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SIMPLE BOUNDED HIGHEST WEIGHT MODULES OF BASIC CLASSICAL LIE SUPERALGEBRAS

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Abstract. We classify all simple bounded highest weight modules of a basic classical Lie superalgebra $\mathfrak{g}$. In particular, our result leads to the classification of the simple weight modules with finite weight multiplicities over all classical Lie superalgebras. We also obtain some character formulas of strongly typical bounded highest weight modules of $\mathfrak{g}$.

Introduction

The representation theory of Lie superalgebras has been extensively studied in the last several decades. Remarkable progress has been made on the study of the (super)category $\mathcal{O}$, see for example [S1] and the references therein. On the other hand, the theory of general weight modules of Lie superalgebras is still at its beginning stage. An important advancement in this direction was made in 2000 in [DMP] where the classification of the simple weight modules with finite weight multiplicities over classical Lie superalgebras was reduced to the classification of the so-called simple cuspidal modules. This result is the superanalog of the Fernando-Futorny parabolic induction theorem for Lie algebras. The classification of the simple cuspidal modules over reductive finite-dimensional simple Lie algebras was completed by Mathieu, [M], following works of Benkart, Britten, Fernando, Futorny, Lemire, Joseph, and others, [BBL], [BL], [F], [Fu], [Jo]. One important result in [M] is that every simple cuspidal module is a twisted localization of a simple bounded highest weight module, where, a *bounded module* by definition is a module whose set of weight multiplicities is bounded. The maximum weight multiplicity of a bounded module is called the *degree* of the module.

The presentation of the simple cuspidal modules via twisted localization of highest weight modules was extended to the case of classical Lie superalgebras in

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[Gr]. In this way, the classification of simple weight modules with finite weight multiplicities of a classical Lie superlagebra $\mathfrak{k}$ is reduced to the classification of the simple bounded highest weight modules of $\mathfrak{k}$. The latter modules are easily classified for Lie superalgebras of type I. For Lie superalgebras of type II a classification is obtained for Lie supealgebras of Q-type in [GG], for the exceptional Lie superalgebra $D(2,1,a)$ in [H], and for $\mathfrak{osp}(1|2n)$ in [FGG]. The main goal of this paper is to complete the classification in all remaining cases, namely for $\mathfrak{osp}(m|2n)$, $m = 3,4,5,6$. In particular, by classifying the simple bounded highest weight modules for these four series of orthosymplectic superalgebras, we complete the classification of all simple weight modules with finite weight multiplicities over all classical Lie superalgebras.

Apart from the classification of simple weight modules, the category of bounded modules is interesting on its own. We believe that the results in the present paper mark the first step towards the systematic study of this category. Note that in the case of Lie algebras, bounded modules have nice geometric realizations and an equivalence of categories of bounded modules and weight modules of algebras of twisted differential operators was established in [GrS1], [GrS2]. We expect that similar geometric properties of the category of bounded modules of classical Lie superalgebras hold as well. We also expect that, as in the Lie algebra case, the injective objects in the category of bounded modules will be obtained via twisted localization functors.

We remark that in [Co], there is a classification and explicit examples of all simple highest weight modules of degree 1. One should note that in this classification there is a minor gap in the proof for lower-rank cases.

Most of the new results in this paper concern the highest weight bounded modules of the orthosymplectic Lie superalgebras $\mathfrak{osp}(m|2n)$. One should note though that the above mentioned classification is new also for the exceptional Lie superalgebras $F(4)$ an $G(3)$. In addition to the completion of this classification, we prove that the category of $\mathcal{O}$-bounded $\mathfrak{osp}(1|2n)$-modules is semisimple for $n > 1$. Last, but not least, we establish explicit character formula for strongly typical bounded modules over all basic classical Lie superalgebras.

A crucial part in the paper plays the notion of *the nonisotropic algebra* $\mathfrak{g}_{ni}$ associated to a Kac–Moody superalgebra $\mathfrak{g}$. Most of the criteria for boundedness are expressed in terms of the components of $\mathfrak{g}_{ni}$. Also, for our classification we use distinguished sets of simple roots - simple roots that contain at most one isotropic root. One of the tools used in the paper are Enright functors — localization type of functors introduced originally by Enright in [En] for classical Lie algebras and later generalized by [IK] for Kac–Moody superalgebras.

Our main result is Theorem 3.3 which describes simple highest weight bounded modules over basic classical Lie superalgebras in terms of the highest weights with respect to the distinguished Borel subalgebras. For all $\mathfrak{g}$ except for $\mathfrak{g} = \mathfrak{osp}(m|2n), m \geq 5, n \geq 2$, we give a simple criterion, Corollary 3.5.1. On the other hand, Theorem 3.6.1 reduces the remaining case $\mathfrak{osp}(m|2n), m \geq 5, n \geq 2$ to the case $\mathfrak{osp}(m|4)$. In Section 4 we provide character formula and an upper bound of the degree of a strongly typical simple highest weight bounded module for $\mathfrak{osp}(m|2n)$. In Section 5 we obtain an upper bound of the degree of the simple $\mathcal{O}$-bounded modules for the cases $\mathfrak{osp}(m|2n)$ with $m = 3, 4$ or $n = 1$.

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1. Preliminaries

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be one of the Lie superalgebras

$$\mathfrak{sl}(m|n), m \neq n,\ \mathfrak{gl}(n|n),\ \mathfrak{osp}(m|2n),\ D(2,1,a),\ F(4),\ G(3).$$

For the sake of brevity $A(m|n)$ stands for the corresponding Kac–Moody superalgebra: $A(m|n) = \mathfrak{sl}(m|n)$ for $m \neq n$ and $A(n|n) = \mathfrak{gl}(n|n)$. We fix a triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ and consider all compatible triangular decompositions of $\mathfrak{g}$, i.e., $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ with $\mathfrak{n}_0^+ = \mathfrak{n}^+ \cap \mathfrak{g}_0$. Recall that any two compatible triangular decompositions are connected by a chain of odd reflections, see [S2]. We denote by Δ the root system of $\mathfrak{g}$ and by Δ_0 (respectively, by Δ_1) the set of even (respectively, odd) roots. We denote by Π_0 the set of simple roots for $\mathfrak{g}_0$ (Π_0 is fixed, since $\mathfrak{n}_0^+$ is fixed) and by Σ a base of Δ^+.

Note that $\mathfrak{g}$ is an indecomposable Kac–Moody superalgebra. Recall that an indecomposable finite-dimensional Kac–Moody superalgebra is isomorphic either to $\mathfrak{gl}(n|n)$ or to a basic classical Lie superalgebra which is not isomorphic to $\mathfrak{psl}(n|n)$. Recall that $\mathfrak{g}$ admits a non-degenerate invariant bilinear form. In all examples we will use the standard notation for root systems, see [K1].

1.1. Notation

We set

$$\Delta_{ni} := \{\alpha \in \Delta \mid \|\alpha\|^2 \neq 0\}$$

to be the set of nonisotropic roots. For $\alpha \in \Delta_{ni}$ we introduce $\alpha^\vee := 2\alpha/(\alpha, \alpha)$ and the reflection $r_\alpha \in \mathrm{GL}(\mathfrak{h}^*)$ given by $r_\alpha(\mu) := \mu - (\mu, \alpha^\vee)\alpha$. We denote by W the Weyl group of Δ (the group generated by the reflections r_α with $\alpha \in \Delta_{ni}$).

For a base Σ we denote by ρ_Σ its Weyl vector, namely the difference of the half sums of the even positive roots and the odd positive roots. For $\lambda \in \mathfrak{h}^*$ we denote by $L(\Sigma, \lambda)$ the corresponding simple highest weight module. Note that $L(\Sigma, \lambda)$ is a simple highest weight module for any base Σ' (compatible with Π_0). In the case when Σ is fixed, we write ρ for ρ_Σ and $L(\lambda)$ for $L(\Sigma, \lambda)$. By $M(\lambda) = M(\Sigma, \lambda)$ we denote the corresponding Verma module.

For a fixed base Σ we consider the standard partial order on $\mathfrak{h}^*$: $\mu \geq \mu'$ if $\mu - \mu' \in \mathbb{Z}_{\geq 0}\Sigma$.

For a $\mathfrak{g}$-module N we set

$$N_\nu := \{v \in N \mid hv = \nu(h)v, \forall h \in \mathfrak{h}\}, \quad \mathrm{supp}(N) := \{\nu \in \mathfrak{h}^* \mid N_\nu \neq 0\}$$

and say that v has *weight* ν if $v \in N_\nu$. If all weight spaces N_ν are finite-dimensional, we set

$$\mathrm{ch}\, N := \sum_{\nu \in \mathfrak{h}^*} \dim N_\nu e^\nu.$$

A $\mathfrak{g}$-module N is called a *weight module* if $N = \bigoplus_{\nu\in\mathfrak{h}^*} N_\nu$, and N is *bounded* if it is a weight module and there is $s > 0$ such that $\dim N_\nu < s$ for all $\nu \in \mathfrak{h}^*$.

1.1.1. Kac–Moody subalgebras. Fix a nonempty subset $\Sigma' \subset \Sigma$ and denote by $\mathfrak{t}$ the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{g}_{\pm\alpha}, \alpha \in \Sigma'$. We call $\mathfrak{t}$ a *Kac–Moody subalgebra* of $\mathfrak{g}$. Note that $\mathfrak{t}$ is a direct sum of a Kac–Moody superalgebra and several copies of $\mathfrak{sl}(s|s)$; $\mathfrak{t}\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{t}$ and $\Sigma_{\mathfrak{t}} := \Sigma'$ is a base; we denote by $\Delta_{\mathfrak{t}}$ the corresponding root system and by $W(\mathfrak{t})$ the corresponding Weyl group. One has $\Delta_{\mathfrak{t}} = \Delta \cap (\mathbb{Z}\Sigma')$, see [K2, Ex. 1.2].

If Σ' is a connected component of Σ we call $\mathfrak{t}$ a *component* of $\mathfrak{g}$.

1.2. Categories $\mathcal{O}, \mathcal{O}^{\inf}$

We denote by $\mathcal{O}^{\inf}(\mathfrak{g})$ the full category of $\mathfrak{g}$-modules with the following properties:

(C1) $\mathfrak{h}$ acts diagonally (or, equivalently, semisimply);

(C2) $\mathfrak{n}_0^+$ acts locally nilpotently.

We denote by $\mathcal{O}(\mathfrak{g})$ the BGG-category which is the full subcategory of $\mathcal{O}^{\inf}(\mathfrak{g})$ consisting of finitely generated modules. Note that $\mathcal{O}(\mathfrak{g}), \mathcal{O}^{\inf}(\mathfrak{g})$ do not depend on the choice of Σ. Indeed, if Σ, Σ' are compatible with the triangular decomposition of $\mathfrak{g}_0$, then $\mathcal{O}^{\inf}(\mathfrak{g})$ is the same category for Σ and for Σ'.

1.2.1. Pairs of Kac–Moody superalgebras. Let $\mathfrak{g}' = \mathfrak{n}'_- \oplus \mathfrak{h}' \oplus \mathfrak{n}'_+$ be a Kac–Moody superalgebra and $\mathfrak{g}' \subset \mathfrak{g}$; we say that $\mathfrak{g}', \mathfrak{g}$ have *compatible triangular decompositions* if $\mathfrak{h}' \subset \mathfrak{h}, \mathfrak{n}'_\pm \subset \mathfrak{n}^\pm$ and $\mathfrak{h}$ acts diagonally on each root space of $\mathfrak{g}'$. Let us assume now that $\mathfrak{g}' \subset \mathfrak{g}$ is a pair of Kac–Moody superalgebras with compatible triangular decompositions. For $N \in \mathcal{O}^{\inf}(\mathfrak{g})$ one has $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{g}'} N \in \mathcal{O}^{\inf}(\mathfrak{g}')$. On the other hand, $\mathcal{O}(\mathfrak{g})$ does not have this property in general. However, the property holds in the special case $\mathfrak{g}' = \mathfrak{g}_0$.

For each $\lambda \in \mathfrak{h}^*$ we denote by $\lambda_{\mathfrak{g}'}$ the restriction of λ to $\mathfrak{g}'\cap\mathfrak{h}$; we denote by $M_{\mathfrak{g}'}(\lambda_{\mathfrak{g}'}), L_{\mathfrak{g}'}(\lambda_{\mathfrak{g}'})$ the corresponding $\mathfrak{g}'$-modules. We use the similar notation for the case when $\mathfrak{g}'$ is a Kac–Moody subalgebra of $\mathfrak{g}$.

