\mathfrak{g},\mathfrak{p}_{\lambda}}(-1) = (-1)^{\|\lambda\|-\|\mu\|} \sum_{P \in P^{>}(\nu,\lambda)} (-1)^{\mathrm{length}(P)} \tag{5.2}$$

(see [20, Section 3.5] for details). For $K^{\lambda,\nu}_{\mathfrak{g},\mathfrak{b}}(-1)$, one has a similar formula in terms of the decreasing paths in $\widehat{\Gamma}^{\chi}_{\mathrm{st}}$.

5.2.3. Let E be the set of edges in $\Gamma^{\chi}_{\mathrm{st}}$. We introduce $b' \colon E \to \mathbb{Z}$ by

$$b'(\mu \xrightarrow{e} \lambda) := b'(\nu,\lambda).$$

One readily sees that b and b' are decreasingly equivalent. We denote by $\mathcal{P}^{>}_{\nu,\lambda}$ the number of paths from ν to λ in $\Gamma^{\chi}_{\mathrm{st}}$ which are increasing with respect to b'.

Moreover, b' satisfies property (BB). Using Lemma 5.1.5, we conclude that, for a stable weight λ, one has

$$\mathrm{ch}\, L(\lambda) = \sum_{\mu \in \Lambda^{+}_{m|n}} (-1)^{\|\lambda\|-\|\mu\|} d^{\lambda,\mu}_{<} \mathcal{E}_{\mu},$$

where $d^{\lambda,\mu}_{<}$ is the cardinality of $\mathcal{P}^{>}_{\nu,\lambda}$.

5.2.4. Notice that $\Gamma^{\chi}_{\mathrm{st}}$ coincides with the "stable part" (the full subgraph corresponding to the stable vertices) of the component $D^{\chi}_{\mathfrak{g}}$ and that b' corresponds to the marking in this graph. This completes the proof of (4.4).

5.3. Examples. Consider the core-free case: $\mathfrak{g} = \mathfrak{gl}(r|r)$ and $\chi_\lambda = \chi_0$.

5.3.1. Case $r = 1$. In this case, $\mathcal{E}_\lambda = \operatorname{ch} L(\lambda) = e^\lambda$.

5.3.2. Case $r = 2$. Set $\beta_1 := \varepsilon_1 - \delta_2$, $\beta_2 := \varepsilon_2 - \delta_1$.
The weights $\lambda \in \Lambda^+_{2|2}$ with $\chi_\lambda = \chi_0$ are of the form

$$s(\beta_1 + \beta_2) + i\beta_1 \quad \text{for } s \in \mathbb{Z},\ i \in \mathbb{Z}_{\geq 0};$$

we denote such weight by $(s;i)$. The diagram of $(s;i)$ has symbols $\times$ at the positions s and $s+i+1$. One has $\|(s;i)\|_{\mathrm{gr}} = i$ and $\operatorname{tail}(s;i) = 1 + \delta_{i0}$.

The graph $D_{\mathfrak{g}}$ is described in Examples 4.6.5. The decreasing paths are the paths of length at most 1; combining (4.2) and (5.2), we obtain

$$\mathcal{E}_{s;i} = \begin{cases} \operatorname{ch} L(s;0) & \text{if } i = 0, \\ \operatorname{ch} L(s;1) + \operatorname{ch} L(s;0) + \operatorname{ch} L(s-1;0) & \text{if } i = 1, \\ \operatorname{ch} L(s;i) + \operatorname{ch} L(s;i-1) & \text{if } i > 1. \end{cases}$$

For $j > 0$, a vertex $(s;j)$ can be reached by increasing paths from vertices $(s;i)$ for $0 \leq i \leq j$ and from a vertex $(s-1;0)$; in both cases, the path is unique; this gives

$$\operatorname{ch} L(s;j) = (-1)^j \mathcal{E}_{s-1;0} + (-1)^{j-i} \sum_{i=0}^{j} \mathcal{E}_{s;i},$$
$$\operatorname{sch} L(s;j) = \mathcal{E}^-_{s-1;0} + \sum_{i=0}^{j} \mathcal{E}^-_{s;i}.$$

5.4. Comparison with other character formulas. For the $\mathfrak{gl}(2|2)$-case, a weight (s,i) is a Kostant weight only if $i = 0$; thus the Kac–Wakimoto character formula does not hold for $L(s;i)$ with $i \neq 0$. By [12], the restriction of any $L(s,i)$ for $i > 0$ is a sum of four simple $\mathfrak{gl}_0$-modules, so the character of $L(s,i)$ is a sum over four Weyl character formula terms for $\mathfrak{gl}_0$. Any $\mathfrak{gl}(2|2)$-module is always *partially disconnected* (PDC) in the sense of [9]. For PDC weights, the authors establish the following character formula:

$$e^\rho R \cdot \operatorname{ch} L(\lambda) = \frac{(-1)^{|(\lambda^\rho)^\Uparrow - \lambda^\rho|_{S_\lambda}}}{t_\lambda} \mathrm{J}_W\left(\frac{e^{(\lambda^\rho)^\Uparrow}}{\prod_{\beta \in S_\lambda}(1 + e^{-\beta})}\right),$$

where we refer to [9] for the definitions. The number t_λ is two for $(s,0)$ and one for (s,i), $i > 0$. However, already for $\mathfrak{gl}(3|3)$, there are simple modules which are not PDC.

The Su–Zhang formula [46] expresses the character in terms of $\mathrm{KW}(\lambda, S)$, where S is chosen to be maximal (the cardinality of S is equal to the atypicality of L), whereas we take S with cardinality equal to $\operatorname{tail}(L)$.

6. Euler characters for $\mathfrak{gl}(m|n)$

In this section, we define $\mathrm{tail}(\lambda)$ and the $\lambda^\dagger$ which appeared in (iv) in Section 1.2 for the $\mathfrak{gl}(m|n)$-case. In addition, in Corollary 6.5, we deduce property (i) of Section 1.2. In this section, $\mathfrak{g} := \mathfrak{gl}(m|n)$.

6.1. The set $\Lambda^\dagger_{m|n}$. Recall that $\mathrm{wt}(\mathrm{diag}(\lambda)) = \lambda + \rho$; see Section 3.3.3.

6.1.1. We denote by $\Lambda^\dagger$ the set of diagrams with the following properties:

- at most one position contains more than one of the symbols $>$, $<$, $\times$ and, if such a position exists, it contains $\times^i$ for $i > 1$ with no symbols $\times$ which precede this position;
- the symbol $\circ$ appears between two leftmost positions containing $\times$.

For instance, $\Lambda^\dagger$ contains $<\circ>\times^2>\circ\times$ and all weight diagrams with atypicality at most one; $\Lambda^\dagger$ does not contain $\times>\times$, $\times\times$, $\times^2\times$ and $\times\times^2$. We denote by $\Lambda^\dagger_{m|n}$ the subset of the diagrams corresponding to $\mathfrak{gl}(m|n)$ (containing m symbols $>$ and n symbols $<$, where $\times$ is counted as $>$ and as $<$). We identify $\Lambda^\dagger_{m|n}$ with its image $\mathrm{wt}(\Lambda^\dagger)$ in $\Lambda^{\geq}_{m|n}$.

We will use a natural one-to-one correspondence $\lambda \to \lambda^\dagger$ between $\Lambda^+_{m|n}$ and $\Lambda^\dagger$. It is more convenient to describe the inverse map, which can be done as follows. Take a diagram $f^\dagger \in \Lambda^\dagger_{m|n}$ and the weight $\lambda^\dagger := \mathrm{wt}(f^\dagger)$. We construct a "usual weight diagram" f by the following procedure: if $f^\dagger$ has a position containing $\times^i$ for $i > 1$, we move $i-1$ symbols $\times$ from this position to the $i-1$ first non-occupied positions to the left, for instance,

$$f^\dagger := \circ>\circ<>\circ\times^3>\circ\times\circ \mapsto \circ>\times<>\times\times>\circ\times\circ =: f. \tag{6.1}$$

Taking $\lambda \in \Lambda^+_{m|n}$ such that $\mathrm{diag}(\lambda) = f$, we obtain a one-to-one correspondence $\lambda \mapsto \lambda^\dagger$ between $\Lambda^+_{m|n}$ and $\Lambda^\dagger_{m|n}$. For example, for $\mathfrak{gl}(2|4)$, we have

$$f := \mathrm{diag}(0) = -2\times\times<<, \quad f^\dagger = -2\circ\times^2<<, \quad 0^\dagger = -\delta_3 - 2\delta_4;$$

note that $0^\dagger$ is the Weyl vector for $(\varepsilon\delta)^2\delta^2 = \Sigma_{L(0)}$; see Corollary 6.1.3 below.

If λ is a stable weight with $\mathrm{tail}(\lambda) \leq 1$, then $\lambda^\dagger = \lambda + \rho$.

6.1.2. For a diagram $f^\dagger \in \Lambda^\dagger$, consider the leftmost position containing the symbol $\times$; by above, this position contains $\times^i$, and we set $i := \mathrm{tail}(f^\dagger)$; if $f^\dagger$ does not contain $\times$, we set $\mathrm{tail}(f^\dagger) = 0$.

If λ is a stable weight, then we have $\mathrm{tail}(\lambda) = \mathrm{tail}(f^\dagger)$; see the above example. For each $\lambda \in \Lambda^+_{m|n}$, we set $\mathrm{tail}(\lambda) := \mathrm{tail}(f^\dagger)$. For example, $\mathrm{tail}(\lambda) = 3$ if $\mathrm{diag}(\lambda) = f$ as in (6.1).

6.1.3. Corollary. *The following statements hold:*

(i) *if λ is a stable weight, then $\lambda^\dagger = \lambda + \rho_L$ (see Proposition* 4.3 *for notation);*

(ii) *one has* $\mathrm{tail}(\lambda) = \mathrm{tail}(\mathrm{howl}(\lambda))$*;*

(iii) *λ is a Kostant weight if and only if* $\mathrm{tail}(\lambda)$ *is equal to the atypicality of λ.*

6.1.4. Take $f^\dagger \in \Lambda^\dagger$ and set $s := \mathrm{tail}(f)$, $\lambda^\dagger := \mathrm{wt}(f^\dagger)$. Our goal is to describe an iso-set $S_{\lambda^\dagger}$ of cardinality s which is orthogonal to $\lambda^\dagger$. If $s = 0$, then $S_{\lambda^\dagger} = \emptyset$. If $s \neq 0$, let y_0

be the leftmost position in $f^{\dagger}$ which contains $\times$; then $f^{\dagger}(y_0) = \times^s$ and we have

$$(\lambda^{\dagger}|\varepsilon_i) = (\lambda^{\dagger}|\delta_j) - y_0 \quad \text{for } i = p+1, \dots, p+s \text{ and } j = q+1, \dots, q+s.$$

We set

$$S_{f^{\dagger}} := \{\varepsilon_{p+i} - \delta_{q+i}\}_{i=1}^{s}, \quad \Sigma_{f^{\dagger}} := \delta^q \varepsilon^p (\varepsilon\delta)^s \delta^{n-s-q} \varepsilon^{m-s-p}.$$

Notice that $S_{f^{\dagger}} \subset \Sigma_{f^{\dagger}}$.

For example, for $f = \times\times>>$, we have $f^{\dagger} = \circ\times^2>>$ with $S_{f^{\dagger}} = \{\varepsilon_3 - \delta_1, \varepsilon_4 - \delta_2\}$ lying in $\Sigma_{f^{\dagger}}\varepsilon^2(\varepsilon\delta)^2$. For f as in (6.1), we have

$$S_{f^{\dagger}} = \{\varepsilon_3 - \delta_2, \varepsilon_4 - \delta_3, \varepsilon_5 - \delta_4\} \quad \text{and} \quad \Sigma_{f^{\dagger}} = \delta\varepsilon^2(\varepsilon\delta)^3\delta\varepsilon^2;$$

see Section 6.2 below for the additional examples.

For $\lambda \in \Lambda^{+}_{m|n}$ with $f := \mathrm{diag}(\lambda)$, we set $S_{\lambda^{\dagger}} := S_{f^{\dagger}}$ and $\mathrm{KW}(\lambda) := \mathrm{KW}(\lambda^{\dagger}, S_{\lambda^{\dagger}})$.

6.1.5. Example (Stable weight λ). Let $\lambda \in \Lambda^{+}_{m|n}$ be a stable weight with $f := \mathrm{diag}(\lambda)$. In this case, the weight diagram of $\lambda^{\dagger}$ starts from $\times^s$, so $p_s = m$ and $q = 0$. Therefore,

$$S_{\lambda^{\dagger}} = \{\varepsilon_{m-s+i} - \delta_i\}_{i=1}^{s}, \quad \Sigma_{f^{\dagger}} = \Sigma_{L(\lambda)}$$

(where $\Sigma_{L(\lambda)}$ is introduced in Proposition 4.3). Observe that Σ_L is obtained from Σ by odd reflections with respect to the roots of $\mathfrak{g}_{\lambda}$; these reflections do not change the highest weight of L, so the highest weight of L with respect to Σ_L is λ. It is easy to see that $\lambda^{\dagger} = \lambda + \rho_L$. Using Proposition 4.3, we get

$$s!Re^{\rho}\mathcal{E}_{\lambda} = \mathrm{KW}(\lambda) = (-1)^{[\frac{s}{2}]}\mathrm{KW}(\lambda^{\dagger}, S_s).$$

6.1.6. Remark. Using Section 4.9, it is not hard to show that the weight $\lambda^{\dagger}$ is always the ρ-twisted highest weight of $L(\lambda)$ with respect to $\Sigma_{f^{\dagger}}$.

6.1.7. Remark. In the $\mathfrak{osp}$-case, the weight $\lambda^{\dagger} = \lambda + \rho$ has a "vertical tail" ($\mathrm{diag}(\lambda)$ has $\times^s$ in the zero position); in the $\mathfrak{gl}$-case, $\mathrm{diag}(\lambda)$ has a "horizontal tail" and the diagram of $\lambda^{\dagger}$ has a "vertical tail" of the same size.

