Lie Algebras, their Representations and Crystals

by

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Part I

1. Historical Prelude and Outline

1.1. Representation theory developed from harmonic analysis itself the study of the harmonics in the vibration of a harpsicord, violin or piano string. A well-played note is a harmonic or a particularly desirable combination of harmonics. An unpleasant screech is an uncontrolled and so rather arbitrary combination of a large number of harmonics. In mathematical terms the combination of all possible harmonics forms an infinite dimensional vector space in which the harmonics define "irreducible" subspaces.

1.2. Group theory developed from symmetry. A perfectly symmetric object such as a ball will roll smoothly on a plane as its symmetry implies that practically no energy is needed to displace it. This is a "first-order" consequence of symmetry easily understandable to the layman. A much more subtle question is to describe the possible vibrations of a sphere. As for the violin string a decomposition into harmonics is possible. However a new phenomenon arises which is difficult to explain in simple terms. Namely the harmonics do not themselves define irreducible subspaces but come in packets each forming the basis of a finite dimensional irreducible subspace of the full harmonic space. The reason lies in the greater complexity of composition in the possible symmetry transformations of a sphere compared to that of the straight line which a violin string describes. The study of such harmonics forms the basis of group representation theory.

1.3. The symmetry transformations of a line or a sphere compared to that of say a cube have the pecularity that one can make them arbitrarily small. They form a particular class of groups called continuous or Lie groups, after Sophus Lie who first studied them. Their rather complicated multiplication rules may be replaced by simpler formulae which describe the rules of composition of infinitesimal transformations obtained from them by the methods of calculus. The latter form a rather special type of algebra called a Lie algebra. To a large extent one may study the representations of a Lie group through the representations of its Lie algebra.

1.4. The theory of Lie algebras and their representation theory acquires a life of its own since one may consider a Lie algebra not necessarily coming from a Lie group, or again the representation of a Lie algebra not necessarily coming from that of the Lie group. An important example are the Kac-Moody algebras together with their rather specific highest weight representations. A whole range of techniques were developed for their study.

1.5. Group theory involves just one composition law namely multiplication in the group. As such its study is of an elementary nature. Typically problems such as determining the groups of a given order can be analyzed by combinatorial techniques. A group representation space is a vector space equipped with the linear action of a group. It involves a subtle combination of two laws of composition, namely multiplication in the group and addition in the vector space. In this it is a natural extension of arithmetic and indeed number theory was an important source of group theory, whilst group representation theory now provides deep results in number theory.

1.6. Conventional wisdom ruled that combinatorics could not play a useful role in group representation theory because of the enormous complexity in handling simultaneously two laws of composition. Crystal theory which admittedly only applies in a highly structured framework has led to a dramatic revision of this point of view. Several important problems in the representation theory of Kac-Moody algebras are resolved through a combinatorial analysis involving crystals.

1.7. The path from representation theory to crystals together with the globalization techniques necessary to lift combinatorial results on crystals to describe the behaviour of the representation space, is a long and arduous one. Here we shall just confine ourselves to some first glimpses of this remarkable theory. First of all it is necessary to describe and understand the structure of Kac-Moody algebras and their representation theory.

1.8. A Kac-Moody algebra is a special type of graded Lie algebra. Notably it is entirely determined by its Cartan datum which is nothing more than an $n \times n$ matrix with particular integer entries. Certain choices of these matrices lead to finite dimensional Lie algebras. These are exactly the ones classified by Elie Cartan at the beginning of the twentieth century. It is quite miraculous that this particular choice of Cartan datum appears in a number of other quite distinct circumstances, for example in the description of Kleinian singularities or in the unraveling of knots or braids. One may surmise that they must have a very deep or even magical character.

1.9. From a Cartan datum one may extract a root system which involves in particular the Weyl group. This is a finitely generated group and even finite if the Kac-Moody algebra is finite dimensional. Thus it may be viewed as an essentially combinatorial object. A particular example is the symmetric or permutation group on n letters. A further example is the braid group which is infinite.

1.10. A root system and its associated Weyl group also admits a geometric interpretation in terms of tiling or tessilation. This was of particular inspiration to the graphic artist M.C. Eschler. The resulting geometric structure implements a significant simplification in the combinatorial structure of Weyl groups and specifically makes their word problem readily solvable at least in the finite or affine case. We shall further examine the Bruhat order which arose as an inclusion order of certain algebraic varieties parameterized by the Weyl group. We shall be concerned only with its purely combinatorial description though one should appreciate that this is a rather deep construction which has far reaching implications for the representation theory of Kac-Moody algebras.

1.11. Crystals are obtained by taking a $q \to 0$ limit. This has the physical interpretation of taking the temperature to zero and “freezing” the representation space into a particularly simple form. The q parameter itself is introduced by modifying the calculus which led to the Lie algebra in the first place. To understand this, recall that in ordinary calculus differentiation ∂ satisfies the Leibnitz rule

$$\partial(fg) = (\partial f)g + f(\partial g)$$

with respect to appropriate functions f, g. An easy exercise is to show that $\partial x = 1$ implies that $\partial x^n = nx^{n-1}$, for all positive integer n. Skew differentiation involves an automorphism τ of the underlying algebra of functions. Then the Leibnitz rule is modified to give

$$\partial(fg) = (\partial f)g + \tau(f)\partial g.$$

Now if we take $\partial x = 1$ and $\tau(x) = q^2 x$, then an easy induction argument gives

$$\partial x^n = q^{n-1}[n]_q x^{n-1},$$

where $[n]_q$ is the Gaussian integer defined by

$$[n]_q = \frac{(q^n - q^{-n})}{(q - q^{-1})} \ .$$

This already has some amusing features. For example a Gaussian integer may vanish at root of unity and in this manner the theory of skew differentiation mimics that of ordinary differentiation in positive characteristic.

1.12. Taking a $q \to 0$ limit in an algebraic context is a sophisticated process with many traps for the unwary. It involves a choice of lattice and moreover the resulting limit depends on this choice. To be of any use the lattice must be stable under the (modified) Lie algebraic operations. This is a very significant constraint which may be enormously difficult to implement. The result is a crystal. As a combinatorial object it is a particular form of graph. In this the vertices of the graph are the basis elements of the "frozen" representation space and the edges recover the Lie algebraic operations. To be of any use we must be able to obtain a global basis of the original representation space by "defrosting".

1.13. Rest assured that this course will not enter into any more details concerning the $q \to 0$ limit. It suffices to know that the theory (due to M. Kashiwara) exists. Incidentally it is quite elementary but perhaps the most complex induction reasoning in the whole of Mathematics. Rather we shall take the Royal Route to crystals which can be traced with only reference to the Cartan datum. Then we shall describe the Kashiwara model of a crystal which is based on tensor product and the Littelmann path model of a crystal which is based on the structure of Bruhat order. A high point in these lectures will be the proof that the Littelmann and Kashiwara crystals coincide. However the former admits a higher level of generality as it applies also to the "non-symmetrizable" case.

1.14. In order to better appreciate crystals and to understand that they are indeed the combinatorial manifestation of a representation space, we shall first describe the classical representation theory of Kac-Moody algebras. This had its origins in the work of I. Schur and H. Weyl. In particular we shall do just enough to prove the Weyl character formula. It is one of the most remarkable results in mathematics. Indeed in the hands of V. Kac who generalized it from the finite dimensional case it recovers as special cases identities from number theory going back to Fermat, Euler, Gauss and Jacobi. We shall describe the Demazure character formula whose correct proof and extension to symmetrizable Kac-Moody algebras was a particular success of crystal theory. Then we shall switch to the crystal theory which will culminate in deriving combinatorial versions of these character formulae. Hopefully the student will not only learn some basic Lie algebra theory but will also be guided quickly to the cutting edge of contemporary research.

2. Lie Algebras and their Representations

2.1. A Lie algebra $\mathfrak{g}$ is a vector space over a field k equipped with a bilinear map $(x, y) \mapsto [x, y]$ of $\mathfrak{g} \times \mathfrak{g}$ into $\mathfrak{g}$ satisfying

$$[x, y] + [y, x] = 0, \ \forall \ x, y \in \mathfrak{g}, \tag{1}$$

$$[[x, y], z] + [[y, z], x] + [[z, x], y] = 0, \ \forall \ x, y, z \in \mathfrak{g}. \tag{2}$$

One calls $[\ ,\ ]$ the Lie product or Lie bracket. By (1) a Lie algebra can only be commutative (outside characteristic 2) if all Lie products are zero. One calls (2) the

Jacobi identity. It means that a Lie algebra fails to be associative; but this failure is not too serious.

2.2. As for an associative algebra, a homomorphism of Lie algebras is a linear map preserving multiplication, in this case defined by the Lie product. Similarly the notions of subalgebra, quotient algebra and ideals are defined. By (1) an ideal $\mathfrak{i}$ a Lie algebra $\mathfrak{g}$ is necessarily two-sided and one may form the quotient algebra $\mathfrak{g}/\mathfrak{i}$. A Lie algebra is called simple, if it admits no proper ideals.

2.3. Let A be an associative algebra. One calls Lie A the vector space with Lie product

$$[x, y] = xy - yx, \ \forall \ x, y \in A.$$

One calls the right hand side above the commutator product. Here it is immediate that (1) (of 2.1) holds and one easily checks (2). Thus Lie A is a Lie algebra.

As we shall see (5.5) every Lie algebra occurs as a subalgebra of some Lie A with A associative. It is hence legitimate to ask why Lie algebras are interesting. In fact a dramatic change takes place on passing from A to Lie A. This can be partly understood through the notion of a derivation.

Let B be an algebra (not necessarily) associative. A derivation ∂ of B is a linear map satisfying the Leibnitz rule

$$\partial(bb') = (\partial b)b' + b(\partial b').$$

The product $\partial\partial'$ of derivations ∂, ∂' of B defined by composition (that is $(\partial\partial')(b) = \partial(\partial' b)$) is still a linear map B to itself; but it is *not* a derivation of B. However one may easily check that their commutator product $\partial\partial' - \partial'\partial$ is again a derivation. Thus the set Der B of all derivations of B is a Lie algebra for the commutator product.

Let $\mathfrak{g}$ be a Lie algebra and $x \in \mathfrak{g}$. Define a linear map $ad\ x : y \mapsto [x, y]$ on $\mathfrak{g}$. Using in particular the Jacobi identity one checks that $ad\ x$ is a derivation of $\mathfrak{g}$.

2.4. Let V be a vector space over k. One denotes by $\mathfrak{gl}(V)$ the Lie algebra Lie End V. If V is finite dimensional, $\mathfrak{gl}(V)$ admits a Lie subalgebra

$$\mathfrak{sl}(V) = \{x \in \mathfrak{gl}(V) \Big| \, tr\ x = 0\}.$$

Then if $char\ k = 0$, one checks that $\mathfrak{sl}(V)$ is a simple Lie algebra of dimension $(dim\ V)^2 - 1$. It is an example to which one may refer in discussing any finite dimensional simple Lie algebra.

2.5. Let $\mathfrak{g}$ be a Lie algebra. A representation of $\mathfrak{g}$ is a vector space V together with a Lie algebra homomorphism θ of $\mathfrak{g}$ into Lie End V. This means that the map $(x, v) \mapsto \theta(x)v,\ x \in \mathfrak{g}, v \in V$ is a bilinear map of $\mathfrak{g} \times V$ into V satisfying

$$([x, y], v) = (x, (y, v)) - (y, (x, v)),\ \forall\ x, y \in \mathfrak{g}, v \in V.$$

Omitting θ we may write the image of (x, v) simply as xv. This rule (and bilinearity) is said to define an action of $\mathfrak{g}$ on V. A vector space V admitting such a $\mathfrak{g}$ action is called a $\mathfrak{g}$ module.

Suppose $n := dim\ V < \infty$ and let $x_1, x_2, \cdots, x_n$ be a basis for V. Then the action of $\mathfrak{g}$ on V extends by the Leibnitz rule to an action of $\mathfrak{g}$ on the polynomial algebra $S(V) = k[x_1, x_2, \cdots, x_n]$. In this the subspace $S_m(V)$ of homogeneous polynomials of degree m is stable under this action and so again is a $\mathfrak{g}$ module. These $\mathfrak{g}$ modules are all very different to the original one, namely V. This construction is quite impossible for *End* V itself and illustrates that $\mathfrak{gl}(V)$ has a very much richer representation theory then End V.

The above construction is also of great importance in physics. For example one may intepret the basis elements $x_1, x_2, \cdots, x_n$ as elementary particles and a given monomial as a composite of the corresponding elementary particles.

2.6. Let $\mathfrak{g}$ be a Lie algebra. A homomorphism of $\mathfrak{g}$ modules is a linear map preserving the $\mathfrak{g}$ action. Similarly the notions of a submodule and quotient module are defined.

A $\mathfrak{g}$ module V is said to be simple if it admits no proper submodules. For example if $dim\ V < \infty$ and $char\ k = 0$, then $S_m(V)$ is a simple $\mathfrak{gl}(V)$ module for every positive integer m, though this is not completely obvious. One may also refer to a simple module as defining an irreducible representation. These describe the (fundamental) harmonics of an infinite string when $\mathfrak{g}$ is one-dimensional.

2.7. Let $\mathfrak{g}$ be a Lie algebra. One checks using in particular the Jacobi identity, that the map $x \mapsto ad\ x$ is a homomorphism of $\mathfrak{g}$ into Lie End $\mathfrak{g}$. It is called the adjoint representation of $\mathfrak{g}$. The corresponding action of $\mathfrak{g}$ on itself is called the adjoint action. It is the analogue of the conjugacy action of a group on itself.

2.8. The basic questions in Lie algebra theory are to classify all Lie algebras and all Lie algebra modules, up to isomorphism. This task is quite hopeless unless some finite dimensionality is involved, for example that the Lie algebra or the module is finite dimensional. Here we study a wide class of Lie algebras and their modules which can be naturally broken into finite dimensional pieces. In this theory a crystal gives a natural way of both parameterizing basis elements and describing the action.

3. Classification of the finite dimensional $\mathfrak{sl}(2)$ modules

3.1. In this section and mainly from now on we shall assume that the base field k is algebraically closed of characteristic zero.

3.2. The space of trace zero 2×2 matrices has basis

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

The commutator products between these elements are easily computed to be

$$[h, e] = 2e,\ \ [h, f] = -2f,\ \ [e, f] = h.$$

3.3. Notice that e (resp. f) is an $ad\ h$ eigenvector of eigenvalue 2 (resp. -2). This leads to a particular simplification in the study of any $\mathfrak{sl}(2)$ module V which admits a decomposition into h eigenspaces. An exercise in linear algebra using the non-vanishing of the Van der Mond determinant shows that any such decomposition is automatically direct. Then we can write

$$V = \bigoplus_{c \in k} V_c$$

where

$$V_c = \{v \in V \bigm| hv = cv\}.$$

Such a module is called an h weight (or simply, a weight) module. If $V_c \neq 0$, we call c a weight of V. One calls *dim* V_c the multiplicity of the weight c. If $v \in V_c$, then $hev = (he - eh)v + ehv = 2ev + ecv = (2+c)v$. Thus $eV_c \subset V_{c+2}$ and similarly $fV_c \subset V_{c-2}$. In particular we conclude that

$$(*) \qquad \bigoplus_{n \in \mathbb{Z}} V_{c+2n} \quad \textit{is a submodule of} \quad V.$$

Thus any weight module for $\mathfrak{sl}(2)$ is a direct sum of submodules in which its weights differ by only even integers.

3.4. Let V be a $\mathfrak{sl}(2)$ module. Then e, f are endomorphisms of V and hence so are their powers. Moreover as endomorphisms of V we have the following identity for all positive integer n

$$\begin{aligned} ef^{n+1} - f^{n+1}e &= \sum_{i=0}^{n} f^i h f^{n-i} \\ &= f^n \sum_{i=0}^{n} (h - 2(n-i)) = (n+1)f^n(h-n). \end{aligned}$$

Lemma. *Let V be a simple finite dimensional $\mathfrak{sl}(2)$ module. Then V is a weight module, its weights lie in $\mathbb{Z}$ and have multiplicity 1.*

Proof. Since V is finite dimensional and k is algebraically closed, there exists $c \in k, v \in V \setminus \{0\}$, such that $hv = cv$. Then $V_c \neq 0$ and the simplicity of V implies it to be a direct sum of the form given in 3.3(∗). Since V is finite dimensional we must be able to find $n \in \mathbb{Z}$

such that $V_{c+2n} \neq 0$ and $V_{c+2n+2} = 0$. Then $w \in V_{c+2n} \setminus \{0\}$ satisfies $ew = 0$. Since the $f^i w \in V_{c+2(n-i)} = i \in \mathbb{N}$, lie in distinct direct summands, there is a least positive integer m such that $f^m w = 0$.

Through the identity preceding the lemma, it follows that $w, fw, \cdots, f^{m-1}w$ is a basis for a submodule of V and hence of V itself. Thus V has weights $c+2n, c+2n-2, \cdots, c+2(n-m)+2$ and these occur with multiplcity 1. Finally $tr\ h = tr(ef - fe) = 0$, on V and this forces $mc + m(2n-m+1) = 0$ and so $c + 2n = m - 1$. In particular $c \in \mathbb{Z}$. $\square$

Remark. We call $c+2n$ the highest weight of V. Observe that there is a relation between the highest weight of V (namely $m-1$) and the dimension of V (namely m).

3.5. The above results suggest that for each positive integer m there exists a unique simple $\mathfrak{sl}(2)$ module of dimension m. Both uniqueness and existence derives from the general theory of highest weight module which we postpone till Section 8. A quick proof of existence derives from the construction of 2.5. Consider the column vectors $v = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. One has $ev = 0, hv = v, fv = w, ew = v, hw = -w, fw = 0$. Thus the space of column vectors forms a simple $\mathfrak{sl}(2)$ module of dimension 2 and highest weight 1. The space of homogeneous polynomials in v, w of degree $m-1$ on which e, h, f act by derivations, forms the required simple module of dimension m. We denote this module by $V(m-1)$.

3.6. The classification of finite dimensional $\mathfrak{sl}(2)$ modules is completed by showing that every such module is a direct sum of simple modules. This is a rather remarkable result though it has an analogue for finite groups (Maschke's lemma). Indeed the first proof due to H. Weyl involved show that any such module could be viewed as a (continuous) representation of a compact group. Via integration using the Haar measure it could then be viewed as a unitary representation. Finally such representations admit direct sum decomposition into irreducible components. Algebraists will be relieved to learn that one may also give a purely algebraic proof. This is indicated below.

3.7. Let V be a $\mathfrak{sl}(2)$ module. Any product in e, h, f makes sense as an endomorphism of V. Consider in particular

$$z := 2(ef + fe) + h^2 = h(h+2) + 4fe.$$

One checks that z commutes with e, h, f. On the other hand on a highest weight vector w of highest weight $m-1$ (that is satisfying $ew = 0, hw = (m-1)w$), z acts by $m^2 - 1$. Consequently z acts on $V(m-1)$ by this scalar multiple of the identity. Observe that these scalars are distinct for distinct positive integer m.

Let V be a finite dimensional $\mathfrak{sl}(2)$ module. If V is not itself simple, it must admit a proper submodule W of minimal dimension. Necessarily W is simple. Repeating this argument on the quotient module V/W which has dimension $dim\ V - dim\ W < dim\ V$, it follows that V admits a decreasing sequence $V = V_r \supsetneq V_{r-1} \supsetneq \cdots \supsetneq V_0 = 0$ of submodules with V_s/V_{s-1} simple for all $s = 1, 2, \cdots, r$. One calls this sequence a composition series for V and the resulting quotients its composition factors. Since every simple finite dimensional $\mathfrak{sl}(2)$ is some $V(m-1) : m \in \mathbb{N}^+$, so are all the composition factors of V. For each $m \in \mathbb{N}^+$, we say that V is a multiple of $V(m-1)$ if every composition factor of V is isomorphic to $V(m-1)$.

Lemma. *Let V be a finite dimensional $\mathfrak{sl}(2)$ module. Then V is a direct sum of submodules each of which is a multiple of some $V(m-1) : m \in \mathbb{N}^+$.*

Proof. This follows by the Chinese remainder theorem applied to the fact that z acts by a distinct scalar on the distinct $V(m-1) : m \in \mathbb{N}^+$. □

3.8. It remains to show that a multiple V of $V(m-1)$ is in fact a direct sum of simple modules (each isomorphic to $V(m-1)$). Since V is a finite dimensional module, it follows by the Chinese remainder theorem that V is a direct sum of its generalized h eigenspaces V^c, where by definition

$$V^c = \{v \in V \,\Big|\, (h-c)^\ell v = 0, \quad for\ some\ \ \ell > 0\}.$$

Obviously $V^c \supset V_c$ and one checks from the identities in 3.2 that $eV^c \subset V^{c+2}$, $fV^c \subset V^{c-2}$.

Since V is a multiple of $V(m-1)$ it follows that $eV^{m-1} = 0$ and $f^m V^{m-1} = 0$. Then taking $n = m-1$ in 3.4 we obtain

$$mf^{m-1}(h-(m-1))V^{m-1} = 0.$$

Yet f^{m-1} induces an isomorphism of $V(m-1)_{m-1}$ onto $V(m-1)_{(m-1)}$ and so it induces an isomorphism of V^{m-1} onto $V^{-(m-1)}$. Consequently the above identity implies that $(h-(m-1))V^{m-1} = 0$, that is $V^{m-1} = V_{m-1}$.

Finally each element of a basis of V_{m-1} generates a copy of $V(m-1)$ and these form a direct sum which is furthermore equal to V since $V^{m-1} = V_{m-1}$.

We have thus proved the

Theorem. *Every finite dimensional $\mathfrak{sl}(2)$ module is a direct sum of simple modules.*

Remark. This is quite false in positive characteristic.

4. Graded Lie algebras, Cartan datum and Kac-Moody algebras

4.1. Let A be an algebra (not necessarily associative). A $\mathbb{Z}$ gradation $\mathcal{G}$ on A is a family of subspaces $\{\mathcal{G}_i A\}_{i\in\mathbb{Z}}$ such that

$$1)\ \ A = \bigoplus_{i\in\mathbb{Z}} \mathcal{G}_i A, \qquad 2)\ \ (\mathcal{G}_i A)(\mathcal{G}_j A) \subset \mathcal{G}_{i+j} A,\ \forall\ i,j \in \mathbb{Z}.$$

For example a polynomial algebra A is graded by degree, that is to say we take $\mathcal{G}_i A$ to be the subspace spanned by homogeneous polynomials of degree i. In this case $\mathcal{G}_i A = 0$, for $i < 0$.

A grading is useful if the graded subspaces are finite dimensional. One calls $a \in \mathcal{G}_i A$ a homogeneous element of degree i.

4.2. A Lie algebra $\mathfrak{g}$ with a grading $\mathcal{G}$ is called a graded Lie algebra. Here we shall simply denote $\mathcal{G}_i\mathfrak{g}$ by $\mathfrak{g}_i$. For example consider the polynomial algebra $A = k[x_1, x_2, \cdots, x_n]$ on n variables. Obviously each $\partial/\partial x_i$ acts on A by derivations and one may check that

$$Der\ A = \sum_{i=1}^{n} A \frac{\partial}{\partial x_i} \ .$$

It is thus a Lie algebra for the commutator product. It can be graded by taking the $a\partial/\partial x_i : i = 1, 2, \cdots, n$ to have degree $m-1$ for any homogeneous polynomial a of degree m.

4.3. Let $\mathfrak{g} = \oplus \mathfrak{g}_i$ be a graded Lie algebra. Observe that $\mathfrak{g}_0$ is a Lie algebra of $\mathfrak{g}$ and that each $\mathfrak{g}_i$ is a $\mathfrak{g}_0$ module. The representation theory (of $\mathfrak{g}_0$) thus helps in understanding the structure of $\mathfrak{g}$. In the example of 4.2 one may note that $\mathfrak{g}_0$ is isomorphic to $\mathfrak{gl}(n)$. If $n = 2$, we already obtain from 3.8, that each $\mathfrak{g}_i$ is a direct sum of simple modules under the action of $\mathfrak{sl}(2)$ submodules and it is not difficult to determine which simple modules occur.

4.4. Let $\mathfrak{g} = \oplus \mathfrak{g}_i$ be a graded Lie algebra. Then

$$\mathfrak{g}^+ := \bigoplus_{i>0} \mathfrak{g}_i, \ \mathfrak{g}^- := \bigoplus_{i<0} \mathfrak{g}_i$$

are subalgebras $\mathfrak{g}$ and we have

$$\mathfrak{g} = \mathfrak{g}^+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}^-$$

which is called a triangular decomposition of $\mathfrak{g}$. Notice that $\mathfrak{g}^+, \mathfrak{g}^-$ are rather special Lie algebras which like the Cheshire cat start to disappear as one takes repeated commutators. This proper is called nilpotence.

The eagle-eyed reader will have already noticed that $\mathfrak{sl}(2)$ is a graded Lie algebra with triangular decomposition

$$\mathfrak{sl}(2) = ke \oplus kh \oplus kf.$$

Moreover we implicitly used triangular decomposition to analyze the structure of a finite dimensional module V. Thus we first consider the decomposition of V into $\mathfrak{g}_0$ modules and then considered how $\mathfrak{g}^+$ and $\mathfrak{g}^-$ translated components of the latter. This method can be adapted to any graded Lie algebra. It is most successful if we further assume that $\mathfrak{g}_0$ is commutative and that each $\mathfrak{g}_i$ is a direct sum of one dimensional $\mathfrak{g}_0$ submodules. In this case we write $\mathfrak{g}_0 = \mathfrak{h}$ and refer to it as the Cartan subalgebra of $\mathfrak{g}$. Then for each $\alpha \in \mathfrak{h}^*$ we may define

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,\Big|\, [h, x] = \alpha(h)x, \ \forall \ h \in \mathfrak{h}\}.$$

It is called the root subspace of $\mathfrak{g}$ of root α. By the above hypothesis each $\mathfrak{g}_i$ is a direct sum of the $\mathfrak{g}_\alpha$.

4.5. The above framework is still a little too general. We shall further assume that each $\mathfrak{g}_i : i \in \mathbb{Z}$ (and in particular $\mathfrak{h}$) is finite dimensional. Secondly we assume that $\mathfrak{g}^+$ (resp. $\mathfrak{g}^-$) is generated by $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$). It follows that $\mathfrak{g}$ is generated by its so-called local part, namely

$$\mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}.$$

For example, this is true of $Der \ k[x_1, x_2, \cdots, x_n]$ if $n \geq 2$.

Next we impose a much more subtle constraint. Observe that certain Lie products make sense entirely within the local part. That is $\mathfrak{g}_0$ is a Lie algebra, $\mathfrak{g}_1, \mathfrak{g}_{-1}$ are $\mathfrak{g}_0$ submodules. Finally the map θ of $(x, y) \mapsto [x, y]$ of $\mathfrak{g}_1 \times \mathfrak{g}_{-1}$ to $\mathfrak{g}_0$ is bilinear and satisfies

$$(ad \ h)\theta(x, y) = \theta((ad \ h)x, y) + \theta(x, (ad \ h)y), \ \forall \ h \in \mathfrak{g}_0.$$

We call such an object a local Lie algebra.

Now consider the set S of all graded Lie algebra with the same local part. One may show that S has a unique largest (more precisely, universal) element $\mathfrak{g}_{max}$ of which all other members are quotients. Moreover $\mathfrak{g}_{max}$ admits a unique maximal graded ideal $\mathfrak{r}$

having null intersection with its local part. Thus S admits a unique smallest element to which all other members project, namely $\mathfrak{g}_{min} := \mathfrak{g}_{max}/\mathfrak{r}$.

The construction of $\mathfrak{g}_{max}$ is not particularly instructive and will be omitted. What is important is the nature of $\mathfrak{r}$ and this is determined in a very subtle way by the structure of the local part and is very far from being understood.

4.6. A (symmetrizable) Kac-Moody algebra is obtained by imposing some further constraints which have in particular the advantage that $\mathfrak{r}$ above can be computed. The constraints are summarized by the Cartan datum.

4.7. A Cartan datum consists of a finite dimensional vector space $\mathfrak{h}$ together with a set π of linearly independent elements $\{\alpha_1, \alpha_2, \cdots, \alpha_\ell\}$ of $\mathfrak{h}^*$ and a set $\pi^\vee$ of linearly independent elements $\{\alpha_1^\vee, \alpha_2^\vee, \cdots, \alpha_\ell^\vee\}$ of $\mathfrak{h}$ such that $\alpha_i^\vee(\alpha_j) = 2$ if $i = j$, $\alpha_i^\vee(\alpha_j) \in -\mathbb{N}$ otherwise and $\alpha_i^\vee(\alpha_j) \neq 0 \iff \alpha_j^\vee(\alpha_i) \neq 0$. Set $I = \{1, 2, \cdots, \ell\}$. The matrix A with entries $a_{i,j} := \alpha_i^\vee(\alpha_j) : i, j = 2, \cdots, n$, is called the Cartan matrix. Conversely given any matrix A with entries $a_{i,j} : i, j \in I$ and one may define a vector space $\mathfrak{h}$ with basis $\pi^\vee := \{\alpha_i^\vee\}_{i \in I}$ and elements $\alpha_i \in \mathfrak{h}^* : i \in I$ such that $a_{i,j} = \alpha_i^\vee(\alpha_j)$, $\forall i, j \in I$. Then $\pi := \{\alpha_i\}_{i \in I}$ generates a subspace of $\mathfrak{h}^*$ of dimension $rk\ A$ and so these elements cannot be linearly independent unless $rk\ A = \ell$. However one can simply augment $\mathfrak{h}$ to a vector space of dimension $\ell + (\ell - rk\ A)$ to separate the $\{\alpha_i\}_{i \in I}$. The resulting triple $\{\mathfrak{h}, \pi^\vee, \pi\}$ is called a realization of A. From the point of view of the Lie algebra we construct all realizations are equivalent as will be quite easy to see.

One calls A symmetrizable if there exist strictly positive integers d_i such that the matrix $d_i a_{i,j}$ is symmetric. After Kac (see exercise sheet 4) this is equivalent to the existence of a scalar product $(\ ,\ )$ on $\mathfrak{h}^*$ such that $\alpha_i^\vee = 2\varphi(\alpha_i)/(\alpha_i, \alpha_i)$, where φ is the linear isomorphism $\xi \mapsto (\eta \mapsto (\xi, \eta))$ of $\mathfrak{h}^*$ onto $\mathfrak{h}$. In particular $\alpha_i^\vee(\alpha_j) = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. In this case φ will generally be omitted and $\mathfrak{h}, \mathfrak{h}^*$ identified.

4.8. Given a Cartan datum one defines a local Lie algebra by taking $\mathfrak{g}_0 = \mathfrak{h}$ considered as a commutative Lie algebra and taking $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$) to have basis e_i (resp. f_i) : $i \in I$, satisfying

$$(*) \qquad [h, e_i] = \alpha_i(h)e_i, \ [h, f_i] = -\alpha_i(h)f_i, \ [e_i, f_j] = \alpha_i^\vee \delta_{i,j}$$

where δ is the Kronecker delta.

A Kac-Moody algebra $\mathfrak{g}(A)$, or simply $\mathfrak{g}$, is the unique minimal Lie algebra $\mathfrak{g}_{min}$ with the above local part. It is called symmetrizable if A is symmetrizable.

4.9. There is a very substantial number of results on Kac-Moody algebras and their representations some of them extremely difficult. We shall steer a careful path through this maze just proving a few elementary though deep facts. One may remark that $\mathfrak{g}(A)$ is finite dimensional if and only if A is positive definite and if this holds $\mathfrak{g}$ is simple if and only if A is indecomposable, that is not of block diagonal form with more than one block. All finite dimensional simple Lie algebras arise in this fashion. If A is positive semidefinite, then $\mathfrak{g}(A)$ is said to be affine. Affine Lie algebras can be described in terms of finite dimensional ones. This was a key discovery of Kac and of Moody and underlies the importance of affine Lie algebras.

5. The tensor product and the enveloping algebra

5.1. Let V, W be vector spaces. The Cartesian product $V \times W$ is the set of all pairs $(v, w) : v \in V, w \in W$. Let $k(V \times W)$ denote the vector space with basis consisting of elements of $V \times W$. The tensor product $V \otimes W$ is the quotient of $k(V \times W)$ modulo the subspace of $k(V \times W)$ generated by the elements

$$(*) \quad \begin{aligned} &\alpha(v, w) - (\alpha v, w), \alpha(v, w) - (v, \alpha w), \ \forall\ \alpha \in k, v \in V, w \in W \\ &(v + v', w) - (v, w) - (v', w), (v, w + w') - (v, w) - (v, w'), \ \forall\ v, v' \in V, w, w' \in W. \end{aligned}$$

Let $(v \otimes w)$ denote the image of (v, w) in the quotient space $V \otimes W$. By construction one has

$$\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w, \ \forall \ \alpha \in k, v \in V, w \in W$$

$$(v + v') \otimes w = v \otimes w + v' \otimes w, v \otimes (w + w') = v \otimes w + v \otimes w', \ \forall \ v, v' \in V, w, w' \in W.$$

Let U be a third vector space. From a bilinear map of $V \times W \to U$ one obtains a linear map $V \otimes W \to U$. The tensor product is universal with respect to this property; but we shall not detain ourselves with these details. Suffice to emphasize that universality allows one to overcome difficulties inherent in there not being an unique presentation of elements of $V \otimes W$.

One may remark that if $\{v_i\}$ (resp. $\{w_j\}$) is a basis for V (resp. W) then $v_i \otimes w_j$ is a basis for $V \otimes W$. This introduces an important simplification since we may choose canonical representatives. However for some purposes we may want to replace k by a ring A and then such canonical representatives do not exist.

5.2. The tensor product is what we should expect to obtain if we could form the product of two vector spaces V, W. For example if A is an algebra and V, W are subspaces of A, then we can form the product VW in A being the linear combination of all elements of the form $vw : v \in V, w \in W$. The distributivity law in A together with the universality of tensor product gives a linear map of $V \otimes W$ into VW.