The following lemma will be useful later.

1.2.2. **Lemma**.

(i) *Let $\mathfrak{t}$ be a Kac–Moody subalgebra of $\mathfrak{g}$. The $\mathfrak{t}$-submodule of $L(\lambda)$ generated by a highest weight vector of $L(\lambda)$ is isomorphic to $L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})$.*

(ii) *Let $\mathfrak{g}' \subset \mathfrak{g}$ be a pair of Kac–Moody superalgebras with compatible triangular decompositions. A cyclic $\mathfrak{g}'$-submodule of a bounded $\mathfrak{g}$-module is $\mathfrak{g}'$-bounded.*

Proof. For (i) let v be a highest weight vector of $L(\lambda)$ and L' be the $\mathfrak{t}$-submodule generated by v. Clearly, L' is a quotient of $M_{\mathfrak{t}}(\lambda_{\mathfrak{t}})$. Let $uv \in L'$ be a $\mathfrak{t}$-primitive ($\mathfrak{t}$-singular) vector, i.e., $u \in \mathcal{U}(\mathfrak{n}^-\cap\mathfrak{t})$ is such that $(\mathfrak{t}\cap\mathfrak{n}^+)(uv) = 0$. Take $\alpha \in \Delta^+ \setminus \Delta_{\mathfrak{t}}$. For each $\beta \in \Delta_{\mathfrak{t}}\cap\Delta^+$ one has $\beta - \alpha \notin \Delta^+$ which gives $[\mathfrak{g}_{-\beta}, \mathfrak{g}_\alpha] \subset \mathfrak{n}^+$. This implies $\mathfrak{g}_\alpha(uv) = 0$ and thus uv is a $\mathfrak{g}$-primitive vector. Therefore uv is proportional to v, so L' is simple. This gives (i).

For (ii) let N be a bounded $\mathfrak{g}$-module and let N' be the $\mathfrak{g}'$-submodule generated by a vector $v' \in N$; we may (and will) assume that v' is a weight vector. Recall that $\mathfrak{h}' = \mathfrak{g}'\cap\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}'$. Set

$$\Delta' := \{\alpha \in \Delta \mid \mathfrak{g}_\alpha \subset \mathfrak{g}'\}.$$

Fix $\nu' \in (\mathfrak{h}')^*$ such that $N'_{\nu'} \neq 0$. One has

$$\dim N'_{\nu'} = \sum_{\nu \in X} \dim(N_\nu \cap N'),$$

where

$$X := \{\nu \in \operatorname{supp} N \mid \nu|_{\mathfrak{h}\cap\mathfrak{t}} = \nu',\ N_\nu \cap N' \neq 0\}.$$

For $\nu_1, \nu_2 \in X$ one has $(\nu_1 - \nu_2) \in \mathbb{Z}\Delta'$, since N' is a cyclic $\mathfrak{g}'$-module generated by a weight vector, and $(\nu_1 - \nu_2)|_{\mathfrak{h}'} = 0$. Thus $\nu_1 = \nu_2$, so $X = \{\nu_1\}$ and $\dim N'_{\nu'} \leq \dim N_{\nu_1}$. $\square$

1.3. Root subsystems $\Delta(N), \Delta(\lambda)$

A subset $\Delta' \subset \Delta_0$ is called a *root subsystem* if $r_\alpha\beta \in \Delta'$ for any $\alpha, \beta \in \Delta'$. For a root subsystem Δ' we denote by $W(\Delta')$ the subgroup of W generated by $r_\alpha, \alpha \in \Delta'$. We set $(\Delta')^+ := \Delta' \cap \Delta^+$ and introduce

$$\Pi((\Delta')^+) := \{\beta \in (\Delta')^+ \mid r_\beta((\Delta')^+ \setminus \{\beta\}) = (\Delta')^+ \setminus \{\beta\}\}.$$

The group $W(\Delta')$ is the Coxeter group for $\Pi((\Delta')^+)$ (see, for example, [KT1], 2.2.8–2.2.9).

For $N \in \mathcal{O}^{\inf}(\mathfrak{g})$ we set

$$\Delta(N) := \{\alpha \in \Delta_0 \mid \exists \lambda \in \operatorname{supp}(N) \ \text{ such that } (\lambda, \alpha^\vee) \in \mathbb{Z}\}.$$

If N is indecomposable, then $\Delta(N) = \{\alpha \in \Delta_0 \mid (\lambda, \alpha^\vee) \in \mathbb{Z} \ \ \forall \lambda \in \operatorname{supp}(N)\ \}$, since for $\gamma \in \Delta$ and $\alpha \in \Delta_0$ one has $(\gamma, \alpha^\vee) \in \mathbb{Z}$. For $\lambda \in \mathfrak{h}^*$ we introduce

$$\Delta(\lambda) := \Delta(L(\lambda)) = \{\alpha \in \Delta_0 \mid (\lambda, \alpha^\vee) \in \mathbb{Z}\}.$$

By [K2, Lem. 3.4], for a simple module L each root space $\mathfrak{g}_\alpha$ acts either injectively or locally nilpotently on L. If for each $\alpha \in \Pi_0$ the root space $\mathfrak{g}_{-\alpha}$ acts locally nilpotently on $L(\lambda)$, then $L(\lambda)$ is finite-dimensional. If $\alpha \in \Pi_0$ is such that $\alpha \in \Sigma$ or $\alpha/2 \in \Sigma$, then the root space $\mathfrak{g}_{-\alpha}$ acts locally nilpotently on $L(\lambda)$ if and only if $\alpha \in \Delta(\lambda)$ and $(\lambda, \alpha^\vee) \geq 0$.

Let N be an indecomposable module. One readily sees that $\Delta(N)$ is a root subsystem of Δ_0. We set $W(N) := W(\Delta(N))$, $W(\lambda) := W(\Delta(\lambda))$, and $\Pi(\lambda) := \Pi(\Delta(\lambda)^+)$.

1.3.1. Maximal element in orbit. It is well known that the orbit $W(\mu)\mu$ contains a unique maximal element and that μ is the maximal element in its orbit $W(\mu)\mu$ if and only if $(\mu, \alpha^\vee) \geq 0$ for each $\alpha \in \Delta(\mu)^+$. Moreover, if μ is a maximal element in $W(\mu)\mu$, then $Stab_W\mu$ is generated by the reflections r_α with $\alpha \in \Pi(\mu)$ such that $(\mu, \alpha^\vee) = 0$.

1.4. Enright functors

The Enright functors were introduced in [En]. For Kac–Moody superalgebras the Enright functors were defined in [IK]. We will use these functors in the following context: let $\mathfrak{p}$ be a Lie superalgebra containing an $\mathfrak{sl}_2$-triple (e, f, h) and $\mathcal{M}_a$ be the full subcategory of $\mathfrak{g}$-modules N with the following properties: h acts diagonally with eigenvalues in $a+\mathbb{Z}$ and e acts locally nilpotently. The Enright functor $\mathcal{C}$ is a covariant functor $\mathcal{C}:\mathcal{M}_a\to\mathcal{M}_{-a}$. We will use the Enright functor for $\mathfrak{g}$ and $\mathfrak{sl}_2$-triple corresponding to $\alpha\in\Delta_0$: $f\in\mathfrak{g}_{-\alpha}$, $h\in\mathfrak{h}$, $e\in\mathfrak{g}_\alpha$; in this case we denote this functor by $\mathcal{C}_\alpha$. We retain the notation of §1.1.1. Note that for an indecomposable $N\in\mathcal{O}^{\mathrm{inf}}(\mathfrak{g})$, the condition $\alpha\in\Delta(N)$ is equivalent to $N\in\mathcal{M}_0$.

We will use the the following properties of the Enright functors. For the proofs we refer the reader to [GS].

1.4.1. **Proposition**.

(i) *If $a\not\in\mathbb{Z}$ then $\mathcal{C}:\mathcal{M}_a\xrightarrow{\sim}\mathcal{M}_{-a}$ is an equivalence of categories.*

(ii) *If $\mathfrak{p}\subset\mathfrak{g}$ is a subalgebra containing the $\mathfrak{sl}_2$-triple (e,f,h), then the Enright functors commute with the restriction functor $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}$. Namely, $\mathcal{C}^{\mathfrak{p}}\circ\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}=\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{p}}\circ\mathcal{C}^{\mathfrak{g}}$, where $\mathcal{C}^{\mathfrak{g}},\mathcal{C}^{\mathfrak{p}}$ are Enright functors for $\mathfrak{g},\mathfrak{p}$, respectively.*

(iii) *Let $\alpha\in\Pi_0$ be such that $\alpha\in\Sigma$ or $\alpha/2\in\Sigma$ and let $\lambda\in\mathfrak{h}^*$ be such that $\alpha\not\in\Delta(\lambda)$. Then $\mathcal{C}_\alpha(L(\lambda))=L(r_\alpha(\lambda+\rho)-\rho)$ and $\mathcal{C}_\alpha(L_{\mathfrak{g}_0}(\lambda))=L(r_\alpha(\lambda+\rho_0)-\rho_0)$.*

(iv) *If $N\in\mathcal{O}^{\mathrm{inf}}(\mathfrak{g})$ has a subquotient $L(\lambda)$ and $\alpha\not\in\Delta(N)$, then $\mathcal{C}_\alpha(L(\lambda))$ is a subquotient of $\mathcal{C}_\alpha(N)$.*

2. Bounded modules in the case when $\Delta=\Delta_{ni}$

In this section $\mathfrak{g}$ is an indecomposable finite-dimensional Kac–Moody superalgebra without isotropic roots, i.e., $\mathfrak{g}$ is isomorphic to a simple Lie algebra or to $\mathfrak{osp}(1|2n)$. In this case all finite-dimensional modules are completely reducible and $L(\lambda)$ is finite dimensional if and only if for each simple root α one has $(\lambda,\alpha^\vee)\in\mathbb{Z}_{\geq 0}$.

A finite-dimensional simple Lie algebra $\mathfrak{t}$ admits infinite-dimensional bounded modules $L(\lambda)$ only for $\mathfrak{g}=\mathfrak{sl}_n,\mathfrak{sp}_{2n}$. This result is proven in by [BBL] generalizing the analogous result in [F] for cuspidal modules.

2.1. Bounded modules for $\mathfrak{sp}_{2n}$, $\mathfrak{osp}(1|2n)$

For $\mathfrak{g}=\mathfrak{sp}_2,\mathfrak{osp}(1|2)$ all modules in $\mathcal{O}$ are bounded, since $\dim L(\lambda)_\mu\leq 1$ for each $\lambda,\mu\in\mathfrak{h}^*$.

Consider the case $\mathfrak{g}=\mathfrak{sp}_{2n},\mathfrak{osp}(1|2n)$ with $n>1$. The root system Δ is of type C_n or BC_n and it contains a unique copy of the root system of type D_n. A module $L(\lambda)$ is an infinite-dimensional bounded module if and only if

$$\Delta(\lambda)=\mathsf{D}_n,\ (\lambda+\rho,\alpha^\vee)>0\ \text{ for each }\alpha\in\Delta(\lambda)^+.$$

For $\mathfrak{sp}_{2n}$ this is proven in [M]. For $\mathfrak{osp}(1|2n)$ this is proven in [FGG] and we give another proof in §2.2 below. Writing the set of simple roots for $\mathfrak{sp}_{2n}$ in the form $\{\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,2\delta_n\}$ we obtain that the root subsystem D_n has a set of simple roots $\{\delta_1-\delta_2,\dots,\delta_{n-1}-\delta_n,\delta_{n-1}+\delta_n\}$. Let $\lambda\in\mathfrak{h}^*$, and let $\lambda+\rho=\sum_{i=1}^n y_i\delta_i$. Then $L(\lambda)$ is an infinite-dimensional bounded module if and only if

$$y_1-y_2,y_2-y_3,\dots,y_{n-1}-y_n,y_{n-1}+y_n\in\mathbb{Z}_{>0} \tag{1}$$

and, in addition, $y_n\in\mathbb{Z}+1/2$ for $\mathfrak{sp}_{2n}$, while $y_n\in\mathbb{Z}$ for $\mathfrak{osp}(1|2n)$. Note that $L(\lambda)$ is finite dimensional if and only if (1) holds and, in addition, $y_n\in\mathbb{Z}_{>0}$ for $\mathfrak{sp}_{2n}$, $y_n\in\mathbb{Z}_{>0}+1/2$ for $\mathfrak{osp}(1|2n)$.