6.2. Examples. In the examples below, we will use the notation $\mathcal{E}^{-}_{f}, L(f)$ for $\mathcal{E}^{-}_{\lambda}, L(\lambda)$ with $\mathrm{diag}(\lambda) = f$.

6.2.1. Consider the $\mathfrak{gl}(3|3)$-module $L(\circ\times\times\circ\circ\times)$. Formula (1.3) gives

$$\begin{aligned}
&\mathrm{ch}\, L(\circ\times\times\circ\circ\times) = \mathcal{E}_{\circ\times\times\circ\circ\times} - \mathcal{E}_{\circ\times\times\circ\times\circ} + \mathcal{E}_{\circ\times\times\times\circ\circ} - 2\mathcal{E}_{\times\times\times\circ\circ\circ},\\
&Re^{\rho}\mathcal{E}_{\circ\times\times\circ\circ\times} = \frac{1}{2}\mathrm{KW}(\circ\circ\times^2\circ\circ\times; \{\varepsilon_2 - \delta_1; \varepsilon_3 - \delta_2\}), \qquad \Sigma_{\circ\circ\times^2\circ\circ\times} = \varepsilon(\varepsilon\delta)^2\delta,\\
&Re^{\rho}\mathcal{E}_{\circ\times\times\circ\times\circ} = \frac{1}{2}\mathrm{KW}(\circ\circ\times^2\circ\times\circ; \{\varepsilon_2 - \delta_1; \varepsilon_3 - \delta_2\}), \qquad \Sigma_{\circ\circ\times^2\circ\times\circ} = \varepsilon(\varepsilon\delta)^2\delta,\\
&Re^{\rho}\mathcal{E}_{\circ\times\times\times\circ\circ} = \frac{1}{6}\mathrm{KW}(\circ\circ\circ\times^3\circ\circ; \{\varepsilon_1 - \delta_1; \varepsilon_2 - \delta_2; \varepsilon_3 - \delta_3\}), \quad \Sigma_{\circ\circ\circ\times^3\circ\circ} = (\varepsilon\delta)^3,\\
&Re^{\rho}\mathcal{E}_{\times\times\times\circ\circ\circ} = \frac{1}{6}\mathrm{KW}(\circ\circ\times^3\circ\circ\circ; \{\varepsilon_1 - \delta_1; \varepsilon_2 - \delta_2; \varepsilon_3 - \delta_3\}), \quad \Sigma_{\circ\circ\times^3\circ\circ\circ} = (\varepsilon\delta)^3.
\end{aligned}$$

6.2.2. Consider the $\mathfrak{gl}(5|4)$-module $L := L(>\circ\times<\times>\circ\circ\times)$.
One has $\mathrm{howl}(>\circ\times<\times>\circ\circ\times) = \circ\times\times\circ\circ\times$. Formula (1.3) gives

$$\operatorname{ch} L = \mathcal{E}_{>\circ\times<\times>\circ\circ\times} - \mathcal{E}_{>\circ\times<\times>\circ\times} + \mathcal{E}_{>\circ\times<\times>\times} - 2\mathcal{E}_{>\times\times<\times>},$$

$$Re^{\rho}\mathcal{E}_{>\circ\times<\times>\circ\circ\times} = \frac{1}{2}\,\mathrm{KW}(>\circ\circ<\times^2>\circ\circ\times; \{\varepsilon_{2+i} - \delta_{1+i}\}_{i=1}^{2}),$$

$$\Sigma_{>\circ\circ<\times^2>\circ\circ\times} = \delta\varepsilon^2(\varepsilon\delta)^2\delta\varepsilon,$$

$$Re^{\rho}\mathcal{E}_{>\circ\times<\times>\circ\times} = \frac{1}{2}\,\mathrm{KW}(>\circ\circ<\times^2>\circ\times; \{\varepsilon_{2+i} - \delta_{1+i}\}_{i=1}^{2}),$$

$$\Sigma_{>\circ\circ<\times^2>\circ\times} = \delta\varepsilon^2(\varepsilon\delta)^2\delta\varepsilon,$$

$$Re^{\rho}\mathcal{E}_{>\circ\times<\times>\times} = \frac{1}{6}\,\mathrm{KW}(>\circ\circ<\circ>\times^3, \{\varepsilon_i - \delta_{1+i}\}_{i=1}^{3}),$$

$$\Sigma_{>\circ\circ<\circ>\times^3} = \delta(\varepsilon\delta)^3\varepsilon^2,$$

$$Re^{\rho}\mathcal{E}_{>\times\times<\times>} = \frac{1}{6}\,\mathrm{KW}(>\circ<\times^3>, \{\varepsilon_{1+i} - \delta_{1+i}\}_{i=1}^{3}),$$

$$\Sigma_{>\circ<\times^3>} = \delta\varepsilon(\varepsilon\delta)^3\varepsilon.$$

6.3. Tail conjecture. Consider the set of ρ-twisted highest weights of $L := L(\lambda)$,

$$\mathrm{Hwt}(L) := \{\mathrm{hwt}_{\Sigma'} L \mid \text{where } \Sigma' \text{ is a base of } \Delta\}.$$

For each $\nu \in \Lambda_{m|n}$, let $\mathrm{diag}(\nu)$ be the weight diagram of ν constructed by the same rules as in Section 3.3 (even though ν is not always in $\Lambda^{\geq}_{m|n}$); consider the leftmost position containing the symbol $\times$, and let $\mathrm{tail}'(\nu)$ be the number of $\times$ in this position. In particular, for $\nu \in \Lambda^{\dagger}$, $\mathrm{tail}'(\nu)$ is the size of the "vertical tail" of the diagram.

6.3.1. Conjecture. $\mathrm{tail}(\lambda) = \max_{\nu\in\mathrm{Hwt}(L(\lambda))} \mathrm{tail}'(\nu)$.

6.3.2. Remark (The $\mathfrak{osp}$-case). For the $\mathfrak{osp}$-case, we define $\mathrm{tail}'(\nu)$ to be the number of $\times$ in the zero position.

6.4. Translation functors and KW(λ). Retain notation of Section 4.7.

Lemma. *Take $\lambda \in \Lambda_{m|n}$. Let $a \in \mathbb{Z}$ be such that exactly one of the positions $a, a+1$ in* $\mathrm{core}(\lambda)$ *is empty.*

(i) $(T_{a,a+1}(\lambda))^{\dagger} = T_{a,a+1}(\lambda^{\dagger})$.

(ii) *For $\chi' = T_{a,a+1}(\chi_\lambda)$, one has* $\Theta^{V}_{\chi',\chi_\lambda}(\mathrm{KW}(\lambda)) = \mathrm{KW}(T_{a,a+1}(\lambda))$.

Proof. For (i), note that $(T_{a,a+1}(f))^{\dagger} = T_{a,a+1}(f^{\dagger})$ except for the case $a = y_0(f)$ and $f(a+1) = \circ$; in the latter case, the positions $a, a+1$ in $\mathrm{core}(f)$ are empty. Combining (i) and the formula $(\lambda^{\dagger})^{\dagger} = (\lambda^{\dagger})$, we reduce (ii) to the case when $\lambda = \lambda^{\dagger}$ (i.e. $\lambda \in \Lambda^{\dagger}$). We set $S := S_\lambda$. We will consider the case when

$$\mathrm{core}(\lambda)(a) = >, \quad \mathrm{core}(\lambda)(a+1) = \circ$$

(other cases are similar). In this case, $V = V_{\mathrm{st}}$.

Using the formula $P_\chi(\mathrm{KW}(\lambda)) = \mathrm{KW}(\lambda)$ and W-invariance of $\mathrm{ch}\, V$, we get

$$\Theta^V_{\chi,\chi'}(\mathrm{KW}(\lambda)) = P_{\chi'}\Big(\mathrm{ch}\, V \cdot \mathrm{J}_W\Big(\frac{e^\lambda}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big) = P_{\chi'}\Big(\mathrm{J}_W\Big(\frac{e^\lambda \,\mathrm{ch}\, V}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big),$$

which allows to rewrite (ii) in the following form:

$$P_{\chi'}\Big(\mathrm{J}_W\Big(\frac{e^\lambda \,\mathrm{ch}\, V}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big) = \mathrm{KW}(T_{a,a+1}(\lambda)). \tag{6.2}$$

Recall that $S = \{\varepsilon_{p+i} - \delta_{q+i}\}_{i=1}^s$ for $s := \mathrm{tail}(\lambda)$ and some p, q. Set

$$A := \big\{\gamma \in \{\varepsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n \mid (\gamma, S) = 0\big\}$$

and $S_i := S \setminus \{\varepsilon_{p+i} - \delta_{q+i}\}$ for $i = 1, \dots, s$. Using

$$\mathrm{ch}\, V = \sum_{i=1}^m e^{\varepsilon_i} + \sum_{j=1}^n e^{\delta_j},$$

we get

$$\begin{aligned}\mathrm{J}_W\Big(\frac{e^\lambda \,\mathrm{ch}\, V}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) &= \sum_{\gamma\in A} \mathrm{J}_W\Big(\frac{e^{\lambda+\gamma}}{\prod_{\beta\in S}(1+e^{-\beta})}\Big) + \sum_{i=1}^s \mathrm{J}_W\Big(\frac{e^{\lambda+\varepsilon_{p+i}}}{\prod_{\beta\in S_i}(1+e^{-\beta})}\Big)\\ &= \sum_{\gamma\in A} \mathrm{KW}(\lambda+\gamma; S) + \sum_{i=1}^s \mathrm{KW}(\lambda+\varepsilon_{p+i}; S_i).\end{aligned}$$

Using (A.1), we obtain

$$P_{\chi'}\Big(\mathrm{J}_W\Big(\frac{e^\lambda \,\mathrm{ch}\, V}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big) = \sum_{\gamma\in B} \mathrm{KW}(\lambda+\gamma, S) + \sum_{\gamma\in B_0} \mathrm{KW}(\lambda+\gamma, S_i),$$

where $B := \{\gamma \in A \mid \mathrm{core}(\lambda+\gamma) = T_{a,a+1}(\mathrm{core}(\lambda))\}$ and

$$B_0 := \{\varepsilon_{p+i} \mid i = 1, \dots, s \text{ such that } \mathrm{core}(\lambda+\varepsilon_i) = T_{a,a+1}(\mathrm{core}(\lambda))\}.$$

Denote by f the weight diagram of λ. Recall $f(a) = >$ and $f(a+1) = \times^j$ for some i. Since $\lambda \in \Lambda^\dagger$, f has a "vertical tail" at the position $y := y_0(f)$ (so $j = 1$ if $a+1 \neq y$). Since $f(a) = >$, there exists a unique k such that $(\lambda, \varepsilon_k) = a$. Note that

$$\mathrm{core}(\lambda+\varepsilon_i) = T_{a,a+1}(\mathrm{core}(\lambda))$$

implies $i = k$ and $\mathrm{core}(\lambda+\delta_i) = \mathrm{core}(\lambda)$ implies $(\lambda, \delta_i) = -a-1$. Note that $\varepsilon_k \in A$, so $B_0 = \emptyset$. We get

$$\begin{aligned}&P_{\chi'}\Big(\mathrm{J}_W\Big(\frac{e^\lambda \,\mathrm{ch}\, V}{\prod_{\beta\in S}(1+e^{-\beta})}\Big)\Big)\\ &\quad= \begin{cases}\mathrm{KW}(\lambda+\varepsilon_k, S) & \text{if } a+1 = y,\\ \mathrm{KW}(\lambda+\varepsilon_k, S) + \sum_{i:(\lambda,\delta_i)=-a-1} \mathrm{KW}(\lambda+\delta_i, S) & \text{otherwise.}\end{cases}\end{aligned}$$

If $f(a+1)=\circ$, then we have $(\lambda,\delta_i)\neq -a-1$ for all i and $\lambda+\varepsilon_k=T_{a,a+1}(\lambda)$ with $S_{\lambda+\varepsilon_k}=S$; thus $\mathrm{KW}(\lambda+\varepsilon_k,S)=\mathrm{KW}(T_{a,a+1}(\lambda))$, and this gives (6.2).

Consider the case $f(a+1)=\times$ with $a+1\neq y$. By Lemma A.4.4 (ii), it follows that $\mathrm{KW}(\lambda+\varepsilon_k,S)=0$ (since $(\lambda+\varepsilon_k,\varepsilon_{k-1}-\varepsilon_k)=(S,\varepsilon_{k-1}-\varepsilon_k)=0$). As $a+1\neq y$, there is a unique i such that $(\lambda,\delta_i)=-a-1$. One has $\lambda+\delta_i=T_{a,a+1}(\lambda)$ and $S_{\lambda+\delta_i}=S$, so (6.2) holds.