5.3. Let V be a vector space. The tensor algebra $T(V)$ is the vector space

$$T(V) := k \oplus V \oplus V \otimes V \oplus \cdots ,$$

with product given by $(u_1 \otimes u_2 \otimes \cdots \otimes u_m, v_1 \otimes v_2 \otimes \cdots \otimes v_n) \mapsto u_1 \otimes u_2 \otimes \cdots \otimes u_m \otimes v_1 \otimes v_2 \otimes \cdots \otimes v_n$. This product is clearly associative. If $\{v_i\}_{i \in \mathcal{I}}$ is a basis of V, then $T(V)$ is just the free associative algebra generated by the $v_i : i \in \mathcal{I}$.

The m^{th} fold tensor product of V is denoted simply by $V^{\otimes m}$.

5.4. Let A be an associative algebra and V a generating subspace of A. In the notation of 5.2 this just means that A is the sum of the $V^n : n \in \mathbb{N}$. Universality implies that the identity on V extend to an algebra map of $T(V)$ onto A. In simple terms this is just a consequence of our having ensured that there are no superfluous relations in $T(V)$.

5.5. Let $\mathfrak{g}$ be a Lie algebra. The enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is just the tensor algebra $T(\mathfrak{g})$ modulo the ideal generated by the elements $x \otimes y - y \otimes x - [x, y], \forall\, x, y \in \mathfrak{g}$. Its importance lies in its universality property. In particular given a $\mathfrak{g}$ module V the Lie algebra homomorphism of $\mathfrak{g}$ into Lie End V it defines, extends to an associative algebra homomorphism of $U(\mathfrak{g})$ into End V. This makes V a module for the associative algebra $U(\mathfrak{g})$. By construction the composed map $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$ is a Lie algebra homomorphism. By the PBW theorem (5.8) this map is injective and so makes $\mathfrak{g}$ a Lie subalgebra of Lie $U(\mathfrak{g})$. A disadvantage with the enveloping algebra is that it is always infinite dimensional. However it leads to some important constructions of $\mathfrak{g}$ modules. In particular $U(\mathfrak{g})$ is a left $U(\mathfrak{g})$ module for left multiplication and any simple $U(\mathfrak{g})$ module (and hence simple $\mathfrak{g}$ module) arises as a quotient of $U(\mathfrak{g})$. Of course this easy fact is also not very useful.

5.6. Let V be a vector space. The symmetric algebra $S(V)$ of V is just $U(V)$ when we view V as a commutative Lie algebra. If V is finite dimensional with basis $x_1, x_2, \cdots, x_n$ then $S(V)$ is just the polynomial algebra $k[x_1, x_2, \cdots, x_n]$.

5.7. Let A be an algebra (not necessarily associative). A $\mathbb{Z}$-filtration $\mathcal{F}$ on A is a collection of increasing subspaces $\{\mathcal{F}^i A\}_{i \in \mathbb{Z}}$ satisfying

$$1) \qquad \bigcup \mathcal{F}^i A = A,\ \bigcap \mathcal{F}^i A = 0, \qquad 2)\ \ (\mathcal{F}^i A)(\mathcal{F}^j A) \subset \mathcal{F}^{i+j} A.$$

Given a filtered algebra $(A, \mathcal{F})$ one may define its associated graded algebra $gr_{\mathcal{F}} A$ to be the direct sum $\oplus \mathcal{F}^i A \big/ \mathcal{F}^{i-1} A$ with multiplication induced by that in A. Indeed given $\bar{f} \in \mathcal{F}^i A / \mathcal{F}^{i-1} A, \bar{g} \in \mathcal{F}^j A / \mathcal{F}^{j-1} A$, we may define their product $\bar{f}\bar{g} \in \mathcal{F}^{i+j} A / \mathcal{F}^{i+j-1} A$, by choosing representatives $f \in \mathcal{F}^i A, g \in \mathcal{F}^j A$ and taking the image of fg in $\mathcal{F}^{i+j} A / \mathcal{F}^{i+j-1} A$. One may view $gr_{\mathcal{F}} A$ as an approximation of A.

5.8. Let $\mathfrak{g}$ be a Lie algebra. Its enveloping algebra admits a canonical filtration $\mathcal{F}$ defined by taking $\mathcal{F}^m(U(\mathfrak{g}))$ to be the image of $k \oplus \mathfrak{g} \oplus \mathfrak{g} \otimes \mathfrak{g} \oplus \cdots \oplus \mathfrak{g}^{\otimes m}$ in $U(\mathfrak{g})$. One of the first significant results in enveloping algebra theory is the Poincare-Birkhoff-Witt theorem which asserts that $gr_{\mathcal{F}}U(\mathfrak{g})$ is isomorphic as a graded algebra to $S(\mathfrak{g})$. The classical proof involved realizing $\mathfrak{g}$ as first order differential operators. A later proof valid also in positive characteristic used the diamond lemma. Finally there is a more abstract proof valid for a Lie algebra which is flat over a commutative ring. This uses deformation theory.

5.9. Assume *char* $k = 0$ and that V is finite dimensional. Recall that $\mathfrak{g} = \mathfrak{sl}(V)$ acts on $S_m(V)$ by derivations and that the resulting module is simple (see exercise sheet 2). Now consider the subspace S_m of $S_m(V)$ generated by the $v^m : v \in V$. One easily checks that S_m is a submodule of $S_m(V)$. For example if $m = 2$ one has

$$xv^2 = (xv)v + v(xv) = \frac{1}{2}((xv + v)^2 - (xv - v)^2).$$

Thus $S_m = S_m(V)$, by simplicity of the latter.

Let $x_1, x_2, \cdots, x_n$ be elements of $\mathfrak{g}$ and set $a = x_1 x_2 \cdots x_n$. Let G_n be the group of all permutations of n letters. The symmetrization $\sigma(a)$ of a is the element of $U(\mathfrak{g})$ given by

$$\sigma(a) = \frac{1}{n!} \sum_{g \in G_n} x_{g(1)} x_{g(2)} \cdots x_{g(n)}.$$

Now assume that the $x_i : i = 1, 2, \cdots, n$ are selected from a basis for $\mathfrak{g}$. Then all such monomials form a basis for $S(\mathfrak{g})$. Applying σ to these basis elements and extending by linearity gives a linear map $\sigma : S(\mathfrak{g}) \to U(\mathfrak{g})$. One checks that $\sigma(x^n) = \sigma(x)^n$, for all $x \in \mathfrak{g},\ n \in \mathbb{N}$. The linearity of σ implies that if $\sum c_x x^n = 0 : c_x \in k$, then $\sum c_x \sigma(x)^n = 0$.

Observe that $\sum c_x \sigma(x)^n \in \mathcal{F}^n(U(\mathfrak{g}))$. Since $gr_{\mathcal{F}}U(\mathfrak{g}) = S(\mathfrak{g})$, by the PBW theorem and $S(\mathfrak{g})$ is a domain, it follows that the image $gr_{\mathcal{F}}\sigma(x)^n$ in $\mathcal{F}^n(U(\mathfrak{g}))/\mathcal{F}^{n-1}U(\mathfrak{g}) \cong S_n(\mathfrak{g})$ is just $(gr_{\mathcal{F}}\sigma(x))^n$ which hence has degree n and further identifies with x^n. Since all the elements in the above sum have the same degree n we conclude that its image

$gr(\sum c_x\sigma(x)^n)$ in $S_n(\mathfrak{g})/S_{n-1}(\mathfrak{g})$ is just $\sum c_x gr(\sigma(x)^n) = \sum c_x x^n$. In particular σ is injective and through induction on n can be easily shown to be surjective.

By the result in the first paragraph above σ is determined by the condition that $\sigma(x^n) = \sigma(x)^n$, $\forall x \in \mathfrak{g}, n \in \mathbb{N}$. From this one checks using the Leibnitz rule that $(ad\ x)\sigma(a) = \sigma((ad\ x)a)$, $\forall x \in \mathfrak{g}, a \in S(\mathfrak{g})$.

Set $Y(\mathfrak{g}) = \{a \in S(\mathfrak{g}) \,\big|\, (ad\ x)a = 0,\ \forall\ x \in \mathfrak{g}\}$, $Z(\mathfrak{g}) = \{a \in U(\mathfrak{g}) \,\big|\, (ad\ x)a = 0\}$, $\forall\ x \in \mathfrak{g}$. The latter is just the centre of $U(\mathfrak{g})$.

Lemma. *The symmetrization map restricts to a linear isomorphism of $Y(\mathfrak{g})$ onto $Z(\mathfrak{g})$.*

5.10. Recall 3.7. One may recognize z defined there to be an element of $Z(\mathfrak{g})$ obtained by the symmetrization of an element of $Y(\mathfrak{g})$ of degree 2.

5.11. Let M, N be $\mathfrak{g}$ modules. For each $x \in \mathfrak{g}$ one may define a linear endomorphism of $k(M \times N)$ through

$$(m, n) \mapsto (xm, n) + (m, xn),\ \forall\ n \in M, n \in N.$$

These expressions vanish on the subspaces defined in 5.1(∗) and so the above map factors to a linear endomorphism of $M \otimes N$, which satisfies

$$(*) \qquad m \otimes n \mapsto xm \otimes n + m \otimes xn,\ \forall\ m \in M, n \in N.$$

It is called the Leibnitz action of $x \in \mathfrak{g}$ on $M \otimes N$. (Notice that we cannot choose (∗) as a definition of this action unless one checks that the result is independent of choices of representatives. In effect this is done through our preliminary step.)

One checks that (∗) in fact defines a $\mathfrak{g}$ module structure on $M \otimes N$. This is the same calculation as showing that $Der\ A$ is a Lie algebra for any algebra A. Moreover if M, N are subspaces of an algebra A, then the map $m \otimes n \mapsto mn$ is a homomorphism of $\mathfrak{g}$ modules of $M \otimes N$ onto MN.

In general the $\mathfrak{g}$-module structure of $M \otimes N$ is quite different from that of M and N. The possibility to get new modules from old is a key feature of Lie algebras (and more generally of groups and of Hopf algebras).

6. Root Systems and the Weyl Group

Let $\mathfrak{g}$ be a Kac-Moody algebra.

6.1. Recall (4.4) that we may write $\mathfrak{g}$ as a direct sum of its root subspaces $\mathfrak{g}_\alpha : \alpha \in \mathfrak{h}^*$. We define

$$\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \,\Big|\, \mathfrak{g}_\alpha \neq 0\}.$$

It is called the set of non-zero roots. Because of the rather astute choices made in 4.8 (due to Kac and to Moody) this set has some truly remarkable properties which we examine below.

6.2. Let κ be the antiautomorphism of its local part defined by $\kappa|_{\mathfrak{h}} = 1d|_{\mathfrak{h}},\ \kappa(e_i) = f_i,\ \kappa(f_i) = e_i,\ \forall\ i \in I$. Since the construction of $\mathfrak{g}$ from its local part is canonical, it follows that κ extends to an antiautomorphism of $\mathfrak{g}$. Then clearly

Lemma. *For each $\alpha \in \Delta$, κ restricts to a linear isomorphism of $\mathfrak{g}_\alpha$ onto $\mathfrak{g}_{-\alpha}$.*

6.3. Take $i \in I$ and set $h_i = \alpha_i^\vee$. Observe that the elements e_i, h_i, f_i satisfy the relations of 3.2. In particular they define a $\mathfrak{sl}(2)$ subalgebra $\mathfrak{s}_i$ of $\mathfrak{g}$.

6.4. Take $\alpha, \beta \in \Delta \cup \{0\},\ x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$. Then since *ad* h is a derivation we obtain

$$[h, [x, y]] = \alpha(h)[x, y] + \beta(h)[x, y].$$

Consequently

$$(*) \qquad [\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}.$$

Apart from this being a particularly simple rule, it means that we may picture the Lie product geometrically through the addition of vectors.

Lemma. One has

(i) $\Delta = \Delta^+ \coprod \Delta^-$

(ii) $\Delta^- = -\Delta^+$

(iii) $\Delta^+ \subset \mathbb{N}\pi \setminus \{0\}$

(iv) $\mathfrak{g}_0 = \mathfrak{h}$

(v) $dim \mathfrak{g}_\alpha = 1, \ \forall \ \alpha \in \pi.$

Proof. Let Δ^+ (resp. Δ^-) denote the set of roots corresponding to root subspaces in $\mathfrak{g}^+$. By $(*)$, the definition of π and the relations in 4.8 we obtain (i), (iii), (iv), (v). Finally (iii) follows from 6.2. □

6.5. Perhaps the most significant fact about Kac-Moody algebras are the following relations known as the Serre relations (noted first by J.-P. Serre in the finite dimensional case). Recall that the Cartan matrix A has entries $a_{i,j} : = \alpha_i^\vee(\alpha_j) \in -\mathbb{N}$, if $i \neq j$.

Lemma. *For all $i, j \in \{1, 2, \cdots, \ell\}$ distinct one has*

(i) $(ad\ e_i)^{1-a_{ij}} e_j = 0$

(ii) $(ad\ f_i)^{1-a_{i,j}} f_j = 0.$

Proof. Though 6.2 we need only prove the second relation. We must show that $y_{i,j} := (ad\ f_i)^{1-a_{i,j}} f_j$ lies in the unique maximal ideal $\mathfrak{r}$ of $\mathfrak{g}_{max}$ defined in 4.5. It is clearly $ad\ \mathfrak{h}$ stable and lies in $\mathfrak{g}_{-(2-a_{i,j})}$. Thus it suffices to show $(ad\ e_m) y_{i,j} = 0$, for all m. Since $ad\ e_m$ is a derivation this is immediate from 4.8 if $m \notin \{i, j\}$. Again taking $m = j$ one notes that

$$(ad\ e_j)((ad\ f_i)^{1-a_{i,j}} f_j) = (ad\ f_i)^{1-a_{i,j}} \alpha_j^\vee = a_{j,i} (ad\ f_i)^{-a_{i,j}} f_i$$

which is zero if $-a_{i,j} > 0$. If $a_{i,j} = 0$, then $a_{j,i} = 0$ also by hypothesis.

It remains to consider the case $m = i$. This will follow from the representation theory of $\mathfrak{sl}(2)$ developed in Section 3 applied to the $\mathfrak{s}_i$.

One has $(ad\ h_i)f_j = -a_{i,j}f_j$. Taking $n = -a_{i,j}$ in the identity given in 3.4 we obtain

$$(ad\ e_i)(ad\ f_i)^{1-a_{i,j}}f_j = (1-a_{i,j})(ad\ f_i)^{a_{i,j}}(ad\ h_i + a_{j,i})f_j = 0,$$

as required. $\square$

6.6. Let M be a $\mathfrak{g}$ module. For all $\lambda \in \mathfrak{h}^*$ the subspace $M_\lambda = \{m \in M \mid hm = \lambda(h)m,\ \forall\ h \in \mathfrak{h}\}$ is called the λ-weight subspace of M. One calls M integrable if it is a direct sum of its weight subspaces and if for each i it is a sum of its finite dimensional $\mathfrak{s}_i$ submodules (which by 3.8 one may assume simple and the sum direct).

Corollary. *As a module for the adjoint action $\mathfrak{g}$ is integrable.*

Proof. We have already noted (4.4) that $\mathfrak{g}$ is a direct sum of its $\mathfrak{h}$ weight subspaces (which we called root subspaces in this special case). Now fix $i \in I$. By 4.8($*$) and 6.5 it follows the submodules generated by the action of $\mathfrak{s}_i$ on any e_j, f_j or $h \in \mathfrak{h}$ is finite dimensional. Since $\mathfrak{s}_i$ acts by derivations this also holds for Lie product in these elements. Since they generate $\mathfrak{g}$ this proves the assertion. $\square$

Remark. If $\mathfrak{g}$ is finite dimensional one may classify all simple integrable modules and these are all finite dimensional and hence of "highest weight". Kac classified similarly all highest weight simple integrable modules of a Kac-Moody algebra. We shall describe this result in Sections 8, 10. At the time he noted that $\mathfrak{g}$ itself is *not* highest weight unless it is finite dimensional and noted that there were no other known examples of simple integrable highest weight modules. About fifteen years ago Chari and Pressley classified all simple admissible integrable modules (that is those having finite dimensional weight subspaces) for $\mathfrak{g}$ affine. These are of intense current interest. There still remains the following simply stated

Conjecture. *Let M be a simple integrable module of an affine Lie algebra. Assume that $M_0 \neq 0$. Then M has finite dimensional weight subspaces.*

6.7. For each $\alpha \in \pi$ consider the linear automorphism of $\mathfrak{h}^*$ defined by

$$s_\alpha \lambda = \lambda - \alpha^\vee(\lambda)\alpha, \ \forall \ \lambda \in \mathfrak{h}^*.$$

The group W generated by the $s_\alpha : \alpha \in \pi$ is called the Weyl group. It plays a fundamental role in the structure of $\mathfrak{g}$ and in the theory of integrable modules. If $\alpha = \alpha_i$ we set $\mathfrak{s}_\alpha = \mathfrak{s}_i$ and $\mathfrak{r}_\alpha = \mathfrak{s}_\alpha + \mathfrak{h}$.

Lemma. *For all $\alpha \in \pi$ one has*

(i) $\Delta^+ \setminus \{\alpha\}$ *is s_α stable*

(ii) Δ *is W stable.*

Proof. Since $s_\alpha \alpha = -\alpha$, assertion (ii) results from (i). For (i) set

$$\mathfrak{m}_\alpha := \bigoplus_{\beta \in \Delta^+ \setminus \{\alpha\}} \mathfrak{g}_\beta.$$

Through 4.8(∗) it follows that $\mathfrak{m}_\alpha$ is an $\mathfrak{r}_\alpha$ submodule of $\mathfrak{g}$. Then by 6.6 it is integrable and so by 3.8 a direct sum of simple finite dimensional $\mathfrak{s}_\alpha$ modules which are direct sums of their $\mathfrak{h}$ weight subspaces. Now by 3.5 every such $\mathfrak{s}_\alpha$ module is some $V(m-1)$. Viewed as an $\mathfrak{r}_\alpha$ module, it must have weights of the form $\lambda, \lambda - \alpha, \cdots, \lambda - (m-1)\alpha$, for some $\lambda \in \mathfrak{h}^*$ satisfying $\alpha^\vee(\lambda) = m - 1$. Since $s_\alpha\lambda = \lambda - \alpha^\vee(\lambda)\alpha = \lambda - (m-1)\alpha, s_\alpha(\lambda - \alpha) = \lambda - (m-2)\alpha, \cdots$, its set of weights is s_α stable. Hence so is the set of weights of $\mathfrak{m}_\alpha$, whilst the latter is just $\Delta^+ \setminus \{\alpha\}$. □

6.8. Set $\Delta_{re} = \{w\alpha : w \in W, \alpha \in \pi\}$. By 6.7 one has $\Delta_{re} \subset \Delta$. Kac calls it the set of real roots and its complement $\Delta_{im} = \Delta \setminus \Delta_{re}$ the set of imaginary roots. The description of Δ_{re} is a purely combinatorial question depending only on the Cartan matrix A. Take $\alpha \in \pi$. By 6.7(i) the set $\Delta_{re}^+ \setminus \{\alpha\}$ is s_α stable. In principle this could be shown purely combinatorially, that is not knowing anything of Kac-Moody algebras. As we shall see in the next section it has significant consequences for the structure of W.

6.9. One may remark that the following three assertions are equivalent.

(i) W is a finite group.

(ii) Δ is a finite set.

(iii) $\Delta = \Delta_{re}$.

7. The Structure of the Weyl Group

Let $\mathfrak{g}$ be a Kac-Moody algebra and recall the notation of Sections 4, 6.

7.1. Set $P = \{\lambda \in \mathfrak{h}^* \,\big|\, \alpha^\vee(\lambda) \in \mathbb{Z}\}$, $P^+ = \{\lambda \in P \,\big|\, \alpha^\vee(\lambda) \geq 0\}$. One calls P (resp. P^+) the set of (integral) weights (resp. dominant weights) of $\mathfrak{g}$. For each $i \in I$ choose $\omega_i \in P^+$ such that $\alpha_i^\vee(\omega_j) = \delta_{i,j}$, $\forall\ j \in I$. These are unique if *det* $A \neq 0$. They are called the fundamental weights.

7.2. Define a partial order relation $\geq$ on P through $\mu \geq \nu$, if $\mu - \nu \in \mathbb{N}\pi$. Observe that $s_\alpha \mu \leq \mu$ if $\alpha^\vee(\mu) \geq 0$.

7.3. Given $\alpha = \alpha_i$, we write s_{α_i} simply as s_i. Assume for the moment that $\mathfrak{g}$ is symmetrizable. Recall the scalar product on $\mathfrak{h}^*$ introduced in 4.7.

Lemma. *For all $\lambda, \mu \in \mathfrak{h}^*, w \in W$ one has $(w\lambda, w\mu) = (\lambda, \mu)$.*

Proof. For all $i \in I$ one has

$$(s_i\lambda, \mu) = (\lambda - \alpha_i^\vee(\lambda)\alpha_i, \mu) = \lambda - 2\frac{(\alpha_i, \lambda)(\alpha_i, \mu)}{(\alpha_i, \alpha_i)} = (\lambda, s_i\mu).$$

Since $s_i^2 = 1d$, we obtain the required assertion for $w = s_i$ and hence for all $w \in W$. □

7.4. Take $\beta \in \Delta_{re}$. By definition $\beta = w\alpha$ for some $w \in W, \alpha \in \pi$. By 7.3 we obtain $(\beta, \beta) = (\alpha, \alpha)$, which is a positive even integer. Thus we may define $s_\beta \in Aut\ \mathfrak{h}^*$ through

$$s_\beta \lambda = \lambda - \beta^\vee(\lambda)\beta, \quad where \quad \beta^\vee = 2\beta/(\beta, \beta).$$

Lemma. *Given $\alpha, \beta \in \Delta_{re}$ and $w \in W$ such that $w\alpha = \beta$ one has $ws_\alpha w^{-1} = s_\beta$.*

Proof. Indeed for all $\lambda \in \mathfrak{h}^*$ one has $ws_\alpha w^{-1}\lambda = \lambda - \alpha^\vee(w^{-1}\lambda)w\alpha = \lambda - \beta^\vee(\lambda)\beta$, by 7.3. Hence the assertion. □

Remark. Relax the hypothesis that $\mathfrak{g}$ is symmetrizable and define an action of W on $\mathfrak{h}$ by

$$s_\alpha h = h - \alpha(h)\alpha^\vee, \ \forall\, \alpha \in \pi.$$

Suppose $\alpha, \beta \in \pi$ and $w \in W$ satisfy $\beta = w\alpha$. One may show (see exercise sheet 3) that $\beta^\vee = w\alpha^\vee$ and consequently the proof of the lemma gives $ws_\alpha w^{-1} = s\beta$. This is needed in 7.6.

7.5. For each $\lambda \in P$ set $W_\lambda = \{w \in W \mid w\lambda = \lambda\}$. It is called the stabilizer of λ. It is clearly a subgroup of W.

7.6. Given $w \in W$ then one may write $w = s_{i_1}s_{i_2}\cdots s_{i_r}$. If r takes its minimal value amongst all possible expressions then one calls the right hand side a reduced expression for w and r its reduced length, denoted $\ell(w)$. Of course such notions exist for any group given by generators and relations; but in the case of a Weyl group they take a particularly simple form essentially as a consequence of 6.7(i). This is expressed as below. For each $w \in W$ set $S(w) = \{\alpha \in \Delta^+_{re} \mid w\alpha \in \Delta^+_{re}\} = \Delta^+_{re} \cap w^{-1}\Delta^-_{re}$. Given $\beta \in \Delta_{re}$ define s_β as in 7.4. Let $\rho \in \mathfrak{h}^*$ be chosen to satisfy $\alpha^\vee(\rho) = 1,\ \forall \alpha \in \pi$. Thus ρ is a sum of a choice of fundamental weights.

Lemma. *For all $w \in W, \alpha \in \pi$ one has*

(i) $S(w^{-1}) = -wS(w)$

(ii) $S(s_\alpha w) = \begin{cases} S(w) \cup \{w^{-1}\alpha\} : w^{-1}\alpha \in \Delta^+_{re} \\ S(w) \setminus \{-w^{-1}\alpha\} : w^{-1}\alpha \in \Delta^-_{re} \end{cases}$

(iii) $S(w) = \emptyset \iff w = e$

(iv) $|S(w)| = \ell(w)$

(v) *The stabilizer W_λ of $\lambda \in P^+$ in W is generated by the simple reflections it contains.*

(vi) $\quad \rho - w^{-1}\rho = \sum_{\beta \in S(w)} \beta$

(vii) $\quad$ *The map $w \mapsto S(w)$ is injective.*

Proof. For (i) observe that $wS(w) = w\Delta^+_{re} \cap \Delta^-_{re} = -S(w^{-1})$. For (ii) suppose $\beta \in S(w)$. Then $w\beta \in \Delta^-_{re}$ and so by 6.7(i) we obtain $\beta \notin S(s_\alpha w)$ if and only if $w\beta = -\alpha$. Replacing w by $s_\alpha w$ gives (ii). It is clear from (ii) that $|S(w)| \leq \ell(w)$.

For (iii) suppose $w \neq e$ satisfies $S(w) = \emptyset$. Let $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_r}$ be a reduced decomposition. Set $\alpha = \alpha_r$. Choose j maximal such that $w_j := s_{\alpha_j} \cdots s_{\alpha_r}$ satisfies $w_j\alpha \in \Delta^+_{re}$. Then $j < r$ and so $w_{j+1}\alpha \in \Delta^-_{re}$. By 6.7(i) this forces $-w_{j+1}\alpha = \alpha_{i_j} =: \beta$. Set $y = w_{j+1}s_\alpha$. Then $y\alpha = \beta$ and so $s_\beta = ys_\alpha y^{-1}$ by 7.4, Remark. Hence $w_j = s_\beta y s_\alpha = y$ forcing

$$\ell(w) \leq j - 1 + \ell(y) \leq j - 1 + r - j - 1 \leq r - 2$$

which is a contradiction.

We prove (iv) by induction on $|S(w)|$. The case $|S(w)| = 0$ is just (iii). Set $S_r = \{w \in W \bigm| |S(w)| = r\}$. Assume that (iv) has been established for S_r. Consider $y \in S_{r+1}$. If $y^{-1}\alpha \in \Delta^+$ for all $\alpha \in \pi$, then $S(y^{-1}) = \emptyset$ and so $S(y) = \emptyset$ by (i), which contradicts the choice of y. Yet given $\alpha \in \pi \cap S(y)$ it follows from (ii) that $s_\alpha y \in S_r$. Moreover $1 + \ell(s_\alpha y) \geq \ell(y) \geq |S(y)| = 1 + |S(s_\alpha y)| = 1 + \ell(s_\alpha y)$. Hence $\ell(y) = |S[y)|$.

(v). Recall (7.2) the order relation $\geq$ on P. Given $\lambda \in P^+$ set $\pi' = \{\alpha \in \pi \bigm| \alpha^\vee(\lambda) = 0\}$ and let W' be the subgroup of W generated by the $s_\alpha : \alpha \in \pi'$. We show that $\lambda \geq w\lambda$ with equality if and only if $w \in W'$ by induction on $\ell(w)$. The assertion is trivial if $w = e$. Otherwise by (i), (iv) there exists $\alpha \in \pi, y \in W$ such that $w = ys_\alpha$ with $\ell(y) = \ell(w) - 1$ and $y\alpha \in \Delta^+_{re}$. Then $\lambda - w\lambda = \lambda - y\lambda + \alpha^\vee(\lambda)y\alpha$ from which the assertion follows.

(vi). The proof is by induction on $\ell(w)$ using (ii) and the relation $\rho - (s_\alpha w)^{-1}\rho = \rho - w^{-1}(\rho - \alpha) = \rho - w^{-1}\rho + w^{-1}\alpha, \ \forall \alpha \in \pi$.

(vii). By (v), $W_\rho = \{1d\}$, so the assertion follows from (vi). $\square$

Remark. Set $P^{++} = \{\lambda \in P \mid \alpha^\vee(\lambda) \in \mathbb{N}^+,\ \forall\ \alpha \in \pi\}$. Then $Stab_W\lambda = \{e\}$ for all $\lambda \in WP^{++}$. Such an element is called regular. For example $\rho \in P^{++}$.

7.7. Given a reduced decomposition $w = s_{i_1}s_{i_2}\cdots s_{i_r}$ we may define $\beta_j = w_{j+1}^{-1}\alpha_{i_j}$ where $w_j = s_{\alpha_j}\cdots s_{\alpha_r}$. It is immediate from 7.6 that $S(w) = \{\beta_j\}_{j=1}^r$. Furthermore one may easily check that $s_{\beta_j}w = s_{i_1}\cdots\hat{s}_{i_j}\cdots s_{i_r}$ where $\hat{s}$ denotes the omission of s. We conclude that $\ell(s_\beta w) \leq \ell(w) - 1$ for all $\beta \in S(w)$. Examples show that a strict inequality may hold. Given $w, w' \in W$ and $\beta \in \Delta_{re}^+$ such that $w' = s_\beta w$ and $\ell(w') = \ell(w) - 1$ we write $w' \xleftarrow{\beta} w$, or simply $w' \longleftarrow w$. The Bruhat order $\leq$ on W is defined by setting $w' \leq w$ if $w' = w$ or if there exists a chain $w' = w_1 \longleftarrow w_2 \longleftarrow \cdots \longleftarrow w_n = w$.

Lemma. *Suppose $w' \xleftarrow{\beta} w$. Take $\alpha \in \pi$ distinct from β. Then either $s_\alpha w' \xleftarrow{s_\alpha\beta} s_\alpha w$ or $s_\alpha w' \xleftarrow{\alpha} w' \xleftarrow{\beta} w \xleftarrow{\alpha} s_\alpha w$.*

Proof. If $\ell(s_\alpha w) < \ell(w)$ then by 7.6 we can take $\alpha_{i_1} = \alpha$. Then the hypothesis $\beta \neq \alpha$ means that $w' = s_\beta w = s_{\alpha_1}\cdots\hat{s}_{i_j}\cdots s_{i_r}$ for $j > 1$ so $\ell(s_\alpha w') = \ell(w') - 1 = \ell(w) - 2 = \ell(s_\alpha w) - 1$ so the first conclusion holds. This is also true if $\ell(s_\alpha w) > \ell(w)$ and $\ell(s_\alpha w') > \ell(w')$. Otherwise the second conclusion holds.

8. Highest Weight Modules

Throughout we fix a Kac-Moody algebra $\mathfrak{g}$.

8.1. Let M be a $U(\mathfrak{g})$ module. For all $\lambda \in \mathfrak{h}^*$ recall that

$$M_\lambda = \{m \in M \mid hm = \lambda(h)m,\ \forall\ h \in \mathfrak{h}\}$$

is called the λ-weight subspace of M. One calls M a weight module given

$$M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda.$$

Define $\Omega(M) := \{\lambda \in \mathfrak{h}^* \mid M_\lambda \neq 0\}$. It is called the set of weights of M.

As in the case of $\mathfrak{g} = \mathfrak{sl}(2)$ one checks easily that

$$\mathfrak{g}_\alpha M_\lambda \subset M_{\lambda+\alpha}, \ \forall \ \lambda \in \mathfrak{h}^*, \ \alpha \in \Delta.$$

Consequently any weight module can be written as a direct sum of weight modules satisfying

$$\Omega(M) \subset \lambda + \mathbb{Z}\pi$$

for some $\lambda \in \mathfrak{h}^*$.

8.2. A weight module is said to be admissible if

$$dim \ M_\lambda < \infty, \ \forall \ \lambda \in \Omega(M).$$

In this case we may define the formal character $ch \ M$ of M through

$$ch \ M = \bigoplus_{\lambda \in \Omega(M)} (dim \ M_\lambda) e^\lambda.$$

Here e^λ is just λ when $\mathfrak{h}^*$ is viewed as a multiplicative group (so then $e^\lambda e^\mu = e^{\lambda+\mu}$). For the moment this sum has only the formal meaning of describing the set of dimensions of the weight spaces M_λ of M.

8.3. A weight module is said to be of bounded weight if

$$\Omega(M) \subset \lambda - \mathbb{N}\pi$$

for some $\lambda \in \mathfrak{h}^*$. If M is also admissible, then one may view $ch \ M$ as a formal power series in the $e^{-\alpha} : \alpha \in \pi$.

8.4. Let M, N be weight modules. Recalling Sect. 5 one may form their tensor product $M \otimes N$ which admits an action of $\mathfrak{g}$ through the Leibnitz formula (5.11). In particular given $m \in M_\mu, \ n \in M_\nu$ one has

$$(*) \qquad h(m \otimes n) = hm \otimes n + m \otimes hn = (\mu + \nu)(h)(m \otimes n), \ \forall \ h \in \mathfrak{h},$$

and so $M_\mu \otimes M_\nu \subset M_{\mu+\nu}$. Furthermore by the last part of 5.1, the images of the $M_\mu \otimes M_\nu$: $\mu + \nu = \xi$ in $(M \otimes N)_\xi$ form a direct sum and each have dimension $dim\ M_\mu\ dim\ M_\nu$.

Now suppose M, N are admissible and of bounded weight. Then there can only be finitely many contributions in the above to a given $(M \otimes N)_\xi$. From the given dimension formula one concludes that

$$(**) \qquad ch(M \otimes N) = ch\ M\ ch\ N,$$

where the product in the right hand side is in the sense of formal power series.

8.5. A module M is said to be of highest weight $\lambda \in \mathfrak{h}^*$ if there exists a vector $m_\lambda \in M$ of weight λ such that $\mathfrak{g}^+ m_\lambda = 0$. One calls m_λ a highest weight vector of highest weight λ. Through triangular decomposition we obtain

$$U(\mathfrak{g})m_\lambda = U(\mathfrak{g}^-)m_\lambda.$$

In particular as a $U(\mathfrak{g}^-)$ module, a highest weight module is an image of $U(\mathfrak{g}^-)$. More precisely it is isomorphic to $U(\mathfrak{g}^-)\big/ Ann_{U(\mathfrak{g}^-)} m_\lambda$.