2.2.

Here we give another proof of the above-mentioned result for $\mathfrak{osp}(1|2n)$.

Theorem. *Let* $\mathfrak{g} = \mathfrak{osp}(1|2n), n > 1$. *A module* $L(\lambda)$ *is an infinite-dimensional bounded module if and only if* $\Delta(\lambda) = \mathsf{D}_n$ *and*

$$(\lambda + \rho, \alpha^\vee) > 0 \quad \textit{for each } \alpha \in \Delta(\lambda)^+. \tag{2}$$

Proof. One has $\mathfrak{g}_0 = \mathfrak{sp}_{2n}$ and $\mathfrak{g}$ admits a unique base Σ compatible with Π_0 (since Δ does not have isotropic roots). One has

$$\Sigma = \Pi' \cup \{\delta_n\},\ \Pi_0 = \Pi' \cup \{2\delta_n\},\ \Pi(\mathsf{D}_n) = \Pi' \cup \{\delta_{n-1} + \delta_n\},$$

where $\Pi' := \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}$. Let v be a highest weight vector of $L(\lambda)$.

Assume that $L(\lambda)$ is bounded. A $\mathfrak{g}_0$-module generated by v is a quotient of $M_{\mathfrak{sp}_{2n}}(\lambda)$, so $L_{\mathfrak{sp}_{2n}}(\lambda)$ is a subquotient of $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{g}_0} L(\lambda)$. Therefore $L_{\mathfrak{sp}_{2n}}(\lambda)$ is bounded.

If $L_{\mathfrak{sp}_{2n}}(\lambda)$ is finite-dimensional, then for each $\alpha \in \Pi_0$ the root space $\mathfrak{g}_{-\alpha}$ acts nilpotently on v and so $L(\lambda)$ is finite-dimensional, see §1.3.

If $L_{\mathfrak{sp}_{2n}}(\lambda)$ is an infinite-dimensional bounded module, then, by §2.1, $\Delta(\lambda) = \mathsf{D}_n$ and thus $\mathfrak{g}_{-\delta_n} v \neq 0$. Since $\mathfrak{n}_0^+(\mathfrak{g}_{-\delta_n} v) = 0$, the module $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{g}_0} L(\lambda)$ has a primitive vector of weight $\lambda - \delta_n$ and thus has a subquotient isomorphic to $L_{\mathfrak{sp}_{2n}}(\lambda - \delta_n)$. Hence $L_{\mathfrak{sp}_{2n}}(\lambda - \delta_n)$ is bounded. Since $\Delta(\lambda - \delta_n) = \Delta(\lambda) = \mathsf{D}_n$, the boundedness of $L_{\mathfrak{sp}_{2n}}(\lambda - \delta_n)$ gives

$$(\lambda + \rho, \alpha^\vee) = (\lambda - \delta_n + \rho_0, \alpha^\vee) > 0 \quad \text{for each } \alpha \in \Pi(\mathsf{D}_n),$$

see §2.1. This establishes the "only if" part.

Now assume that $\Delta(\lambda) = \mathsf{D}_n$ and that (2) holds. Let us show that $L(\lambda)$ is bounded, i.e., that $M := \operatorname{Res}^{\mathfrak{g}}_{\mathfrak{g}_0} L(\lambda)$ is bounded. Since $M \in \mathcal{O}(\mathfrak{g}_0)$, it has a finite length.

Therefore it is enough to show that any simple subquotient of M is a bounded module. Let $L_{\mathfrak{g}_0}(\mu)$ be a subquotient of M. One has $\Delta(\mu) = \Delta(\lambda) = \mathsf{D}_n$. By §2.1 it suffices to show that $(\mu + \rho_0, \alpha) > 0$ for $\alpha \in \Pi(\mathsf{D}_n)$. Take $\alpha \in \Pi'$. By (2) the root space $\mathfrak{g}_{-\alpha}$ acts nilpotently on v and thus locally nilpotently on $L(\lambda)$ and on $L_{\mathfrak{g}_0}(\mu)$. Therefore $(\mu + \rho_0, \alpha) > 0$. It remains to verify that $(\mu + \rho_0, \delta_{n-1} + \delta_n) > 0$. Note that $\Delta(\lambda) = \Delta(\mu)$ does not contain $2\delta_n$. Using Proposition 1.4.1 for $\alpha = 2\delta_n$ we obtain that $\mathcal{C}_{2\delta_n}(L_{\mathfrak{g}_0}(\mu)) = L_{\mathfrak{g}_0}(r_{\delta_n}(\mu + \rho_0) - \rho_0)$ is a subquotient of $\mathcal{C}_{2\delta_n}(L(\lambda)) = L(r_{\delta_n}(\lambda + \rho) - \rho)$. Since $\delta_{n-1} - \delta_n \in \Sigma$ and

$$(r_{\delta_n}(\lambda + \rho), \delta_{n-1} - \delta_n) = (\lambda + \rho, \delta_{n-1} + \delta_n) \in \mathbb{Z}_{>0}$$

the root space $\mathfrak{g}_{\delta_n - \delta_{n-1}}$ acts locally nilpotently on $L(r_{\delta_n}(\lambda + \rho) - \rho)$ and thus on $L_{\mathfrak{g}_0}(r_{\delta_n}(\mu + \rho_0) - \rho_0)$. Hence

$$0 < (r_{\delta_n}(\mu + \rho_0), \delta_{n-1} - \delta_n) = (\mu + \rho_0, \delta_{n-1} + \delta_n)$$

as required. This completes the proof. □

We remark that the reasoning used to prove the boundedness of $L_{\mathfrak{g}_0}(\mu)$ at the end of the last proof is similar to the one used for the classification of the simple highest weight bounded modules of $\mathfrak{sp}_{2n}$, see [M, Lem. 9.2].

2.3. Category $\mathcal{B}(\mathfrak{g})$

We retain the notation of §1.2 and denote by $\mathcal{B}(\mathfrak{g})$ the Serre subcategory of $\mathcal{O}^{\text{inf}}(\mathfrak{g})$ generated by simple bounded modules, i.e., the full subcategory of $\mathcal{O}^{\text{inf}}(\mathfrak{g})$ consisting of modules N such that each simple subquotient of N is bounded.[1] We note that $\mathcal{B}(\mathfrak{g})$ is an example of a category of "snowflake modules" in the terminology of [GS].

If E is a cyclic submodule of $N \in \mathcal{B}(\mathfrak{g})$, then $E \in \mathcal{O}$, so E has a finite length and thus E is bounded. As a result, $\mathcal{B}(\mathfrak{g})$ is the full subcategory of $\mathcal{O}^{\text{inf}}(\mathfrak{g})$ consisting of modules N such that each cyclic submodule of N is bounded. Using Lemma 1.2.2(ii), we obtain the following.

2.3.1. **Corollary**. *Let $\mathfrak{g}' \subset \mathfrak{g}$ be a pair of Kac–Moody superalgebras with compatible triangular decompositions. If $N \in \mathcal{B}(\mathfrak{g})$, then $\operatorname{Res}^{\mathfrak{g}}_{\mathfrak{g}'} N \in \mathcal{B}(\mathfrak{g}')$.*

The following result is a particular case of a more general result about the so-called "snowflake modules" in [GS].

2.3.2. **Proposition**. *Let $\mathfrak{g} = \mathfrak{osp}(1|2n)$ or $\mathfrak{g} = \mathfrak{sp}_{2n}$, $n > 1$.*

(i) *The category $\mathcal{B}(\mathfrak{g})$ is semisimple.*

(ii) *If $\mathfrak{g} \subset \mathfrak{g}''$ are Kac–Moody superalgebras with compatible triangular decompositions, then for each $N \in \mathcal{B}(\mathfrak{g})$ the module $\operatorname{Res}^{\mathfrak{g}''}_{\mathfrak{g}} N$ is completely reducible.*

Proof. Note that part (ii) follows from part (i) and Corollary 2.3.1. One easily shows (see, for example, [GK, Lem. 1.3.1]) that to prove (i) it is enough to verify that each module in $\mathcal{B} = \mathcal{B}(\mathfrak{g})$ has a simple submodule and that

$$\operatorname{Ext}^1_{\mathcal{B}}(L(\mu), L(\mu')) = 0$$

if $L(\mu), L(\mu')$ are bounded. Take any $N \in \mathcal{B}$ and let M be a cyclic submodule of N. Then M lies in the category $\mathcal{O}$ and thus admits a simple submodule. Hence N admits a simple submodule.

Let $L(\mu), L(\mu')$ are bounded and $\operatorname{Ext}^1_{\mathcal{B}}(L(\mu), L(\mu')) \neq 0$. By Theorem 4.2 in [DGK] (the statement and the proof are the same for $\mathfrak{osp}(1|2n)$), this implies

$$\mu' + \rho \in W(\mu)(\mu + \rho).$$

Since $\mathfrak{h}$ acts diagonally on the modules in $\mathcal{B}$, one has $\mu' \neq \mu$.

Let $L(\mu), L(\mu')$ be bounded modules. Using the assumption on $\mathfrak{g}$ and §1.3.1, §2.1, Theorem 2.2, we conclude that $\mu + \rho$ (respectively, $\mu' + \rho$) is the unique maximal element in $W(\mu)(\mu + \rho)$ (respectively, in $W(\mu')(\mu' + \rho)$). Since $\mu \neq \mu'$, one has $\mu' + \rho \notin W(\mu)(\mu + \rho)$, which leads to a contradiction. This completes the proof of (i). □

We remark that in the cases $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathfrak{g} = \mathfrak{osp}(1|2)$ the category $\mathcal{B}(\mathfrak{g})$ is not semisimple. Indeed, take for example an extension of the trivial module $L(0)$ by $L(r_\alpha.0)$, $\alpha \in \Pi_0$ if $\mathfrak{g} = \mathfrak{sl}_n$, and the Verma module $M(0)$ with highest weight $\lambda = 0$ if $\mathfrak{g} = \mathfrak{osp}(1|2)$.

[1] Note that $\operatorname{Ext}^1_{\mathcal{B}(\mathfrak{g})}(M, N) = \operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}(M, N)$ for modules M, N in $\mathcal{B}(\mathfrak{g})$, where $\operatorname{Ext}^1_{\mathfrak{g},\mathfrak{h}}$ is the Ext^1-functor on the category of weight modules.

3. Bounded modules

In this section $\mathfrak{g}$ is an indecomposable finite-dimensional Kac–Moody superalgebra.

3.1. The nonisotropic algebra $\mathfrak{g}_{ni}$

Let $\widetilde{\Sigma}$ be the set of all bases (compatible with Π_0) of $\mathfrak{g}$. Recall that all bases in $\widetilde{\Sigma}$ are connected by chains of odd reflections; in particular, $\Delta_{ni}\cap\Delta^+$ does not depend on the choice of $\Sigma'\in\widetilde{\Sigma}$. Set

$$\Pi_{ni} := \bigcup_{\Sigma'\in\widetilde{\Sigma}} (\Sigma'\cap\Delta_{ni}).$$

If $\mathfrak{g}$ does not have non-isotropic odd roots, then $\Pi_{ni}=\Pi_0$; if $\mathfrak{g}$ has odd non-isotropic roots, then $\mathfrak{g}=\mathfrak{osp}(2m+1|2n)$ or $G(3)$ and Π_0 contains a unique root α with $\alpha/2\in\Delta$; in this case, $\Pi_{ni}=\Pi_0\setminus\{\alpha\}\cup\{\alpha/2\}$.

Consider a Kac–Moody superalgebra $\mathfrak{g}_{ni}$ with the set of simple roots Π_{ni}, parity function $p:\Pi_{ni}\to\mathbb{Z}_2$ given by the restriction of $p:\Delta\to\mathbb{Z}_2$ to Π_{ni}, and the Cartan matrix $a_{ij}:=(\alpha_i^\vee,\alpha_j)$ for $\alpha_i,\alpha_j\in\Pi_{ni}$.