In the remaining case, $a+1=y$. Then $f(a+1)=\times^s$ and $k=p+s+1$. Note that $(\lambda+\varepsilon_k,\varepsilon_i)=a+1$ if and only if $i=p+1,\dots,p+s+1$ and $(\lambda+\varepsilon_k,\delta_i)=-a-1$ if and only if $i=q+1,\dots,q+s$. Set

$$\mu:=(a+1)\bigg(\sum_{i=1}^{s+1}\varepsilon_{p+i}-\sum_{i=1}^{s}\delta_{q+i}\bigg).$$

Let

$$W_\mu\cong S_{s+1}\times S_s\subset W$$

be the group of permutations of $\varepsilon_{p+1},\dots,\varepsilon_{p+s+1}$ and of $\delta_{q+1},\dots,\delta_{q+s}$. Notice that $\lambda+\varepsilon_k$ is W_μ-invariant. Choosing any set of representatives in W/W_μ, we have

$$\mathrm{KW}(\lambda+\varepsilon_k,S)=\mathrm{J}_W\bigg(\frac{e^{\lambda+\varepsilon_k}}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg)=\mathrm{J}_{W/W_\mu}\bigg(e^{\lambda+\varepsilon_k}\,\mathrm{J}_{W_\mu}\bigg(\frac{1}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg)\bigg).$$

Comparing the denominator identities for $\mathfrak{gl}(s+1|s)$ with respect to the bases $(\varepsilon\delta)^s\varepsilon$ and $\varepsilon(\varepsilon\delta)^s$, we get

$$\mathrm{J}_{W_\mu}\bigg(\frac{1}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg)=\mathrm{J}_{W_\mu}\bigg(\frac{e^{-\sum_{\beta\in S'}\beta}}{\prod_{\beta\in S'}(1+e^{-\beta})}\bigg),$$

where $S':=\{\varepsilon_{p+1+i}-\delta_{q+i}\}_{i=1}^{s}$. This gives

$$\begin{aligned}\mathrm{J}_W\bigg(\frac{e^{\lambda+\varepsilon_k}}{\prod_{\beta\in S}(1+e^{-\beta})}\bigg)&=\mathrm{J}_{W/W_\mu}\bigg(e^{\lambda+\varepsilon_k}\,\mathrm{J}_{W_\mu}\bigg(\frac{e^{-\sum_{\beta\in S'}\beta}}{\prod_{\beta\in S'}(1+e^{-\beta})}\bigg)\bigg)\\&=\mathrm{J}_W\bigg(\frac{e^{\lambda'}}{\prod_{\beta\in S'}(1+e^{-\beta})}\bigg),\end{aligned}$$

where $\lambda':=\lambda+\varepsilon_k-\sum_{\beta\in S'}\beta$. One readily sees that $\lambda'=T_{a,a+1}(\lambda)$ and $S'=S_{\lambda'}$. This completes the proof. □

6.5. Corollary. *The following statements hold.*

(i) *Let $T\colon \mathcal{F}\mathrm{in}^{\chi}\xrightarrow{\sim}\mathcal{F}\mathrm{in}^{\chi'}$ be a composition of the translation functors $T^V_{\chi,\chi'}$ which are equivalence of categories, and let $\Theta_{\chi,\chi'}\colon\mathcal{R}_\Sigma\to\mathcal{R}_\Sigma$ be the corresponding composed map. If $T(L(\lambda-\rho))=L(\lambda'-\rho)$, then $\Theta_{\chi,\chi'}(\mathrm{KW}(\lambda))=\mathrm{KW}(\lambda')$.*

(ii) *For each $\lambda\in\Lambda^+_{m|n}$, one has $\mathrm{tail}(\lambda)!Re^{\rho}\mathcal{E}_\lambda=\mathrm{KW}(\lambda)$.*

6.5.1. Denote by $K(\lambda)$ the Kac module of the highest weight λ. Take λ' as in Corollary 6.5 (i). From [39, Theorem 5.1], it follows that $T(K(\lambda))=K(\lambda')$. This gives the following formula:

$$\Theta_{\chi,\chi'}(Re^{\rho}K(\lambda-\rho))=K(\lambda'-\rho)$$

which will be used later (this formula can be also proven as Lemma 6.4 (ii)).

7. Euler supercharacters and the Duflo–Serganova functor

Let $\mathrm{Sch}(\mathfrak{g})$ be the ring of supercharacters of $\mathfrak{g}$. Recall that DS_x induces for any x a homomorphism $\mathrm{Sch}(\mathfrak{g}) \to \mathrm{Sch}(\mathfrak{g}_x)$ which depends only on the rank of x. We denote this homomorphism by ds_j, where j is the rank of x. We always assume that $j > 0$ ($\mathrm{ds}_0 = \mathrm{Id}$).

In this section, $\mathfrak{g}$ stands for $\mathfrak{gl}(m|n)$ or $\mathfrak{osp}(M|N)$.

7.1. Euler supercharacters. Recall that π is the involution of $\mathbb{Z}[\Lambda_{m|n}]$ given by

$$\pi(e^{\lambda}) := (-1)^{p(\lambda)} e^{\lambda};$$

we extend this involution to the ring of fractions of $\mathbb{Z}[\Lambda_{m|n}]$. Recall that

$$\mathrm{sch}\, L(\lambda) = (-1)^{p(\lambda)} \pi(\mathrm{ch}\, L(\lambda)).$$

For each $\nu \in \Lambda^+_{m|n}$, set

$$\mathcal{E}^-_{\nu} := (-1)^{p(\nu)} \pi(\mathcal{E}_{\nu}).$$

Using the character formulas and Remark 3.5.1, we obtain the following formulas.

7.1.1. Corollary. *For $\lambda \in \Lambda^+_{m|n}$, one has*

$$\mathrm{sch}\, L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} (-1)^{p(\lambda-\mu)+\|\lambda\|-\|\mu\|} d^{\lambda,\mu}_{<} \mathcal{E}^-_{\mu},$$

where $d^{\lambda,\mu}_{<}$ is the number of increasing paths from μ to λ in $D_{\mathfrak{g}}$.

If λ is stable and $t \neq 2$, then $\mathrm{sch}\, L(\lambda) = \sum_{\mu \in \Lambda^+_{m|n}} d^{\lambda,\mu}_{<} \mathcal{E}^-_{\mu}$.

The main result of this section is the following theorem, which will be proven in Section 7.3 below.

7.2. Theorem. *Take $\lambda \in \Lambda^+_{m|n}$. If* $\mathrm{tail}(\lambda) < j$, *then* $\mathrm{ds}_j(\mathcal{E}^-_{\lambda}) = 0$. *If* $\mathrm{tail}(\lambda) \geq j$, *let $\lambda' \in \Lambda^+_{m-j|n-j}$ be such that* $\mathrm{diag}(\lambda')$ *is obtained from* $\mathrm{diag}(\lambda)$ *by the removal the first j leftmost symbols $\times$ (and keeping the sign if* $\mathrm{diag}(\lambda')$ *requires the sign). Then*

$$\mathrm{ds}_x(\mathcal{E}^-_{\lambda}) = \begin{cases} \mathcal{E}^-_{\lambda'} & \text{if } \mathrm{tail}(\lambda) > j, \\ \mathcal{E}^-_{\lambda'} & \text{if } \mathrm{tail}(\lambda) = j,\ \mathfrak{g} = \mathfrak{osp}(2m+1|2n), \\ \mathcal{E}^-_{\lambda'} & \text{if } \mathrm{tail}(\lambda) = j,\ \mathfrak{g} = \mathfrak{osp}(2j|2n), \\ \mathrm{sch}\, K(\lambda') & \text{if } \mathrm{tail}(\lambda) = j,\ \mathfrak{g} = \mathfrak{gl}(m|n), \end{cases}$$

where $K(\lambda')$ is the Kac $\mathfrak{g}'$-module with the even highest weight vector of weight λ'.

For $\mathfrak{osp}(2m|2n)$ with $m > j =$ $\mathrm{tail}(\lambda)$, *one has*

$$\mathrm{ds}_x(\mathcal{E}^-_{\lambda}) = \begin{cases} \mathcal{E}^-_{\lambda'} & \text{if } (\lambda')^{\sigma} = \lambda', \\ \mathcal{E}^-_{\lambda'} + \mathcal{E}^-_{(\lambda')^{\sigma}} & \text{if } (\lambda')^{\sigma} \neq \lambda'. \end{cases}$$

7.2.1. Remark. For a typical module N, one has $\mathrm{DS}_x(N) = 0$ for each $x \neq 0$. If $L(\lambda)$ is typical, then $Re^{\rho}\, \mathrm{ch}\, L(\lambda) = \mathrm{KW}(\lambda+\rho, \emptyset)$ and

$$\mathrm{sch}\, L(\lambda) = \mathcal{E}^-_{\lambda} = (\pi(R))^{-1} \sum_{w \in W} (-1)^{p(\lambda+\rho-w(\lambda+\rho))} \mathrm{sgn}\, w \cdot e^{w(\lambda+\rho)}.$$

By above, $\mathrm{ds}_x(\mathcal{E}^-_{\lambda}) = 0$ (since $\mathrm{DS}_x(L(\lambda)) = 0$). In particular, $\mathrm{ds}_x(\mathcal{E}^-_{\lambda}) = 0$ if $\lambda \notin \Lambda^+_{m|n}$.

7.2.2. Weight λ'. If $\operatorname{tail}(\lambda) \geq j$ for $\mathfrak{osp}$ or $\operatorname{tail}(\lambda) > j$ for $\mathfrak{gl}$, then

$$\operatorname{tail}(\lambda') = \operatorname{tail}(\lambda) - j$$

and $\lambda \mapsto \lambda'$ corresponds to the "tail-cutting": for example, for $\mathfrak{osp}(13|12)$ with

$$\operatorname{diag}(\lambda) = (+)\times^4{>}{<}\times,$$

we have $\operatorname{diag}(\lambda') = (+)\times^2{>}{<}\times$ for $j = 2$ and $\operatorname{diag}(\lambda') = \circ{>}{<}\times$ for $j = 4$; for the $\mathfrak{gl}(4|4)$-case with $j = 2$, we have

$$\operatorname{diag}(\lambda) = {>}\times\times{<}\times\times{>}\circ\ldots, \qquad f^\dagger = {>}\circ\circ{<}\circ\times^4{>}\circ\ldots,$$
$$\operatorname{diag}(\lambda') = {>}\circ\circ{<}\times\times{>}\circ\ldots, \qquad (f')^\dagger = {>}\circ\circ{<}\circ\times^2{>}\circ\ldots.$$

7.2.3. Remark. Let j be the rank of x. We take $x \in \sum_{\beta\in S_j} \mathfrak{g}_\beta$ and identify $\mathfrak{g}_x$ with a subalgebra of $\mathfrak{g}$ as in [13,22]. In the $\mathfrak{osp}$-case, $\lambda' = \lambda|_{\mathfrak{h}_x}$; for $\mathfrak{gl}$, this holds if λ is stable.

7.2.4. Remark. In the $\mathfrak{osp}$-case, the "tail-cutting" is "reversible" (we can reconstruct the tail if our dog is still alive): $\operatorname{diag}(\lambda)$ is obtained by adding j symbols $\times$ to the zero position in the diagram of $\operatorname{diag}(\lambda')$. Therefore, $\operatorname{ds}_j(\mathcal{E}^-_\lambda) = \operatorname{ds}_j(\mathcal{E}^-_\nu) \neq 0$ implies $\lambda = \nu$. (The same holds for the $\mathfrak{gl}$-case if $\operatorname{tail}(\lambda) > j$.)

This gives the following corollary.

7.2.5. Corollary. *In the $\mathfrak{osp}$-case, $\{\mathcal{E}^-_\lambda \mid \operatorname{tail}(\lambda) \leq j\}$ is a basis of the kernel of* ds_j.

7.2.6. Remark. Take $\lambda \in \Lambda^+_{m|n}$, which is assumed to be stable for the $\mathfrak{gl}$-case. Using the notation of Section 4.2, we introduce

$$\mathcal{E}_{\lambda,i} := R^{-1}e^{-\rho}\,\mathrm{J}_W\left(\frac{e^{\lambda+\rho}}{\prod_{\alpha\in\Delta(\mathfrak{p}^{(i)})^-_1}(1+e^\alpha)}\right), \quad \mathcal{E}^-_{\lambda,i} := (-1)^{p(\lambda)}\pi(\mathcal{E}_{\lambda,i}).$$

Note that $\mathcal{E}_\lambda = \mathcal{E}_{\lambda,\operatorname{tail}(\lambda)}$. Take $\lambda \in \Lambda^+_{m|n}$, which is assumed to be stable in the $\mathfrak{gl}$-case and retain notation of Theorem 7.2. If $\operatorname{tail}(\lambda) \geq j$ for $\mathfrak{osp}$ or $\operatorname{tail}(\lambda) > j$ for $\mathfrak{gl}$, then one has $\mathcal{E}^-_{\lambda',\operatorname{tail}(\lambda)-j} = \mathcal{E}^-_{\lambda'}$. In the $\mathfrak{gl}$-case with $\operatorname{tail}(\lambda) = j$, one has $\mathcal{E}^-_{\lambda',0} = \operatorname{sch} K(\lambda')$.

7.2.7. Let χ be a central character of atypicality $j = \operatorname{rank} x$. Let ν be the weight of $\mathfrak{g}_x$ with the diagram equal to the diagram of χ. This means that, for each $\lambda \in \Lambda^+_{m|n}$ with $\chi_\lambda = \chi$, one has $\nu = \lambda|_{\mathfrak{h}_x}$ (the diagram of ν is obtained from the diagram of λ by removing all symbols $\times$). Note that ν is a typical weight, so $\mathcal{E}^-_\nu = \operatorname{sch} L(\nu)$. We put $L^{\text{core}} = L(\nu)$ in the $\mathfrak{gl}$ and $\mathfrak{osp}(2m+1|2n)$-case and

$$L^{\text{core}} := \begin{cases} L(\nu) & \text{if } \nu \text{ is } \sigma\text{-invariant},\\ L(\nu)\oplus L(\nu)^\sigma & \text{else,}\end{cases}$$

in the $\mathfrak{osp}(2m|2n)$-case. The notion of L^{core} was first introduced in [24], but there, L^{core} always equals $L(\nu)$ and therefore differs from ours in the $\mathfrak{osp}(2m|2n)$ in case ν is not σ-invariant.

7.2.8. Extend sdim to a linear function on the Grothendieck ring $\operatorname{Ch}_{\mathfrak{k}}(\mathfrak{g})$. Clearly, sdim gives a linear function on $\operatorname{Sch}(\mathfrak{g})$.

