

WEYL DENOMINATOR IDENTITY FOR FINITE-DIMENSIONAL LIE SUPERALGEBRAS

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ABSTRACT. Weyl denominator identity for the basic simple Lie superalgebras was formulated by V. Kac and M. Wakimoto and was proven by them for the defect one case. In this paper we prove the identity for the rest of the cases.

0. INTRODUCTION

The basic simple Lie superalgebras are finite-dimensional simple Lie superalgebras, which have a reductive even part and admit an even non-degenerate invariant bilinear form. These algebras were classified by V. Kac in [K1] and the list (excluding Lie algebra case) consists of four series: $A(m, n)$, $B(m, n)$, $C(m)$, $D(m, n)$ and the exceptional algebras $D(2, 1, a)$, $F(4)$, $G(3)$.

Let \mathfrak{g} be a basic simple Lie superalgebra with a fixed triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, and let $\Delta_+ = \Delta_{+,0} \amalg \Delta_{+,1}$ be the corresponding set of positive roots. The Weyl denominator associated to the above data is

$$R := \frac{\prod_{\alpha \in \Delta_{+,0}} (1 - e^{-\alpha})}{\prod_{\alpha \in \Delta_{+,1}} (1 + e^{-\alpha})}.$$

If \mathfrak{g} is a finite-dimensional simple Lie algebra (i.e., $\Delta_1 = \emptyset$), then the Weyl denominator is given by the Weyl denominator identity

$$Re^\rho = \sum_{w \in W} \text{sgn}(w) e^{w\rho},$$

where ρ is the half-sum of the positive roots, W is the Weyl group, i.e. the subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections with respect to the roots, and $\text{sgn}(w) \in \{\pm 1\}$ is the sign of $w \in W$. This identity may be viewed as the character of trivial representation of the corresponding Lie algebra.

The Weyl denominator identities for superalgebras were formulated and partially proven (for $A(m-1, n-1)$, $B(m, n)$, $D(m, n)$ with $\min(m, n) = 1$ and for $C(n)$, $D(2, 1, a)$, $F(4)$, $G(3)$) by V. Kac and M. Wakimoto in [KW]. In order to state the Weyl denominator identity for basic simple Lie superalgebras we need the following notation. Let $\Delta^\#$ be the ‘‘largest’’ component of Δ_0 , see 2.2 for the definition. Let W be the Weyl group of \mathfrak{g}_0 , i.e. the

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subgroup of $GL(\mathfrak{h}^*)$ generated by the reflections with respect to the even roots Δ_0 , and let $\text{sgn}(w)$ be the sign of w . One has $W = W_1 \times W_2$, where $W^\#$ be the Weyl group of root system $\Delta^\#$, i.e. the subgroup of W generated by the reflections with respect to the roots from $\Delta^\#$. Set $\rho_0 := \sum_{\alpha \in \Delta_{+,0}} \alpha/2$, $\rho_1 := \sum_{\alpha \in \Delta_{+,1}} \alpha/2$, $\rho := \rho_0 - \rho_1$. A subset Π of Δ_+ is called a *set of simple roots* if the elements of Π are linearly independent and $\Delta_+ \subset \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$. For each functional f the corresponding set $\Delta_+(f)$ contains a unique system of simple roots, which we denote by $\Pi(f)$. A subset S of Δ is called *maximal isotropic* if the elements of S form a basis of a maximal isotropic space in V . By [KW], Δ contains a maximal isotropic subset and each maximal isotropic subset is a subset of a set of simple roots (for a certain functional f). Fix a maximal isotropic subset $S \subset \Delta$, choose a set of simple roots Π containing S , and choose a functional $f : V \rightarrow \mathbb{R}$ in such a way that $\Pi = \Pi(f)$. Let R be the Weyl denominator for this choice of f . The following Weyl denominator identity was suggested by V. Kac and M. Wakimoto in [KW]:

$$(1) \quad Re^\rho = \sum_{w \in W^\#} \text{sgn}(w) w \left(\frac{e^\rho}{\prod_{\beta \in S} (1 + e^{-\beta})} \right).$$

If S is empty (i.e., $(-, -)$ is a positive/negative definite) the denominator identity takes the form $Re^\rho = \sum_{w \in W} \text{sgn}(w) e^{w\rho}$. In this case either Δ_1 is empty (i.e., \mathfrak{g} is a Lie algebra), or $\mathfrak{g} = \mathfrak{osp}(1, 2l)$ (type $B(0, l)$). The Weyl denominator identity for the case $\mathfrak{osp}(1, 2l)$ was proven in [K2]; for the case when S has the cardinality one the identity was proven in [KW].

The Weyl denominator identity for root system of Lie (super)algebra \mathfrak{g} can be again naturally interpreted as the character of one-dimensional representation of \mathfrak{g} . The proofs in abovementioned cases ([K2],[KW]) are based on a analysis of the highest weights of irreducible subquotients of the Verma module $M(0)$ over \mathfrak{g} . In this paper we give a proof of the Weyl denominator identity (1) for the case when S has the cardinality greater than one. A similar proof works for the case when the cardinality of S is one. Unfortunately, our proof does not use representation theory, but requires an analysis of the roots systems. The proof is based on a case-by-case verification of the following facts:

(i) the monomials appearing in the right-hand side of (1) are of the form $e^{\rho-\nu}$, where $\nu \in Q^+ := \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$ and that the coefficient of e^ρ is one;

(ii) the right-hand side of (1) is W -skew-invariant (i.e., $w \in W$ acts by the multiplication by $\text{sgn}(w)$).

Taking into account that for any $\lambda \in V$ the stabilizer of λ in W is either trivial or contains a reflection, and that if the stabilizer is trivial and $W\lambda \subset (\rho_0 - \sum_{\alpha \in \Delta_+} \mathbb{Z}_{\geq 0} \alpha)$, then $\lambda = \rho_0$, we easily deduce the identity (1) from (i), (ii).