For each $\lambda \in \mathfrak{h}^*$ there exists a unique up to isomorphism universal highest weight module, called the Verma module $M(\lambda)$ of highest weight λ. It is characterized by the property that

$$(*) \qquad Ann_{U(\mathfrak{g}^-)} m_\lambda = 0.$$

It is constructed through the tensor product and then $(*)$ results from the PBW theorem. Any highest weight module with highest weight λ is an image of $M(\lambda)$.

8.6. It is easy to compute $ch\ M(\lambda)$. We may view $U(\mathfrak{g}^-)$ and $S(\mathfrak{g}^-)$ as $U(\mathfrak{h})$ modules under adjoint action. By the PBW theorem they are isomorphic and from 8.5$(*)$ we obtain

$$ch\ M(\lambda) = e^\lambda\ ch\ U(\mathfrak{g}^-) = e^\lambda\ ch\ S(\mathfrak{g}^-).$$

Now for each $\alpha \in \Delta^+$, let $\{x^i_{-\alpha}\}$ be a basis of $\mathfrak{g}_{-\alpha} \subset \mathfrak{g}^-$. Then $S(\mathfrak{g}_{-\alpha})$ is a polynomial algebra on the $x^{(i)}_{-\alpha} : i = 1, 2, \cdots, dim\ \mathfrak{g}_{-\alpha}$ and so a tensor product of the one variable polynomial algebras $k\left[x^{(i)}_{-\alpha}\right]$. Again $S(\mathfrak{g})$ is a tensor product of the $S(\mathfrak{g}_{-\alpha}) : \alpha \in \Delta^+$. Since $ch\ k\left[x^{(i)}_{-\alpha}\right] = 1 + e^{-\alpha} + e^{-2\alpha} + \cdots = (1 - e^{-\alpha})^{-1}$, we conclude from 8.4(**) that

$$ch\ S(\mathfrak{g}^-) = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-dim \mathfrak{g}_{-\alpha}}.$$

Setting $m_\alpha = dim\ \mathfrak{g}_{-\alpha}$ gives

Lemma. *For all $\lambda \in \mathfrak{h}^*$ one has*

$$ch\ M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{m_\alpha}} .$$

Remark. A basic open question is to determine the m_α. If $\alpha \in \Delta^+_{re}$ one has $m_\alpha = 1$ by 6.9 and 6.4(iii). Since $\Delta_{re} = \Delta$ for $\mathfrak{g}$ semisimple, this case is settled. The m_α are easily computable for $\mathfrak{g}$ affine.

8.7. Let M be a proper submodule of $M(\lambda)$. Then clearly M is an admissible bounded weight module and furthermore satisfies $M_\lambda = 0$. Clearly these properties are shared by any sum of proper submodules which must again be proper. In particular the sum $\overline{M(\lambda)}$ of all proper submodules of $M(\lambda)$ is proper and by construction contains any proper submodule. We conclude that $\overline{M(\lambda)}$ is the unique maximal submodule of $M(\lambda)$. Set $V(\lambda) = M(\lambda)\big/\overline{M(\lambda)}$ which a simple highest weight module of highest weight λ. By construction and the universality of $M(\lambda)$, it is the image of any highest weight module of highest weight λ. This gives the following

Lemma. *The map $\lambda \mapsto V(\lambda)$ is a bijection of $\mathfrak{h}^*$ onto the set of isomorphism classes of simple highest weight modules.*

8.8. The above construction is the exact analogue of the construction of Kac-Moody algebras themselves. Its subtlety derives from the nature of $Ann_{U(\mathfrak{g}^-)} m_\lambda$, where m_λ now denotes the highest weight vector of $V(\lambda)$. In general it is very hard to describe and Verma is famous for his false theorem attempting to give such a formula. A similar hard problem is to describe $ch\ V(\lambda)$.

8.9. Define a partial order $\geq$ on $\mathfrak{h}^*$ through $\lambda \geq \mu$ if $\lambda - \mu \in \mathbb{N}\pi$. From 8.6 it follows that $dim\ M(\lambda)_\mu \geq 1$ implies $\lambda \geq \mu$ and that equality holds if $\lambda = \mu$. Since $V(\lambda)$ is a quotient of $M(\lambda)$, this must hold for $V(\lambda)$ also.

Let $e^{-\mathbb{N}\pi}$ denote the multiplication semigroup formed from the elements $e^{-\beta} : \beta \in \mathbb{N}\pi$. From the above $\{ch\ M(\mu) : \mu \in \lambda - \mathbb{N}\pi\}$ and $\{ch\ L(\mu) : \mu \in \lambda - \mathbb{N}\pi\}$ both form free $\mathbb{Z}$ bases of the $e^{-\mathbb{N}\pi}$ module $\mathbb{Z}e^{\lambda-\mathbb{N}\pi}$. Consequently there exists a triangular matrix with entries $m_{\lambda,\mu} \in \mathbb{Z}$ such that

$$(*) \qquad ch\ V(\lambda) = \sum_{\mu\in\lambda-\mathbb{N}\pi} m_{\lambda,\mu} ch\ M(\mu).$$

A basic problem is to compute the $m_{\lambda,\mu}$. We shall do this in the case that $V(\lambda)$ is integrable and $\mathfrak{g}$ is symmetrizable in Sect. 10.

8.10. The set of all admissible bounded weight modules forms an abelian category called the $\mathcal{O}$ category. Its study was initiated by J. Bernstein, I. Gelfand and S. Gelfand in the case when $\mathfrak{g}$ is semisimple. Its analysis plays a significant role in the determination of $ch\ V(\lambda)$ for arbitrary $\lambda \in \mathfrak{h}^*$.

9. The Casimir Invariant

In this and Section 10, we assume $\mathfrak{g}$ symmetrizable.

9.1. Combining the triangular decomposition of $\mathfrak{g}$ with the PBW theorem one obtains a triangular decomposition $U(\mathfrak{g}) = U(\mathfrak{g}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^+)$ of $U(\mathfrak{g})$. Expressed precisely this means that the map $u \otimes h \otimes v \mapsto uhv$ extends to a linear isomorphism of $U(\mathfrak{g}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^+)$ onto $U(\mathfrak{g})$.

9.2. Recall 8.9. For all $\gamma \in \mathbb{N}\pi$ we set

$$U^\gamma(\mathfrak{g}) : = \bigoplus_{0 \le \beta \le \gamma} U(\mathfrak{g}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^+)_\beta$$

which we identify with its image in $U(\mathfrak{g})$. Via this direct sum decomposition one has a projection of $U^\gamma(\mathfrak{g})$ onto $U^{\gamma'}(\mathfrak{g})$ if $\gamma' \le \gamma$. Thus the projective limit

$$U^{\mathcal{O}}(\mathfrak{g}) = \varprojlim U^\gamma(\mathfrak{g})$$

can be defined and is called the $\mathcal{O}$ completion of $U(\mathfrak{g})$. Via direct sum decomposition one can identify $U^{\mathcal{O}}(\mathfrak{g})$ as the space of all infinite sums

$$\sum_{\beta \in \mathbb{N}\pi} a(\beta) b_\beta \,\Big|\, a(\beta) \in U(\mathfrak{g}^-) \otimes U(\mathfrak{h}),\ b_\beta \in U(\mathfrak{g}^+)_\beta.$$

In multiplying out two such expressions one observes that only finitely many terms contribute to a given summand $U(\mathfrak{g}^-) \otimes U(\mathfrak{h}) \otimes U(\mathfrak{g}^+)_\beta$. Thus the multiplication in $U(\mathfrak{g})$ extends to $U^{\mathcal{O}}(\mathfrak{g})$ giving the latter an algebra structure. Again if M belongs to the $\mathcal{O}$ category only finite many summands are non-zero on a given weight subspace of M. In this fashion M admits a $U^{\mathcal{O}}(\mathfrak{g})$ module structure.

9.3. Recall 4.7. Through the vector space isomorphism $\varphi : \mathfrak{h}^* \xrightarrow{\sim} \mathfrak{h}$ obtained from the scalar product on $\mathfrak{h}^*$ one obtains a scalar product K on $\mathfrak{h}$ defined explicitly by

$$K(\varphi(\lambda), \varphi(\mu)) = (\lambda, \mu),\ \forall\ \lambda, \mu \in \mathfrak{h}^*.$$

9.4. A bilinear form K on a Lie algebra $\mathfrak{g}$ is said to be invariant if

$$K((ad\ x)y, z) + K(y, (ad\ x)z) = 0,\ \forall\ x, y, z \in \mathfrak{g}.$$

This formula can be understood by formally exponentiating $ad\ x$ to an element $g = exp\ ad\ x$ which may be formally viewed as an element of $Aut\ \mathfrak{g}$. This gives the more familiar condition for invariance, namely

$$K(gy, gz) = K(y, z),\ \forall\ y, z \in \mathfrak{g}.$$

A typical example of such a form is the Killing form defined on any finite dimensional Lie algebra by the formula

$$K(y,z) = tr\ ad\ y\ ad\ z,\ \forall\ y,z \in \mathfrak{g}.$$

A significant classical result is that $\mathfrak{g}$ is semisimple (and hence a direct sum of simple Lie algebras) if and only if K is non-degenerate on $\mathfrak{g}$.

9.5. Of course if $\mathfrak{g}$ is infinite dimensional, the Killing form is not defined. Kac circumvented this difficulty by simply extending K defined in 9.3 to $\mathfrak{g}$ itself by imposing invariance. Here one first checks that K extends to the local Lie algebra defined by 4.8 where invariance is taken with respect to the Lie relations defined on the local part - see 4.5. We shall leave the details as an exercise but note that a crucial point is that the Cartan matrix $\{\alpha_i^\vee(\alpha_j)\}_{i,j\in I}$ is symmetrizable. Moreover one obtains

$$(*) \qquad K(e_i, f_j) = \frac{2}{(\alpha_i,\alpha_i)}\delta_{i,j}.$$

Then a straightforward induction argument shows that K extends to an invariant bilinear form on $\mathfrak{g}_{max}$, which is furthermore symmetric. Finally an almost formal argument shows that the kernel K on $\mathfrak{g}_{max}$ is just the ideal $\mathfrak{r}$ defined in 4.5. Thus K factors to a non-degenerate bilinear symmetric invariant form on the Kac-Moody algebra $\mathfrak{g}_{min} = \mathfrak{g}_{max}/\mathfrak{r}$. Details are left as an exercise.

9.6. The non-degeneracy of K combined with $\mathfrak{h}$ invariance implies that K restricts on each $\gamma \in \Delta^+$ to a non-degenerate pairing $\mathfrak{g}^-_{-\gamma} \times \mathfrak{g}^+_\gamma \to k$. This implies the existence of a unique element $J \in \prod\limits_{\alpha\in\Delta^+} (\mathfrak{g}^-_{-\gamma} \otimes \mathfrak{g}^+_\gamma)$ satisfying

$$(K \times K)(J, x\otimes y) = K(x,y),\ \forall\ x \in \mathfrak{g}^+, y \in \mathfrak{g}^-.$$

Indeed J can be written down as an appropriate infinite sum involving dual bases defined by the above pairing. In particular J maps to an element of $U^{\mathcal{O}}(\mathfrak{g})$. It is not quite invariant but satisfies

Lemma. *For all $i \in I$ one has*

(i) $(ad\ e_i)J = \frac{1}{2}(\alpha_i, \alpha_i)(\alpha_i^\vee \otimes e_i)$

(ii) $(ad\ f_i)J = -\frac{1}{2}(\alpha_i, \alpha_i)(f_i \otimes \alpha_i^\vee)$.

Proof. Set

$$\mathfrak{m}^\pm = \sum_{\beta \in \Delta^+ \setminus \pi} \mathfrak{g}^\pm_{\pm\beta}.$$

In virtue of 9.5(∗) and the definition of I we obtain

$$J - \sum_{i \in I} \frac{1}{2}(\alpha_i, \alpha_i)(f_i \otimes e_i) \in \mathfrak{m}^- \otimes \mathfrak{m}^+.$$

Noting that $(ad\ e_i)\mathfrak{m}^- \subset \mathfrak{n}^-$, for all $i \in I$ one checks that the invariance of K implies that

$$(K \times K)((ad\ e_i)J, \mathfrak{g}^+ \otimes \mathfrak{m}^-) = 0.$$

Consequently

$$(ad\ e_i)J = \sum_{j \in I} \frac{1}{2}(\alpha_j, \alpha_j)(ad\ e_i)f_j \otimes e_j = -\frac{1}{2}(\alpha_i, \alpha_i)(\alpha_i^\vee \otimes e_i)$$

which is (i). The proof of (ii) is similar. □

9.7. Recall (7.6) the definition of $\rho \in \mathfrak{h}^*$. Let h_ρ be the unique element of $\mathfrak{h}^\varphi$ satisfying $K(h, h_\rho) = \rho(h),\ \forall\ h \in \mathfrak{h}$. Let $\{h_j\}$ be an orthonormal basis for $\mathfrak{h}^*$ with respect to K. One has

$$\begin{aligned}(ad\ e_i)(h_\rho + \frac{1}{2}\sum_j h_j^2) &= -(\rho, \alpha_i)e_i - \frac{1}{2}\sum_j \alpha_i(h_j)(e_i h_j + h_j e_i) \\ &= -\frac{(\alpha_i, \alpha_i)}{2} e_i + \frac{1}{2}\sum_j \alpha_i(h_j)\alpha_i(h_j)e_i - 2\alpha_i(h_j)h_j e_i \\ &= -\frac{(\alpha_i, \alpha_i)}{2}\alpha_i^\vee e_i\ .\end{aligned}$$

Similarly or through the Chevalley antiautomorphism

$$(ad\ f_i)(h_\rho + \frac{1}{2}\sum_j h_j^2) = \ -\ \frac{(\alpha_i, \alpha_i)}{2}\ f_i \alpha_i^\vee\ .$$

Identify J with its image in $U^{\mathcal{O}}(\mathfrak{g})$ and set $C = J + \frac{1}{2}\sum h_j^2 + h_\rho + \frac{1}{2}(\rho, \rho)$. Combining 9.6 and the above relations one concludes that C commutes with generators $e_i, f_i : i \in I$ of $\mathfrak{g}$ and hence is central in $U^{\mathcal{O}}(\mathfrak{g})$. It is called the Casimir invariant. Its construction in the above general context of Kac-Moody algebras is due to Kac.

9.8. The importance of the Casimir invariant obtains from the following

Lemma. *For all $\lambda \in \mathfrak{h}^*$ one has*

$$C\Big|_{M(\lambda)} = \frac{1}{2}(\lambda + \rho, \lambda + \rho)1d\Big|_{M(\lambda)}\ .$$

Proof. Since C is central it is enough to check this property on kv_λ, where v_λ is a choice of highest weight vector of weight λ. Then

$$\begin{aligned} Cv_\lambda &= \left(\frac{1}{2}\sum_j \lambda(h_j)^2 + \lambda(h_\rho) + \frac{1}{2}(\rho, \rho)\right) v_\lambda \\ &= \left(\frac{1}{2}(\lambda, \lambda) + (\lambda, \rho) + \frac{1}{2}(\rho, \rho)\right) v_\lambda \\ &= \frac{1}{2}(\lambda + \rho, \lambda + \rho) v_\lambda\ , \end{aligned}$$

as required. □

10. The Weyl Character Formula

10.1. The Casimir invariant permits us to assert that certain coefficients occurring in 8.9(∗) must vanish. The precise reasoning is given in the lemmas below. Set $\mathfrak{b} = \mathfrak{g}^+ \oplus \mathfrak{h}$.

Lemma. *Suppose $M \in Ob\mathcal{O}$. Then for all $m \in M$ one has dim $U(\mathfrak{b})m < \infty$.*

Proof. Since M is a direct sum of its weight subspaces we can assume that m is a weight vector without loss of generality, say of weight μ. Then

$$U(\mathfrak{b})m_\mu \subset \bigoplus_{\nu\in\mu+\mathbb{N}\pi} M_\nu \ .$$

On the other hand $\Omega(M) \subset \lambda - \mathbb{N}\pi$ so $M_\nu \neq 0$ in the above implies that $\nu \in \mu+\mathbb{N}\pi \cap \lambda - \mathbb{N}\pi$, which is a finite set. Since $dim\ M_\nu < \infty$, by the admissibility of M, the required assertion follows. $\square$

10.2. Let $\mathcal{O}_f$ denote the subcategory of $\mathcal{O}$ of finitely generated modules.

Lemma. *Given $M \in Ob\mathcal{O}_f$, there exists a finite dimensional $U(\mathfrak{b})$ invariant subspace $N \subset M$ satisfying $U(\mathfrak{g}^-)N = M$.*

Proof. By hypothesis there exists a finite dimensional subspace N_0 of M such that $U(\mathfrak{g})N_0 = M$. By 10.1 and triangular decomposition, $N := U(\mathfrak{b})N_0$ has the required properties. $\square$

10.3. Take $M \in Ob\mathcal{O}_f$ and choose N as in 10.2. Since $\Omega(M)$ is bounded above, there exists $\mu \in \Omega(M)$ maximal. Then $\mathfrak{g}^+ M_\mu = 0$. Furthermore $dim\ M_\mu < \infty$ and so M_μ generates a finite direct sum M_1 of highest weight submodules of M of highest weight μ. Since $U(\mathfrak{g}^-)N = M$, it follows that $N_\mu = M_\mu$. Now the image of N in M/M_1 is generated by $N/N \cap M_1$ which by the above has dimension strictly less than N. Induction on $dim\ N$, then proves the following

Lemma. *Suppose $M \in Ob\mathcal{O}_f$. There exists a finite filtration $M = M_n \supsetneqq M_{n-1} \cdots \supsetneqq M_1 \supsetneqq 0$ of M by $U(\mathfrak{g})$ modules such that each quotient is a highest weight module.*

10.4. Combining 9.8 and 10.3 we obtain the following

Corollary. *Suppose $M \in Ob\mathcal{O}_f$. There exists a polynomial p_M such that $p_M(C)$ vanishes on M.*

10.5. Applying the Chinese remainder theorem to the conclusion of 10.4 gives a block decomposition of $\mathcal{O}_f$. This means that we can write each $M \in Ob\mathcal{O}_f$ as a finite direct sum such that each summand is annihilated by some $(C-(\lambda_i+\rho,\lambda_i+\rho))^{n_i} : \lambda_i \in \mathfrak{h}^*, n_i \in \mathbb{N}^+$. These are of course just the factors in p_M corresponding to a given root. Now by 9.8, $M(\lambda)$ and $V(\mu)$ must belong to a fixed block and this can be the same block if and only if $(\lambda+\rho,\lambda+\rho)=(\mu+\rho,\mu+\rho)$. We conclude that in 8.9(∗) one has

$$(*) \qquad m_{\lambda,\mu} \neq 0 \Longrightarrow (\lambda+\rho,\lambda+\rho)=(\mu+\rho,\mu+\rho).$$

10.6. Recall 9.7 and define the translated action of W on $\mathfrak{h}^*$ through

$$w.\lambda = w(\lambda+\rho)-\rho.$$

Observe that 10.5 is satisfied if $\mu \in W.\lambda$ Thus the translated action is the natural one to use with respect to the behaviour of the blocks defined by C. This is strengthened by the following remarkable

Lemma. *Take $\lambda \in P^+$. Suppose $\mu \in \mathfrak{h}^*$ satisfies $y.\mu \leq \lambda$, $\forall\, y \in W$ and $(\mu+\rho,\mu+\rho) = (\lambda+\rho,\lambda+\rho)$. Then there exists $w \in W$ such that $w.\mu = \lambda$.*

Proof. By hypothesis $W.\mu$ is bounded above by λ and so admits a maximal element ν. Then $s_\alpha.\nu \leq \nu$, for all $\alpha \in \pi$, which implies that $\nu+\rho \in P^+$. The second hypothesis implies that $(\lambda-\nu,\nu+\rho)+(\lambda+\rho,\lambda-\nu)=0$. Since $\lambda,\nu+\rho \in P^+$, both terms are positive hence zero. Yet $\lambda+\rho \in P^{++}$ and so $(\lambda+\rho,\lambda-\nu)=0$ implies $\lambda=\nu$, as required. □

10.7. Recall (6.6) the notion of an integrable module. We shall characterize those integrable modules belonging to $\mathcal{O}$. As before set $m_\alpha = dim\ \mathfrak{g}_{-\alpha}$.

Take $\lambda \in \mathfrak{h}^*$ and let $Q(\lambda)$ be a highest weight module with highest weight λ.

Theorem. *Suppose $Q(\lambda)$ is integrable. Then*

(i) $\lambda \in P^+$.

(ii) $Q(\lambda)$ *is a quotient of*

$$M(\lambda)\Big/\sum_{\alpha\in\pi} M(s_\alpha.\lambda),$$

and the latter is also integrable.

(iii) $\quad ch\ Q(\lambda) = \frac{\sum_{w \in w}(-1)^{\ell(w)} e^{w.\lambda}}{\prod_{\alpha \in \Delta^+}(1-e^{-\alpha})^{m_\alpha}}$

(iv) $\quad Q(\lambda)$ *is simple, inparticular* $Q(\lambda) = M(\lambda) \Big/ \sum_{\alpha \in \pi} M(s_\alpha.\lambda).$

(v) *Every integrable module Q belonging to $\mathcal{O}$ is a direct sum of the* $Q(\lambda) : \lambda \in \mathfrak{h}^*$.

Proof. Let v_λ be a highest weight vector for $Q(\lambda)$. Fix $\alpha_i \in \pi$. Integrability forces $f_i^{n+1} v_\lambda = 0$, for some $n \in \mathbb{N}$ which we can assume minimal. By 3.3 this forces $\alpha_i^\vee(\lambda) = n$. Hence (i).

Take $\lambda \in P^+$ and let m_λ be a highest weight vector for $M(\lambda)$. By universality the map $m_\lambda \mapsto v_\lambda$ extends to a surjection of $M(\lambda)$ onto $Q(\lambda)$ of $U(\mathfrak{g})$ modules. Take $\alpha_i \in \pi$ and set $n_i = \alpha_i^\vee(\lambda)$. We have seen that the image of $f_i^{n_i+1} m_\lambda$ in $Q(\lambda)$ is zero. Clearly $e_j f_i^{n_i+1} m_\lambda = 0$ if $j = i$; but this also holds for $j = i$, through the identity in 3.4. Thus $f_i^{n_i+1} m_\lambda$ is a highest weight vector in $M(\lambda)$ of weight $\lambda - (n_i+1)\alpha_i = s_i.\lambda$. Consequently it generates an image of $M(s_i.\lambda)$ and moreover $Q(\lambda)$ is an image of the resulting quotient $M(\lambda)/M(s_i.\lambda)$.

Now set

$$\mathfrak{m}_i^- = \sum_{\beta \in \mathbb{N}\pi \setminus \{\alpha_i\}} \mathfrak{g}_\beta^- .$$

As in 6.7 it follows that $\mathfrak{m}_i^-$ is a subalgebra of $\mathfrak{g}^-$ and is $\mathfrak{s}_i$ stable. Moreover by 6.6 it follows that $U(\mathfrak{m}_i)$ is a sum of finite dimensional $\mathfrak{s}_i$ modules under adjoint action.

Let V_i denote the $\mathfrak{s}_i$ submodule of $M(\lambda)/M(s_i.\lambda)$ generated by the image of m_λ. Since $M(\lambda) = U(\mathfrak{g}^-)m_\lambda$, it follows that $M(\lambda)/M(s_i.\lambda) = U(\mathfrak{m}_i^-)V_i$. By the previous paragraph the latter is again a sum of finite dimensional $\mathfrak{s}_i$ modules and hence so is $Q(\lambda)$.

Since $i \in I$ is arbitrary in the above, (ii) results.

The integrability of $Q(\lambda)$ implies as in 6.7 that $s_\alpha ch\ Q(\lambda) = ch\ Q(\lambda) : \alpha \in \pi$. Hence

$ch\ Q(\lambda)$ is W stable. From 8.6 and 6.7(i) one checks that

$$(*) \qquad s_\alpha ch\ M(w.\lambda) = -ch\ M((s_\alpha w).\lambda),\ \forall\ \alpha \in \pi.$$

By 9.7 we may write

$$ch\ Q(\lambda) = \sum_{\mu \leq \lambda} a_\mu\ ch\ M(\mu).$$

Through block decomposition we further have that $a_\mu \neq 0$ implies $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$. By the W invariance of $ch\ Q(\lambda)$ and $(*)$ we must have $a_{y.\mu} = (-1)^{\ell(y)} a_\mu$ and this forces $y.\mu \leq \lambda$ given $a_\mu \neq 0$. Then by 10.6 we conclude that $a_\mu \neq 0$ implies $\mu = w.\lambda$ and furthermore $a_{w.\lambda} = (-1)^{\ell(w)} a_\lambda$. Thus (iii) results.

The unique simple quotient $V(\lambda)$ of $M(\lambda)$ must be an image of $Q(\lambda)$. Since both are integrable their character is given by (iii) in both cases. Consequently they must coincide. This proves (iv).

Since a module is a sum of its finitely generated submodules, it is enough to prove (v) for $Q \in Ob\mathcal{O}_f$. By 10.3, Q admits a finite filtration with quotients which are highest weight modules, so integrable and hence simple by (iv). It remains to show that the above sum must be direct, equivalently that there can be no extensions between simple integrable modules in $\mathcal{O}$.

Let X be a $U(\mathfrak{g})$ module having $V(\lambda)$ as a submodule such that quotient $X/V(\lambda)$ is isomorphic to $V(\mu)$. We assume that both $V(\lambda), V(\mu)$ are integrable, equivalently $\lambda, \mu \in P^+$. We can assume that $V(\lambda), V(\mu)$ lie in the same block for otherwise X is a direct sum.

Let v_μ be a highest weight vector for $V(\mu) = X/V(\lambda)$ and let $\hat{v}_\mu$ be an inverse image of v_μ in X. If $\hat{v}_\mu$ is still a highest weight vector, then it generates an integrable highest weight module with highest weight μ, so isomorphic to $V(\mu)$. It follows easily that X is a direct sum of this copy of $V(\mu)$ and its submodule $V(\lambda)$. If $\hat{v}_\mu$ is not a highest weight vector, then $\mu+\alpha$ is a weight of X for some $\alpha \in \pi$ and hence a weight of $V(\lambda)$. This forces

$\lambda > \mu$. Yet $\lambda, \mu \in P^+$ and satisfy $(\lambda+\rho, \lambda+\rho) = (\mu+\rho, \mu+\rho)$. This forces $\lambda = \mu$ through the equality used in the proof of 10.6. $\square$

10.8. The formula in 10.7(iii) is known as the Weyl-Kac character formula for the simple module $V(\lambda)$ of highest weight $\lambda \in P^+$. Weyl's original proof valid only for a finite dimensional semisimple Lie algebra was far more computational and indeed inspired by the intricate determinantal identities of Schur who first established the result in the case $\mathfrak{g} = \mathfrak{sl}(n)$. Kac's proof followed an argument of Bernstein-Gelfand-Gelfand. At first sight it may seem a little fortuitous. However one may clarify the argument by proving a stronger result on the $\mathfrak{g}^-$ homology spaces $H_i(\mathfrak{g}^-, V(\lambda))$ of $V(\lambda)$. An argument using the codimension one subalgebras $\mathfrak{m}_i^-$ and the Casimir invariant shows that the weight subspaces of $H_i(\mathfrak{g}^-, V(\lambda))$ satisfy

$$(*) \qquad dim\ H_i(\mathfrak{g}^-, V(\lambda))_\mu = \begin{cases} 1 \ : \ \mu = w.\lambda,\ i = \ell(w) \\ 0 \ : \quad otherwise. \end{cases}$$

Then some combinatorics of W combined with a formal argument gives a long exact sequence

$$\longrightarrow M_i \longrightarrow \cdots\cdots \longrightarrow M(\lambda) \longrightarrow V(\lambda) \longrightarrow 0$$

where

$$M_i = \bigoplus_{\ell(w)=i} M(w.\lambda).$$

It is called the BGG resolution of $V(\lambda)$. It gives 10.7(iii) and indeed is a refinement of the latter. Applying $(*)$ to the trivial module one may show that the Serre relations generate $\mathfrak{r}$ (of 4.5).

10.9. The W invariance of $ch\ V(\lambda)$ implies that for each $w \in W$, there is a unique up to scalars vector $v_{w\lambda} \in V(\lambda) \setminus \{0\}$ of weight λ. The $U(\mathfrak{b})$ submodule

$$F(w\lambda) = U(\mathfrak{b})v_{w\lambda}$$

is called a Demazure module. They have some particularly interesting properties. For example $F(w\lambda) \subset F(y\lambda) \iff w \leq y$ (Bruhat order). For $\mathfrak{g}$ semisimple, Demazure described $ch\ F(w\lambda)$; but unfortunately his proof was based on a(nother!) false lemma of Verma. Andersen first gave a correct proof for $\mathfrak{g}$ semisimple. A major achievement of crystal theory was to extend this formula for all $\mathfrak{g}$ symmetrizable Kac-Moody. Here we shall just examine a combinatorial version of this formula at the level of crystals which is a prelude to the proof of the module theoretic formula.

A hard open problem is to compute the homology spaces $H_i(\mathfrak{g}^-, F(w\lambda))$.

10.10. Suppose $\lambda = 0$, then $Q(\lambda)$ is one dimensional. Moreover $ch\ Q(0) = e^0$. Substitution into 10.7(iv) gives the famous Weyl-Kac denominator formula, namely

$$\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho - \rho} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{m_\alpha}.$$

Notice that this gives a purely combinatorial interpretation of Δ^+ and of the $m_\alpha : \alpha \in \Delta^+$. It is not known if this extends to the non-symmetrizable case.

The above formula is of particular interest when the m_α can be computed in another fashion. This is true for affine Lie algebras and the resulting combinatorial identity generalizes classical results developed in number theory to solve Waring's problem.

Lie Algebras, their Representations and Crystals

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Part II

11. The Notion of a Crystal

11.1. A crystal is a purely combinatorial object which recovers the essential features of an integrable module (for a symmetrizable Kac-Moody algebra) stripped of its linear structure. Taking a $q \to 0$ limit of the corresponding quantized module one obtains a basis B formed from weight vectors. The elements of B form the vertices of the corresponding crystal graph. The Lie algebra elements $e_i, f_i : i \in I$ implement transitions between the elements of B, changing weights by the corresponding α_i. They are represented by labelled edges in the graph. Integrability is realized by the condition that suppressing all labels except one leads to a disjoint union of linear graphs. This is in correspondence with the conclusion of 3.8. Furthermore a normality condition is imposed so that these linear graphs are stable under the reflection corresponding to the remaining index making the original graph W invariant.

11.2. From a graph theoretic point of view an entirely new feature introduced by the present theory is the possibility to take the tensor product of two crystals B_1, B_2. As a set it is just the Cartesian product $B_1 \times B_2$, as one might expect from 5.1. Weights add to be compatible with 8.4($*$). The edges in the corresponding graph are determined label by label. The rule due to Kashiwara was obtained through a $q \to 0$ limit. It recovers the decomposition of the tensor product of two simple finite dimensional $\mathfrak{sl}(2)$ modules.

11.3. The notion of a highest weight module lends itself naturally to the notion of a highest weight crystal. However because of the loss of linear structure, these are less constrained than their module theoretic counterparts. This is counterbalanced by imposing conditions on the structure of their tensor product which eventually leads to a uniqueness theorem for a family of highest weight normal crystals closed in a natural sense under tensor product.

11.4. We present two models for the unique family alluded to in 11.3. The first due to Kashiwara is based on a repeated tensor product construction inspired by the Bott-

Samelson resolution of the flag variety. It leads one to introduce the "Kashiwara function" which governs the "filling-up" of this tensor product. One obtains a crystal analogue, denoted as $B(\infty)$, of a "universal" Verma module from which the members of the above family can be recovered much as in the conclusion of 10.7.

A disadvantage of the Kashiwara model is that the construction of $B(\infty)$, like the Bott-Samelson resolution, depends on a choice of reduced decomposition on the unique longest element of the Weyl group (in the case that the latter is finite). Moreover like a Verma module, $B(\infty)$ should be a rather uncomplicated object, but this is completely unobvious from its presentation. Indeed one may say that an important open problem is to find a simple presentation of $B(\infty)$ and more specifically to describe the set of values that the Kashiwara function may take on it.

11.5. The second model for the above family is the Littelmann path model. It directly constructs the members of the family through sequences of Bruhat ordered elements combined with rational partitions of unity. A nice feature is that tensor product is realized by concatenation of paths. A remarkable fact is the independence of the crystal under deformation of the dominant path (within the dominant chamber). We call this the Littelmann path independence theorem.

An advantage of the Littelmann path model is its independence on choices. However it is necessarily very complex because the crystals corresponding to integrable quotients of Verma modules are being directly described.

11.6. The Littelmann path independence theorem gives purely combinatorial proofs of a number of results which had been previously obtained by Kashiwara through a $q \to 1$ limit. This includes the seemingly innocuous fact that $B(\infty)$ is an "upper normal" crystal (see 12.5). It establishes the equivalence of the Kashiwara and Littelmann highest weight families as well as a decomposition for the tensor product of members of this family.

11.7. The deeper meaning of the Littelmann model becomes apparent through a process of blowing up the crystal by taking multiple tensor products and changing parameters. In this fashion the Kashiwara and Littelmann models become to resemble one another and this provides a further way of proving their equivalence discovered independently by Kashiwara and by Zelikson.

11.8. The Demazure algorithm has a particularly precise meaning in terms of the Kashiwara-Littelmann family of highest weight crystals. This leads to combinatorial versions of the Demazure and Weyl character formulae. It leads to the notion of a Demazure "crystal". A decomposition theorem for (certain) tensor products of Demazure crystals is obtained. This yields a combinatorial version (in the symmetrizable Kac-Moody case) of the Demazure flag (or excellent filtration) theorem established using geometric techniques through the work of Mathieu, Polo and van der Kallen (for $\mathfrak{g}$ semisimple). Finally the Kashiwara involution on $B(\infty)$ (whose existence has not yet a purely combinatorial proof) is used to describe the highest weight elements in the tensor product of Kashiwara-Littelmann crystals.