Corollary. *For $\lambda \in \Lambda^+_{m|n}$,* $\operatorname{sdim} \mathcal{E}^-_\lambda = \operatorname{sdim}(L(\lambda))^{\mathrm{core}}$ *if λ is a Kostant weight and $L(\lambda)$ has the maximal atypicality;* $\operatorname{sdim} \mathcal{E}^-_\lambda = 0$ *for other weights.*

Proof. Since $\operatorname{sdim}(\mathrm{DS}_x(N)) = \operatorname{sdim} N$ (see [13,43]), the homomorphism ds_x preserves sdim. Take x of the maximal rank ($= \min(m,n)$). By Theorem 7.2, $\mathrm{ds}_x(\mathcal{E}^-_\lambda) = 0$ if

$$\operatorname{tail}(\operatorname{howl}(\lambda)) = \operatorname{tail}(\lambda) < \min(m,n).$$

Hence $\mathrm{ds}_x(\mathcal{E}^-_\lambda) \neq 0$ implies that $\operatorname{tail}(\operatorname{howl}(\lambda)) = \min(m,n)$, which means that λ is a Kostant weight and $L(\lambda)$ has the maximal atypicality. Now let λ be a Kostant weight, and $L(\lambda)$ has the maximal atypicality. The algebra $\mathfrak{g}_x$ is either a Lie algebra ($\mathfrak{gl}_{|m-n|}$, $\mathfrak{o}_{2(m-n)}$, $\mathfrak{o}_{2(m-n)}$, $\mathfrak{sp}_{2n-2m}$ or $\mathfrak{osp}(1|2(n-m))$ and L^{core} is a $\mathfrak{g}_x$-module with $\operatorname{sch} L^{\mathrm{core}} = \mathcal{E}^-_{\lambda'}$ except for the case when $\mathfrak{g}_x = \mathfrak{o}_{2(m-n)} \neq 0$ and $\operatorname{sch} L^{\mathrm{core}} = \mathcal{E}^-_{\lambda'} + \mathcal{E}^-_{(\lambda')^\sigma}$. □

7.2.9. Corollary. *Take $\lambda \in \Lambda^+_{m|n}$. If the rank of x is equal to the atypicality of χ_λ, then*

$$\mathrm{DS}_x L(\lambda) = \Pi^{\|\lambda\| - \|\lambda_0\| + p(\lambda-\lambda_0)} (L^{\mathrm{core}})^{\oplus m(\lambda)},$$

where $m(\lambda)$ is equal to the number of increasing paths from the Kostant weights to λ.

7.3. Proof of Theorem 7.2. Using (2.1), we reduce the assertions to the case $j := 1$. Set $s := \operatorname{tail}(\lambda)$.

7.3.1. First, we consider the $\mathfrak{osp}$-case and the $\mathfrak{gl}$-case with a stable weight λ. Take $\beta_0 \in S_1$ ($\beta_0 = \pm(\varepsilon_m - \delta_n)$ for the $\mathfrak{osp}$-case and $\beta_0 = \varepsilon_m - \delta_1$ for $\mathfrak{gl}(m|n)$). We take $x \in \mathfrak{g}_{\beta_0}$. Set $\mathfrak{g}' := \mathrm{DS}_x(\mathfrak{g})$. By [13] (and [40]), we can identify $\mathfrak{g}'$ with a subalgebra of $\mathfrak{g}$ such that $\mathfrak{h} \cap \mathfrak{g}'$ is a Cartan subalgebra of $\mathfrak{g}'$ and a base Σ' for $\Delta(\mathfrak{g}')$ satisfies

$$\Delta^+ = \Delta^+(\Sigma') \coprod \{\beta_0\} \coprod B \coprod \{\alpha + \beta_0 \mid \alpha \in B\} \tag{7.1}$$

for some $B \subset \Delta^+$. Let ρ' and R' be the Weyl vector and denominator, respectively, for $\mathfrak{g}'$ with respect to Σ'. As in Section A.6, we define

$$\operatorname{pr}(e^\nu) = c_\nu e^{\nu|_{\mathfrak{h}'}},$$

where $c_\nu := e^{-\pi i (\nu|\delta_q)}$ with $q = n$ for the $\mathfrak{osp}$-case and $q = 1$ for the $\mathfrak{gl}$-case $(\beta_0|\delta_q) \neq 0$. By (7.1), one has

$$\rho - \rho' \in \mathbb{Z}\beta_0, \quad \operatorname{pr}(R(1 + e^{-\beta_0})) = R'.$$

By Section A.6.1, we have

$$\mathrm{ds}_x(\mathcal{E}^-_\lambda) = (\pi \operatorname{pr} \pi)(\mathcal{E}^-_\lambda) = (-1)^{p(\lambda)} (\pi \operatorname{pr})(\mathcal{E}_\lambda),$$

which allows to rewrite the required formula as follows: $\operatorname{pr}(\mathcal{E}_\lambda) = 0$ if $s = 0$ and

$$(-1)^{p(\lambda)-p(\lambda')} \operatorname{pr}(\mathcal{E}_\lambda) \tag{7.2}$$

$$= \begin{cases} \mathcal{E}_{\lambda'} & \text{if } s > 1, \\ \mathcal{E}_{\lambda'} & \text{if } s = 1,\ \mathfrak{g} = \mathfrak{osp}(2m+1|2n), \\ \mathcal{E}_{\lambda'} & \text{if } s = 1,\ \mathfrak{g} = \mathfrak{osp}(2|2n), \\ \operatorname{ch} K(\lambda') & \text{if } s = 1,\ \mathfrak{g} = \mathfrak{gl}(m|n), \\ \mathcal{E}_{\lambda'} & \text{for } \mathfrak{osp}(2m|2n),\ m > s = 1,\ (\lambda')^\sigma = \lambda', \\ \mathcal{E}_{\lambda'} + \mathcal{E}^-_{(\lambda')^\sigma} & \text{for } \mathfrak{osp}(2m|2n),\ m > s = 1,\ (\lambda')^\sigma \neq \lambda'. \end{cases}$$

In the $\mathfrak{gl}$-case, take $\lambda^\dagger$ as in Section 6.1; in the $\mathfrak{osp}$-case, we have $\lambda^\dagger = \lambda + \rho$. Using Proposition 4.3, we get

$$j_s \operatorname{pr}(\mathcal{E}_\lambda) = \operatorname{pr}\big(R^{-1}e^{-\rho}\,\mathrm{KW}(\lambda^\dagger, S_s)\big) = c_{-\rho}\cdot(R'e^{\rho'})^{-1}\operatorname{pr}\big((1+e^{-\beta_0})\,\mathrm{KW}(\lambda^\dagger, S_s)\big).$$

For $s = 0$, formula (A.4) gives $\operatorname{pr}(\mathcal{E}_\lambda) = 0$ as required. From now on, we assume $s > 0$. The pair $(\lambda^\dagger, S_s)$ satisfies the assumptions of Proposition A.6.3; using this proposition and taking into account that, for the $\mathfrak{gl}$-case,

$$R'e^{\rho'}\operatorname{ch} K(\lambda') = \mathrm{KW}(\lambda' + \rho', \emptyset),$$

we see that (7.2) holds up to a non-zero scalar a_λ which can be computed directly. Instead of performing such computation, we can employ the following reasoning. One has

$$\operatorname{sch}\mathrm{DS}_x(L(\lambda)) = \mathrm{ds}_x(\mathcal{E}^-_\lambda) + \sum_{\nu<\lambda} d^{\lambda,\nu}_< \mathrm{ds}_x(\mathcal{E}^-_\nu).$$

By above, $\mathrm{ds}_x(\mathcal{E}^-_\nu)$ is proportional to $\mathcal{E}^-_{\nu'}$ (or to $\mathcal{E}^-_{\nu'} + \mathcal{E}^-_{(\nu')^\sigma}$ for $\mathfrak{osp}(2m|2n)$), where $\nu' := \nu|_{\mathfrak{h}'}$. By Corollary 4.9.1,

$$\operatorname{supp}(\mathcal{E}^-_{\nu'}) \subset \nu' - \mathbb{N}\Sigma',$$

where $\mathcal{E}^-_{\nu'}$ is viewed as element of $\mathcal{R}_{\Sigma'}$; see Section A.2. The inequality $\nu < \lambda$ means that $\lambda - \nu \in \mathbb{N}\Sigma$, which implies $\nu' \in \lambda' - \mathbb{N}\Sigma'$ by (7.1). Hence the coefficient of $e^{\lambda'}$ in $\mathrm{ds}_x(\mathcal{E}^-_\lambda)$ is equal to $\operatorname{sdim}\mathrm{DS}_x(L(\lambda))_{\lambda'}$. Using the same reasoning for the formula

$$\operatorname{sch} L(\lambda') = \sum_{\nu'} d^{\lambda',\nu'}_< \mathcal{E}^-_{\nu'},$$

we conclude the coefficient of $e^{\lambda'}$ in $\mathcal{E}^-_{\lambda'}$ is 1. Combining $(\lambda, \beta_0) = 0$ and $\beta_0 \in \Sigma$, one readily sees that $\mathrm{DS}_x(L(\lambda))_{\lambda'} = \mathbb{C}$, so $\operatorname{sdim}\mathrm{DS}_x(L(\lambda))_{\lambda'} = 1$. Hence the coefficients of $e^{\lambda'}$ in $\mathrm{ds}_x(\mathcal{E}^-_\lambda)$ and in $\mathcal{E}^-_{\lambda'}$ are equal, so $a_\lambda = 1$.

7.3.2. Consider the case when $j = 1$ and $\mathfrak{g} = \mathfrak{gl}(m|n)$. If λ is stable, the required formula is established in Section 7.3.1. Using the fact that DS_x commutes with translation functors, we deduce from the stable case the required formula for the non-stable case taking into account Corollary 6.5 and Section 6.5.1 for $s > 1$ and $s = 1$, respectively. □

7.4. Examples. In the examples below, we will use the notation $\mathcal{E}^-_f, L(f)$ for $\mathcal{E}^-_\lambda, L(\lambda)$ with $\operatorname{diag}(\lambda) = f$. We will demonstrate the compatibility of the formulas in Theorem 7.2 with the descriptions of $\mathrm{DS}_x(L)$ given in [21, 26].

7.4.1. One has

$$\begin{aligned}
&\operatorname{sch} L_{\mathfrak{gl}(3|3)}(0) = \mathcal{E}^-_{\times\times\times 1}, \quad \mathrm{ds}_1(\mathcal{E}^-_{\times\times\times 1}) = \mathcal{E}^-_{\times\times 1} = \operatorname{sch} L_{\mathfrak{gl}(2|2)}(0),\\
&\operatorname{sch} L_{\mathfrak{gl}(4|3)}(0) = \mathcal{E}^-_{\times\times\times >1}, \quad \mathrm{ds}_1\,\mathcal{E}^-_{\times\times\times >1} = \mathcal{E}^-_{\times\times >1} = \operatorname{sch} L_{\mathfrak{gl}(3|2)}(0).
\end{aligned}$$

7.4.2. Consider the $\mathfrak{osp}(7|6)$-module

$$L := L(+\times^2\circ\times).$$

Combining the Gruson–Serganova formula and Theorem 7.2, we obtain

$$\begin{aligned}\operatorname{sch} L &= \mathcal{E}^-_{+\times^2\circ\times} + \mathcal{E}^-_{-\times^2\circ\times} + \mathcal{E}^-_{+\times^2\times} + \mathcal{E}^-_{-\times^2\times} + 2\mathcal{E}^-_{+\times^3} + 3\mathcal{E}^-_{-\times^3},\\ \mathrm{ds}_1(\operatorname{sch} L) &= \mathcal{E}^-_{+\times\circ\times} + \mathcal{E}^-_{-\times\circ\times} + \mathcal{E}^-_{+\times\times} + \mathcal{E}^-_{-\times\times} + 2\mathcal{E}^-_{+\times^2} + 3\mathcal{E}^-_{-\times^2} = \operatorname{sch} L(+\times\circ\times),\\ \mathrm{ds}_2(\operatorname{sch} L) &= \mathcal{E}^-_{\circ\circ\times} + \mathcal{E}^-_{\circ\times} + 2\mathcal{E}^-_{+\times} + 3\mathcal{E}^-_{-\times}, \quad \mathrm{ds}_3(\operatorname{sch} L) = 3.\end{aligned}$$

The results of [21] give $\mathrm{DS}_1(L(+\times^2\circ\times)) = L(+\times\circ\times)$,

$$\mathrm{DS}_2(L(+\times^2\circ\times)) = \mathrm{DS}_1(L(+\times\circ\times)) = L(+\times) \oplus L(\circ\circ\times)$$

and $\mathrm{DS}_3(L(+\times^2\circ\times)) = \mathbb{C}^{\oplus 3}$. The Gruson–Serganova formula gives

$$\operatorname{sch} L(+\times) = \mathcal{E}^-_{+\times} + \mathcal{E}^-_{-\times}, \quad \operatorname{sch} L(\circ\circ\times) = \mathcal{E}^-_{\circ\circ\times} + \mathcal{E}^-_{\circ\times} + \mathcal{E}^-_{+\times} + 2\mathcal{E}^-_{-\times},$$

which establishes the compatibility for ds_2.

7.4.3. Consider the $\mathfrak{gl}(2|2)$-module

$$L(\varepsilon_1 - \delta_2) = L(\times 0 \times) = \Pi\, \mathrm{Ad}(\mathfrak{psl}(2|2)).$$

One has

$$\begin{aligned}\operatorname{sch} L(\times 0\times) &= \mathcal{E}^-_{\times 0\times} + \mathcal{E}^-_{\times\times 1} + \mathcal{E}^-_{\times\times 0},\\ \mathrm{ds}_1(\operatorname{sch} L(\times 0\times)) &= \operatorname{sch} K(0\times) + \mathcal{E}^-_{\times 1} + \mathcal{E}^-_{\times 0},\\ \mathrm{ds}_2(\operatorname{sch} L(\times 0\times)) &= 2\end{aligned}$$

and $\mathrm{DS}_1(\mathrm{Ad}(\mathfrak{psl}(2|2)) = \mathrm{Ad}(\mathfrak{psl}(1|1))$, $\mathrm{DS}_2(\mathrm{Ad}(\mathfrak{psl}(2|2)) = \Pi\mathbb{C}^{\oplus 2}$. This gives

$$\mathrm{DS}_1(L(\times 0\times) = L(\varepsilon_1 - \delta_1) \oplus L(\delta_1 - \varepsilon_1) = L(0\times) \oplus L(\times 0).$$

Notice that $\operatorname{sch} L(\times 0) = \mathcal{E}^-_{\times 0}$ and

$$\operatorname{sch} L(0\times) = \mathcal{E}^-_{0\times} = \operatorname{sch} K(0\times) + \mathcal{E}^-_{\times 0}$$

since $L(0\times) \cong K(0\times)/\Pi(L(\times 0))$.