If $\mathfrak{g}$ does not have non-isotropic odd roots, then $\mathfrak{g}_{ni}\cong[\mathfrak{g}_0,\mathfrak{g}_0]$ and we identify these algebras. For $\mathfrak{g}=\mathfrak{osp}(1|2n)$ one has $\Pi_{ni}=\Sigma$ and $\mathfrak{g}_{ni}\cong\mathfrak{g}$; we identify these algebras. For $\mathfrak{g}=G(3),\mathfrak{osp}(2s+1|2n)$ with $s>0$, one has $\mathfrak{g}_0=\mathfrak{t}\times\mathfrak{sp}_{2n}$ and $\mathfrak{g}_{ni}=\mathfrak{t}\times\mathfrak{osp}(1|2n)$, where $\mathfrak{t}=G_2,\mathfrak{o}_{2s+1}$ respectively; in these cases, $\mathfrak{g}_{ni}$ is not a subalgebra of $\mathfrak{g}$.

Using the above identifications, we have $(\mathfrak{g}_{ni})_0=[\mathfrak{g}_0,\mathfrak{g}_0]$ and we fix $\mathfrak{h}\cap[\mathfrak{g}_0,\mathfrak{g}_0]$ to be the Cartan subalgebra of $\mathfrak{g}_{ni}$. We identify the root system of $\mathfrak{g}_{ni}$ with Δ_{ni}.

Observe also that, with the terminology of §1.1.1, the connected components of $\mathfrak{g}_0$ are the even parts of the connected components of $\mathfrak{g}_{ni}$.

3.1.1. Distinguished bases. A base $\Sigma'\in\widetilde{\Sigma}$ is called *distinguished*, if Σ' contains at most one isotropic root. It is easy to check that each connected component Π' of Π_{ni} lies in a certain distinguished base $\Sigma'\in\widetilde{\Sigma}$. For instance, for $\mathfrak{osp}(7|4)$ one has $\Pi_{ni}=\Pi'\coprod\Pi''$, where

$$\begin{aligned}
\Pi' &= \{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3,\varepsilon_3\}, & \Sigma' &= \{\delta_1-\delta_2,\delta_2-\varepsilon_1,\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3,\varepsilon_3\},\\
\Pi'' &= \{\delta_1-\delta_2,\delta_2\}, & \Sigma'' &= \{\varepsilon_1-\varepsilon_2,\varepsilon_2-\varepsilon_3,\varepsilon_3-\delta_1,\delta_1-\delta_2,\delta_2\}.
\end{aligned}$$

3.1.2. Base $\Sigma_{\mathfrak{t}}$. Let $\mathfrak{t}$ be a component of $\mathfrak{g}_{ni}$ and $\Pi(\mathfrak{t})$ be the corresponding connected component of Π_{ni}. For $\mathfrak{g}=A(m|n)$ we choose one distinguished set of simple roots Σ and set $\Sigma_{\mathfrak{t}}:=\Sigma$ for all components $\mathfrak{t}$ of $\mathfrak{g}_{ni}$. If $\mathfrak{g}\neq A(m|n)$ we denote by $\Sigma_{\mathfrak{t}}$ a distinguished base containing $\Pi(\mathfrak{t})$.

Then $\Pi(\mathfrak{t})\subset\Sigma_{\mathfrak{t}}$ and $\mathfrak{t}$ is a subalgebra of $\mathfrak{g}$. For instance, $\mathfrak{g}=\mathfrak{osp}(2s+1|2n)$ does not contain $\mathfrak{g}_{ni}=\mathfrak{o}_{2s+1}\times\mathfrak{osp}(1|2n)$, but contains subalgebras isomorphic to $\mathfrak{o}_{2s+1}$ and $\mathfrak{osp}(1|2n)$.

3.1.3. Example. Take $\mathfrak{g}=\mathfrak{osp}(5|4)$. We have $\mathfrak{g}_{ni}=\mathfrak{o}_5\times\mathfrak{osp}(1|4)$. Then

$$\Sigma_{\mathfrak{o}_5}=\{\delta_1-\delta_2,\delta_2-\varepsilon_1,\varepsilon_1-\varepsilon_2,\varepsilon_2\},\quad \Sigma_{\mathfrak{osp}(1|4)}=\{\varepsilon_1-\varepsilon_2,\varepsilon_2-\delta_1,\delta_1-\delta_2,\delta_2\}.$$

Recall the notation $\lambda_{\mathfrak{t}}$ from §1.1.1. For $\lambda=x_1\varepsilon_1+x_2\varepsilon_2+y_1\delta_1+y_2\delta_2$ we have $\lambda_{\mathfrak{o}_5}=x_1\varepsilon_1+x_2\varepsilon_2$ and $\lambda_{\mathfrak{osp}(1|4)}=y_1\delta_1+y_2\delta_2$.

3.2.

Proposition. *Let* $\mathfrak{t}$ *be a component of* $\mathfrak{g}_{ni}$ *such that* $\mathfrak{t}_0 \neq A_n$. *Let* Σ *contain* $\Pi(\mathfrak{t})$. *If* $L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})$ *is bounded, then* $\mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}} L(\lambda) \in \mathcal{B}(\mathfrak{t})$.

Proof. Set $M := \mathrm{Res}^{\mathfrak{g}}_{\mathfrak{t}} L(\lambda)$. Note that $M \in \mathcal{O}^{\inf}(\mathfrak{t})$.

Let v be a highest weight vector of $L(\lambda)$. By Lemma 1.2.2(i), the $\mathfrak{t}$-submodule of $L(\lambda)$ generated by v is isomorphic to $L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})$.

If $L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})$ is finite-dimensional, then for each $\alpha \in \Pi(\mathfrak{t})$ the root spaces $\mathfrak{g}_{-\alpha} = \mathfrak{t}_{-\alpha}$ acts nilpotently on v and thus acts locally nilpotently on M. Then M is a direct sum of finite-dimensional simple $\mathfrak{t}$-modules, and thus $M \in \mathcal{B}(\mathfrak{t})$.

Now assume that $L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})$ is infinite-dimensional. Since this module is bounded and $\mathfrak{t}_0 \neq A_n$, the algebra $\mathfrak{t}$ is $\mathfrak{sp}_{2n}$ or $\mathfrak{osp}(1|2n)$ with $n > 1$. Moreover, $\Delta(L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})) = \mathsf{D}_n$ and $(\lambda_{\mathfrak{t}} + \rho(\mathfrak{t}), \alpha^\vee) > 0$ for each $\alpha \in \Pi(\mathsf{D}_n)$, where

$$\Pi_0 = \Pi' \cup \{2\delta_n\},\ \Pi(\mathsf{D}_n) = \Pi' \cup \{\delta_{n-1} + \delta_n\},\quad \Pi' := \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n\}.$$

Since $\mathfrak{t}$ is a component of $\mathfrak{g}_{ni}$ and $\Pi(\mathfrak{t}) \subset \Sigma$ one has

$$\rho_{\mathfrak{t}} = \rho(\mathfrak{t}),\quad (\lambda + \rho, \alpha) = ((\lambda + \rho)_{\mathfrak{t}}, \alpha) = (\lambda_{\mathfrak{t}} + \rho_{\mathfrak{t}}, \alpha)\ \text{ for } \alpha \in \Delta(\mathfrak{t}).$$

Therefore

$$(\lambda + \rho, \alpha^\vee) > 0\ \text{ for each } \alpha \in \Pi(\mathsf{D}_n). \tag{3}$$

Let $L_{\mathfrak{t}}(\mu)$ ($\mu \in (\mathfrak{h} \cap \mathfrak{t})^*$) be a simple subquotient of M. Let us show that $L_{\mathfrak{t}}(\mu)$ is bounded. One has

$$\mu \in \mathrm{supp}(M) = \{\nu_{\mathfrak{t}} \mid \nu \in \mathrm{supp}(L(\lambda))\},$$

so for $\alpha \in \Delta(\mathfrak{t})$ one has

$$(\mu, \alpha^\vee) \subset (\lambda, \alpha^\vee) + \mathbb{Z} = (\lambda_{\mathfrak{t}}, \alpha^\vee) + \mathbb{Z}.$$

Therefore $\Delta(L_{\mathfrak{t}}(\lambda_{\mathfrak{t}})) = \Delta(L_{\mathfrak{t}}(\mu)) = \mathsf{D}_n$. By §2.1 it sufficies to show that $(\mu + \rho(\mathfrak{t}), \alpha) > 0$ for $\alpha \in \Pi(\mathsf{D}_n)$. Take $\alpha \in \Pi'$. By (3) the root space $\mathfrak{g}_{-\alpha}$ acts nilpotently on v and thus locally nilpotently on $L(\lambda)$ and on $L_{\mathfrak{t}}(\mu)$. Therefore $(\mu + \rho(\mathfrak{t}), \alpha) > 0$. By above, $\Delta(L_{\mathfrak{t}}(\mu)), \Delta(L(\lambda))$ do not contain $2\delta_n$. Using Proposition 1.4.1 for $\alpha = 2\delta_n$ we obtain that $\mathcal{C}_{2\delta_n}(L_{\mathfrak{t}}(\mu)) = L_{\mathfrak{t}}(r_{\delta_n}(\mu + \rho(\mathfrak{t})) - \rho(\mathfrak{t}))$ is a subquotient of $\mathcal{C}_{2\delta_n}(L(\lambda)) = L(r_{\delta_n}(\lambda + \rho) - \rho)$. Since $\delta_{n-1} - \delta_n \in \Pi(\mathfrak{t}) \subset \Sigma$ and

$$(r_{\delta_n}(\lambda + \rho), \delta_{n-1} - \delta_n) = (\lambda + \rho, \delta_{n-1} + \delta_n) \in \mathbb{Z}_{>0}$$

the root space $\mathfrak{g}_{\delta_n - \delta_{n-1}}$ acts locally nilpotently on $L(r_{\delta_n}(\lambda + \rho) - \rho)$ and thus on $L_{\mathfrak{t}}(r_{\delta_n}(\mu + \rho(\mathfrak{t}) - \rho(\mathfrak{t}))$. Therefore

$$0 < (r_{\delta_n}(\mu + \rho(\mathfrak{t})), \delta_{n-1} - \delta_n) = (\mu + \rho(\mathfrak{t}), \delta_{n-1} + \delta_n)$$

as required. Hence $L_{\mathfrak{t}}(\mu)$ is bounded. We conclude that $M \in \mathcal{B}(\mathfrak{t})$ as required. $\square$

3.3.

Retain the notation of §1.1.1. For a simple $\mathfrak{g}$-module L in $\mathcal{O}$ and a component $\mathfrak{t}$ of $\mathfrak{g}_{ni}$, by $\lambda^{\mathfrak{t}} \in \mathfrak{h}^*$ we denote the highest weight of L with respect to $\Sigma_{\mathfrak{t}}$, i.e., $L = L(\Sigma_{\mathfrak{t}}, \lambda^{\mathfrak{t}})$.

Theorem. *Let $\mathfrak{g}$ be a finite-dimensional Kac–Moody superalgebra and let $L \in \mathcal{O}$ be a simple $\mathfrak{g}$-module. The module L is bounded if and only if the module $L_{\mathfrak{t}}((\lambda^{\mathfrak{t}})_{\mathfrak{t}})$ is bounded for each component $\mathfrak{t}$ of $\mathfrak{g}_{ni}$.*

3.3.1. Example. We apply the theorem to the case we are mostly interested in: $\mathfrak{g} = \mathfrak{osp}(m|2n)$. Take $\lambda \in \mathfrak{h}^*$ and write $\lambda + \rho = \sum_{i=1}^{s} x_i\varepsilon_i + \sum_{i=1}^{n} y_i\delta_i$, where $s = \lfloor m/2 \rfloor$. Assume that $x_i + y_j \neq 0$ for all i, j. Then for any base Σ' we have $L(\Sigma, \lambda) = L(\Sigma', \lambda')$, where $\lambda + \rho = \lambda' + \rho'$, $\rho' = \rho_{\Sigma'}$. The Theorem above states that $L(\lambda)$ is bounded if and only if $L_{\mathfrak{t}}((\lambda^{\mathfrak{t}})_{\mathfrak{t}})$ is bounded for each component $\mathfrak{t}$ of $\mathfrak{g}_{ni}$. We have $\Pi(\mathfrak{t}) \subset \Sigma_{\mathfrak{t}}$, so

$$\rho_{\Sigma_{\mathfrak{t}}} = \rho_{\mathfrak{t}}.$$

Hence, for λ as above, $L(\lambda)$ is bounded if and only if $L_{\mathfrak{t}}((\lambda + \rho)_{\mathfrak{t}} - \rho_{\mathfrak{t}})$ is bounded for each $\mathfrak{t}$. One has $\mathfrak{g}_{ni} = \mathfrak{o}_m \times \mathfrak{t}'$, where $\mathfrak{t}' = \mathfrak{sp}_{2n}$ if m is even and $\mathfrak{t}' = \mathfrak{osp}(1|2n)$ if m is odd.