11.9. From crystals one may return to the modules by a globalization procedure due to Kashiwara. The resulting global basis $\{G(b) : b \in B\}$ is indexed by B. It coincides with Lusztig's canonical basis obtained by completely different and much more sophisticated methods. A key open problem is to relate the action of the crystal operators on the $b \in B$ to the Lie algebra operators on the $G(b) : b \in B$. Here one has significant results of Kashiwara, Lusztig and others; but the solution to certain problems demands more information, for example concerning the existence of Demazure flags.

11.10. The blowing up procedure mentioned in 11.6 as a technical device to establish the equivalence of the Kashiwara and Littelmann crystals has a significant interpretation in terms of quantum Frobenius splitting. This in particular enabled Littelmann to write down a basis for the dual modules which can be interpreted as global sections of certain invertible sheaves on the flag variety, so in particular as homogeneous functions on the latter. This solved a long-time open problem of finding a "standard monomial basis" for the algebra of homogeneous functions on the flag variety.

11.11. Crystals are a powerful tool when no extensions or higher homology are involved. Thus for example the direct sum decomposition described in 10.7(v) is naturally realized as disjoint union (of crystals) so not surprisingly they are well adapted to studying integrable modules. However they have so far proved useless in describing the characters of arbitrary simple highest weight modules (which admit non-trivial extensions) or in describing the higher homologies of Demazure modules - see 10.9. Whether crystals can be adapted to these wider problems is a key open question.

11.12. Fix a Cartan datum (4.7) and recall 7.1.

Definition. *A crystal B is a set with maps*

(i) $wt : B \to P,\ \varepsilon_i, \varphi_i : B \to \mathbb{Z} \cup \{-\infty\},$

(ii) $e_i, f_i : B \to B \cup \{0\},\ e_i 0 = f_i 0 = 0,$

satisfying

(C1) $\varphi_i(b) = \varepsilon_i(b) + \alpha_i^\vee(wt\ b),$

(C2) *If $b \in B$ and $e_i b \in B$ then*

$$wt(e_i b) = wt\ b + \alpha_i,\ \ \varepsilon_i(e_i b) = \varepsilon_i(b) - 1,\ \ \varphi_i(e_i b) = \varphi_i(b) + 1.$$

If $b \in B$ and $f_i b \in B$ then

$$wt(f_i b) = wt\ b - \alpha_i,\ \ \varepsilon_i(f_i b) = \varepsilon_i(b) + 1,\ \ \varphi_i(f_i b) = \varphi_i(b) - 1.$$

(C3) *For all $b, b' \in B$ one has $b' = e_i b \Longleftrightarrow b = f_i b'$,*

(C4) *For $b \in B$, if $\varphi_i(b) = -\infty$, then $e_i b = f_i b = 0$, for all $i \in I$.*

So far the matrix $\alpha_i^\vee(\alpha_j)$ imposes no particular condition. However a crystal is called upper (resp. lower) normal if $\varepsilon_i(b) = max\{n \,\big|\, e_i^n b \neq 0\}$ (resp. $\varphi_i(b) = max\{n \,\big|\, f_i^n b \neq 0\}$).

If both hold then a crystal is called normal. Note that normality implies that $\alpha_i^\vee(\alpha_i) = 2$, given $b, e_i b \in B$. Set $wt_i(b) = \alpha_i^\vee(wt\ b)$.

11.13. Let B be a crystal. For any $\omega \in P$ set $B_\omega = \{b \in B \,\big|\, wt\ b = \omega\}$. As in 8.2 view P as a multiplicative group and define the formal character $ch\ B$ of B

$$ch\ B = \sum_{\omega \in P} \# B_\omega e^\omega.$$

This belongs to $\mathbb{Z}P$. Recall 6.7.

Lemma. *Suppose B is a normal crystal. Then $ch\ B$ is W invariant.*

Proof. Fix $\omega \in P$ and $i \in I$. Assume $\alpha_i^\vee(\omega) \geq 0$ and take $b \in B_\omega$. Normality implies that $\varepsilon_i(b) \geq 0$ and so $\varphi_i(b) \geq \alpha_i^\vee(\omega)$ by (C1). Then by (C2) and normality $f_i^{\alpha_i^\vee(\omega)} b \neq 0$ and has weight $\omega - \alpha_i^\vee(\omega) = s_i\omega$. Finally by (C3) the map $b \mapsto f_i^{\alpha_i^\vee(\omega)} b$ is a bijetion of B_ω onto $B_{s_i\omega}$. A similar result holds if $\alpha_i^\vee(\omega) \leq 0$.

11.14. From 11.13 we can expect the normal crystal to be infinite whenever W is infinite. Of course there is always the singleton crystal of character e^0.

Example. Assume $\ell = 2$ and that $\alpha_i^\vee(\alpha_j) = 2$ if $i = j$ and $\alpha_i^\vee(\alpha_j) = -2$ if $i \neq j$. Then $s_1\omega_1 = \omega_1 - \alpha_1$, $s_2 s_1 \omega_1 = \omega_1 - \alpha_1 - 2\alpha_2$, $(s_1 s_2)^n s_1 \omega_1 = \omega_1 - [(n+1)^2\alpha_1 + n(n+1)\alpha_2] :$ $n \geq 1$. In this case any normal crystal with an element of weight ω_1 is infinite.

11.15. A crystal $B(\lambda)$ is said to be of highest weight $\lambda \in P$ if there exists $b_\lambda \in B(\lambda)$ of weight λ such that

1) $e_i b_\lambda = 0,\ \forall\ i \in I$.

2) $B(\lambda)$ is generated by the $f_i : i \in I$ acting on b_λ.

One checks that an element satisfying 1), 2) is unique and we call it the primitive element of $B(\lambda)$. It is clear that for a normal crystal condition 1) imposes that $\lambda \in P^+$. For a highest weight normal crystal of weight λ to exist we must have $\alpha_i^\vee(\alpha_j) \leq 0$ whenever $\alpha_i^\vee(\lambda) > 0$ and $\alpha_j^\vee(\lambda) = 0$. Indeed $s_i\lambda = \lambda - \alpha_i^\vee(\lambda)\alpha_i$, so $s_j s_i \lambda = \lambda - \alpha_i^\vee(\lambda)\alpha_i +$

$\alpha_i^\vee(\lambda)\alpha_j^\vee(\alpha_i)\alpha_j$. Setting $\alpha_i^\vee(\lambda) = r > 0$, $\alpha_j^\vee(\alpha_i) = s$ and assuming $s > 0$ we conclude that $e_j^{rs} f_i^r b_\lambda \neq 0$ contradicting 2).

Lemma. *Suppose that there exists a normal highest weight crystal for every $\lambda \in P^+$. Then $\alpha_i^\vee(\alpha_j) \leq 0$, $\forall\ i, j$ distinct. Furthermore $\alpha_i^\vee(\alpha_j) \neq 0 \Longleftrightarrow \alpha_j^\vee(\alpha_i) \neq 0$.*

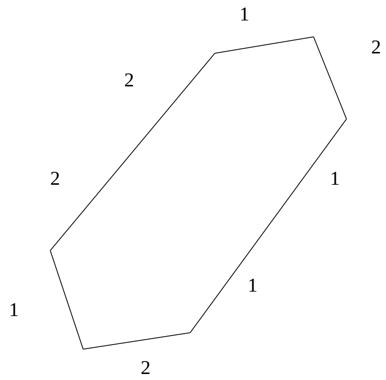
Proof. It remains to verify the last part. Suppose $-\alpha_j^\vee(\alpha_i) = s > 0$ and $\alpha_i^\vee(\alpha_j) = 0$. Then $(s_j s_i)^2 \omega_i = \omega_i + s\alpha_j$ and we obtain a contradiction as before. □

11.16. One may note the remarkable fact that the existence of enough normal highest weight crystals exactly requires that the matrix A with entries $\alpha_i^\vee(\alpha_j)$ defines a (not necessarily symmetrizable) Kac-Moody algebra. In fact any matrix A gives an algebra $\mathfrak{g}(A)$ defined by general procedure outlined in 4.5 - 4.8. However the above Kac-Moody conditions were introduced in order to be able to obtain the W group as a subgroup of *Aut* $\mathfrak{g}$. Symmetrizability is needed to obtain a Casimir invariant, though much of the theory goes through without it. For example for $\lambda \in P^+$ the quotient $Q(\lambda) := M(\lambda) \Big/ \sum_{\alpha \in \pi} M/s_\alpha.\lambda)$ occurring in 10.7(ii) is clearly integrable. A rather difficult result proved independently by S. Kumar and O. Mathieu shows that it satisfies the conclusion of 10.7(iii). However it is still an open problem to show that $Q(\lambda)$ is simple (when symmetrizability is not imposed).

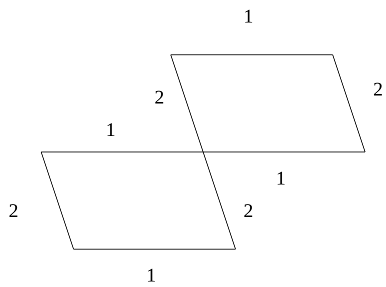
The Casimir invariant and hence the symmetrizability of A is needed to obtain the quantum analogue of these modules and so is needed in Kashiwara's theory which takes their $q \to 0$ limit. However it is not needed in Littelmann's purely combinational theory of crystals.

11.17. A normal crystal of given highest weight is not uniquely determined.

Example. Take $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. The corresponding Kac-Moody Lie algebra is $\mathfrak{sl}(3)$. Its adjoint representation has highest weight $\lambda = \omega_1 + \omega_2$ and leads to the crystal



having 8 elements, two of which have zero weight. On the other hand



is a highest weight normal crystal of the same highest weight $\omega_1 + \omega_2$; but has only 7 elements. In this case $f_1 f_2 b_\lambda = f_2 f_1 b_\lambda$. Thus had the above crystal arisen from a finite dimensional module over a Kac-Moody Lie algebra we would need $\alpha_1^\vee(\alpha_2) = 0$, that is $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and the Lie algebra is $\mathfrak{sl}(2) \times \mathfrak{sl}(2)$. However no module for this algebra leads to such a crystal.

11.18. A morphism ψ of crystals is a map $B_1 \cup \{0\} \to B_2 \cup \{0\}$ with $\psi(0) = 0$ satisfying for all $i \in I$,

(i) ψ *commutes with* $wt,\ \varepsilon_i,\ \varphi_i$,

(ii) $\psi(e_i b) = e_i \psi(b), \quad if \quad e_i b \neq 0,$

(iii) $\psi(f_i b) = f_i \psi(b), \quad if \quad f_i b \neq 0.$

Notice that (ii) permits $\psi(e_i b) \neq 0$ when $e_i b = 0$. One calls B_1 a subcrystal if ψ is an embedding. A subcrystal B_1 is exactly obtained by deleting from the graph of B_2 all vertices in $B_2 \setminus B_1$ and all edges not joining vertices in B_1. An embedding is called full if $e_i \psi(b) \in B_2 \iff e_i b \in B_1$, equivalently if e_i commutes with ψ. A morphism ψ is called strict if it commutes with $e_i, f_i, \ \forall\, i \in I$. A strict subcrystal is the image of a strict embedding. In this case no edges are deleted.

12. Tensor Product and the Kashiwara Function

12.1. Let B_1, B_2 be crystals. Then one may give the Cartesian product $B_1 \times B_2$ the structure of a crystal denoted by $B_1 \otimes B_2$ and called the tensor product of B_1, B_2. The tensor product is not commutative; but it is associative. This is made transparent by directly writing down the crystal rules for the multiple tensor product. They are described below.

Let $B_1, B_2, \cdots, B_n$ be crystals. Given $b = b_1 \otimes b_2 \otimes \cdots \otimes b_n \in B_1 \otimes B_2 \otimes \cdots \otimes B_n$ define the Kashiwara functions $k \mapsto r^i_k(b)$, with $i \in I, \ k \in \{1, 2, \cdots, n\}$ through

$$r^i_k(b) = \varepsilon_i(b_k) - \sum_{1 \leq j < k} \alpha_i^\vee(wt \ b_j).$$

For each i, let $p^i(b)$ (resp. $g^i(b)$) be the smallest (resp. largest) value of k such that $r^i_k(b)$ takes its maximum value $R^i(b)$. Then

1) $\varepsilon_i(b) = R^i(b), \ wt \ b = \sum wt \ b_k, \ \varphi_i(b) = \varepsilon_i(b) + \alpha_i^\vee(wt \ b).$

2) $e_i(b_1 \otimes b_2 \otimes \cdots \otimes b_n) = b_1 \otimes \cdots \otimes e_i b_p \otimes \cdots \otimes b_n$, where $p = p^i(b)$, that is e_i goes into the $p^i(b)$-th place of b.

3) f_i goes in the $g^i(b)$-th place of b.

12.2. Observe that if $e_j b \neq 0$, then

$$r^i_k(e_j b) = \begin{cases} r^i_k(b) & : \quad k < p^j(b) \\ r^i_k(b) - 1 & : \quad k = p^j(b) \\ r^i_k(b) - \alpha^\vee_i(\alpha_j) & : \quad k > p^j(b) \end{cases} .$$

Since $\alpha^\vee_i(\alpha_i) = 2$, it follows that $R^i(e_i b) = R^i(b) - 1$, as required by (C2). Again $g^i(e_i b) = p^i(b)$ and so if $e_i b \neq 0$ one obtains $f_i e_i b = b$, as required by (C3).

Again if $f_j b \neq 0$, then

$$r^i_k(f_j b) = \begin{cases} r^i_k(b) & : \quad k < g^j(b) \\ r^i_k(b) + 1 & : \quad k = g^j(b) \\ r^i_k(b) + \alpha^\vee_i(\alpha_j) & : \quad k > g^j(b) \end{cases} .$$

Thus $R^i(f_i b) = R(b) + 1$ and $p^i(f_i b) = g^i(b)$. From this and the previous computation, it follows that $B_1 \otimes B_2 \otimes \cdots \otimes B_n$ is a crystal.

One easily checks that when $n = 2$ one has

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i b_1 \otimes b_2 & : \quad \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes e_i b_2 & : \quad otherwise \end{cases}$$

$(*)$

$$f_i(b_1 \otimes b_2) = \begin{cases} f_i b_1 \otimes b_2 & : \quad \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes f_i b_2 & : \quad otherwise \end{cases}$$

Indeed $\varphi_i(b_1) = \varepsilon_i(b_1) + \alpha^\vee_i(b_1) = r^i_1(b) + \alpha^\vee_i(b_1) = r^i_2(b) + \alpha^\vee_i(b_1) = \varepsilon_i(b_2)$, from which the assertion follows.

12.3. The main interest of the multiple tensor product is that one may build a complicated crystal from simpler ones. The key building blocks are the crystals defined below.

For each $i \in I$ let B_i be the crystal with elements $\{b_i(n) : n \in -\mathbb{N}\}$ satisfying the rules

$$wt\ b_i(n) = n\alpha_i,\ \varphi_i(b_i(n)) = n,\ e_i b_i(n) = b_i(n+1),\ f_i b_i(n) = b_i(n-1)$$

$$\varphi_j(b_i(n)) = -\infty,\ e_j(b_i(n)) = 0,\ f_j(b_i(n)) = 0, \quad for \quad j \neq i.$$

These crystals are very far from normal. However when $|I| = 1$, this crystal is at least upper normal. In this case it is modelled on the Verma module $M(0)$ for $\mathfrak{sl}(2)$, more precisely on its $\mathcal{O}$-dual $\delta M(0)$. We now describe an analogue of $\delta M(0)$ for an arbitrary Kac-Moody algebra $\mathfrak{g}$.

12.4. Starting from the right, fix any countable sequence $\mathcal{S} = (\cdots, i_2, i_1)$ with $i_j \in I$ in which each index occurs infinitely many times and all nearest neighbours are distinct. Let $\hat{B}_{\mathcal{S}}(\infty)$ denote the subset of $\cdots \otimes B_{i_2} \otimes B_{i_1}$, in which the j^{th} entry differs from $b_{i_j}(0)$ only finitely many times. One easily checks from the rules in 12.1 that the multiple tensor product rules give $\hat{B}_{\mathcal{S}}(\infty)$ the structure of a crystal.

Clearly the element b_∞ with entries $b_{i_j}(0),\ \forall\, j \in \mathbb{N}^+$, lies in $\hat{B}_{\mathcal{S}}(\infty)$. Let $B_{\mathcal{S}}(\infty)$ be the subcrystal of $\hat{B}_{\mathcal{S}}(\infty)$ generated by b_∞.

12.5. We shall prove some very remarkable results about $B_{\mathcal{S}}(\infty)$. First it is independent of the choice of $\mathcal{S}$ and we shall simply denote it by $B(\infty)$, which for the moment we can regard as taking a fixed $\mathcal{S}$. Secondly it is an upper normal crystal. This sounds innocuous; but is a deep fact. Indeed one easily checks given $b \in \hat{B}_{\mathcal{S}}(\infty)$ one has

$$\varepsilon_i(b) = 0,\ \forall\, i \in I \Longleftrightarrow b = b_\infty.$$

Thus the upper normality of $B(\infty)$ implies that given $b \in B(\infty)$ one has

$$e_i b = 0,\ \forall\, i \in I \Longleftrightarrow b = b_\infty.$$

Obviously this implies that 11.15(1) holds with respect to b_∞ and one checks from the crystal rules that 11.15(2) also holds. Thus $B(\infty)$ is a highest weight crystal with primitive element b_∞ of weight 0.

A truly remarkable fact is that the formal character $ch\ B(\infty)$ of $B(\infty)$ coincides with that of the Verma module $M(0)$, though it is better to view $B(\infty)$ as a crystal model for $\delta M(0)$. However whereas the construction of $\delta M(0)$ is very simple - it is just (for $\mathfrak{g}$ semisimple) the space of regular functions on the open Bruhat cell - the above combinatorial construction of $B(\infty)$ is rather complex. Indeed it is not all obvious how it sits in a given $\hat{B}_{\mathcal{S}}(\infty)$.

12.6. There is also an important simplification in the crystal rules for $B(\infty)$, namely the $e_i : i \in I$ (resp. $f_j : j \in I$) satisfy the Verma identities on $B(\infty)$. For example $f_i f_j^2 f_i = f_j f_i^2 f_j$ if $\alpha_i^\vee(\alpha_j) = \alpha_j^\vee(\alpha_i) = -1$. However unlike the case of the Verma module one does *not* have $e_i f_j = f_j e_i \ \ (i \neq j)$ on $B(\infty)$.

12.7. For each $\lambda \in P^+$ let $S(\lambda) = \{s_\lambda\}$ be the singleton crystal defined by the rules $wt\ s_\lambda = \lambda,\ \varepsilon_i(s_\lambda) = -\alpha_i^\vee(\lambda),\ \forall\ i \in I.$

Set $\mathcal{E}_i = \bigcup\limits_{n\in\mathbb{N}} e_i^n$ (resp. $\mathcal{F}_i = \bigcup\limits_{n\in\mathbb{N}} f_i^n$) and $\mathcal{E}$ (resp. $\mathcal{F}$) the monoid they generate.

Admit that $B(\infty)$ is upper normal.

Lemma. *The element $b_\infty \otimes s_\lambda$ of $B(\infty) \otimes S(\lambda)$ generates a strict subcrystal $B(\lambda)$ which is furthermore normal and of highest weight λ. In particular $B(\lambda) = \mathcal{F}(b_\infty \otimes s_\lambda)$.*

Proof. From 12.2$(*)$ $f_i(b \otimes s_\lambda) = f_i b \otimes s_\lambda \iff \varphi_i(b) + \alpha_i^\vee(\lambda) > 0$. Take $b \otimes s_\lambda \in B(\lambda)$. Then $\varphi_i(b) + \alpha_i^\vee(\lambda) \geq 0$ and so $\varepsilon_i(b \otimes s_\lambda) = max\{\varepsilon_i(b), -\alpha_i^\vee(\lambda) - \alpha_i^\vee(wt\ b)\} = \varepsilon_i(b) \geq 0$, by the upper normality of $B(\infty)$. Thus $\varphi_i(b \otimes s_\lambda) = \varphi_i(b) + \alpha_i^\vee(\lambda) \geq 0$. Moreover equality holds if and only if $f_i(b \otimes s_\lambda) = b \otimes f_i s_\lambda = 0$. Thus $B(\lambda)$ is lower normal. Again since $\varphi_i(b) + \alpha_i^\vee(\lambda) \geq 0$, we have $e_i(b \otimes s_\lambda) = e_i b \otimes s_\lambda$, whilst $\varepsilon_i(b \otimes s_\lambda) = \varepsilon_i(b)$. Thus the upper normality of $B(\infty)$ implies that of $B(\lambda)$. Finally $b_\infty \otimes s_\lambda$ is the unique primitive element of $B(\lambda)$ through the corresponding property for $B(\infty)$ noted in 12.5. □

12.8. Kashiwara obtained $B(\infty)$ by taking a $q \to 0$ limit of the quantized enveloping algebra obtaining its upper normality in the process. Thus we may view the above as containing the essence of Kashiwara's construction of a family $\mathcal{B} = \{B(\lambda) : \lambda \in P^+\}$ of normal highest weight crystals.

12.9. The family $\mathcal{B}$ has two remarkable properties. First the character of $B(\lambda)$ coincides with that of the simple highest module $V(\lambda)$ and is hence given by the Weyl character formula. Secondly $B(\lambda) \otimes B(\mu)$ is a disjoint union of some of the $B(\nu) : \nu \in P^+$ occurring with the same multiplicity as the occurrence of $V(\nu)$ in $V(\lambda) \otimes V(\mu)$. All this may seem natural and indeed obtained from the $q \to 0$ limit; but it is extremely difficult to prove purely combinatorially.

12.10. A further remarkable fact is that $B(\infty)$ admits an involution $\star$ obtained (by Kashiwara) through a rather simple antiautomorphism on $U_q(\mathfrak{n}^-)$ and taking the $q \to 0$ limit. Through 12.7 we can view $B(\lambda)$ as a subset of $B(\infty)$. Slightly surprisingly $\star$ does not leave $B(\lambda)$ stable; but we can regard $B(\lambda)^\star$ as a subset of $B(\infty)$. Further regarding $B(\mu)$ as a subset of $B(\infty)$ through 12.7 we can consider $B(\mu) \cap B(\lambda)^\star$.

From the tensor product rules it follows easily that every element $c \in B(\lambda) \otimes B(\mu)$ satisfying $\mathcal{E}c = \{0\}$ takes the form $b_\lambda \otimes b$, for some $b \in B(\mu)$. One can show (but it is not obvious) that $\mathcal{F}c$ is a subcrystal of $B(\lambda) \otimes B(\mu)$, hence of highest weight. We show that the set of primitive elements of $B(\lambda) \otimes B(\mu)$ is exactly $\{b_\lambda \otimes b \,\big|\, b \in B(\mu) \cap B(\lambda)^\star\}$.

13. A Uniqueness Theorem

13.1. Let $\mathcal{B} = \{B(\lambda) : \lambda \in P^+\}$ be a family of normal highest weight crystals with b_λ the primitive element of $B(\lambda)$. We say that $\mathcal{B}$ is closed if for all $\lambda, \mu \in P^+$, the subset $\mathcal{F}(b_\lambda \otimes b_\mu)$ of $B(\lambda) \otimes B(\mu)$ is a crystal (necessarily having $b_\lambda \otimes b_\mu$ as its primitive element so of highest weight $\lambda + \mu$) and furthermore isomorphic to $B(\lambda + \mu)$. We noted (without proof) in 12.9 that the Kashiwara family is closed.

Let $\mathcal{B} = \{B(\lambda) : \lambda \in P^+\}$ be a closed family of normal highest weight crystals. We show that $\mathcal{B}$ is unique up to isomorphism.

13.2. The proof of the above assertion obtains by constructing from $\mathcal{B}$ an upper normal crystal $B(\infty)$ whose structure is canonically determined and from which $\mathcal{B}$ can be recovered. To a large extent this just reverses the procedure described in Sects. 12.3 - 12.9.

13.3. Let $\succcurlyeq$ be the order relation on P defined by $\lambda \succcurlyeq \mu$ if $\lambda - \mu \in P^+$.

Lemma. *Take $\lambda, \mu \in P^+$. The map $\psi_{\lambda,\lambda+\mu} : b \mapsto b \otimes b_\mu$ is an injection of $B(\lambda)$ into $B(\lambda+\mu)$, commuting with the ε_i, $e_i : i \in I$.*

Proof. Normality implies that $\varepsilon_i(b_\mu) = 0$ and $\varphi_i(b) \geq 0$, $\forall i \in I, b \in B(\lambda)$. Thus $e_i(b \otimes b_\mu) = e_i b \otimes b_\mu$. Moreover if $f_i b \neq 0$, then $\varphi_i(b) > 0$ and $f_i(b \otimes b_\mu) = f_i b \otimes b_\mu$. This implies the required assertions. □

13.4. As a set $B(\infty)$ can now be defined as the inductive limit

$$B(\infty) = \varinjlim B(\lambda)$$

with respect to the inclusions $\psi_{\lambda,\mu} : \mu \succcurlyeq \lambda$. Then $\psi_\lambda : b \mapsto \varinjlim \psi_{\lambda,\mu}(b)$ is an injection of $B(\lambda)$ into $B(\infty)$. The crystal operations on $B(\infty)$ are defined as follows.

Given $b \in B(\infty)$, there exists $\lambda \in P^+$ such that $b \in Im\psi_\lambda$. Let us write $b = \psi_\lambda(c)$ and set $e_i b = \psi_\lambda(e_i c), \varepsilon_i(b) = \varepsilon_i(c)$. By 13.3 this is independent of the choice of λ. By construction ψ commutes with the e_i, ε_i, $\forall i \in I$. In particular $B(\infty)$ is upper normal. Again replacing λ by $\lambda+\lambda'$ with $\alpha_i^\vee(\lambda') > 0$ we can assume $f_i c \neq 0$ and we set $f_i b = \psi_\lambda(f_i c)$, which is again independent of choices (by the first result and (C3)).

We set $wt\ b = wt\psi_\lambda(c) := wt\ c - \lambda$. This is again independent of choices. Finally we define $\varphi_i : i \in I$ on $B(\infty)$ through (C1).

13.5. Recall the crystal S_λ defined in 12.7.

Lemma. *For all $\lambda \in P^+$, the map $\Psi_\lambda : b \mapsto \psi_\lambda(b) \otimes s_\lambda$ is a strict monomorphism of $B(\lambda)$ into $B(\infty) \otimes S_\lambda$.*

Proof. One has $wt\Psi_\lambda(b) = wt\Psi_\lambda(b) + \lambda = wt\ b$. Again $\varphi_i(\psi_\lambda(b)) = \varepsilon_i(\psi_\lambda(b)) + \alpha_i^\vee(wt\psi_\lambda(b)) = \varepsilon_i(b)+\alpha_i^\vee(wt\ b)-\alpha_i^\vee(\lambda) = \varphi_i(b)-\alpha_i^\vee(\lambda)$ and so $\varphi_i(\psi_\lambda(b))-\varepsilon_i(s_\lambda) = \varphi_i(b) \geq 0$, by the normality of $B(\lambda)$. Hence $\varepsilon_i(\Psi_\lambda(b)) = \varepsilon_i(\psi_\lambda(b)\otimes s_\lambda) = max\{\varepsilon_i(b), \varepsilon_i(b)-\varphi_i(b)\} = \varepsilon_i(b)$. For a similar reason $e_i(\Psi_\lambda(b)) = e_i(\psi_\lambda(b) \otimes s_\lambda) = e_i(\psi_\lambda(b)) \otimes s_\lambda = \psi_\lambda(e_ib) \otimes s_\lambda = \Psi_\lambda(e_ib)$.

Finally suppose $f_ib \neq 0$. Then $\varphi_i(\psi_\lambda(b)) - \varepsilon_i(s_\lambda) = \varphi_i(b) > 0$ as so by 12.2($*$), $f_i(\Psi_\lambda(b)) = f_i(\psi_\lambda(b) \otimes s_\lambda) = f_i(\psi_\lambda(b)) \otimes s_\lambda = \psi_\lambda(f_ib) \otimes s_\lambda = \Psi_\lambda(f_ib)$. On the other hand if $f_ib = 0$, then $\varphi_i(b) = 0$ and so by 12.2($*$), $f_i(\Psi_\lambda(b)) = \psi_\lambda(b) \otimes f_is_\lambda = 0$, also. □

13.6. Recall the crystals $B_i : i \in I$ defined by 12.3.

Proposition. *For all $i \in I$, there exists a strict monomorphism of $B(\infty)$ into $B(\infty)\otimes B_i$, sending b_∞ to $b_\infty \otimes b_i(0)$.*

Proof. Fix $f \in \mathcal{F}, i \in I$. One calls f' an i-submonomial (resp. submonomial) of f if it can be obtained by deleting f_i factors (resp. f_j factors for any $j \in I$).

Take $\lambda \in P^+$ such that $\alpha_i^\vee(\lambda) = 0$ and such that for all $j \in I \setminus \{i\}$ and every submonomial f'' of f one has $\alpha_j^\vee(wt\ f''b_\lambda) > 0$. Take $\mu \in P^+$ such that $\alpha_j^\vee(\mu) = 0 : j \neq i$.

We claim that for all $\lambda, \mu \in P^+$ chosen as above there exists an i-submonomial f' of f, and $m \in \mathbb{N}$ such that $f(b_\lambda \otimes b_\mu) = f'b_\lambda \otimes f_i^m b_\mu$.

For any submonomial f'' of f one has

$$\begin{aligned}\varphi_i(f''b_\lambda) &= \alpha_i^\vee(wt\ f''b_\lambda) + \varepsilon_i(f''b_\lambda)\\ &= \alpha_i^\vee(wt\ f''b_\lambda) - \alpha_i^\vee + \varepsilon_i(f''b_\lambda)\\ &= \varphi_i(f''b_\infty), \quad \textit{since} \quad \alpha_i^\vee(\lambda) = 0.\end{aligned}$$

Consequently $\varphi_i(f''b_\lambda)$ is independent of λ.

Take $j \in I \setminus \{i\}$ and $m' \in \mathbb{N}$ such that $f_i^{m'}b_\mu \neq 0$. Then $wt(e_jf_i^{m'}b_i) = \alpha_j + \mu - m'\alpha_i \notin \mu - \mathbb{N}'$ and so is not a weight of $B(\mu)$. Hence $e_jf_i^{m'}b_\mu = 0$. Consequently

$\varepsilon_j(f_i^{m'}b_\mu) = 0$ by normality. Again by normality for any submonomial f'' of f one has $\varphi_j(f''b_\lambda) = \varepsilon_j(f''b_\lambda) + \alpha_j^\vee(wt\ f''b_\lambda) \geq \alpha_j^\vee(wt\ f''\beta_\lambda) > 0$, through the choice of λ.

We conclude that $\varphi_j(f''b_\lambda) - \varepsilon_j(f_i^{m'}b_\mu) > 0$ and so our claim results via 12.2$(*)$ and induction on the length of f. Furthermore $\varepsilon_j(b_i(-m')) = -\infty < \varphi_j(f''b_\infty)$ and $\varepsilon_i(b_i(-m')) = m' = \varepsilon_i(f_i^{m'}b_\mu)$. Thus $f(b_\infty \otimes b_i(0)) = f'b_\infty \otimes b_i(-m)$ with f', m as above.

Since $\mathcal{B}$ is closed $b_\lambda \otimes b_\mu$ and $b_{\lambda+\mu}$ generate the same crystal. Moreover the above conditions on λ, μ still allow the $\alpha_k^\vee(\lambda+\mu)$ to be large enough relative to the given $f \in \mathcal{F}$. From the definition of $B(\infty)$ we can view $b_{\lambda+\mu}$ as b_∞ with respect to any finite subset $\mathcal{F}^0 \in \mathcal{F}$. The previous calculation shows that for all $f_1, f_2 \in \mathcal{F}^0$ the condition $f_1 b_\infty = f_2 b_\infty$ implies $f_1(b_\infty \otimes b_i(0)) = f_2(b_\infty \otimes b_i(0))$. Since $B(\infty) = \mathcal{F}b_\infty$ we obtain a map $\psi : fb_\infty \mapsto f(b_\infty \otimes b_i(0))$ of $B(\infty)$ into $B(\infty) \otimes B_i$ commuting with the $f_k : k \in I$. In addition $f(b_\infty \otimes b_i(0)) = f'b_\infty \otimes b_i(-m)$, which can only be zero if $f'b_\infty = 0$ and so ψ is injective. As $\varepsilon_k(b_i(0)) \leq 0$ and $\varphi_k(b_\infty) = 0,\ \forall\ k \in I$ one has $\varepsilon_k(b_\infty \otimes b_i(0)) = \varepsilon_k(b_\infty)$. Through (C2) and the above we conclude that ψ commutes with ε_k. Again $wt\ b_\infty = 0$, so ψ commutes with wt.

Take $b \in B(\infty)$. If $e_k b \neq 0$, then $\psi(b) = \psi(f_k e_k b) = f_k \psi(e_k b)$. Hence $e_k\psi(b) = \psi(e_k b)$. If $e_k b = 0$, then $\varepsilon_k(b) = 0$ since $B(\infty)$ is upper normal. Thus $\varepsilon_k(\varphi(b)) = 0$. From the above we can write $\psi(b) = f'b_\infty \otimes b_i(-m)$. Now $\varepsilon_k(f'b_\infty) \geq 0$, by upper normality and $0 = \varepsilon_k(\psi(b)) = max\{\varepsilon_k(f'b_\infty), \varepsilon_k(b_i(-m)) - \alpha_k^\vee(wt\ f'b_\infty)\}$. We conclude that $\varepsilon_k(f'b_\infty) = 0$ and $e_k\psi(b) = e_k f'b_\infty \otimes b_i(-m) = 0$, again by the tensor product rule and the upper normality of $B(\infty)$. □

13.7. Let us observe that 13.6 implies that $B(\infty)$ embeds into the crystal $\hat{B}_{\mathcal{S}}(\infty)$ constructed in 12.4. Take $f \in \mathcal{F}$. Suppose $\mathcal{S} = \{\cdots, i_2, i_1\}$. We can write $f = f_{i_s}^{k_s} \cdots f_{i_1}^{k_1}$ for some $i_j \in I, k_j \in \mathbb{N}$. Since $\varphi_i(b_\infty) = 0$ and $\varepsilon_i(b_i(0)) = 0$, it follows from 12.2$(*)$ that $\psi(f_i^k b_\infty) = f_i^k(b_\infty \otimes b_i(0)) = b_\infty \otimes b_i(-k)$. Repeated application of this gives $k_1', k_2', \cdots, k_s' \in \mathbb{N}$ and a strict monomorphism $\psi : B(\infty) \to B(\infty) \otimes B_{i_s} \otimes B_{i_{s-1}} \otimes \cdots B_{i_1}$

with $\psi(fb_\infty) = f(b_\infty \otimes b_{i_s}(0) \otimes \cdots \otimes b_{i_1}(0_1)) = b_\infty \otimes b_{i_s}(-k'_s) \otimes \cdots \otimes b_{i_1}(-k'_1) \subset \hat{\mathcal{B}}_{\mathcal{S}}(\infty)$, as required. Notice this also proves that $B(\infty)$ is independent of $\mathcal{S}$.