7.4.4. Consider the $\mathfrak{gl}(3|2)$-module $L(\times\circ{>}\times)$ (note that $\mathrm{howl}(\times\circ{>}\times) = \times\circ\times$; see the previous example). In all formulas in this example, we assume that the symbol $>$ has the same coordinate. One has

$$\mathrm{DS}_1(L(\times\circ{>}\times)) = \Pi(L(\times\circ{>})) \oplus L({>}\times)$$

and $\mathrm{DS}_2(L(\times\circ{>}\times)) = \Pi\mathbb{C}^{\oplus 2}$. Theorem 7.2 gives

$$\begin{aligned}\operatorname{sch} L(\times\circ{>}\times) &= \mathcal{E}^-_{\times\circ>\times} - \mathcal{E}^-_{\times\times>} - \mathcal{E}^-_{\times\times\circ>},\\ \mathrm{ds}_1(\operatorname{sch} L(\times\circ{>}\times)) &= \operatorname{sch} K({>}\times) - \mathcal{E}^-_{\times>} - \mathcal{E}^-_{\times\circ>},\\ \mathrm{ds}_2(\operatorname{sch} L(\circ\times\circ\times)) &= -2.\end{aligned}$$

We have the following formulas for the $\mathfrak{gl}(2|1)$-modules:

$$\operatorname{sch} L(\times\circ{>}) = \mathcal{E}_{\times\circ>} \quad \text{and} \quad \operatorname{sch} K({>}\times) = \mathcal{E}^-_{\times>} + \mathcal{E}^-_{>\times}$$

since $L({>}\times) \cong K({>}\times)/L(\times{>})$.

7.4.5. Consider the $\mathfrak{gl}(3|3)$-module $L(\circ\times\times\circ\circ\times)$; see Section 6.2.1. Using (1.3) and Theorem 7.2, we obtain

$$\begin{aligned}
\operatorname{sch} L(\circ\times\times\circ\circ\times) &= \mathcal{E}^-_{\circ\times\times\circ\circ\times} + \mathcal{E}^-_{\circ\times\times\circ\times\circ} + \mathcal{E}^-_{\circ\times\times\times\circ\circ} + 2\mathcal{E}^-_{\times\times\times\circ\circ\circ},\\
\mathrm{ds}_1(\operatorname{sch} L(\circ\times\times\circ\circ\times)) &= \mathcal{E}^-_{\circ\circ\times\circ\circ\times} + \mathcal{E}^-_{\circ\circ\times\circ\times\circ} + \mathcal{E}^-_{\circ\circ\times\times\circ\circ} + 2\mathcal{E}^-_{\circ\times\times\circ\circ\circ},\\
\mathrm{ds}_2(\operatorname{sch} L(\circ\times\times\circ\circ\times)) &= \operatorname{sch} K(\circ\circ\circ\circ\times) + \operatorname{sch} K(\circ\circ\circ\times\circ) + \mathcal{E}^-_{\circ\circ\circ\times\circ\circ} + 2\mathcal{E}^-_{\circ\circ\times\circ\circ\circ},\\
\mathrm{ds}_3(\operatorname{sch} L(\circ\times\times\circ\circ\times)) &= 3\operatorname{sch} K(\circ\circ\circ\circ\circ\circ) = 3.
\end{aligned}$$

On the other hand, the results of [26] give

$$\begin{aligned}
\mathrm{DS}_1(L(\circ\times\times\circ\circ\times)) &= L(\circ\circ\times\circ\circ\times) \oplus L(\circ\times\times\circ\circ\circ),\\
\mathrm{DS}_2(L(\circ\times\times\circ\circ\times)) &= L(\circ\circ\circ\circ\circ\times) \oplus L(\circ\circ\times\circ\circ\circ)^{\oplus 2},\\
\mathrm{DS}_3(L(\circ\times\times\circ\circ\times)) &= \mathbb{C}^{\oplus 3}.
\end{aligned}$$

Let us check that the above formulas are compatible. Using (1.3), we get

$$\operatorname{sch} L(\circ\times\times\circ\circ\circ) = \mathcal{E}^-_{\circ\times\times\circ\circ\circ}, \quad \operatorname{sch} L(\circ\circ\times\circ\circ\times) = \mathcal{E}^-_{\circ\circ\times\circ\circ\times} + \mathcal{E}^-_{\circ\circ\times\circ\times\circ} + 2\mathcal{E}^-_{\circ\times\times\circ\circ\circ},$$

which establishes the compatibility for ds_1. For the $\mathfrak{gl}(1|1)$-modules, we have

$$\operatorname{sch} L(\circ\circ\times\circ\circ\circ) = \mathcal{E}^-_{\circ\circ\times\circ\circ\circ}.$$

Taking R' and ρ' to be the Weyl denominator and the Weyl vector, respectively, for $\mathfrak{gl}(1|1)$ and $\mu := \mathrm{wt}(\circ\circ\circ\circ\times)$, we get

$$\begin{aligned}
&\pi(R'e^{\rho'})\big(\operatorname{sch} K(\circ\circ\circ\circ\times) + \operatorname{sch} K(\circ\circ\circ\times\circ) + \mathcal{E}^-_{\circ\circ\circ\times\circ\circ}\big)\\
&\quad = \mathrm{J}_W\left(e^{\mu} + e^{\mu-\beta} + \frac{e^{\mu-2\beta}}{1-e^{-\beta}}\right) = \mathrm{J}_W\left(\frac{e^{\mu}}{1-e^{-\beta}}\right)\\
&\quad = \pi(R'e^{\rho'})\mathcal{E}^-_{\circ\circ\circ\circ\circ\times} = \pi(R'e^{\rho'})\operatorname{sch} L(\circ\circ\circ\circ\circ\times);
\end{aligned}$$

this establishes the compatibility for ds_2.

8. Superdimensions and modified superdimensions

We discuss modified nontrivial trace and dimension functions on the thick ideal I_k generated by the irreducible representations of atypicality k, and how they can be calculated explicitly by means of the Duflo–Serganova functor. We do this for the $\mathfrak{osp}(m|2n)$ and the $\mathrm{OSp}(m|2n)$-case. For the $\mathfrak{gl}$-case, see [26].

8.1. The core of a block. Recall that $\widetilde{\mathcal{F}} = \mathrm{Rep}(\mathrm{SOSp}(m|2n))$. Exactly as for $\mathfrak{gl}(m|n)$ (see Section 2.1), we have a decomposition

$$\widetilde{\widetilde{\mathcal{F}}} = \widetilde{\mathcal{F}} \oplus \Pi\widetilde{\mathcal{F}}$$

into two subcategories which are equivalent by the parity shift Π. We use the notation

$$\widetilde{\mathcal{F}}' = \widetilde{\mathcal{F}}'(m|2n) = \mathrm{Rep}(\mathrm{OSp}(m|2n))$$

for the finite-dimensional algebraic representations of $\mathrm{OSp}(m|2n)$. As for $\tilde{\mathcal{F}}$, the category decomposes $\tilde{\mathcal{F}}' = \mathcal{F}' \oplus \Pi\mathcal{F}'$ into two equivalent subcategories.

The irreducible typical module L^{core} (as defined in Section 7.2.7) attached to a block of atypicality k in $\tilde{\mathcal{F}}$ is both an $\mathfrak{osp}(m-2k|2n-2k)$ and $\mathrm{OSp}(m-2k|2n-2k)$-module. Therefore, the core can be defined in the $\tilde{\mathcal{F}}'$-case as well.

The DS_x functor on $\tilde{\mathcal{F}}$ induces a functor

$$\mathrm{DS}_x\colon \tilde{\mathcal{F}}'(m|2n) \to \tilde{\mathcal{F}}'(m-2k|2n-2k),$$

where $k = \mathrm{rk}(x)$ (see [11] for details).

If the rank of x equals $\mathrm{def}(\mathfrak{osp}(m|2n))$, we obtain

- $\mathfrak{g}_x = \mathfrak{o}(m-2n|0)$, $m > 2n$;
- $\mathfrak{g}_x = \mathfrak{sp}(0|2n-2m)$, $2n > m$ even;
- $\mathfrak{g}_x = \mathfrak{osp}(1|2n-2m)$, $2n-m$ odd.

In the OSp-case, we obtain representations of the groups $G_x = \mathrm{O}(m-2n)$, $\mathrm{Sp}(2n-2m)$ (considered as odd) and $\mathrm{OSp}(1|2n-2m)$.

8.2. Superdimensions. If we apply DS_x to an irreducible representation $L(\lambda)$ with atypicality equal to $\mathrm{rk}(x)$, then $\mathrm{DS}_x(L(\lambda))$ does not depend on the choice of x. Indeed, the induced morphism on the supercharacter ring does not depend on x and $\mathrm{DS}_x(L(\lambda))$ is semisimple. We simply write DS_k in this case.

The parity rule of [21] yields

$$\mathrm{DS}_k(L(\lambda)) \in \Pi^{\|\mathrm{howl}(\lambda)\|}\, \mathcal{F}(\mathfrak{g}_x)$$

and hence

$$\mathrm{DS}_k(L(\lambda)) = \Pi^{\|\mathrm{howl}(\lambda)\|}(L^{\mathrm{core}})^{\oplus m(\lambda)}$$

for the positive integer $m(\lambda)$ defined in Corollary 7.2.9 (the number of increasing paths from λ to the weights with adjacent $\times$'s).

8.2.1. OSp-modules. We first consider $\mathfrak{g} = \mathfrak{osp}(2m|2n)$. By [14, Proposition 4.11], the simple $\mathrm{OSp}(2m|2n)$-modules are either of the form $L(\lambda)$ if $\lambda \in \Lambda^+_{m|n}$ is σ-invariant or $L(\lambda) \oplus L(\lambda^\sigma)$. Thus the simple $\mathrm{OSp}(2m|2n)$-modules are in one-to-one correspondence with the unsigned $\mathfrak{osp}(2m|2n)$-diagrams. For $\mathfrak{osp}(2m+1|2n)$ and any $\lambda \in \Lambda^+_{m|n}$, there are two irreducible $\mathrm{OSp}(2m+1|2n)$-modules $L(\lambda,+)$ and $L(\lambda,-)$ which restrict to $L(\lambda)$. We will often simply write $L_{\mathrm{OSp}}(\lambda)$ for an irreducible representation of OSp. The diagram

$$\begin{tikzcd}
\mathcal{F}'(G) \arrow[dd, "\mathrm{DS}_x"'] \arrow[dr, "\mathrm{Res}"] & \\
 & \tilde{\mathcal{F}}(\mathfrak{g}) \arrow[dl, "\mathrm{DS}_x"] \\
\mathcal{F}'(G_x) &
\end{tikzcd}$$

commutes for any x since $\mathrm{DS}_x(L(\lambda))$ is in $\mathcal{F}'(G_x)$. It follows from this diagram that the multiplicity of L^{core} in $\mathrm{DS}_x(L_{\mathrm{OSp}}(\lambda)$ is the same as for $\mathrm{Res}(L_{\mathrm{OSp}}(\lambda))$ if the restriction is irreducible and is twice the multiplicity of $\mathrm{DS}(L(\lambda))$ if the restriction decomposes into two irreducible summands.

Since DS is a symmetric monoidal functor, it preserves the superdimension.

8.2.2. Corollary. *For $L(\lambda) \in \tilde{\tilde{\mathcal{F}}}$ of atypicality k,*

$$\operatorname{sdim} L(\lambda) = (-1)^{\|\mathrm{howl}(\lambda)\|} m(\lambda) \operatorname{sdim} L^{\mathrm{core}}.$$

In particular, $\operatorname{sdim} L(\lambda) \neq 0$ *if and only if λ is maximal atypical.*

8.3. Modified traces. In this section, $\tilde{\tilde{\mathcal{F}}}$ means either $\tilde{\tilde{\mathcal{F}}}$ or $\tilde{\tilde{\mathcal{F}}}'$ unless otherwise specified. If $\mathrm{at}(L(\lambda)) < n$, $\mathrm{sdim}(L) = 0$. However, one can define a modified superdimension for L as follows. Recall that a thick (tensor) ideal I in $\tilde{\tilde{\mathcal{F}}}$ is a subset of objects which is closed under tensor products with arbitrary objects and closed under direct summands. A trace on I is by definition a family of linear functions

$$t = \{t_V : \operatorname{End}_{\tilde{\tilde{\mathcal{F}}}}(V) \to k\},$$

where V runs over all objects of I such that following two conditions hold.

(i) If $U \in I$ and W is an object of $\tilde{\tilde{\mathcal{F}}}$, then for any $f \in \operatorname{End}_{\tilde{\tilde{\mathcal{F}}}}(U \otimes W)$, we have

$$t_{U \otimes W}(f) = t_U(t_R(f))$$

for the right trace $\mathrm{tr}_R(\ )$.

(ii) If $U, V \in I$, then for any morphisms $f : V \to U$ and $g : U \to V$, we have

$$t_V(g \circ f) = t_U(f \circ g).$$

For such a trace on I, we define

$$\dim^I(X) = t_X(\mathrm{id}_X), \quad X \in I,$$

the modified dimension of (I, t). For an object $J \in \tilde{\tilde{\mathcal{F}}}$, let I_J be the thick ideal generated by J. By Kujawa [33, Theorem 2.3.1], the trace on the ideal I_L, L irreducible, is unique up to multiplication by an element of $\mathbb{C}$.