I. M. Musson informed us that he has an unpublished proof of the Weyl denominator identity for basic simple Lie superalgebras.

The Weyl denominator identity for the affinization of a simple finite-dimensional Lie superalgebra with non-zero Killing form was also formulated by V. Kac and M. Wakimoto in [KW] and was proven for the defect one case. We prove this identity in [G].

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1. THE ALGEBRA \mathcal{R}

In this section we introduce the algebra \mathcal{R} . Since the main technical difficulty in our proof of the denominator identity comes from the existence of different triangular decompositions, we illustrate how the proof works when there is only one triangular decomposition (see 1.3,1.4).

1.1. Notation. Denote by Q the root lattice of \mathfrak{g} and by Q^+ the positive part of root lattice $Q^+ := \sum_{\alpha \in \Pi} \mathbb{Z}_{\geq 0} \alpha$; we introduce the following partial order on V :

$$\mu \leq \nu \text{ if } (\nu - \mu) \in \sum_{\alpha \in \Delta_+} \mathbb{R}_{\geq 0} \alpha.$$

Introduce the height function $\text{ht} : Q^+ \rightarrow \mathbb{Z}_{\geq 0}$ by

$$\text{ht}\left(\sum_{\alpha \in \Pi} m_\alpha \alpha\right) := \sum_{\alpha \in \Pi} m_\alpha.$$

We use the following notation: for $X \subset \mathbb{R}, Y \subset V$ we set $XY := \{xy | x \in X, y \in Y\}$; for instance, $Q^+ := \mathbb{Z}_{\geq 0} \Pi$.

Note that Δ_0 is the root system of the reductive Lie algebra \mathfrak{g}_0 . In particular, all isotropic roots are odd. Both sets Δ_0, Δ_1 are W -stable: $W\Delta_i = \Delta_i$.

1.2. The algebra \mathcal{R} . Denote by $\mathbb{Q}[e^\nu, \nu \in V]$ the algebra of polynomials in $e^\nu, \nu \in V$. Let \mathcal{R} be the algebra of rational functions of the form

$$\frac{X}{\prod_{\alpha \in \Delta_+} (1 + a_\alpha e^{-\alpha})^{m_\alpha}},$$

where $X \in \mathbb{Q}[e^\nu, \nu \in V]$ and $a_\alpha \in \mathbb{Q}, m_\alpha \in \mathbb{Z}_{\geq 0}$. Clearly, \mathcal{R} contains the rational functions of the form $\frac{X}{\prod_{\alpha \in \Delta} (1 + a_\alpha e^{-\alpha})^{m_\alpha}}$, where X, a_α, m_α as above. The group W acts on \mathcal{R} by the automorphism mapping e^ν to $e^{w\nu}$. Say that $P \in \mathcal{R}$ is W -invariant (resp., W -skew-invariant) if $wP = P$ (resp., $wP = \text{sgn}(w)P$) for every $w \in W$.

1.2.1. For a sum $Y := \sum b_\mu e^\mu$, $b_\mu \in \mathbb{Q}$ introduce the *support* of Y by the formula

$$\text{supp}(Y) := \{\mu \mid b_\mu \neq 0\}.$$

Any element of \mathcal{R} can be uniquely expanded in the form

$$\frac{\sum_{i=1}^m a_i e^{\nu_i}}{\prod_{\alpha \in \Delta_+} (1 + a_\alpha e^{-\alpha})^{m_\alpha}} = \sum_{i=1}^s \sum_{\mu \in Q^+} b_\mu e^{\nu_i - \mu}, \quad b_\mu \in \mathbb{Q}.$$

For $Y \in \mathcal{R}$ denote by $\text{supp}(Y)$ the support of its expansion; by above, $\text{supp}(Y)$ lies in a finite union of cones of the form $\nu - Q^+$.

1.2.2. We call $\lambda \in V$ *regular* if $\text{Stab}_W \lambda = \{\text{id}\}$; we call the orbit $W\lambda$ regular if λ is regular (so the orbit consists of regular points).

It is well-known that for $\lambda \in V$ the stabilizer $\text{Stab}_W \lambda$ is either trivial or contains a reflection (see Lemma 4.1.1 (ii)). Therefore the stabilizer of a non-regular point $\lambda \in V$ contains a reflection. As a result, the space of W -skew-invariant elements of $\mathbb{Q}[e^\nu]$, $\nu \in V$ is spanned by $\sum_{w \in W} \text{sgn}(w) e^{w\lambda}$, where $\lambda \in V$ is regular. In particular, the support of a W -skew-invariant element of $\mathbb{Q}[e^\nu]$, $\nu \in V$ is a union of regular W -orbits.

1.3. **Lie algebra case.** The denominator identity for Lie algebra is

$$Re^\rho = \prod_{\alpha \in \Delta_+} (1 - e^{-\alpha}) e^\rho = \sum_{w \in W} \text{sgn}(w) e^{w\rho}.$$

There are several proofs of this identity. The proof which we are going to generalize is the following. Observe that $Re^\rho \in \mathbb{Q}[e^\nu]$, $\nu \in V$ and $\text{supp}(Re^\rho) \subset (\rho - Q^+)$. Moreover, $Re^\rho = \prod_{\alpha \in \Delta_+} (e^{\alpha/2} - e^{-\alpha/2})$ is W -skew-invariant, so $\text{supp}(Re^\rho)$ is a union of regular orbits lying in $(\rho - Q^+)$. However, $W\rho$ is the only regular orbit lying entirely in $(\rho - Q^+)$, see Lemma 4.1.1 (iii). Hence Re^ρ is proportional to $\sum_{w \in W} \text{sgn}(w) e^{w\rho}$. Since the coefficient of e^ρ in the expression Re^ρ is 1, the coefficient of proportionality is 1.