Since $B(\infty)$ is canonically determined through the construction of 12.4 and that $B(\lambda)$ can be recovered from $B(\infty)$ through 12.6 which just reverses 13.5 we obtain the

Theorem. *A closed family $\mathcal{B} = \{B(\lambda) : \lambda \in P^+\}$ of normal highest weight crystals is determined up to isomorphism.*

14. The Littelmann Path Model

14.1. We shall describe the Littelmann path model of a family $\mathcal{B} = \{B(\lambda) : \lambda \in P^+\}$ of normal highest weight crystals making as close as possible an analogy with the Kashiwara theory. All intervals are taken in the rationals $\mathbb{Q}$. For example $[0,1] := \{a \in \mathbb{Q} \big| 0 \leq a \leq 1\}$.

14.2. Let $\mathbb{P}$ denote the set of piecewise linear paths $b : [0,1] \to \mathbb{Q}P$ satisfying $b(0) = 0,\ b[1) \in P$.

Let n be a positive integer and $b : [0,n] \to \mathbb{Q}P$ a piecewise linear path satisfying $b(0) = 0,\ b(n) \in P$. A reparametrization is a piecewise linear increasing surjective map $f : [0,n] \to [0,1]$. It turns b into bf which is an element of $\mathbb{P}$. We identify paths which become equal under reparametrization.

Given $b_1, b_2 \in \mathbb{P}$ we define their concatenation as the path defined by

$$(b_1 \otimes b_2)(t) = \begin{cases} b_1(t) & : \quad 0 \leq t \leq 1 \\ b_1(1) + b_2(t-1) & : \quad 1 \leq t \leq 2 \end{cases}$$

and reparametrization.

14.3. For any $\lambda \in \mathbb{Q}P$, let b_λ be the path defined by $b_\lambda(t) = t\lambda$. Then we may view $b \in \mathbb{P}$ as a concatenation $b = b_{\lambda_1} \otimes b_{\lambda_2} \otimes \cdots \otimes b_{\lambda_n} : \lambda_k \in \mathbb{Q}P$. Such a presentation is analogous to that of the Kashiwara crystal $\hat{B}_{\mathcal{S}}(\infty)$.

14.4. We define the Littelmann functions $t \mapsto h^i_b(t) := -\alpha^\vee_i(b(t)) : i \in I$, on [0,1] and set $m^i_b = max\{h^i_b(t) \bigm| t \in [0,1]\}$. Define $e^i_\pm(b),\ f^i_\pm(b)$ as follows, omitted indices being understood.

Let e_+ (resp f_+) be the smallest (resp. largest) element of [0,1] such that $h(e_+) = m$ (resp. $h(f_+) = m$). If $m \geq 1$ (resp $m - h(1) \geq 1$) let e_- (resp. f_-) be the largest (resp. smallest) element of $[0, e_+]$ (resp. $[f_+, 1]$) such that $h(t) \leq m-1,\ \forall\ t \in [0, e_-]$ (resp. $h(t) \leq m-1,\ \forall\ t \in [f_-, 1]$).

We say that $t \in [0,1]$ breaks $b = b_{\lambda_1} \otimes b_{\lambda_2} \otimes \cdots \otimes b_{\lambda_n}$ at $k \in \{1, 2, \cdots, n\}$ if the interval [0,t] is the maximal one on which $b = b_{\lambda_1} \otimes b_{\lambda_2} \otimes \cdots \otimes b_{\lambda_k}$. We call t a breaking point of b. Obviously every t for which $h(t)$ has a local maximum is a breaking point of b. In particular e_+ breaks b at some k_+. However any $t \in [0,1]$ can be made into a breaking point by simply replacing the appropriate b_{λ_k} by $b_{\lambda_k} \otimes b_{\lambda_k}$ with a suitable parametrization. In this sense e_- breaks b at some k_-.

Set $e_i b = 0$ if $m < 1$. Otherwise let u_r be the r^{th} local maximum of $t \mapsto h(t)$ in $[e_-, e_+]$ counting from the left. If $u_r < e_+$, let v_r be the smallest element of $]u_r, e_+]$ for which $h(v_r) = h(u_r)$. Set $u_0 = 0, v_0 = e_-$. Choose $s \in \mathbb{N}$ so that $u_{s+1} = e_+$ and set $v_{s+1} = 1$. Then define $e_i b$ by leaving b unchanged in the regions $[u_r, v_r] : r \in \{0, \cdots, s+1\}$ and reflecting b by s_i elsewhere. Taking the $v_r : r \in \{0, 1, \cdots, s\}$ to be breaking points of b, this just means replacing λ_k by $s_i \lambda_k$ in each of the regions $[v_r, u_{r+1}] : r \in \{0, 1, \cdots, s\}$. Notice that if λ_k is replaced by $s_i\lambda_k$ then we must have had $\alpha_i^\vee(\lambda_k) \leq 0$ and $k_- < k \leq k_+$. However the converse is false.

An analogous definition is given for $f_i b$ which is similarly doctored in the region $[f_+, f_-]$. More precisely if b^* denotes the dual path $b^*(t) := b(-t) - b(1)$ of b we set $f_i b = (e_i b^*)^*$.

14.5. One easily checks that if $e_i b \neq 0$, then $m^i_{e_i b} = m^i_b - 1,\ (e_i b)(1) = b(1) + \alpha_i$. Thus $f^i_+(e_i b) = e^i_-(b)$ and $f^i_-(e_i b) = e^i_+(b)$. Moreover the above convoluted definition of $e_i b, f_i b$ ensures that $f_i e_i b = b$. By duality if $f_i b \neq 0$, then $m^i_{f_i b} = m^i_b + 1,\ (f_i b)(1) = b(1) - \alpha_i$

and $f_i e_i b = b$. Set $wt\ b = b(1)$. Then taking $\varepsilon_i(b)$ to be the integer part of $m_i(b)$ makes $\mathbb{P}$ a normal crystal.

14.6. Set $\mathbb{P}_0 = \{b \in \mathbb{P} \,\big|\, m_b^i \in \mathbb{Z},\ \forall\ i \in I\}$. This restriction is needed for a number of purposes. For a start it implies that $\varepsilon_i(b) = m_b^i$ and then almost by construction the Kashiwara tensor product rules (12.2(∗)) hold with respect to $b_1 \otimes b_2 : b_1, b_2 \in \mathbb{P}_0$ defined by concatenation. It is interesting to remark that since concatenation is associative the tensor product rules on $\mathbb{P}_0$ are visibly associative, whereas in Kashiwara's more general framework this though still true is not so obvious.

14.7. A second place where the integrality of the m_b^i is needed occurs in the following continuity result. It is also here that the particular definition of e_i, f_i plays a crucial role.

Set

$$c = \max_{i,j \in I} \{|\alpha_i^\vee(\alpha_j)|\}.$$

Given $b, b' \in \mathbb{P}$ set

$$d(b, b') = \max_{t \in [0,1], i \in I} \{|\alpha_i^\vee(b(t) - b'(t)|\}.$$

Lemma. *Take $b, b' \in \mathbb{P}$. Assume $d(b, b') \leq \varepsilon < \frac{1}{2}$. Then*

(i) *If $b, b' \in \mathbb{P}_0$, then $m_b^i = m_{b'}^i$, $\forall\ i \in I$.*

(ii) *If $e_i b \neq 0$ and $e_i b' \neq 0$, then $d(e_i b, e_i b') \leq 2c\varepsilon$.*

(iii) *If $f_i b \neq 0$, and $f_i b' \neq 0$, then $d(f_i b, f_i b') \leq 2c\varepsilon$.*

Proof. Fix $i \in I$. We can suppose $m_b \geq m_{b'}$. Then $|m_b - m_{b'}| = (h_b(e_+(b)) - h_{b'}(e_+(b)) - (h_{b'}(e_+(b')) - h_{b'}(e_+(b))) \leq d(b, b') < \varepsilon < 1$. So $m_b = m_{b'}$ by integrality. Hence (i).

Take $0 \leq u < v \leq 1$. We say that $\varphi : [u, v] \to \mathbb{R}$ is non-constant at $t \in [u, v]$ if $\varphi(t)$ is non-constant in some neighbourhood of t.

Fix $i \in I$. Since $e_i b$ obtains from b by replacing b_{λ_k} by $b_{s_i \lambda_k}$ only when $s_i \lambda_k - \lambda_k \in \mathbb{N}\alpha_i$, it follows that $(e_i b)(t) - b(t) = \varphi(t)\alpha_i$, where φ is a non-decreasing function. Moreover

$\varphi(t) \geq h_b(t)$, with equality when φ is non-constant at t. Similar assertions hold for φ' defined by $e_i b' - b' = \varphi' \alpha_i$.

Recall that $|m_b - m_{b'}| \leq \varepsilon$. Assume $e_i b' \neq 0$ and suppose $e_+(b) \leq e_-(b')$. Then $h_{b'}(e_+(b)) \leq m_{b'} - 1$, by definition of $e_-(b')$, so $\varepsilon \geq d(b,b') \geq \Big| \ h_b(e_+(b)) - h_{b'}(e_+(b)) \Big| \geq \Big| \ m_b - (m_{b'} - 1) \Big| \geq 1 - \varepsilon$ which contradicts the choice of ε. Assume further that $e_i b \neq 0$, then similarly $e_+(b') > e_-(b)$. Writing $J = [e_-(b), e_+(b)]$, $J' = [e_-(b'), e_+(b')]$, it follows that $|J \cap J'|$ is of the form $[u,v]$ with $min(e_-(b), e_-(b')) \leq u < v \leq \max(e_+(b), e_+(b'))$. Outside $J \cap J'$, either φ or φ' are constant and so one easily concludes that $|\varphi(t) - \varphi'(t)|$ is non-decreasing in $[0,u]$ and non-increasing in $[v,1]$.

Consider $t \in [u,v]$. If $\varphi(t)$ is non-constant at t then $\varphi(t) = h_b(t)$ and so $\varphi(t) - \varphi'(t) \leq \varepsilon$. Since φ (resp. φ') is non-constant at the end-points of J (resp. J'), neither φ nor φ' can be constant on $J \cap J'$. We conclude that $|\varphi(t) - \varphi'(t)| \leq \varepsilon$ on $[u,v]$ and hence by the first part on [0,1].

Finally

$$d(e_i b, e_i b') \leq d(b,b') + c \max_{t \in [0,1]} |\varphi(t) - \varphi'(t)| \leq 2c\varepsilon,$$

which gives (ii). Assertion (iii) obtains from (ii) through duality. □

14.8. We now describe Lakshmibai-Seshadri (or simply, LS) paths on which notably the definitions of e_i, f_i simplify.

Take $\lambda \in P$. We describe an order relation on $W\lambda$ which is just the reverse Bruhat order on W if $\lambda \in P^{++}$.

Given $\mu, \nu \in W\lambda, \beta \in \Delta^+$, write $\mu \xrightarrow{\beta} \nu$ if $\mu = s_\beta \nu$ and $\beta^\vee(\nu) > 0$. Observe that β is uniquely determined by the pair μ, ν and so we may just write $\mu \longrightarrow \nu$. A sequence $\mu = \nu_1 \longrightarrow \nu_2 \longrightarrow \cdots \longrightarrow \nu_{s+1} = \nu$ of maximal length joining μ, ν is called a Bruhat sequence and one writes $\ell(\mu,\nu) = s$ and $\mu \overset{B}{\leq} \nu$. Moreover if this holds $\nu_i \longrightarrow \nu_{i+1} \longrightarrow \cdots \longrightarrow \nu_j$ for $i \leq i \leq j \leq s$ is again a Bruhat sequence. Note that $\mu \overset{B}{\leq} \nu \Longleftrightarrow \mu \leq \nu$ but the converse is false. The following is a version of the exchange lemma, see 7.6(iii) and 7.7.

Lemma. *Suppose* $\mu \xrightarrow{\beta} \nu$ *with* $\ell(\mu,\nu) = 1$. *If* $\alpha \in \pi$ *satisfies* $\alpha^\vee(\mu) < 0$ *and* $\alpha^\vee(\nu) \geq 0$, *(or* $\alpha^\vee(\mu) \leq 0$ *and* $\alpha^\vee(\nu) > 0$*), then* $\alpha = \beta$.

Proof. If $\alpha \neq \beta$, then $s_\alpha\beta \in \Delta^+$. Moreover $(s_\alpha\beta)^\vee(s_\alpha\mu) = \beta^\vee(\mu) < 0$. Suppose $\alpha^\vee(\mu) < 0$. Then $\mu \xrightarrow{s_\alpha} s_\alpha\mu \xrightarrow{s_\alpha\beta} s_\alpha\mu$. Since $\alpha^\vee(s_\alpha\nu) \leq 0$ this implies $\ell(\mu,\nu) \geq 2$, a contradiction. The second case is similar. □

14.9. The previous result extends a characteristic property of Bruhat order to Bruhat sequences.

Corollary. *Suppose* $\mu \overset{B}{\leq} \nu$ *and* $\alpha \in \pi$.

(i) *If* $\alpha^\vee(\mu) < 0, \alpha^\vee(\nu) \geq 0$, *then* $s_\alpha\mu \overset{B}{\leq} \nu$ *and* $\ell(s_\alpha\mu,\nu) = \ell(\mu,\nu) - 1$.

(ii) *If* $\alpha^\vee(\mu) \leq 0, \alpha^\vee(\nu) > 0$, *then* $\mu \overset{B}{\leq} s_\alpha\nu$ *and* $\ell(\mu,s_\alpha\nu) = \ell(\mu,\nu) - 1$.

(iii) *If* $\alpha^\vee(\mu), \alpha^\vee(\nu) > 0$ *(or* $\alpha^\vee(\mu), \alpha^\vee(\nu) < 0$*) then* $s_\alpha\mu \overset{B}{\leq} s_\alpha\nu$ *and* $\ell(s_\alpha\mu, s_\alpha\nu) = \ell(\mu,\nu)$.

Proof. (i). Since $\mu \xrightarrow{s_\alpha} s_\alpha\mu$ it is immediate that $\ell(\mu,\nu) \geq \ell(s_\alpha\mu,\nu) + 1$. For the converse let $\mu = \nu_1 \xrightarrow{\beta_1} \nu_2 \longrightarrow \cdots \longrightarrow \nu_{s+1} = \nu$, be a Bruhat sequence. By the hypothesis and 14.8 we must have $\alpha = \beta_i$ for some $i \in \{1,2,\cdots,s\}$. Take i minimal with this property. Then $s_\alpha\mu = s_\alpha\nu_1 \xrightarrow{s_\alpha\beta_1} s_\alpha\mu_2 \longrightarrow \cdots s_\alpha\nu_i = \nu_{i+1} \longrightarrow \nu_{i+2} \longrightarrow \cdots \longrightarrow \nu_{s+1} = \nu$, which implies that $\ell(s_\alpha\mu,\nu) \geq s - 1 = \ell(\mu,\nu) - 1$.

(ii). Since $s_\alpha\nu \xrightarrow{\alpha} \nu$, one has $\ell(\mu,\nu) \geq \ell(\mu,s_\alpha\nu) + 1$. For the converse define a Bruhat sequence as in (i). Then by 14.8 we must have $\alpha = \beta_i$, for some $i \in \{1,2,\cdots,s\}$. Take i maximal with this property. Then as above $\ell(\mu,s_\alpha\nu) \geq s - 1 = \ell(\mu,\nu) - 1$.

(iii) follows from (i) and (ii). □

14.10. Take $\lambda \in P^+$. Let $\boldsymbol{\nu} = (\nu_1 \xrightarrow{\beta_1} \nu_2 \longrightarrow \cdots \xrightarrow{\beta_s} \nu_{s+1})$ be a Bruhat sequence of elements in $W\lambda$ and $\mathbf{a} = (0 = a_0 \leq a_1 \leq \cdots \leq a_{s+1} = 1)$ a sequence of rational numbers satisfying $a_i\beta_i^\vee(\nu_i) \in \mathbb{Z}$, for all $i \in \{1,2,\cdots,s\}$. Let $C(\lambda)$ denote the set of all such pairs.

Given $(\mathbf{a}, \boldsymbol{\nu}) \in C(\lambda)$, the LS path $b_{\boldsymbol{\nu},\mathbf{a}}$ of class λ defined by the pair $(\boldsymbol{\nu}, \mathbf{a})$ is given by

$$(*) \qquad b_{(\boldsymbol{\nu},\mathbf{a})}(t) = \sum_{i=1}^{j-1}(a_i - a_{i-1})\nu_i + (t - a_{j-1})\nu_j, \ \forall\, t \in [a_{j-1}, a_j]$$

for all $j = 1, 2, \cdots, s+1$. In the case $s = 0$, then $\boldsymbol{\nu} = (\nu_1)$, $\mathbf{a} = (0, 1)$ and $b_{(\boldsymbol{\nu},\mathbf{a})}$, or simply b_ν, is given by $b_\nu(t) = \nu t$.

Since $\nu_{i+1} = s_{\beta_i}\nu_i = \nu_i - \beta_i^\vee(\nu_i)\beta_i$ we obtain

$$\begin{aligned} b_{(\boldsymbol{\nu},\mathbf{a})}(a_j) = &\ a_j\nu_{j+1} + \sum_{i=1}^{j} a_i(\nu_i - \nu_{i+1}) \\ = &\ a_j\nu_{j+1} - \sum_{i=1}^{j} a_i\beta_i^\vee(\nu_i)\beta_i. \end{aligned}$$

In particular

$$wt(b_{\boldsymbol{\nu},\mathbf{a}}) = b_{\boldsymbol{\nu},\mathbf{a}}(1) \in P.$$

Moreover the previous formula immediately gives the

Lemma. *For all $j = 1, 2, \cdots, s$, one has*

$$\beta_j^\vee(b_{\boldsymbol{\nu},\mathbf{a}}(a_j)) \in \mathbb{Z}.$$

14.11. Take $b \in \mathbb{P}$ and $i \in I$. Call t_0 a left (resp. right) local maximum of h_b^i if for some $t_1 < t_0 < t_2$, the function $h_b^i(t)$ is strictly increasing (resp. decreasing) in $[t_1, t_0]$ (resp. $[t_0, t_2]$) and decreasing (resp. increasing) in $[t_0, t_2]$ (resp. $[t_1, t_0]$).

Lemma. *$b = b_{(\boldsymbol{\nu},\mathbf{a})}$ be an LS path. If t_0 is a left (resp. right) local maximum for h_b^i, then $t_0 = a_j$ for some $j \in \{1, 2, \cdots, s\}$ and $-\alpha_i^\vee(\nu_j) > 0, -\alpha_i^\vee(\nu_{j+1}) \leq 0$ (resp. $-\alpha_i^\vee(\nu_j) \geq 0$, $-\alpha_i^\vee(\nu_{j+1}) < 0$). Moreover $\alpha_i = \beta_j$ and $h_b^i(t_0) \in \mathbb{Z}$.*

Proof. The first part is immediate from definitions. The second follows from the first and 14.8. Then the last part follows from 14.10. □

Remark. Consequently h_b^i is integer at a local maximum.

14.12. A path $b \in \mathbb{P}$ is called monotone if $m_b^i < 1$ or the function h_b^i is strictly increasing in $[e_-^i(b), e_+^i(b)]$, and $m_b^i - h_b^i(1) < 1$ or h_b^i is strictly decreasing in $[f_+^i(b), f_-^i(b)]$. For such paths the definitions of e_i, f_i are much simpler. From 14.11 it is immediate that

Corollary. *Any* LS *path is integral and monotone.*

Remark. Note that for an integral, monotone path b the definitions of e_i (resp. f_i) simplify. Namely one just reflects b in the region $[e_-^i(b), e_+^i(b)]$ (resp. $[f_+^i(b), f_-^i(b)]$) by s_i. Unfortunately for this simplified definition more general paths fail to satisfy 14.7 and 15.13.

14.13. We can write $b_{\boldsymbol{\nu},\mathbf{a}}$ as the concatenation $b_{\nu_1} \otimes b_{\nu_2} \otimes \cdots \otimes b_{\nu_{s+1}}$, where the parametrization is chosen so 14.10(∗) is satisfied. Fix $i \in I$ and suppose $m_b^i \geq 1$ and define k_+ as in 14.4, but now defining k_- to be maximal such that $a_{k_-} \leq e_-(b)$. Observe that by definition, $e_+^i(b)$ is a left (local) maximum of h_b^i and by 14.11, h_b^i is strictly increasing in $[e_-^i(b), e_+^i(b)]$.

Lemma. *Let $b = b_{\boldsymbol{\nu},\mathbf{a}}$ be an* LS *path. Take $i \in I$ and suppose $m_b^i \geq 1$. Then $e_i b_{\boldsymbol{\nu},\alpha} = b_{\boldsymbol{\nu}',\mathbf{a}'}$ where*

$$(\nu'_j, a'_j) = \begin{cases} (\nu_{k_-+1}, e_-) & : \quad j = k_- + 1 \\ (s_i\nu_{j-i}, a_{j-1}) & : \quad k_- + 2 \leq j \leq k_+ + 1 \\ (\nu_j, a_j) & : \quad \text{otherwise.} \end{cases}$$

Moreover $(\boldsymbol{\nu}', \mathbf{a}') \in C(\lambda)$.

Proof. First let us show that $\alpha_i^\vee(\nu_k) < 0$, for all $k : k_- < k \leq k_+$. (This might fail if $a_k = a_{k+1}$.) It holds for $k = k_- + 1, k_+$, by definition of $e_\pm^i(b)$. Let $k : k_- + 1 < k < k_+$ be minimal with the property that $\alpha_i^\vee(\nu_k) \geq 0$. By 14.8, we obtain $\alpha_i = \beta_k$ and so by 14.10, $h_b^i(a_k) \in \mathbb{Z}$. Yet $e_-^i(b) < a_k < e_+^i(b)$, so this is impossible. We conclude that $\boldsymbol{\nu}'$ is obtained from $\boldsymbol{\nu}$ by replacing the (Bruhat) subsequence $\nu_{k_-+1} \longrightarrow \cdots \longrightarrow \nu_{k_++1}$, by the sequence

$\nu_{k_-+1} \longrightarrow s_i\nu_{k_-+1} \longrightarrow \cdots \longrightarrow s_i\nu_{k_+} = \nu_{k_++1}$ of the same length. Thus $\boldsymbol{\nu}'$ has the same length as $\boldsymbol{\nu}$ with the same end-points and is hence a Bruhat sequence. The integrality conditions $a'_j\beta_j'^\vee(\nu'_j) \in \mathbb{Z}$ hold trivially if $j \notin [k_- + 1, k_+ + 1]$. If $j \in [k_- + 2, k_+ + 1]$, then $\beta'_j = s_i\beta_{j-1}$ and so $a'_j\beta_j'^\vee(\nu'_j) = a_{j-1}(s_i\beta_{j-1})^\vee(s_i\nu_{j-1}) = a_{j-1}\beta_{j-1}^\vee(\nu_{j-1}) \in \mathbb{Z}$ since $(\boldsymbol{\nu}, \mathbf{a}) \in C(\lambda)$. Finally

$$
\begin{aligned}
a'_{k_-+1}\beta_{k_-+1}^\vee(\nu'_{k_-+1}) &= e_-\alpha_i^\vee(\nu_{k_-+1}) \\
&= -h_b^i(e_-) + \sum_{j=1}^{k_-} a_j\beta_j^\vee(\nu_j)\alpha_i^\vee(\beta_j) \in \mathbb{Z},
\end{aligned}
$$

since $h_b^i(e_-) = m_b^i - 1 \in \mathbb{Z}$, by 14.12. □

14.14. Let $b = b_{\boldsymbol{\nu},\mathbf{a}}$ be an LS path and suppose $m_b^i - h_b^i(1) \geq 1$. Write b as in 14.13 and suppose $f_\pm$ breaks b at k_+ and where k_- is minimal such that $a_{k_-} \geq f_-(b)$. A similar calculation gives the

Lemma. *Let $b = b_{\boldsymbol{\nu},\mathbf{a}}$ be an* LS *path. Take $i \in I$ and suppose $m_b^i - h_b^i(1) \geq 1$. Then $f_ib_{\boldsymbol{\nu},\mathbf{a}} = b_{\boldsymbol{\nu}',\mathbf{a}'}$ where*

$$
(\nu'_j, a'_j) = \begin{cases} (s_i\nu_{j+1}, a_{j+1}) & : \quad k_+ \leq j \leq k_- - 2 \\ (s_i\nu_{k_-}, f_-) & : \quad j = k_- - 1 \\ (\nu_j, a_j) & : \quad \text{otherwise.} \end{cases}
$$

Moreover $(\boldsymbol{\nu}', \mathbf{a}') \in C(\lambda)$.

14.15. Theorem. *Take $\lambda \in P$. Then $B(\lambda) := \{b_{\boldsymbol{\nu},\mathbf{a}} : (\boldsymbol{\nu}, \mathbf{a}) \in C(\lambda)\}$ is a normal subcrystal of $\mathbb{P}_0$. If $\lambda \in P^+$, then $B(\lambda)$ is a highest weight crystal with b_λ its unique primitive element.*

Proof. It remains to show the last part. For it suffices to show that b_λ is the unique primitive element of $B(\lambda)$. Its primitivity follows because $\lambda \in P^+$.

Now consider $b = b_{(\boldsymbol{\nu},\mathbf{a})}$. We can suppose $a_1 > a_0 = 0$. If $\nu_1 \neq \lambda$, then $\nu_1 \notin P^+$ and there exists $i \in I$ such that $\alpha_i^\vee(\nu_1) < 0$. Then $\alpha_i^\vee(b(a_1)) = a_1\alpha_i^\vee(\nu_1) < 0$, so $m_b^i > 0$.

Hence $m_b^i \geq 1$ by 14.12 and so $e_i b \neq 0$. Thus if $b_{(\boldsymbol{\nu},\mathbf{a})}$ is primitive, the Bruhat sequence $\boldsymbol{\nu}$ collapes to λ, that is $b_{(\boldsymbol{\nu},\mathbf{a})} = b_\lambda$. $\square$

14.16. The above constructs the Littelmann family $\mathcal{B} = \{\mathcal{B}(\lambda) \mid \lambda \in P^+\}$ of normal highest weight crystals. We remark that since in a given sequence $\boldsymbol{a}$ the a_i need not be all distinct, different $(\boldsymbol{a}, \boldsymbol{\nu}) \in C(\lambda)$ may describe the same path. In describing the crystal only distinct paths are counted. Alternatively one may identify elements of $C(\lambda)$ which yield the same path. Littelmann gave a slightly different description of $C(\lambda)$ which avoids this difficulty. It means deleting a_{i+1} in the $\boldsymbol{a}$ sequence and ν_{i+1} in $\boldsymbol{\nu}$ sequence whenever $a_{i+1} = a_i$, until all the a_i become distinct. The advantage is the resulting ν_i are supposed to have a geometric meaning. So far Brion and Littelmann has found a geometric meaning for the end-points of $\boldsymbol{\nu}$, namely ν_1, ν_{s+1} in terms of the support of the corresponding Littelmann standard monomial assigned to the path. The price to pay is the integrality conditions must still be phrased in terms of some Bruhat sequence containing the Littelmann $\boldsymbol{\nu}$ as a subsequence (for example the original one).

15. The Littelmann Path Independence Theorem

15.1. A key step in the proof of the path independent theorem is to show that 14.12, 14.15 hold for a slightly more general set of paths. In particular given $\lambda \in P^+, \mu \in P$ such that $\lambda + \mu \in P^+$, these paths are used to deform $b_\lambda \otimes b_\mu$ into $b_{\lambda+\mu}$. Then the continuity lemma 14.7 is used to show that the crystals they generate are isomorphic.

15.2. Take $b, b' \in \mathbb{P}$ and $0 \leq s \leq s' \leq 1$. Let $b * \theta_s^{s'} * b'$ be the path defined by

$$(b * \theta_s^{s'} * b')(t) = \begin{cases} b(t) : 0 \leq t \leq s \\ b(s) : s \leq t \leq s' \\ b(s) + b'(t) - b'(s') : s' \leq t \leq 1 \end{cases}$$

It is called the join of b, b' across $[s, s']$. It can also be viewed as a concatenation of truncated paths. In particular it almost satisfies the tensor product rules for $b_1 \otimes b_2$ and indeed does so exactly with respect to e_i, f_i when $h_b^i(s) \in \mathbb{Z}$. When s, s' are fixed we will simply write $\theta_s^{s'}$ as θ. When $s = s'$ we omit θ altogether.

15.3. Fix $\lambda, \mu \in P$ and for all $x \in [0,1]$, set $b^x = (1-x)(b_\lambda \otimes b_\mu) + xb_{\lambda+\mu}$. This family deforms $b_\lambda \otimes b_\mu$ into $b_{\lambda+\mu}$. Set $\delta = (1-x)\lambda + \frac{1}{2}x(\lambda+\mu)$, $\delta' = (1-x)\mu + \frac{1}{2}x(\lambda+\mu)$. Then $\delta + \delta' = \lambda + \mu$. Moreover $b^x = b_\delta \otimes b_{\delta'}$ in a suitable parametrization. Although $\delta, \delta' \notin P$, for each fixed x we can choose $r \in \mathbb{N}^+$ such that $r\delta, r\delta' \in P$. Then up to suitable parametrization $b^x = b_\delta \otimes b_{\delta'} = b_{r\delta} * \theta_{1/r}^{1-1/r} * b_{r\delta'}$.

15.4. Given $\mu \in \mathbb{Q}P$ set $S(\mu) = \{\beta \in \Delta_{re}^+ \,\big|\, \beta^\vee(\mu) < 0\}$, $S_0(\mu) = \{\beta \in \Delta_{re}^+ \,\big|\, \beta^\vee(\mu) \leq 0\}$. Given $\mu, \nu \in \mathbb{Q}P$ one writes following Kashiwara that $\nu \supset \mu$ if $S(\nu) \subset S_0(\mu)$. This relation may be interpreted as follows. Through 7.6 and induction one checks that any *finite* subset S of Δ_{re}^+ satisfying $\alpha, \beta \in S,\ \alpha+\beta \in \Delta_{re}^+ \implies \alpha+\beta \in S$ and $\alpha, \beta \in \Delta^+ \setminus S,\ \alpha+\beta \in \Delta_{re}^+ \implies \alpha+\beta \in \Delta^+ \setminus S$, takes the form $S = S(w)$, for some $w \in W$. Thus if $S(\nu)$ (resp. $S_0(\mu)$) is finite there exists $w_\nu^{min} \in W$ (resp. $w_\mu^{max} \in W$) such that $S(\nu) = S(w_\nu^{min})$, $S_0(\mu) = S(w_\mu^{max})$. Moreover w_ν^{min} (resp. w_μ^{max}) is the unique element of W of shortest (resp longest) length such that $w_\nu^{min}\nu \in \mathbb{Q}P^+$ (resp. $w_\mu^{max}\mu \in \mathbb{Q}P^+$). Let us write $w \subset w'$ if $S(w) \subset S(w')$. One may remark that $w \subset w' \implies w \leq w'$ and indeed for this reason $\subset$ is sometimes called the weak (left) Bruhat order. Here the condition $\nu \supset \mu$ implies $w_\nu^{min} \subset w_\mu^{max}$ and so can be viewed as coming from the reverse weak Bruhat order. In particular if $\mu, \nu \in W\lambda$ for some $\lambda \in P^{++}$, then $\mu \subset \nu$ implies $\mu \overset{B}{\leq} \nu$.

15.5. In our definition (14.10) of an LS path $b_{\boldsymbol{\nu},\mathbf{a}} : (\boldsymbol{\nu}, \mathbf{a}) \in C(\lambda) : \nu_1 \to \nu_2 \to \cdots \to \nu_{s+1},\ 0 = a_0 \leq a_1 \leq \cdots \leq a_{s+1} = 1$ there is no loss of generality in assuming $0 = a_0 < a_1$ and $a_s < a_{s+1} = 1$. Then $s = \ell(\nu_1, \nu_{s+1}) =: \ell(\boldsymbol{\nu}, \mathbf{a})$ depends only on the path (see 14.16). We shall retain this convention below.

15.6. Suppose $(\boldsymbol{\nu}, \mathbf{a}), (\boldsymbol{\mu}, \mathbf{b}) \in C(\lambda)$ with $a_\ell \leq b_1$ and $\nu_{\ell+1} = \mu_1$, where $\ell = \ell(\boldsymbol{\nu}, \mathbf{a})$. Let $\nu * \mu$ be the Bruhat sequence $\nu_1 \to \nu_2 \to \cdots \to \nu_{\ell+1} = \mu_1 \to \mu_2 \to \cdots \to \mu_{\ell'+1} \ :\ \ell' = \ell(\boldsymbol{\mu}, \boldsymbol{\nu})$ and $\mathbf{a} * \mathbf{b}$ the sequence $0 = a_0 < a_1 \leq \cdots \leq a_\ell \leq b_1 \leq \cdots \leq b_{\ell'} < b_{\ell'+1} = 1$. Then

$(\boldsymbol{\nu} * \boldsymbol{\mu}, \mathbf{a} * \mathbf{b}) \in C(\lambda)$. Moreover $b_{\boldsymbol{\nu}*\boldsymbol{\mu},\mathbf{a}*\mathbf{b}} = b_{\boldsymbol{\nu},\mathbf{a}} * b_{\boldsymbol{\mu},\mathbf{b}}$. If $a_1 = a_2 = \cdots = a_\ell = s'$ (resp. $b_1 = b_2 = \cdots = b_{\ell'} = s$) we shall simply write $\mathbf{a} = s', \mathbf{b} = s$. Observe that in this case $b_{\boldsymbol{\nu},\mathbf{a}} = b_{\nu_1}$ (resp. $b_{\boldsymbol{\mu},\mathbf{b}} = b_{\mu_{\ell'+1}}$) and are LS paths.