8.4. The generalized Kac–Wakimoto conjecture. Let I_k be the thick ideal generated by all irreducible representations of atypicality k. The ideal I_0 coincides with Proj. The following theorem was proven for $\mathfrak{gl}(m|n)$ by Serganova [43] and for $\mathfrak{osp}(m|2n)$ by Kujawa [33]. We give a slightly different simplified proof. Moreover, we explain how to compute these modified superdimensions.

8.4.1. Theorem (Generalized Kac–Wakimoto conjecture). *The ideal I_k admits a non-trivial modified trace function. For irreducible $L(\lambda)$, the associated dimension function*

$$\dim^k := \dim^{I_k}$$

satisfies $\dim^k L(\lambda) \neq 0$ *if and only if the atypicality of $L(\lambda)$ is k.*

It was shown in [16, Theorem 1.3.1] that if an ideal I carries a modified trace function, all indecomposable objects in I are ambidextrous in the sense of [16]. Since the I_k define an exhaustive filtration of $\tilde{\tilde{\mathcal{F}}}$, the conjecture implies that every simple module in $\tilde{\tilde{\mathcal{F}}}$ is ambidextrous.

8.5. A trace on I_k. There are two different ways to see that $\mathrm{Proj} \subset \widetilde{\mathcal{F}}'$ carries a nontrivial trace function. It was proven in [17, Theorem 4.8.2] that $\mathrm{Proj} \subset \widetilde{\mathcal{F}}$ has such a trace function. This implies that $\mathrm{Proj} \subset \widetilde{\mathcal{F}}'$ has one as well using the restriction rules of Ehrig–Stroppel and the argument of [33].

Alternatively, it follows from [27] that $\mathrm{Proj} \subset \widetilde{\mathcal{F}}'$ carries such a trace function. Note that it is unique up to a scalar: any $P \in \mathrm{Proj}$ satisfies $\langle P \rangle = \mathrm{Proj}$. Indeed, $\langle P \rangle \subset \mathrm{Proj}$ is clear, and $\mathrm{Proj} \subset P$ follows since Proj is the smallest thick ideal [11]. We denote any normalization of this trace function by Tr^0.

8.5.1. Proposition. *The thick ideal $I_k \subset \widetilde{\mathcal{F}}$ carries a nontrivial modified trace function* Tr^k.

Proof. Let $L(\lambda) \in \widetilde{\mathcal{F}}$. Then we have $\mathrm{DS}_x(L(\lambda)) \in \widetilde{\mathcal{F}}'$ for all x by [21]. Let $X \in I_k$ and $f \in \mathrm{End}(X)$. Then we define

$$\mathrm{Tr}^k(f) = \mathrm{Tr}^0(\mathrm{DS}_k(f)).$$

Then $\mathrm{DS}_k(X)$ is typical and therefore projective. Since DS_k is a symmetric monoidal functor, this defines a trace function. We claim that it is nontrivial. For $X = L(\mu)$, we obtain

$$\mathrm{DS}_k(f) \in \mathrm{End}(\Pi^{\|\mathrm{howl}(\mu)\|}(L^{\mathrm{core}})^{\oplus m(\lambda)}).$$

Since the parity is either even or odd and Tr^0 is nontrivial for any typical module, we compute, for $f \in \mathrm{End}(X)$,

$$\mathrm{Tr}^k(\mathrm{id}_L) = \mathrm{Tr}^0_{\mathrm{DS}_k(L)}(\mathrm{id}_{\mathrm{DS}_k(L)}) = m(\lambda)\mathrm{Tr}^0_{\Pi^{\|\mathrm{howl}(\mu)\|}L^{\mathrm{core}}}(\mathrm{id}_{\Pi^{\|\mathrm{howl}(\mu)\|}L^{\mathrm{core}}}) \neq 0.$$

The same proof works for $L_{\mathrm{OSp}}(\lambda)$. □

8.5.2. Remark. It can be shown [33] that I_k is in fact generated by an arbitrary irreducible representation of atypicality k. Therefore, the above trace is the unique modified trace up to a scalar.

Since $\mathrm{DS}_k(L) = 0$ for any L of atypicality less than k, we obtain, for the modified superdimension, $\mathrm{sdim}^k(X) := \mathrm{Tr}^k(\mathrm{id}_X)$

8.5.3. Corollary. *Let $L(\lambda)$ be a representation of atypicality at most k. Then we have* $\mathrm{sdim}^k(L(\lambda)) \neq 0$ *if and only if* $at(L(\lambda)) = k$.

A. Kac–Wakimoto terms and the rings $\mathcal{R}$, $\mathcal{R}_{\Sigma'}$

In this section, $\mathfrak{g}$ is $\mathfrak{gl}(m|n)$, $\mathfrak{osp}(M|N)$ or one of the exceptional Lie superalgebras $F(4)$, $G(3)$, $D(2|1,a)$. We use the standard notation for the roots of $\mathfrak{g}_0$ and denote by Π_0 a standard set of simple roots. In what follows, we consider only bases Σ of Δ which are compatible with Π_0, that is $\Delta^+(\Sigma)_0 = \Delta^+(\Pi_0)$. By [40], all such bases are connected by chains of odd reflections. In the $\mathfrak{gl}$ and $\mathfrak{osp}$-cases, these bases can be encoded by words consisting of m letters ε and n letters δ.

A.1. Notation. We denote by W the Weyl group of $\mathfrak{g}_0$. We denote by Δ the set of roots of $\mathfrak{g}$ and set

$$\mathfrak{h}^*_{\mathrm{int}} := \{\lambda \in \mathfrak{h}^* \mid \lambda - w\lambda \in \mathbb{Z}\Delta \text{ for all } w \in W\},$$
$$P(\mathfrak{g}_0) := \{\lambda \in \mathfrak{h}^* \mid \lambda - w\lambda \in \mathbb{Z}\Delta_0 \text{ for all } w \in W\}.$$

For each non-isotropic root α, let $r_\alpha \in W$ be the reflection with respect to α. For any subset $Y \subset W$, we denote by J_Y the linear operator

$$P \mapsto \sum_{w \in Y} \operatorname{sgn}(w) w(P),$$

where $\operatorname{sgn}\colon W \to \mathbb{Z}_2$ is the standard sign homomorphism (given by $\operatorname{sgn} r_\alpha = -1$).

A.1.1. Choice of the Weyl vector. We denote by ρ_0 a Weyl vector of $\mathfrak{g}_0$ which is an element of $\mathfrak{h}^*$ satisfying $r_\alpha \rho_0 = \rho_0 - \alpha$ for each $\alpha \in \Pi_0$. Note that ρ_0 is unique if Δ_0 spans $\mathfrak{h}^*$, i.e. for $\mathfrak{g} \neq \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n)$. We choose the Weyl vector ρ by the rule

$$\rho := \rho_0 - \rho_1, \quad \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha.$$

If $\beta \in \Sigma$ is isotropic and $\Sigma' = r_\beta \Sigma$, we have $\rho' := \rho + \beta$. Using [42] (or a short case-by-case reasoning), we obtain $\rho \in \mathfrak{h}^*_{\mathrm{int}}$. We introduce

$$R_0 := \prod_{\alpha \in \Delta_0^+} (1 - e^{-\alpha}), \quad R_1(\Sigma) := \prod_{\alpha \in \Delta_1^+(\Sigma)} (1 + e^{-\alpha}), \quad R(\Sigma) := \frac{R_0}{R_1(\Sigma)}.$$

Note that the following term is W-invariant and does not depend on the choice of Σ:

$$e^{\rho_0 - \rho} R_1(\Sigma) = \prod_{\alpha \in \Delta_1^+(\Sigma)} (e^{\alpha/2} + e^{-\alpha/2}).$$

Hence, for each Σ' satisfying $\Delta_0^+ \subset \Delta^+(\Sigma)$, we have

$$R(\Sigma') e^{\rho'} = R e^{\rho}, \quad \text{where } R := R(\Sigma).$$

A.2. Rings $\mathcal{R}$ and $\mathcal{R}_\Sigma$. For a sum of the form $\sum_{\nu \in \mathfrak{h}^*} a_\nu e^\nu$ with $a_\nu \in \mathbb{Q}$, we define the support by the formula

$$\operatorname{supp}\Big(\sum a_\nu e^\nu\Big) = \{\nu \in \mathfrak{h}^* \mid a_\nu \neq 0\}.$$

Let $\mathcal{R}_\Sigma$ be the set consisting of the sums $\sum_{\nu \in \mathfrak{h}^*} a_\nu e^\nu$ with $a_\nu \in \mathbb{Q}$ and such that

$$\operatorname{supp}\Big(\sum a_\nu e^\nu\Big) \subset \bigcup_{i=1}^{k} (\nu_i - \mathbb{N}\Sigma)$$

for some k. Clearly, $\mathcal{R}_\Sigma$ is a ring. This ring contains $\operatorname{ch} N$ and $\operatorname{sch} N$ for any N in the BGG-category $\mathcal{O}$.

A.2.1. Denote by $\mathcal{R}$ the ring of rational functions of the form P/Q, where P lies in the group ring $\mathbb{Q}[\mathfrak{h}^*]$ and Q is a product of the factors of the form $1 \pm e^{-\alpha}$ for $\alpha \in \Delta$. Using the formula

$$1 \pm e^{-\alpha} = 1 \mp e^{-\alpha} + e^{-2\alpha} \mp e^{-3\alpha} + \cdots,$$

we will view the element $P/Q \in \mathcal{R}$ as a series $\mathcal{R}_\Sigma$; we will call this series the Σ-expansion of P/Q. For instance, $R(\Sigma')$, $R(\Sigma')^{-1} \in \mathcal{R}$ for any base Σ' and Σ-expansion of $R(\Sigma')^{-1}$ is equal to the character of a Verma module of the highest weight 0 (defined with respect to the base Σ).

A.2.2. Lemma. *For any base Σ' satisfying $\Delta^+(\Sigma')_0 = \Delta^+(\Sigma)_0$, one has*

$$\mathrm{J}_W\left(\frac{e^{\rho'}}{\prod_{\alpha\in\Delta_1^+(\Sigma')}(1+e^{-\alpha})}\right) = Re^{\rho}.$$

Proof. By above, $e^{\rho_0-\rho'}R_1(\Sigma')$ is W-invariant, so

$$\mathrm{J}_W\left(\frac{e^{\rho'}}{\prod_{\alpha\in\Delta_1^+(\Sigma')}(1+e^{-\alpha})}\right) = \mathrm{J}_W\left(\frac{e^{\rho'}}{R_1(\Sigma')}\right) = \frac{\mathrm{J}_W(e^{\rho_0})}{e^{\rho_0-\rho'}R_1(\Sigma')}.$$

The Weyl character formula for the trivial $\mathfrak{g}_0$-module gives $\mathrm{J}_W(e^{\rho_0}) = R_0 e^{\rho_0}$. Using the above identity $R(\Sigma')e^{\rho'} = Re^{\rho}$, we obtain the required formula. □

A.3. Projection P_χ. Let $\mathcal{O}^\chi$ be the full subcategory of the category $\mathcal{O}$ corresponding to a central character χ. For $N \in \mathcal{O}$, let N^χ be the projection of N to $\mathcal{O}\chi$. The character of N^χ can be expressed via $\operatorname{ch} N$ by the following procedure.

By above, $\mathcal{R}_{\Sigma'}$ contains the terms $\operatorname{ch} N$, $Re^{\rho}\operatorname{ch} N$ for any module $N \in \mathcal{O}$. It is well known that, for $N \in \mathcal{O}^\chi$, the Σ'-expansion of $Re^{\rho}\operatorname{ch} N$ satisfies

$$\operatorname{supp}(Re^{\rho}\operatorname{ch} N) \subset \{\mu+\rho \mid \chi_\mu = \chi\}.$$

Introducing a projection $P_\chi\colon \mathcal{R}_{\Sigma'} \to \mathcal{R}_{\Sigma'}$ by $P_\chi\big(\sum a_\mu e^\mu\big) = \sum_{\mu:\chi_{\mu-\rho}=\chi} a_\mu e^\mu$, we get

$$Re^{\rho}\operatorname{ch} N^\chi = P_\chi(Re^{\rho}\operatorname{ch} N).$$

A.3.1. For a finite-dimensional module V, a translation functor $T^V_{\chi,\chi'}\colon \mathcal{O}^\chi \to \mathcal{O}^{\chi'}$ is given by $T^V_{\chi,\chi'}(N) := (N\otimes V)^{\chi'}$. By above,

$$Re^{\rho}\operatorname{ch}(T^V_{\chi,\chi'}(N)) = \Theta^V_{\chi,\chi'}(Re^{\rho}\operatorname{ch} N),$$

where $\Theta^V_{\chi,\chi'}\colon \mathcal{R}_{\Sigma'} \to \mathcal{R}_{\Sigma'}$ is given by

$$\Theta^V_{\chi,\chi'}\Big(\sum a_\mu e^\mu\Big) := P_{\chi'}\Big(\operatorname{ch} V \cdot P_\chi\Big(\sum a_\mu e^\mu\Big)\Big).$$

A.4. The terms $\mathbf{KW}(\lambda, S)$. We say that a subset $S \subset \Delta_1$ is an *iso-set* if S is a basis of an isotropic subspace of $\mathfrak{h}^*$, i.e. S is linearly independent and $(S|S) = 0$.

For $\lambda \in \mathfrak{h}^*_{\mathrm{int}}$ and an iso-set $S \subset \Delta_1$ satisfying $(\lambda|S) = 0$, we set

$$\mathrm{KW}(\lambda, S) := \mathrm{J}_W\left(\frac{e^\lambda}{\prod_{\beta\in S}(1+e^{-\beta})}\right).$$

A.4.1. Remark. For an arbitrary weight $\lambda \in \mathfrak{h}^*$, the group W should be substituted by the "λ-integral" subgroup; see [23, Section 11].