1.4. **Case $Q(n)$.** For the case $\mathfrak{g} := Q(n)$ one has $\mathfrak{g}_0 = \mathfrak{gl}(n)$, $W = S_n$, $\Delta_{0+} = \Delta_{+,1} = \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq n\}$ and so $\rho_0 = \rho_1$, $\rho = 0$. The Weyl denominator is $R = \prod_{\alpha \in \Delta_{+,0}} \frac{1 - e^{-\alpha}}{1 + e^{-\alpha}}$. For each $S \subset \Delta_{+,1}$ define

$$A(S) := \{w \in S_n \mid wS \subset \Delta_{+,0}\}, \quad a(S) := \sum_{w \in A(S)} \text{sgn}(w).$$

1.4.1. **Proposition.** *For each $S \subset \Delta_{+,1}$ one has*

$$a(S)R = \sum_{w \in S_n} \text{sgn}(w) \frac{1}{\prod_{\beta \in S} (1 + e^{-w\beta})}.$$

Proof. Observe that the support of both sides of the formula lies in $-Q^+$ and that the coefficients of $1 = e^0$ in both sides are equal to $a(S)$. Multiplying both sides of the formula by the W -invariant expression $\prod_{\alpha \in \Delta_{+,1}} (1 + e^{-\alpha}) e^{\rho_1} = \prod_{\alpha \in \Delta_{+,1}} (e^{\alpha/2} + e^{-\alpha/2})$ we obtain

$$\begin{aligned} Y : &= \prod_{\alpha \in \Delta_{+,0}} (1 - e^{-\alpha}) e^{\rho_0} - \prod_{\alpha \in \Delta_{+,1}} (1 + e^{-\alpha}) e^{\rho_1} \sum_{w \in S_n} \operatorname{sgn}(w) \frac{1}{\prod_{\beta \in S} (1 + e^{-w\beta})} \\ &= \prod_{\alpha \in \Delta_{+,0}} (1 - e^{-\alpha}) e^{\rho_0} - \sum_{w \in S_n} \operatorname{sgn}(w) w \left(\prod_{\alpha \in \Delta_{+,0} \setminus S} (1 + e^{-\alpha}) e^{\rho_0} \right). \end{aligned}$$

Since $\prod_{\alpha \in \Delta_{+,0}} (1 - e^{-\alpha}) e^{\rho_0} = \prod_{\alpha \in \Delta_{+,0}} (e^{\alpha/2} - e^{-\alpha/2})$ is W -skew-invariant, Y is also W -skew-invariant. Clearly, $Y \in \mathbb{Q}[e^\nu, \nu \in V]$, so, by 1.2.2, $\operatorname{supp}(Y)$ is a union of regular W -orbits. By above, $\operatorname{supp}(Y) \subset (\rho_0 - Q^+) \setminus \{\rho_0\}$. However, by Lemma 4.1.1 (iii), any regular W -orbit intersects $\rho_0 + Q^+$. Hence $Y = 0$ as required. \square

1.4.2. In order to obtain a formula for the Weyl denominator R , we choose S such that $a(S) \neq 0$. Taking $S = \{\varepsilon_1 - \varepsilon_n, \varepsilon_2 - \varepsilon_{n-1}, \dots, \varepsilon_{\lfloor n/2 \rfloor} - \varepsilon_{n+1 - \lfloor n/2 \rfloor}\}$ and using Lemma 1.4.3, we obtain the following formula

$$R = \frac{1}{\lfloor n/2 \rfloor!} \sum_{w \in S_n} \operatorname{sgn}(w) \frac{1}{\prod_{\beta \in S} (1 + e^{-w\beta})},$$

which appears in [KW] (7.1) (up to a constant factor).

Note that such S has a minimal cardinality: if the cardinality of S is less than $\lfloor n/2 \rfloor$, then $a(S) = 0$. Indeed, if the cardinality of S is less than $\lfloor n/2 \rfloor$, then there is a root $\varepsilon_i - \varepsilon_j$, which does not belong to the span of S , and thus $s_{\varepsilon_i - \varepsilon_j} W(S) = W(S)$; since $\operatorname{sgn}(w) + \operatorname{sgn}(s_{\varepsilon_i - \varepsilon_j} w) = 0$, this forces $a(S) = 0$.

1.4.3. **Lemma.** *Set $A := \{\sigma \in S_n \mid \forall i \leq \lfloor n/2 \rfloor \quad \sigma(i) > \sigma(n+1-i)\}$. Then*

$$\sum_{\sigma \in A} \operatorname{sgn}(\sigma) = \lfloor n/2 \rfloor!$$

Proof. For each $\sigma \in A$ let $P(\sigma)$ be the set of pairs $\{(\sigma(1), \sigma(n)); (\sigma(2), \sigma(n-1)); \dots\}$, that is $P(\sigma) := \{(\sigma(j), \sigma(n+1-j))\}_{j=1}^{\lfloor n/2 \rfloor}$. Let $B := \{\sigma \in A \mid P(\sigma) = \{(j, j+1)\}_{j=1}^{\lfloor n/2 \rfloor}\}$. Define an involution f on the set $A \setminus B$ as follows: for $\sigma \in A \setminus B$ set $f(\sigma) := (i, i+1) \circ \sigma$, where i is minimal such $(i, i+1) \notin P(\sigma)$. Since $\operatorname{sgn}(\sigma) + \operatorname{sgn}(f(\sigma)) = 0$, we get $\sum_{\sigma \in A \setminus B} \operatorname{sgn}(\sigma) = 0$. One readily sees that B has $\lfloor n/2 \rfloor!$ elements and that $\operatorname{sgn}(\sigma) = 1$ for each $\sigma \in B$. Hence $\sum_{\sigma \in A} \operatorname{sgn}(\sigma) = \lfloor n/2 \rfloor!$ as required. \square