One has $L_{\mathfrak{t}'}((\lambda + \rho)_{\mathfrak{t}'} - \rho_{\mathfrak{t}'}) = L_{\mathfrak{t}'}(\sum_{i=1}^{n} y_i\delta_i - \rho_{\mathfrak{t}'})$. For $n = 1$ this module is bounded. For $n > 1$ the conditions on y_i are given in §2.1.

Consider the module $L_{\mathfrak{o}_m}(\sum_{i=1}^{s} x_i\varepsilon_i - \rho_{\mathfrak{o}_m})$. For $m = 1, 2, 3, 4$ this module is always bounded. For $m > 6$ this module is bounded only if it is finite-dimensional, i.e., if $x_1 - x_2, \ldots, x_{s-1} - x_s, 2x_s \in \mathbb{Z}_{>0}$. For $m = 6$ we have $\mathfrak{o}_6 \cong \mathfrak{sl}_4$ and the boundedness is reduced to the boundedness of a module over $\mathfrak{sl}_4$. For $m = 5$ one has $\mathfrak{o}_5 \cong \mathfrak{sp}_4$ and this module is bounded if and only if either $x_1 - x_2, 2x_2 \in \mathbb{Z}_{>0}$ or $2x_1, 2x_2 \in \mathbb{Z}_{>0}$, $x_1 - x_2 \notin \mathbb{Z}$.

3.4. Proof of Theorem 3.3

We start from the following useful lemma.

3.4.1. **Lemma**.

(i) *A simple $\mathfrak{g}$-module L is bounded if and only if it has a bounded $\mathfrak{g}_0$-submodule.*

(ii) *If $L_{\mathfrak{g}_0}(\lambda - 2\rho_1)$ is bounded, then $L(\lambda)$ is bounded.*

Proof. Let N be a $\mathfrak{g}_0$-module. One has $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N = N \otimes \Lambda\mathfrak{g}_1$ as $\mathfrak{g}_0$-modules. Since the tensor product of a finite-dimensional module and a bounded module is a bounded module, $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N$ is bounded if N is bounded.

To prove (i), let N be a bounded $\mathfrak{g}_0$-submodule of a simple $\mathfrak{g}$-module L. Since

$$\mathrm{Hom}_{\mathfrak{g}_0}(N, L) = \mathrm{Hom}_{\mathfrak{g}}(\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} N, L),$$

the module L is bounded. For (ii), note that the maximal weight of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L_{\mathfrak{g}_0}(\nu)$ is equal to $\nu + \sum_{\alpha \in \Delta_1^+} \alpha = \nu + 2\rho_1$. In particular, $L(\nu + 2\rho_1)$ is a subquotient of $\mathrm{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} L_{\mathfrak{g}_0}(\nu)$. $\square$

3.4.2. Continuation of proof of Theorem 3.3. Assume that L is bounded. Let $\mathfrak{t}$ be a component of $\mathfrak{g}_{ni}$. Let v be a primitive vector of L with respect to the base

$\Sigma_{\mathfrak{t}}$. Then v has weight $\lambda^{\mathfrak{t}}$ and, by Lemma 1.2.2, $\mathcal{U}(\mathfrak{t})v \cong L_{\mathfrak{t}}((\lambda^{\mathfrak{t}})_{\mathfrak{t}})$ is a bounded $\mathfrak{t}$-module. This establishes the "only if" part.

For the "if" part, we assume that $L_{\mathfrak{t}}((\lambda^{\mathfrak{t}})_{\mathfrak{t}})$ is bounded for each component of $\mathfrak{g}_{ni}$. By Lemma 3.4.1 the boundedness of L follows from the existence of a bounded $\mathfrak{g}_0$-submodule. Let us show that L contains a bounded $\mathfrak{g}_0$-submodule. If $\mathfrak{g} = \mathfrak{osp}(1|2n)$, the assertion is tautological. For $\mathfrak{g} = D(2,1,a)$ any $\mathfrak{g}_0$-submodule of $L(\lambda)$ is bounded.

Consider the remaining case when $\mathfrak{g} \neq \mathfrak{osp}(1|2n), D(2,1,a)$. Let $\mathfrak{t}$ be the following component of $\mathfrak{g}_{ni}$: for $\mathfrak{g} = \mathfrak{osp}(m|2n)$ let $\mathfrak{t} = \mathfrak{o}_m$, for $\mathfrak{g} = \mathfrak{gl}(m|n)$ with $m \leq n$ let $\mathfrak{t} = \mathfrak{sl}_n$, and for $\mathfrak{g} = G(3)$ (respectively, $F(4)$) let $\mathfrak{t} = G_2$ (respectively, $\mathfrak{t} = \mathfrak{o}_7$). Set $\Sigma := \Sigma_{\mathfrak{t}}$ and $\lambda := \lambda^{\mathfrak{t}}$. Let v_λ be the highest weight vector of L. Let us show that $\mathcal{U}(\mathfrak{g}_0)v_\lambda$ is a bounded $\mathfrak{g}_0$-module. One has $\mathfrak{g}_{ni} = \mathfrak{t} \times \mathfrak{t}'$ and $[\mathfrak{g}_0, \mathfrak{g}_0] = \mathfrak{t} \times \mathfrak{t}'_0$, where $\mathfrak{t}'_0 = A_1$ for $F(4)$, $G(3)$ and $\mathfrak{t}'_0 = \mathfrak{sp}_{2n}$ (respectively, $\mathfrak{t}'_0 = \mathfrak{t}' = \mathfrak{sl}_m$) for $\mathfrak{g} = \mathfrak{osp}(m|2n)$ (respectively, for $\mathfrak{gl}(m|n)$). Set

$$E := \mathcal{U}(\mathfrak{t})v_\lambda,\ E' := \mathcal{U}(\mathfrak{t}'_0)v_\lambda.$$

By Lemma 1.2.2(i) one has $E = L_{\mathfrak{t}}((\lambda^{\mathfrak{t}})_{\mathfrak{t}})$, so E is a simple bounded $\mathfrak{t}$-module. If $\mathfrak{g} = \mathfrak{gl}(m|n)$, then Σ contains $\Pi(\mathfrak{t}')$, so $E' = L_{\mathfrak{t}'}((\lambda^{\mathfrak{t}'})_{\mathfrak{t}'})$ is a simple bounded $\mathfrak{t}'$-module. If $\mathfrak{t}'_0 \cong A_1$, then any module in $\mathcal{O}(\mathfrak{t}')$ is bounded, so E' is a bounded $\mathfrak{t}'$-module. In the remaining case one has $\mathfrak{t}'_0 = \mathfrak{sp}_{2n}, n > 1$. Since $L_{\mathfrak{t}'}((\lambda^{\mathfrak{t}'})_{\mathfrak{t}'})$ is bounded, Proposition 3.2 implies that $Res^{\mathfrak{t}}_{\mathfrak{t}'} L(\lambda) \in \mathcal{B}(\mathfrak{t}')$ and thus, by Proposition 2.3.2, any cyclic $\mathfrak{t}'_0$-submodule of $L(\lambda)$ is bounded. We conclude that E' is a bounded $\mathfrak{t}'_0$-module.

View $E \otimes E'$ as a $\mathfrak{t} \times \mathfrak{t}'_0$-module by

$$gg'(e \otimes e') := ge \otimes g'e' \ \text{ for } g \in \mathfrak{t}, g' \in \mathfrak{t}', e \in E, e' \in E'.$$

By above, E, E' are bounded. Each weight space of $E \otimes E'$ is of the form $(E \otimes E')_\nu = E_{\nu_1} \otimes E_{\nu_2}$, so $E \otimes E'$ is a bounded $\mathfrak{t} \times \mathfrak{t}'_0$-module.

Set $N := \mathcal{U}(\mathfrak{g}_0)v_\lambda$. Since $\mathfrak{g}_0 = [\mathfrak{g}_0, \mathfrak{g}_0] \times \mathcal{Z}(\mathfrak{g}_0)$ one has

$$N = \mathcal{U}([\mathfrak{g}_0, \mathfrak{g}_0])v_\lambda = \mathcal{U}(\mathfrak{t} \times \mathfrak{t}'_0)v_\lambda.$$

The natural map $\phi : E \otimes E' \to \mathcal{U}(\mathfrak{t} \times \mathfrak{t}'_0)v_\lambda = N$ defined by $uv_\lambda \otimes u'v_\lambda \mapsto uu'v_\lambda$ is a surjective homomorphism of $\mathfrak{t} \times \mathfrak{t}'_0$-modules. Hence N is a bounded $\mathfrak{t} \times \mathfrak{t}'_0$-module and thus N is a bounded $\mathfrak{g}_0$-submodule of L. Now Lemma 3.4.1 completes the proof. $\square$

3.5.

Checking the boundedness of $L_{\mathfrak{t}}((\lambda^{\mathfrak{t}})_{\mathfrak{t}})$ for all $\mathfrak{t}$ could be computationally heavy. These computations could be shortened with the aid of Corollary 3.5.1 and Theorem 3.6.1 below.

It turns out that for $\mathfrak{g} \neq \mathfrak{osp}(m|2n)$, it is enough to consider only one distinguished set of simple roots.

3.5.1. **Corollary**.

(i) *If all components of $\mathfrak{g}_{ni}$ have rank one, then $L(\lambda)$ is bounded for any λ.*

Assume that $\mathfrak{t}$ is a component of $\mathfrak{g}_{ni}$ of rank greater than one. Set $\Sigma := \Sigma_{\mathfrak{t}}$.

(ii) *If $\mathfrak{g} \neq \mathfrak{osp}(m|2n)$, then $L(\lambda)$ is bounded if and only if $L_{\mathfrak{g}_0}(\lambda)$ is bounded.*

(iii) *If $\mathfrak{g} = \mathfrak{osp}(m|2n)$ with $m = 2, 3, 4$ or $n = 1$, then $L(\lambda)$ is bounded if and only if $L_{\mathfrak{t}}(\lambda)$ is bounded.*

Proof. If $\mathfrak{t}'$ is a component of $\mathfrak{g}_{ni}$ of rank one, then any module in $\mathcal{O}(\mathfrak{t}')$ is bounded and (i) follows from Theorem 3.3.

If $\mathfrak{t}$ is a unique component of $\mathfrak{g}_{ni}$ which has rank greater than one, then Theorem 3.3 implies that $L(\lambda)$ is bounded if and only if $L_{\mathfrak{t}}(\lambda)$ is bounded. Note that $\mathfrak{g}_{ni}$ contains more than one component of rank greater than one in the following cases: $\mathfrak{g} = \mathfrak{osp}(m|2n)$ with $m > 4, n > 1$ and $A(m|n)$ with $m, n > 1$; this gives (iii). For $\mathfrak{g} = A(m|n)$ one has $\Sigma_0 \subset \Sigma_{\mathfrak{t}}$, so (ii) follows from Theorem 3.3. $\square$

3.6. Reduction to $n = 2$

Let $\mathfrak{g} = \mathfrak{osp}(m|2n)$. Take

$$\Sigma := \Sigma_{\mathfrak{o}_m}.$$

For $n > 2$ we consider the subalgebra

$$\mathfrak{osp}(m|4) \subset \mathfrak{osp}(m|2n)$$

with the set of simple roots lying in Σ. For instance, for $\mathfrak{osp}(2s+1|2n)$ we have

$$\Sigma = \{\delta_1 - \delta_2, \ldots, \delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \ldots, \varepsilon_{s-1} - \varepsilon_s, \varepsilon_s\}$$

and we take $\mathfrak{osp}(2s+1|4)$ to be the subalgebra with the set of simple roots $\{\delta_{n-1} - \delta_n, \delta_n - \varepsilon_1, \ldots, \varepsilon_{s-1} - \varepsilon_s, \varepsilon_s\}$.