A.4.2. Note that

$$\mathrm{KW}(\lambda, S)\cdot \prod_{\alpha\in\Delta_1^+}(1+e^{-\alpha})\in S(\mathfrak{h}^*),$$

so $\mathrm{KW}(\lambda, S)\in\mathcal{R}$. One readily sees that, for the Σ-expansion of $\mathrm{KW}(\nu, S)$, we have

$$\operatorname{supp}\mathrm{KW}(\nu, S)\subset W(\nu+\mathbb{Z}S).$$

By [31, 44], $\chi_{\mu-\rho}=\chi_{\nu-\rho}$ for each $\mu\in\nu+\mathbb{Z}S$. Thus, for P_χ introduced in Section A.3, we have

$$P_\chi(\mathrm{KW}(\lambda+\rho, S))=\delta_{\chi,\chi_\lambda}\,\mathrm{KW}(\lambda+\rho, S). \tag{A.1}$$

A.4.3. For $\mathfrak{g}=\mathfrak{gl}(m|n), \mathfrak{osp}(M|N)$, one has $S=\{\pm\varepsilon_{p_i}\pm\delta_{q_i}\}_{i=1}^t$, where $p_i\neq p_j$, $q_i\neq q_j$ for $i\neq j$. We denote the intersection of $\mathbb{Z}\Delta$ with the span of $\varepsilon_{p_1},\dots,\varepsilon_{p_t},\delta_{q_1},\dots,\delta_{q_t}$ by $\mathfrak{h}(S)^*$. Notice that S spans a maximal isotropic subspace in $\mathfrak{h}(S)^*$.

A.4.4. Lemma. *The following statements hold:*

(i) $w\,\mathrm{KW}(\lambda, S)=\operatorname{sgn}(w)\,\mathrm{KW}(\lambda, S)=\mathrm{KW}(w\lambda, wS)$*;*

(ii) $\mathrm{KW}(\lambda, S)=0$ *if there exists* $\alpha\in\Delta_0$ *such that* $(\alpha|S)=(\alpha|\lambda)=0$*;*

(iii) $\mathrm{KW}(\lambda-\beta, S)=\mathrm{KW}(\lambda,(S\cup\{-\beta\})\setminus\{\beta\})$ *for each* $\beta\in S$*;*

(iv) *in the* $\mathfrak{osp}$*-case, if* $(\lambda|\mathfrak{h}(S)^*)=0$*, then* $\mathrm{KW}(\lambda-\beta, S)=-\mathrm{KW}(\lambda, S)$ *for each* $\beta\in S$*.*

Proof. (i), (iii) are straightforward and (ii) follows from (i) for $w:=r_\alpha$. For (iv), note that

$$\mathrm{KW}(\lambda, S)+\mathrm{KW}(\lambda-\beta, S)=\mathrm{J}_W\left(\frac{e^\lambda+e^{\lambda-\beta}}{\prod_{\beta\in S}(1+e^{-\beta})}\right)=\mathrm{KW}(\lambda, S\setminus\{\beta\}).$$

Since $\beta=\pm\varepsilon_i\pm\delta_j$ for some i,j, we have $(\lambda|\delta_j)=(S\setminus\{\beta\}|\delta_j)=0$, so (iii) gives

$$\mathrm{KW}(\lambda; S\setminus\{\beta\})=0$$

as required. □

A.4.5. Denominator identity. Let S be an iso-set of the cardinality $\min(m,n)$ and let Σ' be a base of Δ containing S (for instance, $\Sigma'=\Sigma$ for the $\mathfrak{osp}$-case and Σ' corresponding to $(\varepsilon\delta)^m\delta^{n-m}$ for $n\geq m$). By [19], one has $\mathrm{KW}(\rho', S)=jRe^\rho$, where j is a certain integer (j is the order of the "smallest factor" in W, for instance, $j=m!$ for $\mathfrak{gl}(m|n)$ with $m\leq n$).

Consider the case $\mathfrak{g}=\mathfrak{gl}(s|s)$ or $\mathfrak{osp}(2s+t|2s)$. Then we have $j=s!$ for $\mathfrak{gl}(s|s)$, $j=\max(2^{s-1}s!,1)$ for $\mathfrak{osp}(2s|2s)$, and $j=2^s s!$ for $\mathfrak{osp}(2s+t|2s)$ with $t=1,2$. Let Σ' be the base corresponding to the word $(\varepsilon\delta)^s$; this base contains an iso-set $\{\varepsilon_i-\delta_i\}_{i=1}^s$. Note that $w\rho'=\rho'$ for any $w\in S_s\times S_s$; using Lemma A.4.4 (i), we obtain

$$jRe^\rho=\mathrm{KW}(\rho',\{\varepsilon_i-\delta_i\}_{i=1}^s)=(-1)^{[\frac{s}{2}]}\,\mathrm{KW}(\rho',\{\varepsilon_i-\delta_{s+1-i}\}_{i=1}^s).$$

A.5. The term $\frac{\mathrm{KW}(\lambda,S)}{Re^{\rho}}$. Recall that $L_{\mathfrak{g}_0}(\lambda-\rho_0)$ is finite-dimensional if and only if $\lambda\in P^{++}(\mathfrak{g}_0)$, where

$$P^{++}(\mathfrak{g}_0):=\{\lambda\in P(\mathfrak{g}_0)\mid \lambda-w\lambda\in\mathbb{Z}_{\geq 0}\Delta^+ \text{ for all } w\in W,\ \lambda\neq w\lambda\}.$$

The character ring $\mathrm{Ch}(\mathfrak{g}_0)$ has a basis $\{\mathrm{ch}\, L_{\mathfrak{g}_0}(\lambda-\rho_0)\}_{\lambda\in P^{++}(\mathfrak{g}_0)}$. This allows to extend dim to the linear map $\dim\colon \mathrm{Ch}(\mathfrak{g}_0)\to\mathbb{Z}$ having $\dim(\mathrm{ch}\, N)=\dim N$ for any finite-dimensional module N.

The Weyl character and the Weyl dimension formulas give the following.

A.5.1. Lemma. *Take $\lambda\in P(\mathfrak{g}_0)$. One has*

(i) $\mathrm{J}_W(e^{\lambda})=0$ *if and only if* $\lambda\notin WP^{++}(\mathfrak{g}_0)$*;*

(ii) $\frac{\mathrm{J}_W(e^{\lambda})}{R_0e^{\rho_0}}\in\mathrm{Ch}(\mathfrak{g}_0)$*;*

(iii) $\dim\big(\frac{\mathrm{J}_W(e^{\lambda})}{R_0e^{\rho_0}}\big)=\prod_{\alpha\in\Delta_0^+}\frac{(\lambda|\alpha)}{(\rho_0|\alpha)}$.

Proof. If $\lambda\notin WP^{++}(\mathfrak{g}_0)$, then $r_\alpha\lambda=\lambda$ for some $\alpha\in\Delta_0^+$, and thus $\mathrm{J}_W(e^{\lambda})=0$ and both sides of (iii) are equal to zero. Now take $\lambda\in WP^{++}(\mathfrak{g}_0)$, that is $\lambda=w\nu$ for $\nu\in P^{++}(\mathfrak{g}_0)$. Then $\mathrm{J}_W(e^{\lambda})=\mathrm{sgn}(w)\,\mathrm{J}_W(e^{\nu})$. Using the Weyl character formula, we get

$$(R_0e^{\rho_0})^{-1}\,\mathrm{J}_W(e^{\lambda})=\mathrm{sgn}(w)(R_0e^{\rho_0})^{-1}\,\mathrm{J}_W(e^{\nu})=\mathrm{sgn}(w)\,\mathrm{ch}\, L(\nu-\rho_0),$$

which establishes (ii). The Weyl dimension formula gives

$$\begin{aligned}\dim\Big(\frac{\mathrm{J}_W(e^{\lambda})}{R_0e^{\rho_0}}\Big)&=\mathrm{sgn}(w)\dim L(\nu-\rho_0)\\&=\mathrm{sgn}(w)\prod_{\alpha\in\Delta_0^+}\frac{(\nu|\alpha)}{(\rho_0|\alpha)}=\mathrm{sgn}(w)\prod_{\alpha\in w^{-1}\Delta_0^+}\frac{(\lambda|\alpha)}{(\rho_0|\alpha)}.\end{aligned}$$

One has $(-1)^{\#\{\alpha\in w^{-1}\Delta_0^+\cap(-\Delta_0^+)\}}=\mathrm{sgn}(w)$; this gives (iii) for $\lambda\in WP^{++}(\mathfrak{g}_0)$. □

A.5.2. For a subset $U\subset\Delta$, we will use the notation

$$\mathrm{sum}(U):=\sum_{\beta\in U}\beta.$$

Observe that all weights of a finite-dimensional $\mathfrak{g}$-module lie in $P(\mathfrak{g}_0)$. Take $\lambda\in P(\mathfrak{g}_0)+\rho$. Recall that $\rho_0-\rho=\rho_1=\frac{1}{2}\sum_{\alpha\in\Delta_1^+}\alpha$. One has

$$\frac{\mathrm{KW}(\lambda,S)}{Re^{\rho}}=\frac{\mathrm{J}_W\big(e^{\lambda+\rho_1}\prod_{\beta\in\Delta_1^+\setminus S}(1+e^{-\beta})\big)}{R_0e^{\rho_0}}=\sum_{U\subset\Delta_1^+\setminus S}\frac{\mathrm{J}_W(e^{\lambda+\rho_1-\mathrm{sum}(U)})}{R_0e^{\rho_0}}.$$

A.5.3. Corollary. *For each $\lambda\in P(\mathfrak{g}_0)$, the term $\frac{\mathrm{KW}(\lambda,S)}{Re^{\rho}}$ lies in* $\mathrm{Ch}(\mathfrak{g}_0)$ *and*

$$\dim\Big(\frac{\mathrm{KW}(\lambda+\rho,S)}{Re^{\rho}}\Big)=\sum_{U\subset\Delta_1^+\setminus S}\ \prod_{\alpha\in\Delta_0^+}\frac{(\lambda+\rho_0-\mathrm{sum}(U)|\alpha)}{(\rho_0|\alpha)}.$$

Moreover,

$$\frac{\mathrm{KW}(\lambda+\rho,S)}{Re^{\rho}}=\sum n_{\lambda\mu}\,\mathrm{ch}\,L_{\mathfrak{g}_0}(\mu)$$

with the coefficients given by

$$n_{\lambda\mu}=\sum_{U\subset\Delta_1^+\setminus S}\sum_{w\in W}\mathrm{sgn}(w)\delta_{w(\mu+\rho_0),\lambda+\rho_0-\mathrm{sum}(U)}.$$

A.5.4. Example. If $\mathrm{ch}\,L$ is given by the Kac–Wakimoto formula

$$Re^{\rho}\,\mathrm{ch}\,L=j^{-1}\,\mathrm{KW}(\lambda+\rho,S),$$

then

$$\dim L=j^{-1}\sum_{U\subset\Delta_1^+\setminus S}\prod_{\alpha\in\Delta_0^+}\frac{(\lambda+\rho_0-\sum_{\beta\in U}\beta|\alpha)}{(\rho_0|\alpha)}.$$

A.5.5. $L(\lambda)$ as a $\mathfrak{g}_0$-module. A Verma module $M(\lambda)$ has a filtration with the factors of the form $\{M_{\mathfrak{g}_0}(\lambda-\mathrm{sum}(U))\}_{U\subset\Delta_1^+}$. Notice that if $M_{\mathfrak{g}_0}(\lambda-\mathrm{sum}(U))$ has a finite-dimensional quotient, then this quotient is $L_{\mathfrak{g}_0}(\lambda-\mathrm{sum}(U))$. Hence $[L(\lambda):L_{\mathfrak{g}_0}(\lambda-\mu)]\neq 0$ implies $\mu=\mathrm{sum}(U)$ for some $U\subset\Delta_1^+$. The multiplicity $m_{\lambda;U}:=[L(\lambda):L_{\mathfrak{g}_0}(\lambda-\mathrm{sum}(U))]$ can be computed using Corollary A.5.3 as follows:

$$m_{\lambda;U}=\sum_{\mu}(-1)^{\|\lambda\|-\|\mu\|}d_{<}^{\lambda,\mu}\sum_{U'\subset\Delta_1^+\setminus S_\mu}j_\mu^{-1}\sum_{w\in W}\mathrm{sgn}(w)\delta_{w(\lambda+\rho_0-\mathrm{sum}(U)),\mu^{\dagger}+\rho_1-\mathrm{sum}(U')}.\tag{A.2}$$

For the $\mathfrak{osp}$-case, this gives

$$m_{\lambda;U}=\sum_{\mu}(-1)^{\|\lambda\|-\|\mu\|}d_{<}^{\lambda,\mu}\sum_{U'\subset\Delta_1^+\setminus S_\mu}j_\mu^{-1}\sum_{w\in W}\mathrm{sgn}(w)\delta_{w(\lambda+\rho_0-\mathrm{sum}(U)),\mu+\rho_0-\mathrm{sum}(U')}.$$

A.5.6. Remark. A variation of the above reasoning allows to find the *graded multiplicities*

$$[L(\nu)_0:L_{\mathfrak{g}_0}(\mu)]+\xi[L(\nu)_1:L_{\mathfrak{g}_0}(\mu)]$$

using the Gruson–Serganova character formula. In order to do this, we define the graded version of $\mathrm{KW}(\lambda,S)$ by the following procedure.