Now take $(\boldsymbol{\nu}, \mathbf{a}) \in C(\lambda), (\boldsymbol{\nu}', \mathbf{a}') \in C(\lambda')$ and $s, s' : 0 < s \leq s' < 1$. One says that $b = b_{\boldsymbol{\nu},\mathbf{a}}, b' = b_{\boldsymbol{\nu}',\mathbf{a}'}$ can be properly joined across $[s, s']$ by a joining pair μ_{t+1}, μ'_1 if $a_\ell < s \leq s' < a'_1$ (where $\ell = \ell(\boldsymbol{\nu}, \mathbf{a})$) and there exist Bruhat sequences $\nu_{\ell+1} = \mu_1 \to \mu_2 \to \cdots \to \mu_{t+1}$, $\mu'_1 \to \mu'_2 \to \cdots \to \mu'_{t'+1}$ with $\mu'_{t'+1} = \nu'_1$, $\mu_{t+1} \supset \mu'_1$ and $(\boldsymbol{\mu}, s) \in C(\lambda)$, $(\boldsymbol{\mu}', s') \in C(\lambda')$.

Lemma. *Suppose b, b' are LS paths which can be properly joined across $[s, s']$ and set $b'' = b * \theta_s^{s'} * b'$. Then for all $i \in I$, $h_{b''}^i$ takes an integer value at a local maximum. In particular b'' is integral and monotone.*

Proof. Fix $i \in I$. By 14.11, we are reduced to checking the assertion on $[s, s']$ where $h_{b''}$ is constant. We can further assume $h_{b''}$ strictly increases below s, so $\alpha_i^\vee(\nu_{\ell+1}) < 0$ and strictly decreases above s', so $\alpha_i^\vee(v'_1) > 0$. Suppose $\alpha_i^\vee(\mu_{t+1}) \geq 0$ (resp. $\alpha_i^\vee(\mu'_1) \leq 0$). Then s (resp. s') is a local maximum of the LS path $c := b * b_{\mu_{t+1}}$ (resp. $c' := b_{\nu'_1} * b'$). Hence $h_{b''}^i(s) = h_c^i(s) \in \mathbb{Z}$ (resp. $h_{b''}^i(s') = h_{c'}^i(s') \in \mathbb{Z}$). Finally $\alpha_i^\vee(\mu_{t+1}) < 0$ and $\alpha_i^\vee(\mu'_1) < 0$ is excluded by the condition $\mu_{t+1} \supset \mu'_1$. □

15.7. To extend 14.13, 14.14 to properly joined paths we need the following

Lemma. *Suppose $\nu \supset \mu$, and take $\alpha \in \pi$.*

(i) *If* $\alpha^\vee(\nu) < 0$, *then* $s_\alpha \nu \supset s_\alpha \mu$

(ii) *If* $\alpha^\vee(\nu) > 0$, *and* $\alpha^\vee(\mu) \geq 0$, *then* $s_\alpha \nu \supset s_\alpha \mu$.

Proof. This is immediate from 6.7(i). □

15.8. Let $\mathcal{A}$ denote the monoid generated by the $e_i, f_i : i \in I$. Fix $0 < s \leq s' < 1$. Let $\mathcal{P}_s^{s'}$ (or simply, $\mathcal{P}$) be the set of all pairs of LS paths which can be properly joined across $[s, s']$. Set $\mathbb{P}_s^{s'} = \{b * \theta_s^{s'} * b' \in \mathbb{P} \mid (b, b') \in \mathcal{P}_s^{s'}\}$.

Proposition. $\mathbb{P}_s^{s'}$ *is* $\mathcal{A}$ *stable.*

Proof. Fix $(b, b') \in \mathbb{P}$ and $i \in I$. Set $b'' = b * \theta * b'$ and suppose $e_i b'' \neq 0$. Retain the notation in the proof of 15.6. Write α_i simply as α.

(1). If $e_+(b'') < s$ or if $e_+(b'') = s$ and $e_+(c) = s$ (resp. $e_-(b'') > s'$) then $e_i b'' = e_i b * \theta * b'$ (resp. $e_i b'' = b * \theta * e_i b'$) with no change in the joining pair, whilst $e_i b$ (resp. $e_i b'$) is an LS path by 14.13.

(2). Suppose $e_+(b'') = s$. In view of (1) we can assume $\alpha^\vee(\mu_{t+1}) < 0$. Then $m' := h_c(1) > h_c(s) =: m$ with m, m' integers ≥ 1. Then $c'' := e_i^{m'-m} c = b * e_i^{m'-m} b_{\mu_{t+1}} = b * b_{s_i \mu_{t+1}}$. Moreover $e_+(c'') = s$. Then $e_i c'' = e_i b * b_{s_i \mu_{t+1}}$. Similarly if $n := h_b^i(1)$, then $e_i^{n-m+1} b = e_i b''$ on $[0, s]$. Again $\mu_{t+1} \supset \mu_1'$ forces $\alpha^\vee(\mu_1') \leq 0$. Suppose $n' = -s' \alpha^\vee(\mu_1') > 0$, which we recall is integer. The hypothesis $e_+(b'') = s$ forces $e_+(c') = s'$. Hence $e_i^{n'} c' = e_i^{n'} b_{\mu_1'} * b' = b_{s_i \mu_i'} * b'$. By 14.13 and 15.7(i) we conclude that $(e_i^{n-m+1} b, b') \in \mathbb{P}$ with joining pair $s_i \mu_{t+1}, s_i \mu_1'$ and $e_i b'' = e_i^{n-m+1} b * \theta * b' \in \mathbb{P}_s^{s'}$.

(3). Suppose $e_-(b'') = s'$. Then $h_{b''}(s') \in \mathbb{Z}$. Since

$$b'(t) = \begin{cases} b_{\nu_1'}(t) & : \quad 0 \leq t \leq s' \\ b''(t) - b(s) & : \quad s' \leq t \leq 1 \end{cases}$$

it follows that $\alpha^\vee(\nu_1') < 0$ and $e_+(b') = e_+(b'')$. Set $u := (1-s')\alpha^\vee(\nu_1') = h_{b'}(1) - h_{b'}(s') = h_{b''}(1) - h_{b''}(s') \in \mathbb{Z}$, since $b'' \in \mathbb{P}$. Then

$$(e_i^u b')(t) = \begin{cases} b_{s_i \nu_1'}(t) & : \quad 0 \leq t \leq s' \\ (e_i b'')(t) & : \quad s' \leq t \leq 1 \end{cases}$$

Since $\mu_{t'+1}' = \nu_1' \to s_i \nu_1'$ one may adjoin $s_i \nu_1'$ to $\boldsymbol{\mu}'$ from the right. In this both $\mu_{\ell+1}, \mu_1'$ remain unchanged so by 14.13 we may conclude that $(b, e_i^u b') \in \mathcal{P}$ with joining pair $\mu_{\ell+1}, \mu_1'$ and $e_i b'' = b * \theta * e_i^u b' \in \mathbb{P}_s^{s'}$.

(4). Finally suppose $r := e_-(b'') < s \leq s' < e_+(b'') =: r'$. Recall that $b = b_{\nu_{\ell+1}}$ on $[a_\ell, 1]$ with $a_\ell < s$ and $b' = b_{\nu_1'}$ on $[0, a_1']$ with $s' < a_1'$. Hence $\alpha^\vee(\nu_{\ell+1}) < 0$ and

$\alpha^\vee(\nu_1') < 0$. Moreover $h_{b''}$ cannot be integer-valued on $[s,s']$ forcing $\alpha^\vee(\mu_{t+1}) < 0$ and then $n' = -s'\alpha^\vee(\mu_1') \in \mathbb{N}$. Set $h_b(1) = m, h_c(1) = m', h_{b''}(r') = n$. Then $m, m' > n$. Let d be the restriction of the LS path $e_i^{m-n}b$ to $[0,s]$. Then $e_i^{m'-n}c = d * b_{s_i\mu_{t+1}}$. Moreover $e_ib'' = d$ on $[0,s]$. As in (3) one has $e_+(b') = e_+(b'')$ and $u := (1-r')\alpha^\vee(\nu_1') = h_{b'}(1) - h_{b'}(r') = h_{b''}(1) - h_{b''}(r')$ is an integer > 0. Let d' be the restriction of the LS path $e_i^n b'$ to $[s',1]$. Then $e_i^{n'}c' = b_{s_i\mu_1'} * d$. Moreover $e_ib'' = d' - b(s)$ on $[s',1]$. Hence by 14.13 and 15.7(i) we conclude that $(e_i^{m-n}b, e_i^u b') \in \mathcal{P}$ with joining pair $s_i\mu_{t+1}, s_i\mu_1'$ and $e_ib'' = e_i^{m-n}b * \theta * e_i^u b' \in \mathbb{P}_s^{s'}$.

Now suppose $f_ib'' \neq 0$. The proof in this case is basically the same except we must use 15.7(ii) instead of 15.7(i). A very brief summary is made below using the same numbering of the different cases.

(1) If $f_+(b'') > s'$, or if $f_+(b'') = s'$ and $f_+(c') = s'$ (resp. $f_-(b'') < s$) then $f_ib'' = b * \theta * f_ib'$ (resp. $f_ib'' = f_ib * \theta * b'$) with no change in the joining pair.

(2) Suppose $f_+(b'') = s'$. In view of (1) we can assume $\alpha^\vee(\mu_1') > 0$, which forces $\alpha^\vee(\mu_{t+1}) \geq 0$. Then by 15.7(ii) and as before $f_ib'' \in \mathbb{P}_s^{s'}$ with joining pair $s_i\mu_{t+1}, s_i\mu_1'$.

(3) Suppose $f_-(b'') = s$. Then $\alpha^\vee(\mu_{\ell+1}) > 0$ and so $s_i\mu_{\ell+1} \to \mu_{\ell+1} = \nu_1 \to \nu_2 \to \cdots$. Then one may adjoin $s_i\nu_{\ell+1}$ to $\boldsymbol{\mu}$ from the left. As before $f_ib'' \in \mathbb{P}_s^{s'}$ with no change in the joining pair.

(4) Finally we can suppose $f_+(b'') < s \leq s' < f_-(b'')$. Then $\alpha^\vee(\mu_1') > 0$, which forces $\alpha^\vee(\mu_{t+1}) \geq 0$. Then by 15.7(ii) and as before $f_ib'' \in \mathbb{P}_s^{s'}$ with joining pair $s_i\mu_{t+1}, s_i\mu_1'$. □

15.9 Corollary. *Take $\lambda \in P^+, \mu \in P$. For all $x \in [0,1]$, set $b^x := (1-x)(b_\lambda \otimes b_\mu) + xb_{\lambda+\mu}$. The paths in $\mathcal{A}b^x$ are integral and monotone.*

Proof. Apply 15.8, 15.6 to the presentation of b^x given in 5.3. □

15.10. Recall that each $a \in \mathcal{A}$ is a monomial in the $e_i, f_i \in I$, let $\ell(a)$ denote the shortest length of all such monomials and set $\mathcal{A}^n = \{a \in \mathcal{A} \,\big|\, \ell(a) \leq n\}$.

By the graph of a crystal B we shall just mean its set of weighted vertices $\{b \mid b \in B\}$ joined by labelled directed edges $b \xrightarrow{i} b'$, whenever $b' = e_i b$ and with $wt\ b' = \alpha_i + wt\ b$. One can then speak of an isomorphism of graphs (or subgraphs) of a crystal. This just means forgetting the maps ε_i, φ_i. For normal crystals an isomorphism of their graphs implies an isomorphism of the crystals themselves.

Proposition. *Fix $\lambda \in P^+, \mu \in P$ such that $\lambda + \mu \in P^+$. The crystals $\mathcal{A}(b_\lambda \otimes b_\mu)$ and $\mathcal{A}b_{\lambda+\mu}$ are isomorphic.*

Proof. Fix $x, y \in [0,1]$ and recall the notation of 14.7. Clearly $d(b^x, b^y)$ is a multiple of $x - y$. Thus for all $n \in \mathbb{N}^+$, there exists $m \in \mathbb{N}^+$ such that $d(b^x, b^y) < \left(\frac{1}{4c}\right)^n$, for all x, y satisfying $|x - y| < 1/m$. By 14.7 and 15.9 the subgraphs $\mathcal{A}^n b^x, \mathcal{A}^n b^y$ are isomorphic for all $|x - y| < 1/m$. Breaking [0,1] into intervals of length $< 1/m$, we conclude that $\mathcal{A}^n b^0, \mathcal{A}^n b^1$ are isomorphic. Moreover this holds for any $n \in \mathbb{N}^+$. Yet $b^0 = b_\lambda \otimes b_\mu$, whilst $b^1 = b_{\lambda+\mu}$. □

15.11 Corollary. *The Littelmann family $\mathcal{B} = \{B(\lambda) \mid \lambda \in P^+\}$ of normal highest weight crystals is closed.*

Proof. Take $\mu \in P^+$ in the hypothesis of 15.10. □

Remark. This now gives a purely combinatorial proof of the existence of a closed family of highest weight crystals. One may recall (13.7) that such a family is unique up to isomorphism. By 15.11 we now also have a purely combinatorial proof of $B(\infty)$ being upper normal (13.4) and being independent (see 12.5) of the choice of $\mathcal{S}$.

15.12. We digress slightly to discuss the notion of "gonflage" of crystals. This was introduced independently by Kashiwara and by Zelikson to prove the equivalence of the Kashiwara and Littelmann families. This will be discussed more fully in Sect. 18.

Let B be a crystal and m an integer > 0. For all $i \in I$, set $e_i^{(m)} = e_i^m$, $f_i^{(m)} = f_i^m$, $\varepsilon_i^{(m)} = \frac{1}{m}\varepsilon_i$, $\varphi_i^{(m)} = \frac{1}{m}\varphi_i$, $wt^{(m)} = \frac{1}{m}wt$, $B^{(m)} = \{b \in B \mid \varepsilon_i^{(m)}(b), \varphi_i^{(m)}(b) \in \mathbb{Z} \cup \{-\infty\},\ wt^{(m)} b \in P\}$. One easily checks that $B^{(m)}$ is a crystal with respect to these new operators and maps.

A crystal B is said to be expandable if for all $m \in \mathbb{N}^+$ there exists a strict morphism $\psi^{(m)}$ of B into $(B^{\otimes m})^{(m)}$. Notice that if $B = B(\lambda) : \lambda \in P^+$ is a highest weight crystal then there exists a unique element of $B^{\otimes m}$ of weight $m\lambda$, namely $b_\lambda^{\otimes m}$. We conclude that $\psi^{(m)}(b_\lambda) = b_\lambda^{\otimes m}$, so in this case $\psi^{(m)}$, if it exists, is uniquely determined.

15.13. One has the following result which is a rather easy consequently of the (rather complicated) definitions of the Littelmann operators $e_i, f_i : i \in I$. Given $b \in \mathbb{P}, m \in \mathbb{N}^+$, let mb denote the path $t \mapsto mb(t)$. For all $B \subset \mathbb{P}$, set $mB = \{mb \mid b \in B\}$.

Lemma. For all $i \in I$

(i) $e_i^{(m)}(mb) = m(e_i b)$,

(ii) $f_i^{(m)}(mb) = m(f_i b)$.

Moreover if $B \in \mathbb{P}_0$, then $(mB)^{(m)} \cong B$. In particular any subcrystal of $\mathbb{P}_0$ is expandable.

Proof. For the last part use 14.6 and the fact that mb and $b^{\otimes m}$ coincide as paths. □

15.14. We now prove the Littelmann Path Independence Theorem. Set $\mathbb{Q}^+ = \{a \in \mathbb{Q} \,\big|\, a \geq 0\}$. Set $\mathbb{P}^+ = \{b \in \mathbb{P} \,\big|\, b(t) \in \mathbb{Q}^+ P^+, \ \forall\ t \in [0,1]\}$. Any element $b \in \mathbb{P}^+$, can be written in the form $b_{\lambda_1} \otimes b_{\lambda_2} \otimes \cdots \otimes b_{\lambda_n}$, with $\sum\limits_{j=1}^{i} \lambda_i \in \mathbb{Q}^+ P^+$, for all i. Given $m \in \mathbb{N}^+$, let $\mathcal{A}^{(m)}$ denote the monoid generated by the $e_i^{(m)}, f_i^{(m)} : i \in I$.

Theorem. *Take $b \in \mathbb{P}^+$ and set $\mu = wt\ b \in P^+$. The crystals $\mathcal{A}b$ and $\mathcal{A}b_\mu$ are isomorphic. In particular $\mathcal{A}b$ is a highest weight crystal. Moreover $b' \in \mathbb{P}_0$ for all $b' \in \mathcal{A}b$.*

Proof. Write b as above and choose $m \in \mathbb{N}^+$ such that $m\lambda_i \in P$, for all $i = 1, 2, \cdots, n$. One has $mb = b_{m\lambda_1} \otimes b_{m\lambda_2} \otimes \cdots \otimes b_{m\lambda_n}$. Through 15.10 it follows that the crystals $\mathcal{A}(mb)$ and $\mathcal{A}b_{m\mu}$ are isomorphic. In particular $\mathcal{A}^{(m)}(mb)$ and $\mathcal{A}^{(m)}b_{m\mu}$ are isomorphic as graphs with respect to $e_i^{(m)}, f_i^{(m)} : i \in I$ and $wt^{(m)}$. Yet by 15.13 these are just $\mathcal{A}b$ and $\mathcal{A}b_\mu$ respectively as graphs. Then $\mathcal{A}(mb)$ and $\mathcal{A}b_{m\mu}$ must be isomorphic as graphs for *any*

$m \in \mathbb{N}^+$. Since $b' \in \mathbb{P}_0$ for all $b' \in \mathcal{A}b_\mu$ by 15.6, this forces $b' \in \mathbb{P}_0$, for all $b' \in \mathcal{A}b$. Then by 14.7(i) and normality $\mathcal{A}b$ and $\mathcal{A}b_\mu$ are isomorphic as crystals. □

15.15. Let $\mathcal{B} = \{B(\lambda) \mid \lambda \in P^+\}$ be the (unique) closed family of normal highest weight crystals.

Theorem. *Fix* $\lambda, \mu \in P^+$ *and take* $b \in B(\lambda) \otimes B(\mu)$. *Suppose* $e_i b = 0, \ \forall \ i \in I$.

(i) $b = b_\lambda \otimes b'$, *for some* $b' \in B(\mu)$.

(ii) *Viewed as a path* b *belongs to* $\mathbb{P}^+$.

(iii) $\mathcal{A}b = \mathcal{F}b \cong B(wt\ b)$.

Proof. Write $b = b'' \otimes b'$. By (upper) normality and the tensor product rules $0 = \varepsilon_i(b) = max\{\varepsilon_i(b''), \varepsilon_i(b') - \alpha_i^\vee(b'')\} \Longrightarrow \varepsilon_i(b'') = 0 \Longrightarrow e_i b'' = 0 \Longrightarrow b'' = b_\lambda$. Hence (i).

As a path an element of $B(\lambda) \otimes B(\mu)$ lies in $\mathbb{P}_0$ by 15.14. Then $0 = \varepsilon_i(b) = m_b^i = max\{-\alpha_i^\vee(b(t)) : t \in [0,1]\}$. Consequently $b(t) \in \mathbb{Q}^+ P^+, \ \forall \ t \in [0,1]$, that is $b \in \mathbb{P}^+$. Hence (ii).

Given $b \in \mathbb{P}^+$, then $e_i b = 0, \ \forall \ i \in I$. Moreover $\mathcal{A}b$ is isomorphic to $B(wt\ b)$ as a crystal. In particular $\mathcal{A}b$ is a highest weight crystal and so $\mathcal{A}b = \mathcal{F}b$. □

15.16. We now have a purely combinatorial proof of Kashiwara's tensor product decomposition

Corollary. *Fix* $\lambda, \mu \in P^+$. *Then*

$$B(\lambda) \otimes B(\mu) = \coprod_{b \in \mathbb{P}^+} B(wt\ b).$$

16. The Combinatorial Demazure and Weyl Formulae

Unless otherwise stated $\mathcal{B} = \{B(\lambda) \mid \lambda \in P^+\}$ denotes the unique closed family of normal highest weight crystals. Generally we view the elements of $B(\lambda)$ as LS paths (called of type λ). In particular if $b_{\boldsymbol{\nu},\mathbf{a}} \in B(\lambda)$ we retain the notation and conventions of 15.5, and set $c(\boldsymbol{\nu}) = \nu_1,\ d(\boldsymbol{\nu}) = \nu_{s+1}$. The results in this paragraph are mainly due to Littelmann, in turn inspired by the classical theory of Weyl, Kac and Demazure as well as Kashiwara's crystal theory.

16.1. Fix $\lambda \in P^+$ and recall that $ch\ B(\lambda)$ is W invariant. Thus for each $w \in W$ there exists a unique element of $B(\lambda)$ of weight $w\lambda$. The corresponding LS path is $b_{w\lambda}$.

Let $C_w(\lambda) = \{(\boldsymbol{\nu}, \mathbf{a}) \in C(\lambda) \mid c(\boldsymbol{\nu}) \overset{B}{\geq} w\lambda\}$ and set $B_w(\lambda) = \{b_{\boldsymbol{\nu},\mathbf{a}} \mid (\boldsymbol{\nu}, \mathbf{a}) \in C_w(\lambda)\}$. For example $b_{w\lambda} \in B_w(\lambda)$. It is the crystal analogue of the Demazure module $F(w\lambda)$ and may be shown to have the same formal character. We call it the Demazure crystal (with generating weight $w\lambda$) even though it is not a crystal (though it is $\mathcal{E}$ stable by the lemma below) and $b_{w\lambda}$ is not always a generator (that is one may have $\mathcal{E}b_{w\lambda} \subsetneq B_w(\lambda)$) in contrast to the Demazure module.

Lemma. *For all $i \in I$ one has*

(i) $e_i B_w(\lambda) \subset B_w(\lambda) \cup \{0\}$

(ii) $f_i B_w(\lambda) \subset B_w(\lambda) \cup \{0\},\quad$ *if* $\ s_i w < w$.

Proof. (i) results from 14.13. For (ii) suppose $b_{\boldsymbol{\nu},\mathbf{a}} \in B_w(\lambda)$. By 14.14, one may write $f_i b_{\boldsymbol{\nu},\mathbf{a}} = b_{\boldsymbol{\nu}',\mathbf{a}'}$, where either $c(\boldsymbol{\nu}') = c(\boldsymbol{\nu})$ or $c(\boldsymbol{\nu}') = s_i c(\boldsymbol{\nu})$. In the latter case we can further assume $\alpha_i^\vee(c(\boldsymbol{\nu})) > 0$. By 7.6 one has $s_i w < w \Longrightarrow w^{-1}\alpha_i \in \Delta^- \Longrightarrow \alpha_i^\vee(w\lambda) \leq 0$. Then the hypotheses $c(\boldsymbol{\nu}) \overset{B}{\geq} w\lambda, s_i w < w$ imply by 14.9(ii) that $s_i c(\boldsymbol{\nu}) \overset{B}{\geq} w\lambda\}$. □

16.2. Fix $w \in W$ and a reduced decomposition $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$. In the notation of 12.7 set

$$\mathcal{F}_{\mathbf{w}} = \mathcal{F}_{i_1}\mathcal{F}_{i_2}\cdots\mathcal{F}_{i_n}.$$

Lemma. Fix $i \in I$.

(i) *If* $\ s_i w < w,\ $ *then* $\ \{b \in B_w(\lambda) \mid e_i b = 0\} \subset B_{s_i w}(\lambda)$.

(ii) $B_w(\lambda) = \mathcal{F}_{\mathbf{w}} b_\lambda$.

Proof. Consider $b = b_{(\boldsymbol{\nu},\mathbf{a})}$ satisfying $e_i b = 0$. Then $h_b^i(t) = -\alpha_i^\vee(c(\boldsymbol{\nu}))t, \ \forall\ t \in [0, a_1]$. Through the hypothesis and integrality $h_b^i(t) \leq 0, \ \forall\ t \in [0,1]$. Hence $\alpha_i^\vee(c(\boldsymbol{\nu})) \geq 0$. The hypothesis $s_i w < w$ implies $\alpha_i^\vee(w\lambda) \leq 0$ and the hypothesis $b \in B_w(\lambda)$ implies $c(\boldsymbol{\nu}) \overset{B}{\geq} w\lambda$. Then $c(\boldsymbol{\nu}) \overset{B}{\geq} s_i w\lambda$ by 14.9(i). This proves (i).

Observe that (i) implies that $B_w(\lambda) = \mathcal{F}_i B_{s_i w}(\lambda)$. Then (ii) follows by induction on $\ell(w)$. □

16.3. Retain the above notation.

Lemma. *Take* $b \in B_w(\lambda)$. *If* $f_i b \in B_w(\lambda)$, *then* $\mathcal{F}_i b \subset B_w(\lambda)$.

Proof. In view of 6.1(i) we can suppose $s_i w > w$. As in 6.1 we write $b = b_{\boldsymbol{\nu},\mathbf{a}}$. Up to a suitable parametrization $b = b_{\nu_1} \otimes b_{\nu_2} \otimes \cdots \otimes b_{\nu_{s+1}}$, where $w\lambda \overset{B}{\leq} \nu_1 \overset{B}{\leq} \nu_2 \overset{B}{\leq} \cdots \overset{B}{\leq} \nu_{s+1}$. Through the tensor product rules for a multiple tensor product, one checks that if f_i goes in at the j^{th} place, subsequent f_i factors go in at the j^{th} place or at later places. Thus if $j > 1$, we conclude that $f_i^k b_{\boldsymbol{\nu},\mathbf{a}} = b_{\boldsymbol{\nu}_k,\mathbf{a}_k}$ with $c(\nu_k) = c(\nu) \overset{B}{\geq} w\lambda$, and so belongs to $B_w(\lambda)$. If $j = 1$, then $f_i b_{\boldsymbol{\nu},\mathbf{a}} = b_{\boldsymbol{\nu}_1,\mathbf{a}_1}$ with $c(\boldsymbol{\nu}_1) = s_i c(\boldsymbol{\nu})$ and this belongs to $B_w(\lambda)$ only if $s_i c(\boldsymbol{\nu}) \overset{B}{\geq} w\lambda$. Moreover when this holds $f_i^k b_{\boldsymbol{\nu},\mathbf{a}} = b_{\boldsymbol{\nu}_k,\mathbf{a}_k}$ with $c(\boldsymbol{\nu}_k) = c(\boldsymbol{\nu}_1) = s_i c(\boldsymbol{\nu}) \overset{B}{\geq} w\lambda$, and so belongs to $B_w(\lambda)$. □

16.4. Let B be a crystal and $i \in I$. A subset $S \subset B$ of the form $S = \{e_i^s b : s \in \mathbb{N}\}$ is called a α_i-string. Through the crystal rules any $\mathcal{E}_i$ stable subset of a crystal is a disjoint union of α_i strings. Any singleton α_i-string is of the form $\{b \mid e_i b = 0\}$. An α_i-string is called complete if it is also $\mathcal{F}_i$ stable.

These definitions first arose in the context of a finite dimensional simple module M of a semisimple Lie algebra. In the notation of 4.8 set $\mathfrak{b}_i = ke_i + \mathfrak{h}$. An α_i-string submodule of M is one of the form $k\{e_i^s v_\lambda : s \in \mathbb{N}\}$ for some $\mathfrak{h}$ weight vector v_λ. By an argument similar to the proof of Jordan canonical form every $U(\mathfrak{b}_i)$ stable submodule of M is a direct sum of α_i-string submodules. A one-dimensional α_i-string submodule takes the form $k\{v_\lambda \mid e_i v_\lambda = 0\}$. An α_i-string submodule is said to be complete if it is also kf_i stable.

From 16.1(i) and 16.3 it follows that $B_w(\lambda)$ has the remarkable property that for each $i \in I$, it is a disjoint union of α_i-strings which are either singleton or complete. As we shall see below this allows one to calculate $ch\ B_w(\lambda)$ inductively via the Demazure operators $D_i : i \in I$ defined in 16.5.

Long before the theory of crystals Demazure had obtained his character formulae for the $F(w\lambda)$ through a belief that a similar property held for these modules. This is not quite true and only becomes true in the $q \to 0$ limit. Moreover the proof of this latter fact is extremely complex.

16.5. For all $i \in I$ define $D_i \in End_{\mathbb{Z}} \mathbb{Z}e^P$ through

$$D_i e^\nu = (1 - e^{-\alpha_i})^{-1}(e^\nu - e^{s_i\nu - \alpha_i}).$$

One checks that $D_i^2 = D_i$. Furthermore if $n := \alpha_i^\vee(\nu) \in \mathbb{N}$, then

$$(*) \qquad D_i e^\nu = e^\nu + e^{\nu - \alpha_i} + \cdots + e^{\nu - n\alpha_i},$$

whilst $D_i e^\nu = 0$, if $\alpha_i^\vee(\nu) = -1$.

Let B be a normal crystal. If S is singleton α_i-string, then by $(*)$ and normality

$$(**) \qquad ch(\bigcup_{j \in \mathbb{N}} f_i^j S) = D_i ch S.$$

On the other hand $\bigcup_{j \in \mathbb{N}} f_i^j S$ is just the complete α_i-string generated by S. Since $D_i^2 = D_i$, it follows that $(**)$ also holds for complete α_i-strings.

16.6. Given $w \in W$, fix a reduced decomposition $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$ and set $D_{\mathbf{w}} = D_{i_1} \cdots D_{i_n}$. From 16.2(ii), 16.4, 16.5 we obtain the following combinatorial version of the Demazure character formula.

Theorem. *For all $w \in W$, $\lambda \in P^+$ one has*

$$ch\ B_w(\lambda) = D_{\mathbf{w}} e^{\lambda}.$$

16.7. It is immediate from 16.6 that $D_{\mathbf{w}}$ is independent of the choice of $\mathbf{w}$. We write it simply as D_w.

Consider $b_{\boldsymbol{\nu},\mathbf{a}} \in B(\lambda)$. One has $c(\boldsymbol{\nu}) = w\lambda$ for some $w \in W$ and so $b_{\boldsymbol{\nu},\mathbf{a}} \in B_w(\lambda)$. Again from the definition of $B_w(\lambda)$ it follows that $B_w(\lambda) \subset B_{w'}(\lambda)$ if $w' \geq w$. Thus with respect to the Bruhat ordering we may form the direct limit of the $B_w(\lambda)$ and this limit equals $B(\lambda)$. That is

$$B(\lambda) = \varinjlim B_w(\lambda).$$

16.8. Take $\alpha, \beta \in \pi$ distinct. Set $\pi' = \{\alpha, \beta\}$ and $m_{\alpha,\beta} = \alpha^\vee(\beta)\beta^\vee(\alpha)$ which is an integer ≥ 0. One has $det(s_\alpha s_\beta - \lambda 1d) = \lambda^2 + (2 - m_{\alpha,\beta})\lambda + 1$. If $m_{\alpha,\beta} > 4$ this has a real root $\neq \pm 1$ and so $s_\alpha s_\beta$ has infinite order. When $m_{\alpha,\beta} = 4$ one checks that $(s_\alpha s_\beta)^n \alpha = \alpha + 2n(\alpha + \beta)$, so again $s_\alpha s_\beta$ has infinite order. Thus for $m_{\alpha,\beta} \geq 4$, the subgroup $W_{\pi'}$ (resp. subset $\Delta^+_{\pi'}$ of D^+) generated by s_α, s_β (resp. by α, β) is infinite. Otherwise $W_{\pi'}$ admits a unique longest element $w_{\pi'}$ which can have exactly two reduced decompositions. The ensuing relations are called the Coxeter relations. For example $w_{\pi'} = s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$, if $m_{\alpha,\beta} = 1$.

Lemma. *Any two reduced decompositions of $w \in W$ can be transformed into one another by the Coxeter relations.*

Proof. By induction on $\ell(w)$. Suppose w has two distinct reduced decompositions. Then there exists $\alpha, \beta \in \pi$, $w_1, w_2 \in W$ of length $\ell(w) - 1$ such that $w = w_1 s_\alpha = w_2 s_\beta$. By the induction hypothesis it is enough to reduce to the case $\alpha = \beta$. By 7.6 the above expressions imply that $\alpha, \beta \in S(w)$ and so $\Delta^+_{\pi'} \subset S(w)$. Then by 7.6 again $S(w_1) \supset s_\alpha(\Delta^+_{\alpha,\beta} \setminus \{\alpha\})$ and so there exists a reduced decomposition of w_1 of the form $w_1 = w_3 w_3'$, where $w_3' = w_{\{\alpha,\beta\}} s_\alpha$ (written in reduced form) and $\ell(w_3) = \ell(w_1) - \ell(w_3') = \ell(w) - \ell(w_{\{\alpha,\beta\}})$. Then

$w_1 s_\alpha = w_3 w_{\pi'}$ and writing $w_{\pi'}$ in its second reduced form (which ends in s_β) effects the required reduction. $\square$

16.9. Fix $\lambda \in P^+$ and $\alpha \in \pi$. Then $B(\lambda)$ being a normal crystal is a disjoint union of complete α-strings. Moreover any such string S has weights of multiplicity one and $ch\ S$ is s_α invariant. Thus we may define $\hat{s}_\alpha \in Aut\ S$ by the condition $wt(\hat{s}_\alpha b) = s_\alpha(wt\ b)$. Moreover $\hat{s}_\alpha^2 = 1d_S$. In view of the above (unique) decomposition of $B(\lambda)$ one obtains an element $\hat{s}_\alpha \in Aut\ B(\lambda)$ such that $wt(\hat{s}_\alpha b) = s_\alpha(wt\ b)$, $\forall\ b \in B(\lambda)$. Moreover $\hat{s}_\alpha^2 = 1d_{B(\lambda)}$.