2. NOTATION

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a basic Lie superalgebra with a fixed triangular decomposition of the even part: $\mathfrak{g}_0 = \mathfrak{n}_{-,0} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+,0}$. For $A(m-1, n-1)$ -type we put $\mathfrak{g} = \mathfrak{gl}(m|n)$ (one readily sees that the denominator identities for $\mathfrak{gl}(m|n)$ imply the denominator identities for $\mathfrak{sl}(m|n), \mathfrak{psl}(n|n)$). Let Δ_0 (resp., Δ_1) be the set of even (resp., odd) roots of \mathfrak{g} . Set

$V = \mathfrak{h}_{\mathbb{R}}^*$ (so $V = \text{span } \Delta$ for $\mathfrak{g} \neq A(m, n)$ and $V = \mathbb{R} \text{span } \Delta \oplus \mathbb{R}$ for $\mathfrak{g} = A(m, n)$). Denote by $(-, -)$ a non-degenerate symmetric bilinear form on V , induced by a non-degenerate invariant bilinear form on \mathfrak{g} . Retain notation of Section 0 and define the Weyl denominator.

The dimension of a maximal isotropic space in $V = \mathfrak{h}_{\mathbb{R}}^*$ is called the *defect* of \mathfrak{g} . If \mathfrak{g} is a Lie algebra or $\mathfrak{g} = \mathfrak{osp}(1, 2l)$ (type $B(0, l)$) then the defect of \mathfrak{g} is zero; the defect of $A(m-1, n-1), B(m, n), D(m, n)$ is equal to $\min(m, n)$; for $C(n)$ and the exceptional Lie superalgebras the defect is equal to one. Notice that the cardinality of a maximal isotropic set S is equal to defect of \mathfrak{g} .

2.1. Admissible pairs. The set of positive even roots $\Delta_{+,0}$ is determined by the triangular decomposition $\mathfrak{g}_0 = \mathfrak{n}_{-,0} \oplus \mathfrak{h} \oplus \mathfrak{n}_{+,0}$ (i.e., $\Delta_{+,0}$ is the set of weights of $\mathfrak{n}_{+,0}$). Recall that for each maximal isotropic set S there exists a set of simple roots containing S . We call a pair (S, Π) *admissible* if $S \subset \Pi$ is a maximal isotropic set of roots and Π is a set of simple roots such that the corresponding set of positive even roots coincides with $\Delta_{+,0}$:

$$(S, \Pi) \text{ is admissible if } S \subset \Pi \text{ \& } \Delta_+(\Pi) \cap \Delta_0 = \Delta_{+,0}.$$

For a fixed set of simple roots Π we retain notation of 1.1 and 1.2.

2.2. The set $\Delta^\#$. Let Δ_1, Δ_2 be two finite irreducible root systems; we say that Δ_1 is “larger” than Δ_2 if either the rank of Δ_1 is greater than the rank of Δ_2 , or the ranks are equal and $\Delta_1 \subset \Delta_2$.

If the defect of \mathfrak{g} is greater than one, then the root system Δ_0 is a disjoint union of two irreducible root systems. We denote by $\Delta^\#$ the irreducible component, which is not the smallest one, i.e. $\Delta_0 = \Delta^\# \amalg \Delta_2$, where $\Delta^\#$ is not smaller, than Δ_2 , see the following table:

Δ	A(m-1,n-1)		B(m,n)			D(m,n)	
	$m > n$	$m \leq n$	$m > n$	$m < n$	$m = n$	$m > n$	$m \leq n$
$\Delta^\#$	A_{m-1}	A_{n-1}	B_m	C_n	B_m or C_m	D_m	C_n

The notion of $\Delta^\#$ in [KW] coincides with the above one, except for the case $B(m, m)$, where we allow both choices B_m and C_m , whereas in [KW] $\Delta^\#$ is of the type C_m .

Notice that $\mathfrak{g}_0 = \mathfrak{s}_1 \times \mathfrak{s}_2$, where $\mathfrak{s}_1, \mathfrak{s}_2$ are reductive Lie algebras and $\Delta^\#, \Delta_0 \setminus \Delta^\#$ are roots systems of $\mathfrak{s}_1, \mathfrak{s}_2$ respectively. We normalize $(-, -)$ in such a way that $\Delta^\# := \{\alpha \in \Delta_0 \mid (\alpha, \alpha) > 0\}$; then $\Delta_0 \setminus \Delta^\# = \{\alpha \in \Delta_0 \mid (\alpha, \alpha) < 0\}$.

3. OUTLINE OF THE PROOF

3.1. Let \mathfrak{g} be one of the Lie superalgebras $A(m-1, n-1), B(m, n), D(m, n), m, n > 0$.

3.2. Expansion of the right-hand side of (1). Let (S, Π) be an admissible pair. Set

$$(2) \quad X := \sum_{w \in W^\#} \operatorname{sgn}(w) w \left(\frac{e^\rho}{\prod_{\beta \in S} (1 + e^{-\beta})} \right)$$

and rewrite the denominator identity (1) as $Re^\rho = X$.

Expanding X we obtain

$$(3) \quad X = \sum_{w \in W^\#} \sum_{\mu \in \mathbb{Z}_{\geq 0} S} \operatorname{sgn}(w) (-1)^{\operatorname{ht} \mu} e^{\varphi(w) - |w|\mu + w\rho},$$

where

$$\varphi(w) := \sum_{\beta \in S: w\beta < 0} w\beta \in -Q^+$$

and $|w|$ is a linear map $\mathbb{Z}_{\geq 0} S \rightarrow Q^+$ defined on $\beta \in S$ by the formula

$$|w|\beta = \begin{cases} w\beta & \text{for } w\beta > 0, \\ -w\beta & \text{for } w\beta < 0. \end{cases}$$

3.3. Main steps. The proof has the following steps:

(i) We introduce certain operations on the admissible pairs (S, Π) and show that these operations preserve the expressions X and Re^ρ . Consider the equivalence relation on the set of admissible pairs (S, Π) generated by these operations. We will show that there are two equivalence classes for $D(m, n)$, $m > n$ and one equivalence class for other cases.