3.6.1. **Theorem**. *For $n > 2$ the module $L_{\mathfrak{osp}(m|2n)}(\lambda)$ is bounded if and only if the modules $L_{\mathfrak{sp}_{2n}}(\lambda_{\mathfrak{sp}_{2n}})$ and $L_{\mathfrak{osp}(m|4)}(\lambda_{\mathfrak{osp}(m|4)})$ are bounded.*

Proof. Denote by v_λ the highest weight vector of $L(\lambda) := L_{\mathfrak{osp}(m|2n)}(\lambda)$ and set

$$E := \mathcal{U}(\mathfrak{o}_m)v_\lambda, \quad E' := \mathcal{U}(\mathfrak{sp}_{2n})v_\lambda, \quad E'' := \mathcal{U}(\mathfrak{osp}(m|4))v_\lambda, \quad N := \mathcal{U}(\mathfrak{g}_0)v_\lambda.$$

By Lemma 1.2.2, $E'' \cong L_{\mathfrak{osp}(m|4)}(\lambda_{\mathfrak{osp}(m|4)})$. Since E' has the highest weight $\lambda_{\mathfrak{sp}_{2n}}$, the module $L_{\mathfrak{sp}_{2n}}(\lambda_{\mathfrak{sp}_{2n}})$ is a quotient of E'.

If $L(\lambda)$ is bounded, then all modules E, E', E'', N are bounded by Lemma 1.2.2(ii). This implies the "only if" part.

Now assume that $L_{\mathfrak{sp}_{2n}}(\lambda_{\mathfrak{sp}_{2n}})$ and $L_{\mathfrak{osp}(m|4)}(\lambda_{\mathfrak{osp}(m|4)})$ are bounded. By Lemma 3.4.1(i) in order to show that $L_{\mathfrak{osp}(m|2n)}(\lambda)$ is bounded it is enough to verify N is a bounded $\mathfrak{g}_0$-module. Arguing as in the proof of Theorem 3.3, we see that N is a quotient of $E \otimes E'$, where $E \otimes E'$ is viewed as $\mathfrak{g}_0$-module ($\mathfrak{g}_0 = \mathfrak{o}_m \times \mathfrak{sp}_{2n}$) and that the boundedness of N follows from the boundedness of E and of E'. Since $\mathfrak{o}_m \subset \mathfrak{osp}(m|4)$, E is a cyclic $\mathfrak{o}_m$-submodule of $E'' \cong L_{\mathfrak{osp}(m|4)}(\lambda_{\mathfrak{osp}(m|4)})$, so E is bounded by Lemma 1.2.2(ii). It remains to verify the boundedness of E'.

Note that E' is a $\mathfrak{sp}_{2n}$-module generated by its highest weight vector v_λ which is of the weight

$$\lambda' := \lambda_{\mathfrak{sp}_{2n}}.$$

Write $\Pi' := \{\delta_1 - \delta_2, \dots, \delta_{n-1} - \delta_n\}$, $\Pi(\mathfrak{sp}_{2n}) = \Pi' \cup \{2\delta_n\}$. Consider the copy of $\mathfrak{sl}_n$ in $\mathfrak{g}$ with the set of simple roots Π' and the copy of $\mathfrak{sp}_4$ in $\mathfrak{g}$ with the set of simple roots $\{\delta_{n-1} - \delta_n, 2\delta_n\}$. By Lemma 1.2.2, the $\mathfrak{sl}_n$-submodule generated by v_λ is isomorphic to $L_{\mathfrak{sl}_n}(\lambda_{\mathfrak{sl}_n})$. Note that $\mathfrak{sl}_n \subset \mathfrak{sp}_{2n}$ and $\lambda'_{\mathfrak{sl}_n} = \lambda_{\mathfrak{sl}_n}$. By Lemma 1.2.2, the $\mathfrak{sl}_n$-submodule generated by the highest weight vector in $L_{\mathfrak{sp}_{2n}}(\lambda_{\mathfrak{sp}_{2n}})$ is isomorphic to $L_{\mathfrak{sl}_n}(\lambda_{\mathfrak{sl}_n})$. Since $L_{\mathfrak{sp}_{2n}}(\lambda_{\mathfrak{sp}_{2n}})$ is bounded, $L_{\mathfrak{sl}_n}(\lambda_{\mathfrak{sl}_n})$ is finite-dimensional, see §2.1.

Since E'' is bounded and $\mathfrak{sp}_4 \subset \mathfrak{osp}(m|4)$, the $\mathfrak{sp}_4$-submodule generated by v_λ is bounded. We conclude that E' is an $\mathfrak{sp}_{2n}$-module with the following properties:

E' is generated by the highest weight vector $v_{\lambda'}$;

$\mathcal{U}(\mathfrak{sl}_n)v_{\lambda'}$ is a simple finite-dimensional $\mathfrak{sl}_n$-module;

$\mathcal{U}(\mathfrak{sp}_4)v_{\lambda'}$ is a simple bounded $\mathfrak{sp}_4$-module.

By the description of the simple bounded highest weight modules of $\mathfrak{sp}_{2n}$ (see §2.1), E' is bounded. This completes the proof. $\square$

4. Strongly typical modules for $\mathfrak{osp}(m|2n)$

In this section $\mathfrak{g} = \mathfrak{osp}(m|2n)$.

A weight λ is called *strongly typical* if $(\lambda + \rho, \beta) \neq 0$ for each $\beta \in \Delta_1$; the module $L(\lambda)$ is called strongly typical if λ is strongly typical.

4.1. Notation

We set

$$s := \left\lfloor \frac{m}{2} \right\rfloor; \quad p(m) := 0 \text{ if } m \text{ is even, } p(m) := 1 \text{ if } m \text{ is odd.}$$

One has $\mathfrak{g}_{ni} = \mathfrak{o}_m \times \mathfrak{sp}_{2n}$ for even m and $\mathfrak{g}_{ni} = \mathfrak{o}_m \times \mathfrak{osp}_{1|2n}$ for odd m. We write for convenience $\mathfrak{g}_{ni} = \mathfrak{o}_m \times \mathfrak{osp}_{p(m)|2n}$, where $\mathfrak{osp}_{1|2n} = \mathfrak{osp}(1|2n)$ and $\mathfrak{osp}_{0|2n} = \mathfrak{sp}_{2n}$.

4.1.1. Conventions. We will use the standard notations of [K1] for Δ, in particular, $\Delta(\mathfrak{o}_m)$ lies in the span of $\{\varepsilon_i\}_{i=1}^s$ and $\Delta(\mathfrak{sp}_{2n})$ lies in the span of $\{\delta_i\}_{i=1}^n$. We set

$$\mathfrak{h}_\varepsilon := \mathfrak{h} \cap \mathfrak{o}_m, \quad \mathfrak{h}_\delta := \mathfrak{h} \cap \mathfrak{sp}_{2n}.$$

We identify $(\mathfrak{h} \cap \mathfrak{o}_m)^*$ with $\mathfrak{h}_\varepsilon^* := \text{span}\{\varepsilon_i\}_{i=1}^s$ and $(\mathfrak{h} \cap \mathfrak{sp}_{2n})^* = (\mathfrak{h} \cap \mathfrak{osp}(1|2n))^*$ with $\mathfrak{h}_\delta^* := \text{span}\{\delta_i\}_{i=1}^n$. One has

$$\mathfrak{h} = \mathfrak{h}_\varepsilon \oplus \mathfrak{h}_\delta, \quad \mathfrak{h}^* = \mathfrak{h}_\varepsilon^* \oplus \mathfrak{h}_\delta^*.$$

For $\lambda = \sum a_i\varepsilon_i + \sum b_j\delta_j$ we set $\lambda_\varepsilon := \sum a_i\varepsilon_i$, $\lambda_\delta := \sum b_j\delta_j$.

In this section we use the base $\Sigma = \Sigma_{\mathfrak{o}_m}$, i.e.,

$$\Sigma = \{\delta_1 - \delta_2, \delta_2 - \delta_3, \dots, \delta_n - \varepsilon_1, \\ \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{s-1} - \varepsilon_s, \varepsilon_s\} \text{ for } \mathfrak{osp}(2s+1|2n),$$

$$\Sigma = \{\delta_1 - \delta_2, \delta_2 - \delta_3, \dots, \delta_n - \varepsilon_1, \\ \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{s-1} - \varepsilon_s, \varepsilon_{s-1} + \varepsilon_s\} \text{ for } \mathfrak{osp}(2s|2n).$$

Set

$$\xi := \sum_{i=1}^{n} \delta_i.$$

Then $\rho_1 = \rho_0 - \rho = (m/2)\xi$ and $\rho_{\mathfrak{g}_{ni}} - \rho = s\xi$. We set

$$R_0 := \prod_{\alpha\in\Delta_0^+} (1-e^{-\alpha}),\quad R_1 := \prod_{\alpha\in\Delta_1^+} (1+e^{-\alpha}),\quad R := R_0/R_1$$

and define $R_{\mathfrak{o}_m}, R_{\mathfrak{sp}_{2n}}, R_{\mathfrak{g}_{ni}}$ similarly. It is clear that $R_0 = R_{\mathfrak{o}_m}R_{\mathfrak{sp}_{2n}}$.

4.1.2. Stabilizer. For $\mu \in \mathfrak{h}^*$ we set

$$\Delta_0(\mu) := \{\alpha \in \Delta_0(\mathfrak{sp}_{2n}) \mid (\mu,\alpha) = 0\}.$$

It is well known that $\mathrm{Stab}_{W(\mathfrak{sp}_{2n})}\,\mu$ is generated by r_α with $\alpha \in \Delta_0(\mu)$ (this follows from §1.3.1).

We consider the root system B_n of $\mathfrak{o}_n$ with the set of simple roots $\Pi(\mathsf{B}_n) = \{\delta_1-\delta_2,\dots,\delta_n\}$ and denote by $\Delta^+(\mathsf{B}_n)$ the corresponding set of positive roots. We set

$$\mathcal{C}^+ := \{\mu \in \mathfrak{h}^* \mid (\mu,\alpha^\vee) \notin \mathbb{Z}_{<0}\ \text{ for } \alpha \in \Delta^+(\mathsf{B}_n)\}.$$

Note that for any $\lambda \in \mathfrak{h}^*$ there exists $w \in W(\lambda)\cap W(\mathfrak{sp}_{2n})$ such that $w(\lambda+\rho) \in \mathcal{C}^+$.

4.2.

Theorem. *Let $\nu \in \mathfrak{h}^*$ be a strongly typical weight such that $\nu+\rho \in \mathcal{C}^+$ and $(\nu+\rho,\alpha) \neq 0$ for $\alpha = \delta_i+\delta_j$ with $1 \le i,j \le n$. Then for each $z \in W(\nu)\cap W(\mathfrak{sp}_{2n})$ one has*

$$Re^{\rho}\operatorname{ch} L(z.\nu) = R_{\mathfrak{g}_{ni}}e^{\rho_{\mathfrak{g}_{ni}}}\operatorname{ch} L_{\mathfrak{g}_{ni}}(z(\nu+\rho)-\rho_{\mathfrak{g}_{ni}}).$$

4.2.1. Remark. Since $\mathfrak{g}_{ni} = \mathfrak{o}_m \times \mathfrak{osp}_{p(m)|2n}$

$$\operatorname{ch} L_{\mathfrak{g}_{ni}}(\lambda+\rho-\rho_{\mathfrak{g}_{ni}}) = \operatorname{ch} L_{\mathfrak{osp}_{p(m)|2n}}((\lambda+\rho-\rho_{\mathfrak{g}_{ni}})_\delta)\cdot \operatorname{ch} L_{\mathfrak{o}_m}((\lambda+\rho-\rho_{\mathfrak{g}_{ni}})_\varepsilon),$$

so

$$e^{\rho_{\mathfrak{g}_{ni}}-\rho}\cdot \operatorname{ch} L_{\mathfrak{g}_{ni}}(\lambda+\rho-\rho_{\mathfrak{g}_{ni}}) = e^{s\xi}\cdot \operatorname{ch} L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_\delta - s\xi)\cdot \operatorname{ch} L_{\mathfrak{o}_m}(\lambda_\varepsilon). \qquad (4)$$

4.2.2. **Corollary**. *Let $\lambda \in \mathfrak{h}^*$ be such that $(\lambda+\rho,\alpha) \neq 0$ for each $\alpha \in \Delta_1$ and either $\Delta_0(\lambda+\rho) = \varnothing$ or $(\lambda+\rho, 2\delta_i) \in \mathbb{Z}\setminus\{0\}$ for $i = 1,\dots,n$. Then*

$$\operatorname{ch} L(\lambda) = e^{s\xi}\prod_{i=1}^{s}\prod_{j=1}^{n}(1+e^{-\varepsilon_i-\delta_j})(1+e^{\varepsilon_i-\delta_j})\cdot \operatorname{ch} L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_\delta - s\xi)\cdot \operatorname{ch} L_{\mathfrak{o}_m}(\lambda_\varepsilon).$$

4.2.3. **Corollary**. *Let λ be a strongly typical weight. Then $L(\lambda)$ is bounded if and only if $L_1 := L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_\delta - s\xi)$ and $L_2 := L_{\mathfrak{o}_m}(\lambda_\varepsilon)$ are bounded modules (over $\mathfrak{osp}_{p(m)|2n}$ and $\mathfrak{o}_m$ respectively). Moreover, the degree of $L(\lambda)$ is at most $2^{2sn}\deg L_1\cdot \deg L_2$.*

4.3. Proofs of Corollaries

4.3.1. Proof of Corollary 4.2.2. For Corollary 4.2.2 note that take $w \in W(\lambda)$ such that $w(\lambda+\rho) \in \mathcal{C}^+$. It is enough to verify that $\nu := w.\lambda$ satisfies the assumptions of Theorem 4.2. One has $\nu+\rho = w(\lambda+\rho)$. Since λ is strongly typicial, ν is strongly typical.