Proposition. *The $\hat{s}_\alpha : \alpha \in \pi$ satisfy the Coxeter relations and hence define an action of W on $B(\lambda)$.*

Proof. Fix a pair $\alpha, \beta \in \pi$ such that $\Delta_{\{\alpha,\beta\}}$ is finite. Set $\pi' = \{\alpha, \beta\}$. Then $B(\lambda)$ is a disjoint union of normal highest weight crystals defined with π'. Moreover in each such crystal the elements are still LS paths and so belong to the Littelmann family. Thus we can assume $\pi = \pi'$ without loss of generality. In this case $B(\lambda)$ is finite.

Let $\mathbf{w}_1, \mathbf{w}_2$ be the two reduced decompositions of $w_{\pi'}$ and let $\hat{\mathbf{w}}_1, \hat{\mathbf{w}}_2$ be the corresponding products in the $\hat{s}_\alpha : \alpha \in \pi'$. We must show that $\hat{\mathbf{w}}_1 b = \hat{\mathbf{w}}_2 b$, for all $b \in B(\lambda)$. Take $\alpha \in \pi'$. Since $\hat{s}_\alpha^2 = 1d_{B(\lambda)}$, this holds for $\hat{s}_\alpha b$ if it holds on b. Thus we can assume $wt\ b \in P^+$.

Take $c \in B(\lambda)$ and suppose $k = \alpha^\vee(wtc) \geq 0$. Since $\varphi_\alpha(c) = \alpha^\vee(wtc) + \varepsilon_\alpha(c)$, one has $\varphi_\alpha(f_\alpha^{k-\ell}c) = k - \ell + \varepsilon_\alpha(c) > \varepsilon_\alpha(c)$, $\forall\ 0 \leq \ell < k$. Then by 12.2($*$) one has $f_\alpha^m(c \otimes c) = \hat{s}_\alpha c \otimes f_\alpha^{m-k}c$, $\forall\ m \geq k$.

Now take any $d \in B(\lambda)$. Suppose $k > 0$ and fix $m \in \mathbb{N}^+$. Then for all $n \in \mathbb{N}^+$ such that $n > m$ and $k(n-m) \geq \varepsilon_\alpha(d)$ one has $\varphi_\alpha(c^{\otimes n}) \geq kn \geq \varepsilon_\alpha(d) + km$. Then by 12.2($*$) and the previous observation

$$\begin{aligned} f_\alpha^{km+t}(c^{\otimes n} \otimes d) &= f_\alpha^t(f_\alpha^{km} c^{\otimes n} \otimes d) \\ &= f_\alpha^t((\hat{s}_\alpha c)^{\otimes m} \otimes c^{\otimes (n-m)} \otimes d) \\ &= (\hat{s}_\alpha c)^{\otimes m} \otimes f_\alpha^t(c^{\otimes n-m} \otimes d) \end{aligned}$$

for all $t \in \mathbb{N}$. Moreover this also holds if $k = 0$.

Now fix a reduced decomposition $\mathbf{w}_1 = s_1 s_2 \cdots s_r$ of $w \in W$ and set $\hat{\mathbf{w}}_1 = \hat{s}_1\hat{s}_2\cdots\hat{s}_r$. Fix $n \in \mathbb{N}$ and define $f^{(n)}_{\mathbf{w}_1} = f_1^{i_1}\cdots f_r^{i_r} \in \mathcal{F}_{\mathbf{W}}$ inductively by $i_k = \varphi_k(f_{k+1}^{i_{k+1}}\cdots f_r^{i_r} b^{\otimes n})$. Since $wt\ b \in P^+$, it follows that $\alpha_k^\vee(wt(\hat{s}_{k+1}\cdots\hat{s}_r b)) = \alpha_k^\vee(s_{k+1}\cdots s_r(wt\ b)) \geq 0$, for all k. Consequently by our previous calculation there exists $n_{\mathbf{w}_1} \in \mathbb{N}^+$ such that

$$(*) \qquad f^{(n)}_{\mathbf{w}_1}(b^{\otimes n}) \in (\hat{\mathbf{w}}b) \otimes B(\lambda)^{\otimes n-1} : \forall\ n \geq n_{\mathbf{W}}.$$

Now take $\mu \in P^+, \alpha \in \pi, c \in B(\mu)$ and $d \in B(\lambda)^{\otimes n}$. Set $t = max\{0, \varphi_\alpha(c) - \varepsilon_\alpha(d)\}$. By 12.2$(*)$ one has $f_\alpha^{s+t}(c \otimes d) = f_\alpha^t c \otimes f_\alpha^s d$, for all $s \in \mathbb{N}$. Thus if we define $f^{c,(n)}_{\mathbf{W}_1} \in \mathcal{F}_{\mathbf{W}}$ in the same manner as $f^{(n)}_{\mathbf{W}_1}$ but replacing $b^{\otimes n}$ by $c \otimes b^{\otimes n}$, we conclude from $(*)$ that

$$(**) \qquad f^{c,(n)}_{\mathbf{W}_1}(c \otimes b^{\otimes n}) \in B(\mu) \otimes (\hat{\mathbf{w}}_1 b) \otimes B(\lambda)^{\otimes n-1},\ \forall\ n \geq n_{\mathbf{W}_1}.$$

Note in particular that $n_{\mathbf{W}_1}$ does not depend on c.

Now fix $n \geq n_{\mathbf{W}_1}$. Choose $\mu \in P^+$ (sufficiently large) so that as a path $b_\mu \otimes b^{\otimes n} \in \mathbb{P}^+$ and set $\nu = \mu + n(wt\ b)$. By 15.15, $b_\mu \otimes b^{\otimes n}$ is the primitive element b_ν of the highest weight crystal $B(\nu)$. In particular

$$f^{b_\mu,(n)}_{\mathbf{W}_1}(b_\mu \otimes b^{\otimes n}) = f^{b_\mu,(n)}_{\mathbf{W}_1} b_\nu = \hat{\mathbf{w}} b_\nu = b_{w\nu}.$$

Finally take a second reduced decomposition $\mathbf{w}_2$ of w. Fix $n \geq max\{n_{\mathbf{W}_1}, n_{\mathbf{W}_2}\}$. By $(**)$ there exist $c_1, c_1' \in B(\mu),\ c_2, c_2' \in B(\lambda)^{\otimes n-1}$ such that

$$f^{b_\mu,(n)}_{\mathbf{W}_i}(b_\mu \otimes b^{\otimes n}) = c_i \otimes (\hat{\mathbf{w}}_i b) \otimes c_i' : i = 1, 2.$$

Yet both equal $b_{w\nu}$ and so $\hat{\mathbf{w}}_1 b = \hat{\mathbf{w}}_2 b$, as required. □

16.10. If $|W|$ is finite with w_π its unique element of longest length one has $B_{w_\pi}(\lambda) = B(\lambda)$. Moreover one may show that $D_{w_\pi} e^\lambda$ coincides with the Weyl character formula, that is to say with the right hand side of 10.7(iii). It does not seem to be known how to prove this in general. However one may note that since $Q(0)$ is the trivial module, 10.7(iii) implies

$$(*) \qquad \sum_{w\in W}(-1)^{\ell(w)}e^{w\rho-\rho} = \prod_{\alpha\in\Delta^+}(1-e^{-\alpha})^{m_\alpha}.$$

This is known as the Weyl denominator formula. Outside the case $|W|<\infty$, there seems to be no known purely combinatorial proof and indeed the assertion itself is only known to hold when the Cartan matrix is symmetrizable (need to obtain a Casimir invariant, as in Sections 9, 10). On the other hand using $(*)$ we can cast the combinatorial Weyl character formula into the form

$$(**) \qquad ch\ B(\lambda) = \frac{\sum_{w\in W}(-1)^{\ell(w)}e^{w(\lambda+\rho)-\rho}}{\sum_{w\in W}(-1)^{\ell(w)}e^{w\rho-\rho}}\ .$$

This will be proved below.

16.11. Fix $\lambda\in P^+$ and for all $\mu\in P$ set

$$\Omega(\mu) = \{(b,w)\in B(\lambda)\times W \mid b(1)+w\rho=\mu\}.$$

Through 16.9 and left multiplication on W we obtain a diagonal action of W on $B(\lambda)\times W$. For this action $w(\Omega(\mu)) = \Omega(w\mu)$. Thus if we define

$$\mathcal{S}(\mu) = \sum_{(b,w)\in\Omega(\mu)}(-1)^{\ell(w)}e^{b(1)+w\rho}.$$

Then $w(\mathcal{S}(\mu)) = \mathcal{S}(w\mu),\ \forall\ w\in W$.

Lemma. *Suppose $\mu\in P^+$ and $\mu<\lambda+\rho$. Then $\mathcal{S}(\mu)=0$.*

Proof. Take $(b,w)\in\Omega(\mu)$. Since $\mu=b(1)+w\rho$ is dominant, if $b(t)+w\rho$ is regular on [0,1] then it is also dominant on [0,1], by continuity. Then $w\rho=\rho$ and $\alpha_i^\vee(\rho+b(t))>0,\ \forall\ i\in I,\ \forall\ t\in[0,1]$. This translates to give $m_b^i<\alpha_i^\vee(\rho)=1$. Yet $b\in\mathbb{P}_0$, so $e_ib=0,\ \forall\ i\in I$ forcing $b=b_\lambda$. Then $\mu=\lambda+\rho$, which is a contradiction.

We conclude that there exists $a' \in [0,1]$ (resp. $a'' \in [0,1]$) such that $b(t)+w\rho$ is dominant (resp. regular on $[a',1]$ (resp. $]a'',1]$). By continuity $a' \le a''$. Thus we obtain a unique maximal $a \in [a',1]$ such that $b(a)+w\rho$ is dominant, but not regular. Moreover $b(t)+w\rho$ is dominant and regular on $]a,1]$.

Given $(b',w') \in \Omega(\mu)$, set $(b',w') \sim (b,w)$ if $b'(t)+w'\rho = b(t)+w\rho$ on $[a,1]$. One checks that this is an equivalence relation. It is enough to show that the above sums vanish on each equivalence class.

Let Ω be the equivalence class containing (b,w). Fix $\alpha \in \pi$ such that $\alpha^\vee(b(0)+w\rho) = 0$. We define an involution i_α on Ω such that $i_\alpha(b,w) = (b', s_\alpha w)$, for some $b' \in B(\lambda)$. This will imply the required vanishing.

Suppose first that $m = \alpha^\vee(w\rho) > 0$. On $]a,1]$ one has $h_b^\alpha(t) < \alpha^\vee(w\rho) = m = h_b^\alpha(a)$. Consequently $e_\alpha^m b \neq 0$ and $(e_\alpha^m b)(t)+s_\alpha w\rho = b(t)+m\alpha+s_\alpha w\rho = b(t)+w\rho$ on $[a,1]$. In this case we set $i_\alpha(b,w) = (e_\alpha^m b, s_\alpha w) \in \Omega$. If $m = -\alpha^\vee(w\rho) > 0$, a similar calculation shows that $i_\alpha(b,w) := (f_\alpha^m b, s_\alpha w) \in \Omega$. By the crystal rules i_α is an involution, as required. □

16.12 Theorem. *Fix $\lambda \in P^+$. The Littelmann crystal satisfies the combinatorial character formula* 16.10($**$).

Proof. One has

$$\sum_{b\in B(\lambda)} \sum_{w\in W} (-1)^{\ell(w)} e^{b(1)+w\rho} = \sum_{\mu\in P} \sum_{(b,w)\in\Omega(\mu)} (-1)^{\ell(w)} e^{b(1)+w\rho}.$$

By 16.11 the second summation vanishes unless the unique dominant element $\nu \in W\mu \cap P^+$ equals $\lambda+\rho$. Yet $\nu = b(1)+w\rho \le \lambda+\rho$, with a strict inequality unless $w\rho = \rho$ and $b(1) = \lambda$, forcing $b = b_\lambda$. In this case $\Omega(\mu) = \{(b_{w\lambda}, w\rho) : w \in W\}$ and so the resulting sum is just the numerator in the right hand side of 16.10($*$). □

Remark. Assume that the Cartan matrix is symmetrizable. Then $ch\ B(\lambda) = ch\ V(\lambda),\ \forall\ \lambda \in P^+$.

16.13. Recall 13.6 the notion of a submonomial. Given $w \in W$, fix a reduced decomposition $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$, and define $\mathcal{F}_{\mathbf{w}}$ as in 16.2. Fix $\lambda, \mu \in P^+$.

Lemma. *Take* $b \in B(\lambda), b' \in B(\mu)$.

(i) *Each* $\tilde{f}' \in \mathcal{F}_{\mathbf{w}}$ *admits submonomials* $\tilde{f}'', \tilde{f}$ *such that*

$$\tilde{f}'(b \otimes b') = \tilde{f}'' b \otimes \tilde{f} b'.$$

(ii) *Given* $f \in \mathcal{F}$, *there exists* $f', f'' \in \mathcal{F}_{\mathbf{w}}$ *such that*

$$f'(b \otimes b') = f'' b \otimes f b'.$$

Proof. Fix $i \in I$ and set $m = max\{0, \varphi_i(b) - \varepsilon_i(b')\}$. By the tensor product rules $f_i^{m+t}(b \otimes b') = f_i^m b \otimes f_i^t b'$, $\forall\, t \in \mathbb{N}$. From this (i), (ii) follow easily through induction on $\ell(w)$. □

16.14 Proposition. *Take* $\mu \in P^+$ *and* $b \in B(\mu)$. *The subset* $\mathcal{F}_{\mathbf{w}} b$ *of* $B(\mu) \cup \{0\}$ *is independent of the choice of reduced decomposition of* w.

Proof. Choose $\lambda \in P^+$ such that $b_\lambda \otimes b \in \mathbb{P}^+$. By 15.15(iii), the crystal generated by $b_\lambda \otimes b$ is isomorphic to $B(\lambda + wt\ b)$. Let $\mathbf{w}_1, \mathbf{w}_2$ be reduced decompositions of w. By 16.2(ii)

$$(*) \qquad \mathcal{F}_{\mathbf{w}_1}(b_\lambda \otimes b) = B_w(\lambda + wt\ b) = \mathcal{F}_{\mathbf{w}_2}(b_\lambda \otimes b).$$

Given $f \in \mathcal{F}_{\mathbf{w}_1}$ choose $f', f'' \in \mathcal{F}_{\mathbf{w}_1}$ in the conclusion of 16.13(ii). By $(*)$ and 16.13, there exist $\tilde{f}, \tilde{f}', \tilde{f}'' \in \mathcal{F}_{\mathbf{w}_2}$ such that $f'' b_\lambda \otimes f b = f'(b_\lambda \otimes b) = \tilde{f}'(b_\lambda \otimes b) = \tilde{f}'' b_\lambda \otimes \tilde{f} b$. This implies $fb = \tilde{f} b$ and hence $\mathcal{F}_{\mathbf{w}_1} b \subset \mathcal{F}_{\mathbf{w}_2} b$. Interchanging $\mathbf{w}_1, \mathbf{w}_2$ gives the reverse inclusion. □

16.15. In view of 13.4 we obtain the

Corollary. *Take* $b \in B(\infty)$. *The subset* $\mathcal{F}_{\mathbf{w}} b$ *of* B_∞ *is independent of the choice of reduced decomposition of* w.

16.16. We may similarly define $\mathcal{E}_{\mathbf{w}} = \mathcal{E}_{i_1}\mathcal{E}_{i_2}\cdots\mathcal{E}_{i_n}$.

Corollary. *Take $b \in B(\infty)$. The subset $\mathcal{E}_{\mathbf{w}}b$ of $B(\infty) \cup \{0\}$ is independent of the choice of reduced decomposition of w.*

Proof. Take reduced decompositions $\boldsymbol{w}, \boldsymbol{w}'$ of w. If w is not the identity (in which case the assertion is trivial) both $\mathcal{E}_{\mathbf{w}}b, \mathcal{E}_{\mathbf{w}'}b$ contain zero by weight conclusions. Now suppose $e \in \mathcal{E}_{\mathbf{w}}$ and $eb \neq 0$. By the crystal rules there exists $f \in \mathcal{F}_{i_n}\mathcal{F}_{i_{n+1}}\cdots\mathcal{F}_{i_1}$ such that $b = feb$. Yet $f \in \mathcal{F}_{\mathbf{w}^{-1}}$ which coincides with $\mathcal{F}_{\mathbf{w}'^{-1}}$ on $B(\infty)$. Re-expressing f as an element of $\mathcal{F}_{\mathbf{w}'^{-1}}$, the crystal rules provide an element $e' \in \mathcal{E}_{\mathbf{w}'}$ such that $e'b = eb$. $\square$

16.17. Of course we cannot expect the conclusions of 16.15, 16.16 to hold for arbitrary crystals. However since we are mainly interested in the $B(\lambda) : \lambda \in P^+$, their tensor products and $B(\infty)$ we shall simply write $\mathcal{E}_{\mathbf{w}}, \mathcal{F}_{\mathbf{w}}$ as $\mathcal{E}_w, \mathcal{F}_w$.

The D_w (resp. $\mathcal{E}_w, \mathcal{F}_w$) $: w \in W$ are examples of the singular Hecke monoid. In this respect recall that the Hecke algebra is the algebra over $\mathbb{Q}[q, q^{-1}]$ with generators $T_i : i \in I$ satisfying $(T_i - q)(T_i + 1) = 0$, $\forall\, i \in I$ and the Coxeter relations. It has a basis formed from elements $T_w : w \in W$ defined by taking reduced decompositions. Its simplest manifestation (for $|W| < \infty$) obtains as follows. Let G be the points of a semisimple algebraic group over a field of q elements and B a Borel subgroup of G. The set of B double cosets of G is parametrized by W. If we assign T_w to the coset defined by w, then the above relations are those described by the multiplication of double cosets (as sets counted with multiplicity). Partly for this reason the Hecke algebra plays a fundamental role in the theory of simple (*not* necessarily integrable) highest weight modules.

The singular Hecke monoid obtains from the Hecke algebra by setting $q = 0$ and taking the generators to be the $-T_i : i \in I$. As we have seen it occurs in the present crystal theory developed from the theory of integrable highest weight modules. It also occurs in the theory of such modules themselves. For example as in 6.3 let e_i (resp. f_i) $: i \in I$ denote the generators of a Kac-Moody algebra $\mathfrak{g}$ and define $\mathcal{E}_i$ (resp. $\mathcal{F}_i$) to be the linear span of monomials in e_i (resp. f_i) which are vector subspaces $U(\mathfrak{g})$. Given any two subspaces V, V' of $U(\mathfrak{g})$, let VV' denote the linear span of the products $vv' : v \in V, v' \in V'$. One may check that in this sense the $\mathcal{E}_i$ (resp. $\mathcal{F}_i$) generate the singular Hecke monoid, the proof being through the representation theory of integrable highest weight modules.

17. The Parthasarathy-Ranga Rao-Varadarajan Components

17.1. Since $B(\lambda) : \lambda \in P^+$, is a highest weight crystal, $e^{-\lambda} ch\ B(\lambda)$ may be viewed as an element of the formal power series $\mathbb{Z}[[e^{-\alpha} : \alpha \in \pi]]$. Again in the tensor product, weights add, so we may conclude that

$$ch(B(\lambda) \otimes B(\mu)) = ch\ B(\lambda) ch\ B(\mu),$$

in the sense of convergence in the Krull topology. Moreover since $ch\ B(\lambda)$ is W invariant, it can be expressed as a $\mathbb{Z}$-linear combination of the orbit sums

$$\mathcal{O}(\mu) := \sum_{w \in W} e^{w\mu} : \mu \in P^+,$$

with $\mu \leq \lambda$, where the coefficient of $\mathcal{O}(\lambda)$ equals 1. Consequently the $\mathbb{Z}$-linear independence of the $\mathcal{O}(\mu)$ implies that of the $ch\ B(\lambda)$. (If $|W| < \infty$, one can even say they generate the same $\mathbb{Z}$ submodule $(\mathbb{Z}e^P)^W$ of $\mathbb{Z}e^P$. Moreover it is also true that $\mathbb{Z}e^P$ is free over $(\mathbb{Z}e^P)^W$; but less us not digress too far).

We may conclude from the above that the decomposition of $B(\lambda) \otimes B(\mu)$ described in 15.16 is completely determined by the combinatorial Weyl character formula. However this fact (which also holds for the tensor product of integrable modules) is not particularly useful.

17.2. Suppose $|W| < \infty$ and let w_π denote its unique longest element. Parthasarathy, Ranga Rao and Varadarajan (abbreviated as PRV) showed that the tensor product $V(\lambda) \otimes V(\mu)$ of simple finite dimensional modules admits a component with highest weight in $W(\lambda + w_\pi \mu)$. Subsequently this fact played a fundamental role in the classification of unitarizable highest weight modules. They conjectured that their result also holds for arbitrary $w \in W$. Verma refined their conjecture to include multiplicities. Here making a slight improvement on Littelmann's analysis (to take account of multiplicities) we establish a crystal version of this conjecture in the general Kac-Moody set-up. It turns out to be closely linked to studying tensor products of Demazure crystals. For $|W| < \infty$ the corresponding module theoretic result was established independently by Kumar and by Mathieu using Frobenius splitting. Using the Kashiwara globalization technique, the module theoretic version can also be established (in the symmetrizable case).

17.3. Fix $\lambda, \mu \in P^+$. Take $(\boldsymbol{\nu}, \mathbf{a}) \in C(\mu)$ with $a_s < a_{s+1} = 1$. One calls $b := b_{\boldsymbol{\nu},\mathbf{a}}$ almost λ dominant if $\lambda + b(t) \in \mathbb{Q}^+ P^+$, for all $t \leq a_s$. In this case b is λ dominant if and only if $wt\ b \in P^+$. One may remark that $b_{w\lambda} : w \in W$ is almost λ dominant and λ dominant, exactly when $w\lambda = \lambda$.

Lemma. *Suppose $b_{\boldsymbol{\nu},\mathbf{a}} \in B(\mu)$ is almost λ dominant; but not λ dominant. Then there exists $i \in I$ such that $\alpha_i^\vee(d(\boldsymbol{\nu})) < 0$ and $n \in \mathbb{N}^+$ such that $b' := b_{\boldsymbol{\nu}',\mathbf{a}'} := e_i^n b_{\boldsymbol{\nu},\mathbf{a}}$ is almost λ dominant, where $\boldsymbol{\nu}'$ is obtained from $\boldsymbol{\nu}$ by replacing $d(\boldsymbol{\nu})$ by $d(\boldsymbol{\nu}) \to s_i d(\boldsymbol{\nu})$. In particular $d(\boldsymbol{\nu}') = s_i d(\boldsymbol{\nu})$. Finally $e_i^n(b_\lambda \otimes b) = b_\lambda \otimes b'$ and $wt(b_\lambda \otimes b') = s_i wt(b_\lambda \otimes b)$.*

Proof. Choose t_0 maximal such that $\lambda + b$ is dominant in $[0, t_0]$. Then $a_s \leq t_0 < 1$, by the hypotheses. By continuity $\lambda + b$ is not regular at t_0, so there exists $i \in I$ such that $n(t) := -\alpha_i^\vee(\lambda + b(t))$ is zero and strictly increasing at t_0. This forces $\alpha_i^\vee(d(\boldsymbol{\nu})) < 0$ and then that $n := n(1) \in \mathbb{N}^+$.

By the hypothesis and the definition of t_0 one has $h_b^i(t) \leq \alpha_i^\vee(\lambda)$, $\forall\ t \in [0, t_0]$. Moreover h_b^i increases strictly on $[t_0, 1]$ from $\alpha_i^\vee(\lambda)$ to $\alpha_i^\vee(\lambda)+n$. Since $\alpha_i^\vee(\lambda) \geq 0$, it follows that $b' := e_i^n b \neq 0$ and is obtained from b by reflecting it by s_i on $[t_0, 1]$. This establishes the lemma where for the last part one notes that $\varepsilon_i(b) - \varphi_i(b_\lambda) = \alpha_i^\vee(\lambda) + n - \alpha_i^\vee(\lambda) = n$, and $wt(b_\lambda \otimes b') = \lambda + wt\ b + n\alpha_i = s_i(\lambda + wt\ b)$ through the definition of n. $\square$

17.4 Corollary. *Suppose $b_{\boldsymbol{\nu},\mathbf{a}} \in B(\mu)$ is almost λ dominant. Then there exists $b_{\boldsymbol{\nu}',\mathbf{a}'} \in B(\mu)$ which is λ dominant and satisfies*

(i) $\lambda + wt\ b_{\boldsymbol{\nu}',\mathbf{a}'} \in W(\lambda + wt\ b_{\boldsymbol{\nu},\mathbf{a}})$.

(ii) *If* $s > 1$ *then* $\nu_i' = \nu_i$, *for all* $i \leq s$ *(where* $\nu_s = d(\boldsymbol{\nu})$*).*

Proof. This follows from 17.3 since $S(d(\boldsymbol{\nu}))$ is a finite set and its cardinality strictly decreases at each step. □

17.5. Set $W_\mu = \{w \in W \mid w\mu = \mu\}$. Since $\mu \in P^+$ it follows from 7.6(v) that W_μ is generated by the simple reflections $s_j : j \in J$ it contains. The right coset wW_μ contains an element w' of minimal length obtained by successively replacing w by $ws_j : j \in J$ whenever length decreases. By 7.6(ii) one has

$$S(w') = S(w) \setminus (S(w) \cap \sum_{j \in J} \mathbb{N}\alpha_j).$$

(Moreover since the map $w \mapsto S(w)$ is injective, w' is thereby uniquely determined).

Similarly $W_\lambda w$ (resp. $W_\lambda w W_\mu$) contains a unique element of minimal length.

Proposition. *Suppose $b_{\boldsymbol{\nu},\mathbf{a}} \in B(\mu)$ is almost λ dominant. Then there exists a unique element $b_{\boldsymbol{\nu}',\mathbf{a}'} \in B(\mu)$ such that $b_\lambda \otimes b_{\boldsymbol{\nu}',\mathbf{a}'} \in \mathcal{E}(b_\lambda \otimes b_{\boldsymbol{\nu},\mathbf{a}})$ and is primitive. Moreover it satisfies*

(i) $\lambda + wt\ b'_{\boldsymbol{\nu}',\mathbf{a}'} \in W(\lambda + wt\ b_{\boldsymbol{\nu},\mathbf{a}})$

(ii) $c(\boldsymbol{\nu}') = w'\mu$, *given* $c(\boldsymbol{\nu}) = w\mu$ *and where* w' *is the unique element of minimal length in* $W_\lambda w W_\mu$.

Proof. Existence results from 17.3 applied at most $|S(d(\boldsymbol{\nu}))|$ times. Notice that $s = 1$ if and only if $b = b_{w\mu}$. In this case we can assume that the initial steps replace w by w'. From then on the successive $\boldsymbol{\nu}$ differ only in subsequent terms (as described in 17.3) forcing $c(\boldsymbol{\nu}) = w'\mu$. □

17.6. We now prove the combinatorial version of Verma's improved PRV conjecture.

Theorem. *Take $\lambda, \mu \in P^+, w \in W$. Let ν be the unique dominant element in $W(\lambda+w\mu)$. Then $b_\lambda \otimes b_{w\mu}$ lies in a component $B_w(\lambda,\mu)$ of $B(\lambda)\otimes B(\mu)$ of highest weight ν. Moreover the map $w \mapsto B_w(\lambda+\mu)$ factors to a bijection of $W_\lambda \setminus W/W_\mu$ onto its image.*

Proof. Since $b_{w\mu}$ is almost λ dominant, the first part follows from 17.5(i) and the asserted primitivity. The second part follows from 17.5(ii). □

17.7. Let G be a group. The diagonal map $\Delta : G \to G \times G,\ g \mapsto g \times g$ is a group monomorphism. Through it and left multiplication obtain a left action of G on $G\times G$. Let H, K be subgroups of G. One easily checks that the map $x, y \mapsto x^{-1}y$ of $G \times G$ factors to a bijection of $\Delta G \setminus (G/H \times G/K)$ onto $H \setminus G/K$.

Now take $G = W,\ H = W_\lambda,\ K = W_\mu$. Then $G/H \times G/K$ identifies with the $W \times W$ orbit $W\lambda \times W\mu$ and so $\Delta(G)\setminus(G/H\times G/K)$ identifies with the W orbits in $W\lambda\times W\mu$ under diagonal action. By the above remark and 17.6, the set of such orbits is in bijection with the components $B_w(\lambda,\mu)$ of $B(\lambda)\times B(\mu)$. This observation leads to a proof of surjectivity of the generalized Chevalley map defined with respect to a semisimple Lie algebra $\mathfrak{g}$. The latter further implies that the commuting variety $C(\mathfrak{g}) := \{x, y \in \mathfrak{g} \mid [x,y] = 0\}$ is normal (again for $\mathfrak{g}$ semisimple). An open problem is to show that the ideal of definition of $C(\mathfrak{g})$ is quadratically generated.

18. Expandable Crystals and a Second Uniqueness Theorem

18.1. In 15.12 we defined the notion of an expandable crystal and showed that every subcrystal of $\mathbb{P}_0$ is expandable. This was moreover a rather easy result and followed directly from Littelmann's rather contorted definitions of the $e_i, f_i : i \in I$. It implies that the $B(\lambda) : \lambda \in P^+$ are all expandable.

18.2. Let $\mathcal{B} = \{B(\lambda) \mid \lambda \in P^+\}$ denote a family of highest weight crystals (not necessarily the Kashiwara-Littelmann family). One says that $\mathcal{B}$ is weakly closed if for all $\lambda \in P^+$, $\mathcal{E}(b_\lambda \otimes b_\lambda)$ is a strict subcrystal of $B(\lambda)\times B(\lambda)$ and isomorphic to $B(2\lambda)$. When b_λ

is viewed as the path $t \mapsto \lambda t$, the concatenation $b_\lambda \otimes b_\lambda$ is just the path $b_{2\lambda}$. Consequently weak closure (as opposed to closure) is an immediate property of the Littelmann family.

18.3. Recall 12.8 and let $\mathcal{B} = \{B(\lambda) \mid \lambda \in P^+\}$ be the Kashiwara family of highest weight crystals. Here we recall that this family is obtained from 12.7 where the upper normality of $B_{\mathcal{S}}(\infty)$ and its independence of $\mathcal{S}$ is assumed to be known (from quantum group arguments). Let us further assume that $\mathcal{B}$ is known to be weakly closed (from quantum group arguments). Using these properties we may deduce the

Lemma. *The Kashiwara family is expandable.*

Proof. Fix $n \in \mathbb{N}^+$. Let us recall that every element of $\hat{B}_{\mathcal{S}}(\infty)$ takes the form

$$b = \cdots \otimes b_{i_m}(-k_m) \otimes \cdots \otimes b_{i_1}(-k_1)$$

where $i_j \in I,\ k_j \in \mathbb{N}$ and all but finitely many k_j equal zero. Define $\psi^{(n)}b$ by replacing k_j by nk_j. Clearly $wt\ \psi^{(n)}b = nwt\ b$. For each $i \in I$, the Kashiwara function $k \mapsto r^i_k(b)$ is a sum taking values in $\mathbb{Z} \cup \{-\infty\}$. Moreover $r^i_k(\psi^{(n)}b) = nr^i_k(b)$. Then by 12.1(2) one obtains $\varepsilon^{(n)}_i(\psi^{(n)}b) := \frac{1}{n}\varepsilon_i(\psi^{(n)}(b)) = \varepsilon_i(\psi^{(n)}b)$. Moreover taking $i = j$ in the formulae of 12.2 it follows easily that if e_i (resp. f_i) goes into the p^{th} (resp. g^{th} place) of b, then the e_i (resp. f_i) goes into the p^{th} (resp. g^{th} place) of $e^{t-1}_i b$ (resp. $f^{t-1}_i b$), for all $t = 1, 2, \cdots, n$. Consequently $e^{(n)}_i(\psi^{(n)}(b)) := e^n_i \psi^{(n)}(b) = \psi^{(n)}(e_i b)$ and similarly $f^{(n)}_i \psi^{(n)}(b) = \psi^{(n)}(f_i b)$.

Through the hypotheses that $B(\lambda)$ is weakly closed it is enough to show that there exists a strict morphism $\Psi^{(n)}$ of $B(\lambda)$ into $B(n\lambda)^{(n)}$. Through the strict embedding (12.7) we define $\Psi^{(n)}$ by setting

$$\Psi^{(n)}(b \otimes s_\lambda) = \psi^{(n)}(b) \otimes s_{n\lambda},$$

for all $b \in B(\infty)$ satisfying $b \otimes s_\lambda \in Im\ B(\lambda)$. The tensor product rules imply that $e_i(b \otimes s_\lambda) = e_i b \otimes s_\lambda \iff \varphi_i(b) \geq \varepsilon_i(s_\lambda) - 1 \iff \varphi_i(b) + \alpha^\vee_i(\lambda) \geq -1 \iff \varphi_i(\psi^{(n)}(b)) + \alpha^\vee_i(n\lambda) - n \iff e^n_i(\psi^{(n)}(b) \otimes s_{n\lambda}) = e^n_i \psi^{(n)}(b) \otimes s_{n\lambda}$. Otherwise both terms equal zero.

Through our previous calculation we conclude that $\Psi^{(n)}$ commutes with e_i. A similar assertion holds for f_i. Compatibility with $\varepsilon_i, \varphi_i, wt$ is obvious. $\square$

18.4. We assume in the remainder of this section that the Cartan matrix $\alpha_i^\vee(\alpha_j)$ is symmetrizable. Let $(\ ,\ )$ denote the W invariant scaler product on $\mathfrak{h}^*$ which results (4.7). Take $\lambda \in P^+$ and suppose $B(\lambda)$ is a normal highest weight crystal. Set $\Omega(B(\lambda)) = \{\nu \in P \mid B(\lambda)_\nu \neq 0\}$. By 11.13, $\Omega(B(\lambda))$ is W stable.

Lemma. *One has* $(\nu, \nu) \leq (\lambda, \lambda)$, *for all* $\nu \in \Omega(B(\lambda))$.