(ii) We check that $\operatorname{supp}(X) \subset (\rho - Q^+)$ and that the coefficient of e^ρ in X is 1 for a certain choice of (S, Π) (for $D(m, n)$, $m > n$ we check this for (S, Π) and (S', Π) , which are representatives of the equivalence classes).

(iii) We show that X is W -skew-invariant for a certain choice of (S, Π) (for $D(m, n)$, $m > n$ we show this for (S, Π) and (S', Π) , which are representatives of the equivalence classes).

For \mathfrak{g} of $A(n-1, n-1)$ type we change (ii) to (ii'):

(ii') We check, for a certain choice of (S, Π) , that $\operatorname{supp}(X) \subset (\rho - Q^+)$ and that for $\xi := \sum_{\beta \in S} \beta$ the coefficients of $e^{\rho - s\xi}$ in X and in Re^ρ are equal for each $s \in \mathbb{Z}_{\geq 0}$.

The choices of (S, Π) in (ii), (iii) are the same only for $A(m, n)$ case.

3.4. Why (i)–(iii) imply (1). Let us show that (i)–(iii) imply the denominator identity $X = Re^\rho$. Indeed, assume that $X - Re^\rho \neq 0$.

Since $WS \subset \Delta_1$, $X - Re^\rho$ is a rational function with the denominator of the form $\prod_{\beta \in \Delta_1^+} (1 + e^{-\beta})$; we write

$$X - Re^\rho = \frac{Y}{\prod_{\beta \in \Delta_1^+} (1 + e^{-\beta})} = \frac{Y e^{\rho_1}}{\prod_{\beta \in \Delta_1^+} (e^{\beta/2} + e^{-\beta/2})},$$

where $Y \in \mathbb{Q}[e^\nu, \nu \in V]$. One has

$$Re^\rho = \frac{\prod_{\alpha \in \Delta_0^+} (e^{\alpha/2} - e^{-\alpha/2})}{\prod_{\alpha \in \Delta_1^+} (e^{\alpha/2} + e^{-\alpha/2})}$$

and the latter expression is W -skew-invariant, since its numerator is W -skew-invariant and its denominator is W -invariant. Combining (i) and (iii), we obtain that $X - Re^\rho$ is W -skew-invariant. Thus Ye^{ρ_1} is a W -skew-invariant element of $\mathbb{Q}[e^\nu, \nu \in V]$ and so $\text{supp}(Ye^{\rho_1})$ is a union of regular orbits.

Observe that $\text{supp}(Re^\rho) \subset (\rho - Q^+)$ and that the coefficient of e^ρ in Re^ρ is 1. Using (i), (ii) we get $\text{supp}(X - Re^\rho) \subset (\rho - Q^+) \setminus \{\rho\}$. Note that the sets of maximal elements in $\text{supp}(Y)$ and in $\text{supp}(X - Re^\rho)$ coincide. Thus $\text{supp} Y \subset (\rho - Q^+) \setminus \{\rho\}$ that is $\text{supp}(Ye^{\rho_1}) \subset (\rho_0 - Q^+) \setminus \{\rho_0\}$. Hence $\text{supp}(Ye^{\rho_1})$ is a union of regular orbits lying in $(\rho_0 - Q^+) \setminus \{\rho_0\}$.

By 4.2, for $\mathfrak{g} \neq \mathfrak{gl}(n|n)$, the set $(\rho_0 - Q^+) \setminus \{\rho_0\}$ does not contain regular W -orbit, a contradiction.

Let $\mathfrak{g} = \mathfrak{gl}(n|n)$. Choose Π as in 6.3. By 4.2 the regular orbits in $(\rho_0 - Q^+)$ are of the form $W(\rho_0 - s\xi)$ with $s \in \mathbb{Z}_{\geq 0}$, $\xi = \sum_{\beta \in S} \beta$. One has $W\xi = \xi$ and $w\rho_0 \leq \rho_0$, so $\rho_0 - s\xi$ is the maximal element in its W -orbit. Thus a maximal element in $\text{supp} Ye^{\rho_1}$ is of the form $\rho_0 - s\xi$, so a maximal element in $\text{supp} Y$ is $\rho - s\xi$. Then, by above, $\rho - s\xi \in \text{supp}(X - Re^\rho)$, which contradicts to (ii').

4. REGULAR ORBITS

4.1. Let \mathfrak{g} be a reductive finite-dimensional Lie algebra, let W be its Weyl group, let Π be its set of simple roots and let Π^\vee be the set of simple coroots. For ρ , defined as above, one has $\langle \rho, \alpha^\vee \rangle = 1$ for each $\alpha \in \Pi$. Set

$$Q_{\mathbb{Q}} = \sum_{\alpha \in \Pi} \mathbb{Q}\alpha, \quad Q_{\mathbb{Q}}^+ := \sum_{\alpha \in \Pi} \mathbb{Q}_{\geq 0}\alpha.$$

As above, we define partial order on $\mathfrak{h}_{\mathbb{R}}^*$ by the formula $\mu \leq \nu$ if $(\nu - \mu) \in \sum_{\alpha \in \Delta_+} \mathbb{R}_{\geq 0}\alpha$.

Let $P \subset \mathfrak{h}_{\mathbb{R}}^*$ be the weight lattice of \mathfrak{g} , i.e. $\nu \in P$ iff $\langle \nu, \alpha^\vee \rangle \in \mathbb{Z}$ for any $\alpha \in \Pi$, and let P^+ be the positive part of P , i.e. $\nu \in P^+$ iff $\langle \nu, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for any $\alpha \in \Pi$. One has $P \subset Q_{\mathbb{Q}}$.

4.1.1. **Lemma.** (i) $P = WP^+$.

(ii) For any $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ the stabilizer of λ in W is either trivial or contains a reflection.

(iii) A regular orbit in P intersects with the set $\rho + Q_{\mathbb{Q}}^+$.