If $\Delta_0(\lambda+\rho) = \varnothing$, then $\Delta_0(\nu+\rho) = \varnothing$, so ν satisfies the assumptions of Theorem 4.2. Assume that for each $i = 1, \dots, n$ we have $(\lambda+\rho, 2\delta_i) \in \mathbb{Z} \setminus \{0\}$. Since $w\delta_i = \pm\delta_j$ we have

$$(\nu+\rho, \delta_i^\vee) = (w(\lambda+\rho), \delta_i^\vee) \in \mathbb{Z} \setminus \{0\}.$$

Since $\nu+\rho \in \mathcal{C}^+$, this gives $(\nu+\rho, \delta_i^\vee) \in \mathbb{Z}_{>0}$, so ν satisfies the assumptions of Theorem 4.2. $\square$

4.3.2. Proof of Corollary 4.2.3. Let $L(\lambda)$ be bounded. Then $L_2 = L_{\mathfrak{o}_m}(\lambda_\varepsilon)$ is bounded. Take Σ' which contains the set of simple roots for $\mathfrak{osp}_{p(m)|2n}$ and denote by ρ' the corresponding Weyl vector. Then $\rho'_\delta = (\rho_{\mathfrak{g}_{ni}})_\delta$. Since $L(\lambda) = L(\Sigma', \lambda')$ is bounded, $L_{\mathfrak{osp}_{p(m)|2n}}(\lambda'_\delta)$ is bounded. Since λ is strongly typical, one has $\lambda'+\rho' = \lambda+\rho$, so

$$\lambda'_\delta = (\lambda+\rho-\rho_{\mathfrak{g}_{ni}})_\delta.$$

Thus $L_1 = L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_\delta+\rho-\rho_{\mathfrak{g}_{ni}})$ is bounded.

Now let λ be a strongly typical weight such that L_1, L_2 are bounded modules. Since L_1 is bounded, the description of the simple bounded highest weight modules in §2.1 gives $(\lambda+\rho, \delta_i) \in (1/2)\mathbb{Z}$ for $i = 1, \dots, n$. From Corollary 4.2.2 we conclude that $L(\lambda)$ is bounded and has degree at most $2^{2sn} \deg L_1 \cdot \deg L_2$. $\square$

4.4. Central characters

The rest of the section is devoted to the proof of Theorem 4.2.

For a weight $\lambda \in \mathfrak{h}^*$ we define the $\mathfrak{g}$- and $\mathfrak{g}_0$-central characters by

$$\chi_\lambda \colon \mathcal{Z}(\mathfrak{g}) \to \mathbb{C} \ \text{ such that } z|_{L(\lambda)} = \chi_\lambda(z)\mathrm{Id},$$
$$\chi^0_\lambda \colon \mathcal{Z}(\mathfrak{g}_0) \to \mathbb{C} \ \text{ such that } z|_{L_{\mathfrak{g}_0}(\lambda)} = \chi^0_\lambda(z)\mathrm{Id}.$$

We next recall the notion “perfect mate” which was introduced in Section 8 of [G1]. A maximal ideal χ^0 in $\mathcal{Z}(\mathfrak{g}_0)$ is called a *perfect mate* for a maximal ideal χ in $\mathcal{Z}(\mathfrak{g})$ if the following conditions are satisfied.

(i) For any Verma $\mathfrak{g}$-module annihilated by χ, its $\mathfrak{g}_0$-submodule annihilated by a power of χ^0 is a Verma $\mathfrak{g}_0$-module.

(ii) Any $\mathfrak{g}$-module annihilated by χ has a non-zero vector annihilated by χ^0.

If χ^0 is a perfect mate for χ, then [G2, Thm. 1.3.1] establishes an equivalence of the corresponding categories of $\mathfrak{g}$- and $\mathfrak{g}_0$-modules.

4.4.1. **Lemma**. *Let $\nu \in \mathfrak{h}^*$ satisfies the assumptions of Theorem* 4.2. *Then:*

(i) *for each $j \in \mathbb{Z}_{>0}$ one has $\nu+\rho+(j/2)\xi \in \mathcal{C}^+$ and $\Delta_0(\nu+\rho+(j/2)\xi) = \Delta_0(\nu+\rho)$;*

(ii) *χ^0_ν is a perfect mate for χ_ν.*

Proof. Recall that $\nu+\rho\in\mathcal{C}^+$ and $\Delta_0(\nu+\rho)\subset\{\delta_i-\delta_j\}_{i,j=1}^n$.

One has $(\xi,\alpha^\vee)=2$ for $\alpha=\delta_i,\delta_i+\delta_j$ and $(\xi,(\delta_i-\delta_j)^\vee)=0$ for $1\leq i<j\leq n$. For $j\in\mathbb{Z}_{>0}$ this gives that $\nu+\rho+(j/2)\xi\in\mathcal{C}^+$ and $\Delta_0(\nu+\rho+(j/2)\xi)\subset\{\delta_i-\delta_j\}_{i,j=1}^n$, which implies (i). For (ii) we use [G1, Lem. 8.3.4], which asserts that χ_ν^0 is a perfect mate for χ_ν if the following conditions hold:

(1) $\mathrm{Stab}_W(\nu+\rho_0)\subset\mathrm{Stab}_W(\nu+\rho)$;

(2) if $\Gamma\subset\Delta_1^+$ and $w\in W$ are such that

$$w(\nu+\rho_0)=\nu+\rho_0-\sum_{\beta\in\Gamma}\beta, \tag{5}$$

then $\Gamma=\varnothing$.

One has $W=W(\mathfrak{o}_m)\times W(\mathfrak{sp}_{2n})$, so for each $\mu\in\mathfrak{h}^*$

$$\mathrm{Stab}_W(\mu)=\mathrm{Stab}_{W(\mathfrak{o}_m)}\,\mu\times\mathrm{Stab}_{W(\mathfrak{sp}_{2n})}\,\mu=\mathrm{Stab}_{W(\mathfrak{o}_m)}\,\mu_\varepsilon\times\mathrm{Stab}_{W(\mathfrak{sp}_{2n})}\,\mu_\delta.$$

One has $(\nu+\rho)_\varepsilon=(\nu+\rho_0)_\varepsilon$, so

$$\mathrm{Stab}_{W(\mathfrak{o}_m)}(\nu+\rho)_\varepsilon=\mathrm{Stab}_{W(\mathfrak{o}_m)}(\nu+\rho_0)_\varepsilon.$$

By §4.1.2, the group $\mathrm{Stab}_{W(\mathfrak{sp}_{2n})}\,\mu_\delta$ is generated by $r_\alpha,\alpha\in\Delta_0(\mu_\delta)$, so (i) gives

$$\mathrm{Stab}_{W(\mathfrak{sp}_{2n})}(\nu+\rho_0)=\mathrm{Stab}_{W(\mathfrak{sp}_{2n})}(\nu+\rho)$$

and condition (1) follows. Now let us verify condition (2). Take $w\in W$ and $\Gamma\subset\Delta_1^+$ such that (5) holds. Write $w=w_1w_2$ with $w_1\in W(\mathfrak{o}_m)$, $w_2\in W(\mathfrak{sp}_{2n})$, and set

$$\gamma:=\sum_{\beta\in\Gamma}\beta,\ \ \mu:=(\nu+\rho_0)_\delta.$$

Then $\mu-w_2\mu=\gamma_\delta$ and $\gamma_\delta=0$ implies $\Gamma=\varnothing$. Thus it is enough to verify that $\gamma_\delta=0$.

Write $\mu=:\sum_{i=1}^n b_i\delta_i$ and $w_2\mu=:\sum_{i=1}^n b_i'\delta_i$. The assumptions on ν give

$$b_i-b_j\notin\mathbb{Z}_{<0},\ \ b_i+b_j-m\notin\mathbb{Z}_{\leq 0}\ \ \text{for}\ \ 1\leq i<j\leq n. \tag{6}$$

Note that $\gamma_\delta=\sum_{i=1}^n s_i\delta_i$, where $s_i\in\{0,1,\dots,m\}$ for each i, so

$$b_i-b_i'\in\{0,1,\dots,m\}\ \ \text{for}\ \ i=1,\dots,n.$$

Since $W(\mathfrak{sp}_{2n})$ acts on $\{\delta_i\}_{i=1}^n$ by signed permutations, one has $\{|b_i|\}_{i=1}^n=\{|b_i'|\}_{i=1}^n$ as multisets. If for some i,j one has $b_j'=-b_i$, then $b_j+b_i=b_j-b_j'\in\{0,1,\dots,m\}$, a contradiction to (6). Therefore $\{b_i\}_{i=1}^n=\{b_i'\}_{i=1}^n$ as multisets. Since $b_i\geq b_i'$ for each i, one has $b_i=b_i'$, that is $\gamma_\delta=0$ as required. $\square$

4.5. Proof of Theorem 4.2

Recall that $y.\mu := y(\mu+\rho)-\rho$ for $w \in W, \mu \in \mathfrak{h}^*$; we consider other shifted actions of the Weyl group W on $\mathfrak{h}^*$ given by

$$y_\circ\mu := y(\mu+\rho_0)-\rho_0, \quad y_\bullet\mu := y(\mu+\rho_{\mathfrak{g}_{ni}})-\rho_{\mathfrak{g}_{ni}}$$

and note that $y.\mu = y_\circ\mu = y_\bullet\mu$ if $y \in W_{\mathfrak{o}_m}$.

By Lemma 4.4.1, the central character of $\mathfrak{g}_0$-module $M_{\mathfrak{g}_0}(\nu)$ is a perfect mate for the central character of $\mathfrak{g}$-module $M(\nu)$. This gives rise to equivalence of categories, see [G2, Thm. 1.3.1]. The image of $L(z.\nu)$ under this equivalence is $L_{\mathfrak{g}_0}(z_\circ\nu)$, see [FGG, §8.2.1]. Therefore

$$R_0 e^{\rho_0} \operatorname{ch} L_{\mathfrak{g}_0}(z_\circ\nu) = \sum_{y\in W} a_y^z e^{y(\nu+\rho_0)}, \quad Re^\rho \operatorname{ch} L(z.\nu) = \sum_{y\in W} a_y^z e^{y(\nu+\rho)} \tag{7}$$

for certain integers a_y^z, which are given in terms of Kazhdan–Lusztig polynomials for the Coxeter group $W(\nu+\rho_0)$ (note that a_y^z are not uniquely defined if $\Delta_0(\nu) \neq \varnothing$).