Proof. Since $B(\lambda)$ is a highest weight crystal, $\nu \in \Omega(B(\lambda))$ implies $\nu = \lambda - \gamma$, for some $\gamma = \sum\limits_{i \in I} k_i \alpha_i : k_i \in \mathbb{N}$ and we set $|\nu| = \sum k_i$. If $\nu \notin P^+$, there exists $i \in I$ such that $\alpha_i^\vee(\nu) < 0$ and then $|s_i \nu| < |\nu|$. Induction on $|\nu|$ implies that there exists $w \in W$ such that $\nu_0 := w\nu \in P^+ \cap \Omega(B(\lambda))$. Then $(\lambda, \lambda) - (\nu_0, \nu_0) = (\lambda - \nu_0, \nu_0) + (\lambda, \lambda - \nu_0) \geq 0$, so $(\lambda, \lambda) \geq (\nu_0, \nu_0) = (\nu, \nu)$. $\square$

Remark. Of course a similar property holds for the simple integrable highest weight module $V(\lambda)$. More generally we say that a $U(\mathfrak{g})$ module V with weight space decomposition whose set of weights $\Omega(V)$ lies in $\mathbb{R}P$ is bounded if there exists a positive integer N such that $(\nu, \nu) \leq N$, for all $\nu \in \Omega(V)$. For example $V(\lambda) : \lambda \in P^+$, or $\mathfrak{g}$ itself. A bounded module is necessarily integrable. For $\mathfrak{g}$ affine, the simple bounded modules have been classified and notably have the property that they have finite dimensional weight subspaces. Moreover any simple subquotient of $U(\mathfrak{g})$ with respect to the adjoint action is bounded. On the other hand it is not known if every simple integrable module with weights in $\mathbb{Z}\pi$ is bounded.

18.5 Lemma. *Let $B(\lambda) : \lambda \in P^+$ be a normal crystal of highest weight λ. Then for all $w \in W$, there exists a unique element $b_{w\lambda}$ of weight $w\lambda$. Furthermore for all $i \in I$ one has*

(i) *Suppose* $\alpha_i^\vee(w\lambda) \geq 0$. *Then* $\varepsilon_i(b_{w\lambda}) = 0$, $\varphi_i(b_{w\lambda}) = \alpha_i^\vee(w\lambda)$, $f_i^{\alpha_i^\vee(w\lambda)} b_{w\lambda} = b_{s_i w\lambda}$.

(ii) *Suppose* $\alpha_i^\vee(w\lambda) > 0$. *Then* $\varphi_i(b_{w\lambda}) = 0$, $\varepsilon_i(b_{w\lambda}) = -\alpha_i^\vee(w\lambda)$, $e_i^{-\alpha_i^\vee(w\lambda)} b_{w\lambda} = b_{s_i w\lambda}$.

Proof. The first part follows from 11.13. The hypothesis of (i) gives $(\alpha_i + w\lambda, \alpha_i + w\lambda) \geq (w\lambda, w\lambda) + (\alpha_i, \alpha_i) > (\lambda, \lambda)$. Hence $\alpha_i + w\lambda \notin \Omega(B(\lambda))$ by 18.4 and so $e_i(b_{w\lambda}) = 0$. This forces $\varphi_i(b_{w\lambda}) = \alpha_i^\vee(w\lambda)$ and $f_i^{\alpha_i^\vee(w\lambda)} b_{w\lambda} \neq 0$, by normality. Yet the latter element has weight $s_i w\lambda$ and so must equal $b_{s_i w\lambda}$. The proof of (ii) is similar. □

18.6. Fix $r \in \mathbb{N}^+$ and $\lambda_1, \lambda_2, \cdots, \lambda_r \in P^+$. Let $\mathcal{B} = \{B(\lambda_s) : s = 1, 2, \cdots, r\}$ be a family of highest weight normal crystals. For each $w \in W$, let $b_{w\lambda_s}$ be the unique element of weight $w\lambda_s$ in $B(\lambda_s)$. Take $b \in B(\lambda_1) \otimes B(\lambda_2) \otimes \cdots \otimes B(\lambda_r)$ of the form $b = b_{w_1\lambda_1} \otimes b_{w_2\lambda_2} \otimes \cdots \otimes b_{w_r\lambda_r} : w_s \in W$. Observe using 18.15 that the values $r_j^i(b)$ of the Kashiwara function are independent of the family $\mathcal{B}$ chosen.

Lemma. *Fix* $i \in I$ *and suppose there exist* $1 \leq t_1 \leq t_2 \leq r$ *such that* $\alpha_i^\vee(w_s\lambda_s) \geq 0 : t_1 \leq s \leq t_2$. *Set*

$$m = \sum_{s=t_1}^{t_2} \alpha_i^\vee(w_s\lambda_s).$$

Then $f_i^m b$ *is obtained from* b *by replacing* $b_{w_s\lambda_s}$ *by* $b_{s_i w_s \lambda_s}$ *for* $t_1 \leq s \leq t_2$, *if and only if* $r_s^i(b) \leq r_{t_1}^i(b)$, $\forall\ s \leq t_1$ *and* $r_s^i(b) \leq r_{t_2+1}^i(b)$, $\forall\ s > t_2$.

Proof. The condition $r_s^i(b) \leq r_{t_1}^i(b)$, $\forall\ s \leq t$, is exactly that which forces powers of f_i to go into the t_1^{th} or later place of b. Assuming this, the condition $r_s^i(b) \leq r_{t_2+1}^i(b) : s > t_2$ is exactly that which ensures the $f_i^j : j = 1, 2, \cdots, m$, all go into the places of b between t_1, t_2. Finally by 18.5(i) one has $\varepsilon_i(b_{w_s\lambda_s}) = 0$, for $t_1 \leq s \leq t_2$. For $t_1 \leq s < t_2$ this gives

$$r_s^i(b) - r_{s+1}^i(b) = \alpha_i^\vee(w_s\lambda_s) \geq 0.$$

Then by the tensor product rules and 18.5(i) we conclude that $f_i^m b$ differs from b in the stated fashion. □

18.7. Let $\mathcal{B} = \{B(\lambda) \mid \lambda \in P^+\}$ be the Littelmann family of expandable highest weight normal crystals. Fix $\lambda \in P^+$, $w \in W$ and recalling 16.1 observe that $C_w(\lambda)$ is a finite set. Choose $m \in \mathbb{N}^+$ such that $ma_j \in \mathbb{N}$, for every $\mathbf{a}$, for which $(\boldsymbol{\nu}, \mathbf{a}) \in C_w(\lambda)$. Let $\psi^{(m)} : B(\lambda) \to (B(\lambda)^{\otimes m})^{(m)}$ be the strict morphism whose existence was noted in 18.1.

Proposition. *For all $b \in B_w(\lambda)$, there exists $(\boldsymbol{\nu}, \mathbf{a}) \in C_w(\lambda)$ such that*

$$\psi^{(m)}(b) = b_{\nu_1}^{\otimes n_1} \otimes b_{\nu_2}^{\otimes n_2} \otimes \cdots \otimes b_{\nu_{s+1}}^{\otimes n_{s+1}}$$

where $n_j = m(a_j - a_{j-1}) : j = 1, 2, \cdots, s+1$. Moreover this formula is valid for any family of expandable highest weight normal crystals.

Proof. Since $b \in B_w(\lambda)$, there exists $(\boldsymbol{\nu}, \mathbf{a}) \in C_w(\lambda)$ such that $b = b_{\boldsymbol{\nu}, \mathbf{a}}$. Then $\psi^{(m)}(b) = mb_{\boldsymbol{\nu}, \mathbf{a}}$ which is just the path defined by the right hand side above. This proves the first assertion. The second assertion is shown by induction on the length of $f \in \mathcal{F}$ satisfying $b = fb_\lambda$. Assuming it holds for b' we verify it for $b = f_i b' : i \in I$. Through the first part applied to both $b, b' \in B_w(\lambda)$ and 14.14 it follows that the hypotheses of 18.6 hold. Through its conclusion the result obtains. □

18.8. From the above we may deduce the following result shown independently by Kashiwara and by Zelikson.

Theorem. *A family of expandable highest weight normal crystals is unique up to isomorphism.*

Remark. Although we do not need to know a priori that the family is weakly closed, this was used in both our examples to prove expandability.

18.9. One may conclude from the Kashiwara tensor product theorem 15.16 (as shown by using quantum groups) that the Littelmann family also satisfies 15.16. Yet since the Littelmann family consists of expandable crystals and for such crystals, whose elements are viewed as belonging to $\mathbb{P}_0$, the assertion of 15.16 is equivalent to the path independence theorem 15.15. This gives a further proof of that result. However it is not purely combinatorial and is only valid in the symmetrizable case.

19. A Decomposition Theorem for Demazure Crystals

19.1. Let us recall (16.17) that the $\mathcal{E}_i$ (resp. $\mathcal{F}_i$) $i \in I$ generate the singular Hecke monoid whose elements are the $\mathcal{E}_w$ (resp. $\mathcal{F}_w$) $w \in W$. Set $B_w(\infty) = \mathcal{F}_w b_\infty$, for all $w \in W$. Fix $\lambda \in P^+$ and define

$$B^\lambda(\infty) = \{b \in B(\infty) : \mathcal{E}(b_\lambda \otimes b) = 0\}.$$

Since $B(\infty)$ is a direct limit of the $B(\mu) : \mu \in P^+$, it follows from 15.16 that $B(\infty, \lambda + wt\ b) := \mathcal{F}(b_\lambda \otimes b) : b \in B^\lambda(\infty)$ is isomorphic to $B(\infty)$ up to a shift of weights. Moreover $B(\lambda) \otimes B(\infty)$ is a disjoint union of the $B(\infty, \lambda + wt\ b) : b \in B^\lambda(\infty)$.

19.2. Fix $w \in W$ and set $B^\lambda_w(\infty) = B_w(\infty) \cap B^\lambda(\infty)$. Given $b \in B^\lambda_w(\infty)$, set

$$\mathcal{F}^\lambda_{w,b} := \{f \in \mathcal{F} \mid f(b_\lambda \otimes b) \subset b_\lambda \otimes B_w(\infty)\}.$$

Lemma.

$$b_\lambda \otimes B_w(\infty) = \coprod_{b \in B^\lambda_w(\infty)} \mathcal{F}^\lambda_{w,b}(b_\lambda \otimes b).$$

Proof. The inclusion $\supset$ holds by construction. Conversely given $b_\lambda \otimes b' \in B_w(\infty)$, the set $\mathcal{E}(b_\lambda \otimes b')$ equals $b_\lambda \otimes \mathcal{E}b'$. Moreover it contains $\{0\}$ and hence an element of the form $b_\lambda \otimes b : b \in B^\lambda_w(\infty)$. Then $b_\lambda \otimes b' \in \mathcal{F}(b_\lambda \otimes b)$ by the crystal rules and so $b_\lambda \otimes b \in \mathcal{F}^\lambda_{w,b}(b_\lambda \otimes b)$, by the definition of the latter. Finally the union is disjoint by 19.1. □

19.3. Fix a reduced decomposition $\mathbf{w} = s_{i_1} s_{i_2} \cdots s_{i_n}$ of $w \in W$. Given $b \in B_{\mathbf{w}}(\infty)$ we may write

$$b = b_\infty \otimes b_{i_1}(-m_1) \otimes \cdots \otimes b_{i_n}(-m_n) \qquad (*)$$

for some $m_j \in \mathbb{N}$.

Given $b \in B^\lambda_{\mathbf{w}}(\infty)$, set

$$J^\lambda_{\mathbf{w},b} = \{j \in \{1, 2, \cdots, n\} \mid r^{i_j}_j(b) \le \alpha^\vee_{i_j}(\lambda)\}.$$

In the monoid

$$\mathcal{F}_{\mathbf{w}} = \mathcal{F}_{i_1}\mathcal{F}_{i_2}\cdots\mathcal{F}_{i_n}$$

suppress $\mathcal{F}_{i_j}$ for all $j \in J^\lambda_{\mathbf{w},b}$. The resulting element of the singular Hecke monoid can be written as $\mathcal{F}_{y^\lambda_{\mathbf{w},b}}$, for some $y^\lambda_{\mathbf{w},b} \in W$.

Theorem. For all $\lambda \in P^+, w \in W, b \in B^\lambda_w(\infty)$ *one has*

(i) $\mathcal{F}^\lambda_{w,b} \supset \mathcal{F}_{y^\lambda_{\mathbf{w},b}}$.

(ii) $\mathcal{F}^\lambda_{w,b}(b_\lambda \otimes b) = \mathcal{F}_{y^\lambda_{\mathbf{w},b}}(b_\lambda \otimes b)$.

(iii) $y^\lambda_{\mathbf{w},b}$ *is independent of the reduced decomposition* $\mathbf{w}$ *of* w.

Proof. Set $J = J^\lambda_{\mathbf{w},b}$, $y = y^\lambda_{\mathbf{w},b}$ and $N = \{1, 2, \cdots, n\}$.

Let us first prove (i).

Fix $s = \{n, n-1, \cdots, 0\}$ and consider $b' = f^{u_{s+1}}_{i_{s+1}} \cdots f^{u_n}_{i_n} b$ with $u_j \in \mathbb{N}$ and $u_j = 0$ if $j \in J$. Admit that b' differs from b only in the last $n - s$ places.

Since $b \in B^\lambda(\infty)$, it follows that $0 \leq \varepsilon_i(b) \leq \varphi_i(b_\lambda) = \alpha^\vee_i(\lambda)$, for all $i \in I$. Consequently $max_j\{r^i_j(b)\} = \varepsilon_i(b) \leq \alpha^\vee_i(\lambda)$. Thus if $j \in N \setminus J$ we obtain $r^{i_j}_j(b) = \alpha^\vee_{i_j}(\lambda)$.

Now suppose s above belongs to $N \setminus J$. Then by our hypothesis on b', the largest value of $r^{i_t}_t(b') : t \leq s$ is reached at $t = s$ and this value equals $\alpha^\vee_{i_s}(\lambda)$. By 12.1 this forces $\varepsilon_{i_s}(b') \geq \alpha^\vee_{i_s}(\lambda)$ and $g^{i_s}(b') \geq s$. Since $t \mapsto g^{i_s}(f^t_{i_s} b')$ is an increasing function, it follows that powers of f_{i_s} go in at the s^{th} place in b' or at a later place. Consequently $b'' = f^{u_s}_{i_s} b' : u_s \in \mathbb{N}$ differs from b in only the last $n-(s-1)$ places. Moreover since $\varphi_{i_s}(b_\lambda) = \alpha^\vee_{i_s}(\lambda) \leq \varepsilon_{i_s}(b') \leq \varepsilon_{i_s}(f^t_{i_s} b')$, $\forall\, t \in \mathbb{N}$, we obtain $f^{u_s}_{i_s}(b_\lambda \otimes b') = b_\lambda \otimes f^{u_s}_{i_s} b'$, $\forall\, u_s \in \mathbb{N}$.

If $s \in J$, the corresponding assertions are trivial since we require $u_s = 0$. We conclude that $\mathcal{F}_y(b_\lambda \otimes b) \subset b_\lambda \otimes \mathcal{F}_y b \subset b_\lambda \otimes B_w(\infty)$ and so $\mathcal{F}_y \subset \mathcal{F}^\lambda_{w,b}$.

For (ii), suppose $f \in \mathcal{F}^\lambda_{w,b}$ and consider $\tilde{b} := b_\lambda \otimes fb$. We first show that $\mathcal{E}_{w^{-1}}\tilde{b}$ contains a primitive element which by 19.2 is necessarily $b_\lambda \otimes b$.

Take $s \in N$. Assume that we have found $u_1, u_2, \cdots, u_{s-1} \in \mathbb{N}$ such that $e := e_{i_{s-1}}^{u_{s-1}} \cdots e_{i_1}^{u_1}$ satisfies $e\tilde{b} = b_\lambda \otimes b_s$, for some $b_s \in B_w(\infty)$ and consider the condition

$$(*)_s \qquad r_t^{i_t}(b_s) \leq \alpha_{i_t}^\vee(\lambda), \quad \textit{for all} \;\; t \leq s-1.$$

Observe that $b_s = efb$ and so belongs to $B_w(\infty)$.

Obviously $(*)_1$ holds trivially. We show that $(*)_s \Longrightarrow (*)_{s+1}$.

Suppose $r_s^{i_s}(b_s) > \alpha_{i_s}^\vee(\lambda)$. Then $\varepsilon_{i_s}(b_s) \geq r_s^{i_s}(b_s) > \alpha_{i_s}^\vee(\lambda) = \varphi_{i_s}(\lambda)$. This has the following consequences. First by $(*)_s$ we obtain $p^{i_s}(b_s) \geq 0$, second $e_{i_s}(e\tilde{b}) = e_{i_s}(b_\lambda \otimes b_s) = b_\lambda \otimes e_{i_s} b_s$ and finally $e_{i_s} b_s \neq 0$, since $B(\infty)$ is upper normal.

Suppose $p^{i_s}(b_s) > s$. Then $r_t^{i_t}(e_{i_s} b_s) = r_t^{i_t}(b_s)$, for $t \leq s$ and so the above considerations apply to $e_{i_s} b_s$. Since $v \mapsto p^{i_s}(e_{i_s}^v b_s)$ is a decreasing function which eventually strictly decreases there exists $v \in \mathbb{N}^+$ such that $b_s' = e_{i_s}^v b_s$ satisfies $p^{i_s}(b_s') = s$ and $r_s^{i_s}(b_s') = r_s^{i_s}(b_s)$. Setting $u = r_s^{i_s}(b_s') - \alpha_{i_s}^\vee(\lambda)$, we obtain $r_s^{i_s}(e_{i_s}^{u+v} b_s) = \alpha_{i_s}^\vee(\lambda)$, as required.

From the above we obtain $e \in \mathcal{E}_{w^{-1}}$ such that $e\tilde{b} = b_\lambda \otimes b_{n+1}$ satisfies $(*)_{n+1}$. Now $\varphi_i(b_\lambda) = \alpha_i^\vee(\lambda)$ and $\varepsilon_i(b_{n+1}) = max_j\{r_j^i(b_{n+1})\} \leq \alpha_i^\vee(\lambda)$ by $(*)_{n+1}$ (recall that $r_j^i(b_{n+1}) = -\infty$ if $i \neq i_j$). Consequently the tensor product rules imply that $b_\lambda \otimes b_{n+1}$ is primitive and so $b_{n+1} = b$.

Next we show that e in the above belongs to $\mathcal{E}_{y^{-1}}$. First observe that in the above construction we can write (expressing b as in $(*)$)

$$b_s = b_\infty \otimes b_{i_1}(-m_{1,s}) \otimes b_{i_2}(-m_{2,s}) \otimes \cdots \otimes b_{i_s}(-m_{n,s}),$$

where $m_{t,s} = m_t$, for $t < s$ (because $p^{i_r}(b_r) \geq r$, for all $r \in \mathbb{N}$). In particular

$$(**) \qquad r_s^{i_s}(b_t) = m_{s,t} - m_s + r_s^{i_s}(b_s), \; \forall \; t \geq s.$$

Now suppose $s \in J$. To prove our assertion that $e \in \mathcal{E}_{y^{-1}}$, it is enough to show that $r_s^{i_s}(b_s) \geq \alpha_{i_s}^\vee(\lambda)$, leads to a contradiction.

Since $r_s^{i_s}(b) < \alpha_{i_s}^\vee(\lambda)$, by definition of J, it follows from $(**)$ that $m_{s,s} > m_s$. Yet we have $p^{i_s}(b_s) \geq s$ and so as before we can find $u \in \mathbb{N}$ such that $r_s^{i_s}(e_{i_s}^u b_s) = \alpha_{i_s}^\vee(\lambda)$ and $p^{i_s}(e_{i_s}^u b_s) = s$. Since $\alpha_{i_s}^\vee(\lambda) = \varphi_{i_s}(b_s)$, we then obtain $e_{i_s}(b_\lambda \otimes e_{i_s}^u b_s) = 0$. This forces $b_{s+1} = e_{i_s}^u b_s$. Hence $\alpha_{i_s}^\vee(\lambda) = r_s^{i_s}(b_{s+1}) = m_{s,s+1} - m_s + r_s^{i_s}(b)$, by $(**)$. Yet $r_s^{i_s}(b) < \alpha_{i_s}^\vee(\lambda)$ and so $m_{s,s+1} > m_s$, which contradicts our previous equality. We conclude that $r_s^{i_s}(b_s) < \alpha_{i_s}^\vee(\lambda)$, so we can take $u_s = 0$.

Recall that $e\tilde{b} = ef(b_\lambda \otimes b) = b_\lambda \otimes b$. By the crystal rules we can choose $\tilde{f} \in \mathcal{F}_y$ such that $f(b_\lambda \otimes b) = \tilde{f}(b_\lambda \otimes b)$. Combined with (i) this gives (ii).

Finally $b_\lambda \otimes b$ is a primitive vector and so up to a shift of weights $\mathcal{F}_y(b_\lambda \otimes b) = B_y(\infty)$ which by (ii) is independent of the choice of reduced decomposition of $\mathbf{w}$. Hence (iii). □

Remark 1. The independence of $y_{\mathbf{w},b}^\lambda$ on the choice of reduced decomposition is not a priori obvious and is a subtle combination of the properties of the Kashiwara function and singular Hecke monoid. From now on we write $y_{\mathbf{w},b}^\lambda$ simply as $y_{w,b}^\lambda$.

Remark 2. One cannot conclude that $\mathcal{F}_{w,b}^\lambda = \mathcal{F}_{y_{w,b}^\lambda}$. Indeed it can happen that $fb_\infty = f'b_\infty$: with $f \in \mathcal{F}_w$ and $f' \in \mathcal{F}_{w'}$ with $w \neq w'$. For example take $\pi = \{\alpha_1, \alpha_2\}$ of type A_2. Then $f_1f_2f_1b_\infty = f_2f_1^2b_\infty$. Yet $f_1f_2f_1 \notin \mathcal{F}_2\mathcal{F}_1$.

19.4. Retain the above notation. Given $y \in W$, we extend the notation of 19.1 by writing $B_y(\infty, \lambda + wt\ b) = \mathcal{F}_y(b_\lambda \otimes b)$. Through 19.2, 19.3 we obtain the

Corollary. *For all $\lambda \in P^+$, $w \in W$ one has*

$$b_\lambda \otimes B_w(\infty) = \coprod_{b \in B_w^\lambda(\infty)} B_{y_{w,b}^\lambda}(\infty, \lambda + wt\ b).$$

19.5. Now fix $\mu \in P^+$ and recall the strict embedding $B(\mu) \hookrightarrow B(\infty) \otimes S(\mu)$. With respect to this embedding set

$$B_w^\lambda(\mu) = \{b \in B_w^\lambda(\infty) \mid b \otimes s_\mu \in B(\mu)\}.$$

Through this embedding (or directly following the proof of 19.2, 19.3) we obtain the

Theorem. *For all* $\lambda, \mu \in P^+,\ w \in W$ *one has*

$$b_\lambda \otimes B_w(\infty) = \coprod_{b \in B_w^\lambda(\mu)} B_{y_{w,b}^\lambda}(\lambda + wt\ b).$$

19.6. The above result is a combinatorial version of a conjecture made for Demazure modules, namely that the tensor product $k_\lambda \otimes F(w\lambda)$ admits a filtration whose quotients are Demazure modules. This was studied by P. Polo and settled even in positive characteristic by O. Mathieu for $\mathfrak{g}$ semisimple. It is now also known to hold for $\mathfrak{g}$ Kac-Moody when the Cartan matrix is symmetric and simply-laced, that is when the off-diagonal entries are 0 or -1.

20. The Kashiwara Involution and Primitive Elements

20.1. Kashiwara showed that there exists involution $\star : B(\infty) \to B(\infty)$ which commutes with wt (so in particular $b_\infty^\star = b_\infty$). Moreover if one defines $e_i^\star := \star e_i \star$ and $f_i^\star = \star f_i \star$, for all $i \in I$, then every $b \in B(\infty)$ can be uniquely expressed in form $f_i^{\star m} b_0 : m \in \mathbb{N}$, with $e_i^\star b_0 = 0$, such that the map $b \mapsto b_0 \otimes b_i(-m)$ is a strict embedding of $B(\infty)$ in $B(\infty) \otimes B_i$ taking b_∞ to $b_\infty \otimes b_i(0)$. Now by the tensor product rules such an embedding is uniquely determined and so must coincide with the embedding constructed in 13.6. Notice that this implies that if $b_0 \otimes b_i(-m) \in Im\ B(\infty)$, for some $m \in \mathbb{N}$, then this holds for every $m \in \mathbb{N}$. A priori this is far from obvious and indeed there is no known purely combinatorial proof.

20.2. Take $w \in W$ and fix a reduced decomposition $\boldsymbol{w} = s_{i_1} \cdots s_{i_n}$.

Lemma. *Each* $b \in B_w(\infty)$ *can be uniquely written in the form*

$$b = f_{i_1}^{m_1} \cdots f_{i_n}^{m_n} b_\infty : m_j \in \mathbb{N},$$

such that for all $j = 1, 2, \cdots, n$ *one has*

$$(*) \qquad e_{i_j} f_{i_{j+1}}^{m_{j+1}} \cdots f_{i_n}^{m_n} b_\infty = 0.$$

Proof. Existence follows from 16.2(ii). Consider $b \in B_w(\infty)$. Since $B(\infty)$ is upper normal $m_1 := \varphi_{i_1}(b) \in \mathbb{N}$ and moreover $b' : e_{i_1}^{m_1} b \neq 0$, whilst $e_{i_1} b' = 0$. Since $B(\infty)$ is the direct limit of the $B(\lambda) : \lambda \in P^+$, it follows from 16.2(i) that $b' \in B_{s_i w}(\infty)$. By the crystal rules $b = f_{i_1}^{m_1} b'$ and so the assertion follows by induction on $\ell(w)$.

20.3. It would be tempting to imagine that if $b \in B_w(\infty)$ is written in the above canonical form, then

$$(*) \qquad b = b_\infty \otimes b_{i_1}(-m_1) \otimes \cdots \otimes b_{i_n}(-m_n).$$

However this is not so, although one easily checks that every $B_w(\infty)$ can be written in this form for some other $m_j \in \mathbb{N} : j = 1, 2, \cdots, n$. Indeed this is exactly what follows from applied to the special case when $\lambda = 0$. Here one notes that a given factor of f_{i_j} may go into the j^{th} place or at a later place depending on the precise value of the Kashiwara function. So far there is no known simple description for the subset $\{m_j\}_{j \in 1}^n \in \mathbb{N}^n$, for which $b \in \hat{B}(\infty)$ actually belongs to $B(\infty)$. Yet we do have the

Theorem. *Suppose $b \in B_w(\infty)$ takes the form given in 20.3(**). Then*

$$b = f_{i_n}^{\star m_n} f_{i_{n-1}}^{\star m_{n-1}} \cdots f_{i_1}^{\star m_1} b_\infty$$

where for all $j \in \{1, 2, \cdots, n\}$ one has

$$(**) \qquad e_{i_j}^\star f_{i_{j-1}}^{\star m_{j-1}} \cdots f_{i_1}^{\star m_1} b_\infty = 0.$$

In particular $(B_w(\infty))^\star = B_{w^{-1}}(\infty)$.

Proof. The first part follows by iterating Kashiwara's result described in 20.1, Notice that

$$b_j := (f_{i_{j-1}}^{\star m_{j-1}} \cdots f_{i_1}^{\star m_1} b_\infty)^\star = f_{i_{j-1}}^{m_{j-1}} \cdots f_{i_1}^{m_1} b_\infty$$

which gives the last part and so also shows that $(**)$ is equivalent to $\varepsilon_{i_j}(b_j) = 0, \ \forall \ j \in \{1, 2, \cdots, n\}$. □

20.4. Take $b \in B_w(\infty)$ in the form given in 20.3($*$) and write $b_\infty = b_{i_0}(0)$. One has $e_i(b_\lambda \otimes b) = b_\lambda \otimes e_i b \Longleftrightarrow \varepsilon_i(b) > \varphi_i(b_\lambda) = \alpha_i^\vee(\lambda) \geq 0$, so then $e_i b \neq 0$, since $B(\infty)$ is upper normal. Consequently $e_i(b_\lambda \otimes b) = 0 \Longleftrightarrow \alpha_i^\vee(\lambda) \geq \varepsilon_i(b)$. Now

$$\begin{aligned}\varepsilon_i(b) &= \max_{s \in \{0,1,2,\cdots,n\}} \{\varepsilon_i(b_{i_s}(-m_s)) - \sum_{t<s} \alpha_i^\vee(wt\ b_{i_t}(-m_t))\} \\ &= \max_{s \in \{1,2,\cdots,n\} | i_s = i} \{0, m_s + \sum_{t<s} \alpha_i^\vee(\alpha_{i_t}) m_t\}\end{aligned}$$

Since $\alpha_i^\vee(\lambda) \geq 0$, we conclude that

Lemma. *Take $\lambda \in P^+$. Then b, defined by 20.3($*$), belongs to $B(\infty)^\lambda$ if and only if*

$$\alpha_{i_s}^\vee(\lambda) \geq m_s + \sum_{t<s} \alpha_{i_s}^\vee(\alpha_{i_t}) m_t,$$

for all $s \in \{1, 2, \cdots, n\}$.

20.5. Take $b \in B_w(\infty)$ as in 20.3. For all $s = 1, 2, \cdots, n+1$, set $F_s = f_{i_{s-1}}^{m_{s-1}} \cdots f_{i_1}^{m_1}$. Then $b^\star = F_{n+1} b_\infty$. Now by 12.7 $b^\star \otimes S_\lambda$ belongs to the image of $B(\lambda)$ in $B(\infty) \otimes S_\lambda$ if and only if $F_{n+1}(b_\infty \otimes s_\lambda) = F_{n+1} b_\infty \otimes s_\lambda$. Since $\varepsilon_i(s_\lambda) = -\alpha_i^\vee(\lambda)$, for all $I \in I$, the above holds if and only if

$$(*) \qquad \varphi_s(F_s b_\infty) \geq -\alpha_{i_s}^\vee(\lambda) + m_s, \ \forall s \in \{1, 2, \cdots, n\}.$$

On the other hand 5.3($*$) we noted that 20.3($**$) is equivalent to $\varepsilon_{i_s}(F_s b_\infty) = 0$ and so

$$\varphi_{i_s}(F_s b_\infty) = \alpha_{i_s}^\vee(wt\ F_s b_\infty) = -\sum_{t<s} \alpha_{i_s}^\vee(\alpha_{i_t}) m_t.$$

From this one observes that ($*$) is exactly the condition of 20.4. Thus we have shown that

Theorem. *For all $\lambda \in P^+$, $b \in B(\infty)$ one has*

$$b \in B^\lambda(\infty) \Longleftrightarrow b^\star \otimes s_\lambda \in B(\lambda).$$

20.6. From 20.5 it is relatively easy to compute elements of $B^\lambda(\infty)$. A particular nice case is the following one. Take $\lambda \in P^+$ and $\mathbf{w}$ as in 20.2 and set $w_j = s_{i_{j-1}} \cdots s_{i_1}$. Then $m_j := \alpha_{i_j}^\vee(w_j\lambda) \in \mathbb{N}$, since $\mathbf{w}$ is a reduced decomposition. In the notation of 20.5 and choosing the m_i as above, the unique element $b_{w^{-1}\lambda} \in B(\lambda)_{w^{-1}\lambda}$ is given by $b_{w^{-1}\lambda} = F_{n+1}b_\lambda$ and moreover $e_{i_j}F_jb_\lambda = 0 : j = 1, 2, \cdots, n$, by weight space considerations (as in 18.5). Through 13.5 these last relations hold for b_λ replaced by b_∞. Consequently by 20.3 one has

$$b^\star_{w^{-1}\lambda} = b_{i_0}(0) \otimes b_{i_1}(-m_1) \otimes \cdots \otimes b_{i_n}(-m_n) \in B_w(\infty).$$

Moreover one easily checks that

$$r_j^{i_j}(b^\star_{w^{-1}\lambda}) = \alpha_{i_j}^\vee(\lambda), \ \forall\, j = 1, 2, \cdots, n.$$

Moreover $b^\star_{w^{-1}\lambda}$ is the unique element of $B_{\mathbf{w}}(\infty)$ with this property. Recall the notation of 19.3, and observe that the subset $J^\lambda_{\mathbf{w},b} = \emptyset$, when $b = b^\star_{w^{-1}\lambda}$. Thus we conclude from 19.3(ii), that

Corollary. *For all* $\lambda \in P^+$, $w \in W$ *one has* $b^\star_{w^{-1}\lambda} \in B^\lambda_w(\infty)$ *and*

$$\mathcal{F}^\lambda_{w,b^\star_{w^{-1}\lambda}}(b_\lambda \otimes b^\star_{w^{-1}\lambda}) = \mathcal{F}_w(b_\lambda \otimes b^\star_{w^{-1}\lambda}).$$

Moreover $b^\star_{w^{-1}\lambda}$ *is the unique element of* $B^\lambda_w(\infty)$ *with this property.*

20.7. Since $B(\infty)$ is the direct limit of the $B(\lambda) : \lambda \in P^+$, it follows that $ch\ B(\infty)$ is given by the asymptotic value of $e^{-\lambda} ch\ B(\lambda)$. Through 16.10(∗) this gives

$$ch\ B(\infty) = \left(\sum_{w \in W} (-1)^{\ell(w)} e^{w\rho - \rho} \right)^{-1}.$$

Define $Q(\lambda) : \lambda \in P^+$, as in 10.7(ii).

Assuming that the Cartan matrix is symmetrizable, it follows from 10.7(iv), that $Q(0)$ is the trivial module and so $ch\ Q(0) = 1$. Substitution in 10.7(iv) and the above one obtains

$$ch\ B(\infty) = \frac{1}{\prod_{\alpha \in \Delta^+}(1 - e^{-\alpha})^{m_\alpha}}.$$

It is not known if this holds without the symmetrizability condition.

The above formula shows that $ch\ B(\infty)$ takes a relatively simple form and one might expect this to be true of $B(\infty)$ also. However here one should be aware that $ch\ B_w(\infty)$ does not take a particularly simple form, in particular

$$ch\ B_w(\infty) = \prod_{\alpha\in S(w^{-1})} \left(\frac{1}{1-e^{-\alpha}}\right) \quad \textit{is false.}$$