Let ξ be a formal (even) variable satisfying $\xi^2=1$. We denote by $\mathrm{Ch}_\xi(\mathfrak{g})$ the ring of ξ-characters of the finite-dimensional $\mathfrak{g}$-modules and view $\mathrm{Ch}_\xi(\mathfrak{g})$ as a subring of $\mathcal{R}[\xi]$. For $\nu\in\mathbb{Z}\Delta$, consider the map $\Xi\colon e^{\nu}\mapsto\xi^{p(\nu)}e^{\nu}$ and extend this map to the rational functions P/Q, where P,Q are polynomials in e^{ν} with $\nu\in\mathbb{Z}\Delta$. This allows to define for $\lambda\in\mathfrak{h}_{\mathrm{int}}^*$ the term $\mathrm{KW}_\xi(\lambda,S)$ by the formula

$$\mathrm{KW}_\xi(\lambda,S):=e^{\lambda}\,\Xi(e^{-\lambda}\,\mathrm{KW}(\lambda,S)).$$

Note that $\mathrm{KW}_\xi(\lambda, S)$ and $\Xi(R^{\pm 1})$ lie in the ring $\mathcal{R}[\xi]$ and can be viewed as elements of $\mathcal{R}_\Sigma[\xi]$. Taking $\lambda \in P(\mathfrak{g}_0)$, we have

$$\frac{\mathrm{KW}_\xi(\lambda+\rho, S)}{\Xi(R)e^\rho} = \frac{J_W\big(e^{\lambda+\rho_0}\prod_{\beta\in\Delta_1^+\setminus S}(1+\xi e^{-\beta})\big)}{R_0 e^{\rho_0}} \in \mathrm{Ch}_\xi(\mathfrak{g}_0).$$

The graded multiplicity $\big[\frac{\mathrm{KW}(\lambda+\rho,S)}{\Xi(R)e^\rho} : \mathrm{ch}\, L_{\mathfrak{g}_0}(\mu)\big]$ is given by

$$\sum_{U\subset\Delta_1^+\setminus S} \xi^{\# U} \sum_{w\in W} \mathrm{sgn}(w)\delta_{w(\mu+\rho_0),\lambda+\rho_0-\mathrm{sum}(U)}.$$

The Gruson–Serganova formula (1.3) gives the following formula for $\mathrm{ch}_\xi L$:

$$\Xi(R)e^\rho \,\mathrm{ch}_\xi L = \frac{\prod_{\alpha\in\Delta_0^+}(1-e^{-\alpha})}{\prod_{\alpha\in\Delta_1^+}(1+\xi e^{-\alpha})} e^\rho \,\mathrm{ch}_\xi L = \sum_{L'\in\mathrm{Irr}} \pm b_{L,L'}\,\mathrm{KW}(L') \tag{A.3}$$

(where $L = L(\lambda)$, $L' = L(\nu)$ and the sign $\pm$ is given by $(-1)^{p(\lambda^\dagger-\nu^\dagger)}$); combining the above formulas, one obtains an analogue of (A.2) for the graded multiplicity of $L_{\mathfrak{g}_0}(\mu)$ in $L(\lambda)$.

A.6. The map pr. Let $\mathfrak{g}$ be $\mathfrak{gl}(m|n)$ or $\mathfrak{osp}(M|2n)$. Fix an odd root β_0 of the form $\beta_0 = \pm(\varepsilon_p - \delta_q)$. Let e^μ, $\mu \in \mathfrak{h}^*$ be a basis of the group algebra $\mathbb{C}[\mathfrak{h}^*]$. Consider a projection pr: $\mathbb{C}[\mathfrak{h}^*] \to \mathbb{C}[\mathfrak{h}^*]$ given by

$$\mathrm{pr}(e^{a\varepsilon_p}) := 1, \quad \mathrm{pr}(e^{a\delta_q}) := e^{i\pi a}, \quad \mathrm{pr}(e^{a\varepsilon_t}) := e^{a\varepsilon_t}, \quad \mathrm{pr}(e^{a\delta_j}) := e^{a\delta_j}$$

for any $a \in \mathbb{C}$ and the indices $t \neq p$, $j \neq q$. Note that pr is an algebra homomorphism and $\mathrm{pr}(e^{\beta_0}) = -1$. We extend pr to the rational functions of the form P/Q, where $P, Q \in \mathbb{C}[\mathfrak{h}^*]$ are such that $\mathrm{pr}(Q) \neq 0$.

Since pr is an algebra homomorphism for each $\lambda \in \mathfrak{h}^*$, one has

$$\mathrm{pr}\big(\mathrm{KW}(\lambda, \emptyset)(1+e^{-\beta_0})\big) = 0. \tag{A.4}$$

A.6.1. Take a non-zero vector $x \in \mathfrak{g}_{\beta_0}$. Identify $\mathfrak{g}' := \mathrm{DS}_x(\mathfrak{g})$ with the subalgebra of $\mathfrak{g}$ (recall $\mathfrak{g}' = \mathfrak{gl}(m-1|n-1)$ for $\mathfrak{g} = \mathfrak{gl}(m|n)$, $\mathfrak{g}' = \mathfrak{osp}(M-2|2n-2)$ for $\mathfrak{g} = \mathfrak{osp}(M|2n)$). Recall that $\mathfrak{h}' = \mathfrak{g}' \cap \mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$.

Observe that

$$\mathrm{pr}(e^\lambda) = c_\lambda e^{\lambda|_{\mathfrak{h}'}} \quad \text{for } c_\lambda := e^{-\pi i(\lambda|\delta_q)}$$

and that the restriction of $\pi\,\mathrm{pr}\,\pi$ to the supercharacter ring $\mathrm{J}(\mathfrak{g})$ is equal to ds_x (see [8]).

A.6.2. Set

$$\xi := \begin{cases} \frac{1}{2}\left(\sum_{i=1}^m \varepsilon_i - \sum_{i=1}^n \delta_i\right) & \text{for } \mathfrak{osp}(2m+1|2n), \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\xi|_{\mathfrak{h}'}$ is equal to the vector ξ defined for $\mathfrak{g}'$; we denote this vector by ξ'.

We assume that an iso-set S and $\lambda \in \mathfrak{h}^*$ satisfy

$$\beta_0 \in S \subset (S_m \times S_n)\beta_0, \quad (\lambda - \xi | \mathfrak{h}(S)^*) = 0 \tag{A.5}$$

and set

$$S' := S \setminus \{\beta_0\}, \quad \lambda' := \lambda|_{\mathfrak{h}'}.$$

By above, $\mathrm{pr}(e^\lambda) = e^{-\pi i(\lambda|\delta_q)} e^{\lambda'}$; observe that $(\lambda|\delta_q) = 0$ and $(\lambda|\delta_q) = -\frac{1}{2}$ for $\mathfrak{osp}(2m|2n)$ and $\mathfrak{osp}(2m+1|2n)$, respectively.

A.6.3. Proposition. *Let S, λ be as in* (A.5)*. For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ with $m > 1$ and $|S| = 1$, one has*

$$\mathrm{pr}\big(\mathrm{KW}(\lambda, S)(1 + e^{-\beta_0})\big) = \mathrm{KW}(\lambda', \emptyset) + \mathrm{KW}((\lambda')^\sigma, \emptyset).$$

For other cases

$$\mathrm{pr}\big(\mathrm{KW}(\lambda, S)(1 + e^{-\beta_0})\big) = a e^{-\pi i(\lambda|\delta_q)} \mathrm{KW}(\lambda', S'),$$

where $a = |S|$ for $\mathfrak{g} = \mathfrak{osp}(2|2n), \mathfrak{gl}(m|n)$ and $a = 2|S|$ for $\mathfrak{osp}(2m+1|2n), \mathfrak{osp}(2m|2n)$ with $m, |S| > 1$.

Proof. Denote by W' the Weyl group of $\mathfrak{g}'$ and notice that $W' = \mathrm{Stab}_W \beta_0$. Set $s := |S|$ and $c := e^{-\pi i(\lambda|\delta_q)}$. One has

$$\mathrm{pr}\big(\mathrm{KW}(\lambda, S)(1 + e^{-\beta_0})\big) = \sum_{w \in W} \mathrm{sgn}(w) y(w),$$

where

$$y(w) := \mathrm{pr}\left(\frac{e^{w\lambda}(1 + e^{-\beta_0})}{\prod_{\beta \in S}(1 + e^{-w\beta})}\right).$$

Observe that $\mathrm{pr}(1 + e^\alpha) = 0$ for $\alpha \in \Delta$ is equivalent to $\alpha = \pm\beta_0$. Since pr is an algebra homomorphism, this gives $y(w) = 0$ if $\pm\beta_0 \notin wS$. Therefore,

$$\mathrm{pr}\big(\mathrm{KW}(\lambda, S)(1 + e^{-\beta_0})\big) = Y_+ + Y_-,$$

where

$$Y_\pm := \sum_{w \in W : \pm\beta_0 \in wS} \mathrm{sgn}(w) y(w).$$

Each $\beta \in S'$ can be written as $\beta_0 = w_\beta \beta$ for $w_\beta := r_{\varepsilon_i - \varepsilon_p} r_{\delta_j - \delta_q}$. Setting $w_{\beta_0} := \mathrm{Id}$, we have $\mathrm{sgn}(w_\beta) = 1$ and

$$w_\beta \lambda = \lambda, \quad w_\beta \beta_0 = \beta, \quad w_\beta \beta = \beta_0, \quad w_\beta(\beta') = \beta' \quad \text{for } \beta' \in S \setminus \{\beta, \beta_0\},$$

for each $\beta \in S$. The operator pr commutes with the action of w' for $w' \in W'$. This gives

$$y(w' w_\beta) = \frac{\mathrm{pr}(e^{w' w_\beta \lambda})}{\prod_{\beta' \in S'}(1 + e^{-w'\beta'})} = w'\left(\frac{\mathrm{pr}(e^\lambda)}{\prod_{\beta' \in S'}(1 + e^{-\beta'})}\right) \quad \text{for any } w' \in W'.$$

Since $W' = \mathrm{Stab}_W \beta_0$, one has $\{w \in W \mid \beta_0 \in WS\} = W' w_\beta$.

By Section A.6.2, $\operatorname{pr}(e^{\lambda}) = ce^{\lambda'}$. Summarizing, we obtain

$$Y_+ = \sum_{\beta\in S}\sum_{w'\in W'} \operatorname{sgn}(w')y(w'w_\beta) = cs\,\mathrm{J}_{W'}\Big(\frac{e^{\lambda}}{\prod_{\beta'\in S'}(1+e^{-\beta'})}\Big),$$

that is $Y_+ = cs\,\mathrm{KW}(\lambda'; S')$.

For $\mathfrak{g} = \mathfrak{gl}(m|n), \mathfrak{osp}(2|2n)$, the set WS does not contain $-\beta_0$, so $Y_- = 0$; this completes the proof for these cases.

For the remaining cases $\mathfrak{g} = \mathfrak{osp}(M|N)$ with $M > 2$, the set WS contains $-\beta_0$. For $\mathfrak{g} = \mathfrak{osp}(2m|2n)$ with $s > 1$, we fix $\beta_1 := \pm(\varepsilon_i - \delta_j) \in S'$; for $\mathfrak{osp}(2m|2n)$ with $m > 1$ and $s = 1$, we set $i := m$ if $p \neq m$ and $i := m-1$ if $p = m$. We set

$$w_- := \begin{cases} r_{\varepsilon_p} r_{\delta_q} & \text{for } \mathfrak{g} = \mathfrak{osp}(2m+1|2n),\\ r_{\varepsilon_i} r_{\delta_j} r_{\varepsilon_p} r_{\delta_q} & \text{for } \mathfrak{g} = \mathfrak{osp}(2m|2n),\ s > 1,\\ r_{\varepsilon_i} r_{\varepsilon_p} r_{\delta_q} & \text{for } \mathfrak{g} = \mathfrak{osp}(2m|2n),\ s = 1.\end{cases}$$

Notice that $w_- \in W$ and $w_-\beta_0 = -\beta_0$. Therefore,

$$\{w \in W \mid -\beta_0 \in wS\} = \coprod_{\beta\in S} W'w_-w_\beta,$$

and thus

$$Y_- = \sum_{\beta\in S}\sum_{w'\in W'} \operatorname{sgn}(w'w_-)y(w'w_-w_\beta).$$

For $w' \in W'$, we have

$$y(w'w_-w_\beta) = \operatorname{pr}\Big(\frac{e^{w'w_-\lambda}(1+e^{-\beta_0})}{\prod_{\beta\in S}(1+e^{-w'w_-\beta})}\Big) = -w'\operatorname{pr}\Big(\frac{e^{w_-\lambda}}{\prod_{\beta\in S'}(1+e^{-w_-\beta})}\Big).$$

Therefore,

$$Y_- = -s\operatorname{sgn}(w_-)\,\mathrm{J}_{W'}\Big(\operatorname{pr}\Big(\frac{e^{w_-\lambda}}{\prod_{\beta\in S'}(1+e^{-w_-\beta})}\Big)\Big).$$

For $\mathfrak{osp}(2m+1|2n)$, one has $w_-S' = S'$ and $w_-\lambda = \lambda + \beta_0$, that is $\operatorname{pr}(e^{w_-\lambda}) = -ce^{\lambda'}$. Therefore, $Y_- = cs\,\mathrm{KW}(\lambda', S')$ as required.

For $\mathfrak{osp}(2m|2n)$ with $s > 1$, one has

$$w_-\lambda = \lambda, \quad w_-S' = (S' \cup \{-\beta_1\}) \setminus \{\beta_1\}.$$

Using Lemma A.4.4, we get

$$Y_- = -s\,\mathrm{KW}(\lambda', S' \cup \{-\beta_1\}) \setminus \{\beta_1\} = s|\mathrm{KW}(\lambda', S').$$

For the remaining case $\mathfrak{osp}(2m|2n)$ with $m > 1$ and $S = \{\beta_0\}$, we have

$$\operatorname{sgn}(w_-) = -1, \quad \operatorname{pr}(e^{w_-\lambda}) = e^{r_{\varepsilon_j}\lambda'},$$

that is $Y_- = \mathrm{KW}(r_{\varepsilon_i}\lambda', \emptyset)$. Since $r_{\varepsilon_i}\lambda' = (\lambda')^{\sigma}$, this completes the proof. □

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