Proof. The group W is finite. Take $\lambda \in \mathfrak{h}_{\mathbb{R}}^*$ and let $\lambda' = w\lambda$ be a maximal element in the orbit $W\lambda$. Since λ' is maximal, $\langle \lambda', \alpha^\vee \rangle \geq 0$ for each $\alpha \in \Pi$.

For $\lambda \in P$ one has $\lambda' \in P$, so $\langle \lambda', \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for each $\alpha \in \Pi$, that is $\lambda' \in P^+$, hence (i).

For (ii) note that, if $\langle \lambda', \alpha^\vee \rangle = 0$ for some $\alpha \in \Pi$, then $s_\alpha \in \text{Stab}_W \lambda'$, so $s_{w^{-1}\alpha} \in \text{Stab}_W \lambda$. Assume that $\langle \lambda', \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$. Take $y \in W, y \neq \text{id}$ and write $y = y' s_\alpha$ for $\alpha \in \Pi$, where the length of $y' \in W$ is less than the length of y . Then $y' \alpha \in \Delta_+$ (see, for instance, [J], A.1.1). One has $y' \lambda' - y' s_\alpha \lambda' = \langle \lambda', \alpha^\vee \rangle (y' \alpha) > 0$ so $y' \lambda' > y' s_\alpha \lambda'$. Now (ii) follows by the induction on the length of y .

For (iii) assume that λ' is regular, that is $\langle \lambda', \alpha^\vee \rangle > 0$ for all $\alpha \in \Pi$. Since $\lambda' \in P$, one has $\langle \lambda', \alpha^\vee \rangle \in \mathbb{Z}_{\geq 1}$ so $\langle \lambda - \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Pi$. Write $\lambda - \rho = \sum_{\beta \in \Pi} x_\beta \beta$. For a vector $(y_\beta)_{\beta \in \Pi}$ write $y \geq 0$ if $y_\beta \geq 0$ for each β . The condition $\langle \lambda - \rho, \alpha^\vee \rangle \geq 0$ for all $\alpha \in \Pi$ means that $Ax \geq 0$, where $x = (x_\beta)_{\beta \in \Pi}$ and $A = (\langle \alpha^\vee, \beta \rangle)_{\alpha, \beta \in \Pi}$ is the Cartan matrix of \mathfrak{g} . From [K3], 4.3 it follows that $Ax \geq 0$ forces $x \geq 0$. Hence $\lambda - \rho \in \mathbb{Q}_{\geq 0}^+$ as required. \square

4.2. Now let \mathfrak{g} be a basic simple Lie superalgebra, Q be its root lattice and Q^+ be the positive part of Q .

4.2.1. **Corollary.** *Let \mathfrak{g} be a basic simple Lie superalgebra and $\mathfrak{g} \neq C(n), A(m, n)$. A regular orbit in the root lattice Q intersects with the set $\rho_0 + \mathbb{Q}_{\geq 0} \Delta_{+,0}$.*

Proof. For $\mathfrak{g} \neq C(n), A(m, n)$ one has $\mathbb{Q}\Delta_0 = \mathbb{Q}\Delta$ and the \mathfrak{g} -root lattice Q is a subset of weight lattice of \mathfrak{g}_0 . Thus the assertion follows from Corollary 4.1.1. \square

4.2.2. *Case $\mathfrak{g} = \mathfrak{gl}(m|n)$.* For $\mathfrak{gl}(m|n)$ one has

$$\mathbb{Q}\Delta = \mathbb{Q}\Delta_0 \oplus \mathbb{Q}\xi, \quad \text{where } \xi := \sum \varepsilon_i - \frac{m}{n} \sum \delta_j.$$

Choose a set of simple roots as in 6.3.

Lemma. *Let $\mathfrak{g} = \mathfrak{gl}(m|n)$.*

(i) *For $m \neq n$, $W\rho_0$ is the only regular orbit lying entirely in $(\rho_0 - Q^+)$.*

(ii) *For $m = n$ the regular orbits lying entirely in $(\rho_0 - Q^+)$ are of the form $W(\rho_0 - s\xi), s \geq 0$, where $\xi = \sum \varepsilon_i - \sum \delta_j$.*

Proof. Let $\iota : \mathbb{Q}\Delta \rightarrow \mathbb{Q}\Delta_0$ be the projection along ξ (i.e., $\text{Ker } \iota = \mathbb{Q}\xi$). Since ξ is W -invariant, $w\iota(\lambda) = \iota(w\lambda)$. Let $W\lambda \subset (\rho_0 - Q^+)$ be a regular orbit. Since $\iota(Q)$ lies in the weight lattice of \mathfrak{g}_0 , $\iota(W\lambda)$ intersects with $\rho_0 + \mathbb{Q}_{\geq 0} \Delta_{+,0}$, by Corollary 4.1.1. Thus $W\lambda$ contains a point of the form $\rho_0 + \nu + q\xi$, where $\nu \in \mathbb{Q}_{\geq 0} \Delta_{+,0}$ and $q \in \mathbb{Q}$. By above, $\nu + q\xi \in -Q^+$ so $q\xi \in -\mathbb{Q}_{\geq 0} \Delta_+$.

Consider the case $m > n$. In this case, for any $\mu \in \mathbb{Q}_{\geq 0} \Delta_+$ one has $(\mu, \varepsilon_1) \cdot (\mu, \varepsilon_m) \leq 0$. Since $(\xi, \varepsilon_1) = (\xi, \varepsilon_m) \neq 0$, the inclusion $q\xi \in -\mathbb{Q}_{\geq 0} \Delta_+$ implies $q = 0$. Then $\nu \in \mathbb{Q}_{\geq 0} \Delta_{+,0}$ and $\nu = \nu + q\xi \in -Q^+$ so $\nu = 0$. Therefore $W\lambda = W\rho_0$. Hence $W\rho_0$ is the only regular orbit lying entirely in $\rho_0 - Q^+$. Since $\mathfrak{gl}(m|n) \cong \mathfrak{gl}(n|m)$, this establishes (i).