Set

$$\mu := \nu+\rho-\rho_{\mathfrak{g}_{ni}} = \nu - s\xi.$$

Our goal is to show that

$$R_{\mathfrak{g}_{ni}} e^{\rho_{\mathfrak{g}_{ni}}} \operatorname{ch} L_{\mathfrak{g}_{ni}}(z_\bullet\mu) = \sum_{y\in W} a_y^z e^{y(\mu+\rho_{\mathfrak{g}_{ni}})}. \tag{8}$$

For each $y \in W$ one has $(y_\circ\mu)_\delta = y_\circ(\mu_\delta)$, and the analogous formula holds for $y_\bullet$. Since $z \in W(\mathfrak{sp}_{2n})$, one has $(z_\circ\nu)_\varepsilon = \nu_\varepsilon = \mu_\varepsilon = (z_\bullet\mu)_\varepsilon$. Hence we have the following identities:

$$\begin{aligned}
\operatorname{ch} L_{\mathfrak{g}_0}(z_\circ\nu) &= \operatorname{ch} L_{\mathfrak{o}_m}(\nu_\varepsilon)\cdot \operatorname{ch} L_{\mathfrak{sp}_{2n}}(z_\circ\nu_\delta),\\
\operatorname{ch} L_{\mathfrak{g}_{ni}}(z_\bullet\mu) &= \operatorname{ch} L_{\mathfrak{o}_m}(\nu_\varepsilon)\cdot \operatorname{ch} L_{\mathfrak{osp}_{p(m)|2n}}(z_\bullet\mu_\delta),\\
R_{\mathfrak{o}_m} e^{\rho_{\mathfrak{o}_m}}\cdot \operatorname{ch} L_{\mathfrak{o}_m}(\nu_\varepsilon) &= \sum_{x\in W_{\mathfrak{o}_m}} b_x e^{x(\nu+\rho_0)_\varepsilon}\\
&= \sum_{x\in W_{\mathfrak{o}_m}} b_x e^{x(\nu+\rho)_\varepsilon},\\
R_{\mathfrak{sp}_{2n}} e^{\rho_{\mathfrak{sp}_{2n}}}\cdot \operatorname{ch} L_{\mathfrak{sp}_{2n}}(z_\circ\nu_\delta) &= \sum_{u\in W_{\mathfrak{sp}_{2n}}} c_u^z e^{u(\nu_\delta+\rho_{\mathfrak{sp}_{2n}})}\\
&= \sum_{u\in W_{\mathfrak{sp}_{2n}}} c_u^z e^{u(\nu+\rho_0)_\delta},\\
R_{\mathfrak{osp}_{p(m)|2n}} e^{\rho_{\mathfrak{osp}_{p(m)|2n}}}\cdot L_{\mathfrak{osp}_{p(m)|2n}}(z_\bullet\mu_\delta) &= \sum_{u\in W_{\mathfrak{sp}_{2n}}} d_u^z e^{u(\mu_\delta+\rho_{\mathfrak{osp}_{p(m)|2n}})},
\end{aligned} \tag{9}$$

where b_x, c_u^z, d_u^z are certain integers. Therefore for each $x \in W_{\mathfrak{o}_m}, u \in W_{\mathfrak{sp}_{2n}}$ we have

$$a_{xu}^z = b_x c_u^z.$$

Also, one has that $\mu_\delta+\rho_{\mathfrak{osp}_{p(m)|2n}}=(\mu+\rho_{\mathfrak{g}_{ni}})_\delta=(\nu+\rho)_\delta$, so the last formula of (9) can be rewritten as

$$R_{\mathfrak{osp}_{p(m)|2n}}e^{\rho_{\mathfrak{osp}_{p(m)|2n}}}\cdot L_{\mathfrak{osp}_{p(m)|2n}}(z_\bullet\mu_\delta)=\sum_{u\in W_{\mathfrak{sp}_{2n}}}d_u^z e^{u(\nu+\rho)_\delta}.$$

Therefore for each $x\in W_{\mathfrak{o}_m}, u\in W_{\mathfrak{sp}_{2n}}$ we have

$$R_{\mathfrak{g}_{ni}}e^{\rho_{\mathfrak{g}_{ni}}}\operatorname{ch}L_{\mathfrak{g}_{ni}}(z_\bullet\mu)=\sum_{x\in W_{\mathfrak{o}_m},u\in W_{\mathfrak{sp}_{2n}}}b_x d_u^z e^{xu(\nu+\rho)}.$$

Now (8) reduces to the fact that we can choose c_u^z, d_u^z in such a way that $c_u^z=d_u^z$ for each $u\in W_{\mathfrak{sp}_{2n}}$ (note that c_u^z, d_u^z are not uniquely defined if $\Delta_0(\nu+\rho)$ or $\Delta_0(\nu+\rho_0)$ is not empty).

Consider the case when m is odd. Combining (7) for $m=1$ and the weight $\mu_\delta+\rho_{\mathfrak{osp}_{1|2n}}=(\nu+\rho)_\delta$ with the last formula of (9) we get

$$R_{\mathfrak{sp}_{2n}}e^{\rho_{\mathfrak{sp}_{2n}}}\cdot\operatorname{ch}L_{\mathfrak{sp}_{2n}}(z_\circ\mu_\delta)=\sum_{u\in W_{\mathfrak{sp}_{2n}}}d_u^z e^{u(\mu_\delta+\rho_{\mathfrak{sp}_{2n}})}. \tag{10}$$

Note that for even m the formula (10) coincides with the last formula of (9). Hence (10) holds for all m. Compare (10) and the forth formula of (9). In light of [KT2, Prop. 3.9], the required formulae $c_y^z=d_y^z$ follow from the following conditions:

(a) $\nu_\delta-\mu_\delta$ lies in the weight lattice of $\mathfrak{sp}_{2n}$;
(b) $(\nu_\delta+\rho_{\mathfrak{sp}_{2n}})(\alpha^\vee), (\mu_\delta+\rho_{\mathfrak{sp}_{2n}})(\alpha^\vee)\not\in\mathbb{Z}_{<0}$ for each $\alpha\in\Delta(\mathfrak{sp}_{2n})$;
(c) $\Delta_0(\nu_\delta+\rho_{\mathfrak{sp}_{2n}})=\Delta_0(\mu_\delta+\rho_{\mathfrak{sp}_{2n}})$.

Condition (a) follows from $\nu_\delta-\mu_\delta=s\xi$. For (b), (c) notice that

$$\nu_\delta+\rho_{\mathfrak{sp}_{2n}}=(\nu+\rho)_\delta+\frac{m}{2}\xi;\quad \mu_\delta+\rho_{\mathfrak{sp}_{2n}}=(\nu+\rho)_\delta+\frac{p(m)}{2}\xi.$$

Using Lemma 4.4.1(i) we obtain (c) and $\nu_\delta+\rho_{\mathfrak{sp}_{2n}}, \mu_\delta+\rho_{\mathfrak{sp}_{2n}}\in\mathcal{C}^+$; one readily sees that these inclusions imply (b). This completes the proof. □

5. The cases $\mathfrak{osp}(m|2n)$ for $m=3,4$ or $n=1$

Corollary 4.2.3 gives an upper bound for the degree of a simple strongly typical highest weight bounded module. In this section we deduce an upper bound on the degree of a simple highest weight bounded module for the cases $m=3,4$ or $n=1$.

We retain the notation of §4.1. Recall that $\mathfrak{osp}_{p(m)|2n}$ stands for $\mathfrak{sp}_{2n}$ if m is even and for $\mathfrak{osp}(1|2n)$ if m is odd.

5.1.

Theorem. *Let $\mathfrak{g}=\mathfrak{osp}(m|2)$ with the base $\Sigma_{\mathfrak{o}_m}$. The module $L(\lambda)$ is bounded if and only if the $\mathfrak{o}_m$-module $L_{\mathfrak{o}_m}(\lambda_{\mathfrak{o}_m})$ is bounded. The degree of $L(\lambda)$ is at most $2^{2m}\deg L_{\mathfrak{o}_m}(\lambda_{\mathfrak{o}_m})$.*

Proof. Assume that $L_{\mathfrak{o}_m}(\lambda_{\mathfrak{o}_m})$ is bounded. Set

$$\Lambda := \{\nu \in \mathfrak{h}^* \mid \nu_{\mathfrak{o}_m} = \lambda_{\mathfrak{o}_m}\}.$$

If $\nu \in \Lambda$ is strongly typical, then, by Corollary 4.2.3, for each μ one has

$$\dim L(\nu)_{\nu-\mu} \leq 2^{2m} \deg L_{\mathfrak{o}_m}(\lambda_{\mathfrak{o}_m}) \tag{11}$$

since any simple highest weight $\mathfrak{osp}_{p(m)|2}$-module has degree 1 (note that $\mathfrak{osp}_{p(m)|2}$ is isomorphic to $\mathfrak{sl}_2$ for even m, and to $\mathfrak{osp}_{1|2}$ for odd m).

Recall that $\dim L(\nu)_{\nu-\mu}$ is equal to the rank of the Shapovalov matrix $S_\mu(\nu)$, see [Sh]. The Shapovalov matrix is a $k \times k$ matrix (where $k = \dim \mathcal{U}(\mathfrak{n})_\mu$) with entries in $\mathcal{S}(\mathfrak{h})$, and such that for each $\nu \in \mathfrak{h}^*$ the matrix $S_\mu(\nu)$ is a $k \times k$ scalar matrix. Let Λ_{st} be the set of strongly typical weights in Λ. Then Λ_{st} is Zariski dense in Λ. By (11), for $\nu \in \Lambda_{st}$ the rank of $S_\mu(\nu)$ is at most $d := 2^{2m} \deg L_{\mathfrak{o}_m}(\lambda_{\mathfrak{o}_m})$. Hence the rank of $S_\mu(\nu)$ is at most d for each $\nu \in \Lambda$. Thus (11) holds for each $\nu \in \Lambda$. This completes the proof. $\square$

5.2.

Theorem. *Let $\mathfrak{g} = \mathfrak{osp}(m|2n)$ for $m = 3$ or $m = 4$ with the base*

$$\Sigma_{\mathfrak{osp}_{p(m)|2n}} = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \delta_2, \dots, a\delta_n\},$$

where $a = 1$ for $m = 3, a = 2$ for $m = 4$. The module $L(\lambda)$ is bounded if and only if the $\mathfrak{osp}_{p(m)|2n}$-module $L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_{\mathfrak{sp}_{2n}})$ is bounded. The degree of $L(\lambda)$ is at most $2^{2n} \deg L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_{\mathfrak{sp}_{2n}})$.

Proof. Assume that $L_{\mathfrak{osp}_{p(m)|2n}}(\lambda_\delta)$ is bounded. Set

$$\Lambda := \{\nu \in \mathfrak{h}^* \mid \nu_{\mathfrak{sp}_{2n}} = \lambda_{\mathfrak{sp}_{2n}}\}$$

and let Λ_{st} be the set of strongly typical weights in Λ.

Set $\Sigma := \Sigma_{\mathfrak{osp}_{p(m)|2n}}$ and $\Sigma' := \Sigma_{\mathfrak{o}_m}$; denote by ρ (respectively, ρ') the Weyl vector for Σ (respectively, Σ'). Observe that any simple highest weight $\mathfrak{o}_m$-module has degree 1 (since $\mathfrak{o}_3 \cong \mathfrak{sl}_2$ and $\mathfrak{o}_4 \cong \mathfrak{sl}_2 \times \mathfrak{sl}_2$). If $\nu \in \Lambda_{st}$, then $L(\nu) = L(\Sigma', \nu')$ with $\nu' + \rho' = \nu + \rho$ and Corollary 4.2.3 gives

$$\dim L(\nu)_{\nu-\mu} = \dim L(\Sigma', \nu')_{\nu-\mu} \leq 2^{2n} \deg L_{\mathfrak{osp}_{p(m)|2n}} \left(\nu'_{\mathfrak{sp}_{2n}} - \left\lfloor \frac{m}{2} \right\rfloor \sum_{i=1}^{n} \delta_i \right)$$

for each μ. One has

$$\nu'_{\mathfrak{sp}_{2n}} - \left\lfloor \frac{m}{2} \right\rfloor \sum_{i=1}^{n} \delta_i = \nu_{\mathfrak{sp}_{2n}} + \rho_{\mathfrak{sp}_{2n}} - \rho'_{\mathfrak{sp}_{2n}} - \left\lfloor \frac{m}{2} \right\rfloor \sum_{i=1}^{n} \delta_i = \nu_{\mathfrak{sp}_{2n}}.$$

Therefore for $\nu \in \Lambda_{st}$ one has

$$\dim L(\nu)_{\nu-\mu} \leq 2^{2n} \deg L_{\mathfrak{osp}_{p(m)|2n}}(\nu_{\mathfrak{sp}_{2n}}). \tag{12}$$

Since Λ_{st} is Zariski dense in Λ, we can use again the last argument in the proof of Theorem 5.1. Thus (12) holds for each $\nu \in \Lambda$. $\square$

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