Consider the case $m = n$. Set $\beta_i := \varepsilon_i - \delta_i, \beta'_i := \delta_i - \varepsilon_{i+1}$. One has $\xi = \sum_{i=1}^n \beta_i$ and $\Pi = \{\beta_i\}_{i=1}^n \cup \{\beta'_i\}_{i=1}^{n-1}$. The simple roots of $\Delta_{+,0}$ are $\varepsilon_i - \varepsilon_{i+1} = \beta_i + \beta'_i$ and $\delta_i - \delta_{i+1} = \beta'_i + \beta_{i+1}$. Thus $\nu \in \mathbb{Q}_{\geq 0} \Delta_0^+$ takes the form $\nu = \sum_{i=1}^{n-1} b_i(\beta_i + \beta'_i) + c_i(\beta'_i + \beta_{i+1})$ with $b_i, c_i \in \mathbb{Q}_{\geq 0}$. By above, $\nu' := -(\nu + q\xi) \in Q^+$. Therefore $\nu + \nu' \in \mathbb{Q} \sum_{i=1}^n \beta_i$ and $\nu' \in Q^+$. One readily sees that this implies $b_i = c_i = 0$ that is $\nu = 0$. One has $Q^+ \cap \mathbb{Q}\xi = \mathbb{Z}_{\geq 0}\xi$. Hence a regular orbit in $\rho_0 - Q^+$ intersects with the set $\rho_0 - \mathbb{Z}_{\geq 0}\xi$ as required. \square

4.2.3. *Case $C(n)$.* Take $\Pi = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \dots, \varepsilon_n - \delta_1, \varepsilon_n + \delta_1\}$. One has $\mathbb{Q}\Delta = \mathbb{Q}\Delta_0 \oplus \mathbb{Q}\delta_1$. We claim that $W\rho_0$ is the only regular orbit lying entirely in $(\rho_0 - Q^+)$.

Indeed, take a regular orbit lying entirely in $(\rho_0 - Q^+)$. Combining the fact that $W\delta_1 = \delta_1$ and Corollary 4.1.1, we see that this orbit contains a point of the form $\rho_0 + \nu + q\delta_1$, where $\nu \in \mathbb{Q}_{\geq 0} \Delta_0^+$ and $q \in \mathbb{Q}$. Since $-(\nu + q\delta_1) \in Q^+$, one has $q\delta_1 \in -\mathbb{Q}_{\geq 0} Q^+$. However, $2\delta_1$ is the difference of two simple roots ($2\delta_1 = (\varepsilon_n + \delta_1) - (\varepsilon_n - \delta_1)$) so $\mathbb{Q}\delta_1 \cap (-\mathbb{Q}_{\geq 0} Q^+) = \{0\}$ that is $q = 0$. The conditions $\nu \in \mathbb{Q}_{\geq 0} \Delta_0^+, -(\nu + q\delta_1) \in Q^+$ give $\nu = 0$. The claim follows.

5. STEP (I)

Consider the following operations with the admissible pairs (S, Π) . First type operations are the odd reflections $(S, \Pi) \mapsto (s_\beta S, s_\beta \Pi) = (S \setminus \{\beta\} \cup \{-\beta\}, s_\beta \Pi)$ with respect to an element of $\beta \in S$ (see 5.2). By 5.2, these odd reflections preserve the expressions X, Re^ρ . Second type operations are the operations $(S, \Pi) \mapsto (S', \Pi)$ described in Lemma 5.3, where it is shown that these operations also preserve the expressions X, Re^ρ . Consider the equivalence relation on the set of admissible pairs (S, Π) generated by these operations. In 5.4 we will show that there are two equivalence classes for $D(m, n), m > n$ and one equivalence class for other cases. In 5.5 we will show that if $(S, \Pi), (S, \Pi')$ are admissible pairs, then $\Pi = \Pi'$ (on the other hand, there are admissible pairs $(S, \Pi), (S', \Pi)$ with $S \neq S'$, see Lemma 5.3).

5.1. **Notation.** Let us introduce the following operator $F : \mathcal{R} \rightarrow \mathcal{R}$

$$F(Y) := \sum_{w \in W^\#} \text{sgn}(w) wY.$$

Clearly, $F(wY) = \text{sgn}(w)F(Y)$ for $w \in W^\#$, so $F(Y) = 0$ if $wY = Y$ for some $w \in W^\#$ with $\text{sgn}(w) = -1$.

For an admissible pair (S, Π) introduce

$$Y(S, \Pi) := \frac{e^{\rho(\Pi)}}{\prod_{\beta \in S} (1 + e^{-\beta})}, \quad X(S, \Pi) := F(Y(S, \Pi)),$$

where $\rho(\Pi)$ is the element ρ defined for given Π . Note that $X = X(S, \Pi)$ for the corresponding pair (S, Π) .

5.2. Odd reflections. Recall a notion of odd reflections, see [S]. Let Π be a set of simple roots and $\Delta_+(\Pi)$ be the corresponding set of positive roots. Fix a simple isotropic root $\beta \in \Pi$ and set

$$s_\beta(\Delta_+) := \Delta_+(\Pi) \setminus \{\beta\} \cup \{-\beta\}.$$

For each $P \subset \Pi$ set $s_\beta(P) := \{s_\beta(\alpha) \mid \alpha \in P\}$, where

$$\text{for } \alpha \in \Pi \quad s_\beta(\alpha) := \begin{cases} -\alpha & \text{if } \alpha = \beta, \\ \alpha & \text{if } (\alpha, \beta) = 0, \alpha \neq \beta \\ \alpha + \beta & \text{if } (\alpha, \beta) \neq 0. \end{cases}$$

By [S], $s_\beta(\Delta_+)$ is a set of positive roots (i.e., $s_\beta(\Delta_+) = \Delta_+(f)$ for some functional f) and the corresponding set of simple roots is $s_\beta(\Pi)$. Clearly, $\rho(s_\beta(\Pi)) = \rho(\Pi) + \beta$.

Let (S, Π) be an admissible pair. Take $\beta \in S$. Then for any $\beta' \in S \setminus \{\beta\}$ one has $s_\beta(\beta') = \beta'$ so $s_\beta(S) = (S \setminus \{\beta\}) \cup \{-\beta\}$. Clearly, the pair $(s_\beta(S), s_\beta(\Pi))$ is admissible. Since $\rho(s_\beta(\Pi)) = \rho(\Pi) + \beta$, one has $Y(S, \Pi) = Y(s_\beta(S), s_\beta(\Pi))$.

5.3. Lemma. *Assume that γ, γ' are isotropic roots such that*

$$\gamma \in S, \gamma' \in \Pi, \gamma + \gamma' \in \Delta^\#, \quad (\gamma', \beta) = 0 \text{ for each } \beta \in S \setminus \{\gamma\}.$$

Then the pair (S', Π) , where $S' := (S \cup \{\gamma'\}) \setminus \{\gamma\}$, is admissible and $X(S, \Pi) = X(S', \Pi)$.

Proof. It is clear that the pair (S', Π) is admissible. Set $\alpha := \gamma + \gamma'$ and let $s_\alpha \in W^\#$ be the reflection with respect to the root α . Our assumptions imply that

$$(4) \quad s_\alpha \rho = \rho; \quad s_\alpha \gamma' = -\gamma; \quad s_\alpha \beta = \beta \text{ for } \beta \in S \setminus \{\gamma\}.$$

Therefore $s_\alpha(Y(S', \Pi)) = Y(S, \Pi)e^{-\gamma}$ that is $F(Y(S', \Pi)) = F(s_\alpha(Y(S, \Pi)e^{-\gamma}))$. Since $F \circ s_\alpha = -F$, the required formula $X(S, \Pi) = X(S', \Pi)$ is equivalent to the equality $F(Y(S, \Pi)(1 + e^{-\gamma})) = 0$, which follows from the fact that the expression

$$Y(S, \Pi)(1 + e^{-\gamma}) = \frac{e^\rho}{\prod_{\beta \in S \setminus \{\gamma\}} (1 + e^{-\beta})}$$

is, by (4), s_α -invariant. □

5.4. Equivalence classes. We consider the types $A(m-1, n-1), B(m, n), D(m, n)$ with all possible m, n . We express the roots in terms of linear functions ξ_1, \dots, ξ_{m+n} , see [K1], such that

Δ	A(m-1,n-1)	B(m,n)	D(m,n)
$\Delta_{+,0}$	U	$U' \cup \{\xi_i\}_{i=1}^m \cup \{2\xi_i\}_{i=m+1}^{m+n}$	$U' \cup \{2\xi_i\}_{i=m+1}^{m+n}$
Δ_1	$\{\pm(\xi_i - \xi_j)\}_{1 \leq i < m < j \leq m+n}$	$\{\pm\xi_i \pm \xi_j; \pm\xi_j\}_{1 \leq i < m < j \leq m+n}$	$\{\pm\xi_i \pm \xi_j\}_{1 \leq i < m < j \leq m+n}$

and

$$U := \{\xi_i - \xi_j \mid 1 < i < j \leq m \text{ or } m+1 \leq i < j \leq m+n\},$$

$$U' := \{\xi_i \pm \xi_j \mid 1 < i < j \leq m \text{ or } m+1 \leq i < j \leq m+n\}.$$

5.4.1. For a system of simple roots $\Pi \subset \text{span}\{\xi_i\}_{i=1}^{m+n}$ take an element $f_\Pi \in \text{span}\{\xi_i^*\}_{i=1}^{m+n}$ such that $\langle f_\Pi, \alpha \rangle = 1$ for each $\alpha \in \Pi$ (the existence of f_Π follows from linear independence of the elements in Π). Note that f_Π is unique if \mathfrak{g} is not of the type $A(m-1, n-1)$; for $A(m-1, n-1)$ we fix f_Π by the additional condition $\min_{1 \leq i \leq m+n} \langle f_\Pi, \xi_i \rangle = 1$. In the notation of Sect. 0 one has $\Pi = \Pi(f_\Pi)$. We will use the following properties of f_Π :

- (i) $\langle f_\Pi, \alpha \rangle \in \mathbb{Z} \setminus \{0\}$ for all $\alpha \in \Delta$;
- (ii) if $\alpha \in \Delta^+$, then $\langle f_\Pi, \alpha \rangle \geq 1$;
- (iii) a root $\alpha \in \Delta$ is simple iff $\langle f_\Pi, \alpha \rangle = 1$.

Write $f_\Pi = \sum_{i=1}^{m+n} x_i \xi_i$. By (i), the x_i s are pairwise different; by (ii), $x_i \pm x_{i+1} > 0$ for $i \neq m$ ($1 \leq i \leq m+n-1$); by (iii), a root $\xi_i - \xi_j$ is simple iff $x_i - x_j = 1$.

5.4.2. For type $A(m-1, n-1)$ all roots are of the form $\xi_i - \xi_j$. In particular, $\{x_i\}_{i=1}^{m+n} = \{1, \dots, m+n\}$.

Consider the case $B(m, n)$. In this case for each i one has $\xi_i \in \Delta_+$ so $x_i \geq 1$. Therefore, by (iii), a simple root can not be of the form $\pm(\xi_i + \xi_j)$ so $\Pi = \{\xi_{i_1} - \xi_{i_2}, \xi_{i_2} - \xi_{i_3}, \dots, \xi_{i_{m+n-1}} - \xi_{i_{m+n}}, \xi_{i_{m+n}}\}$ and $\{x_i\}_{i=1}^{m+n} = \{1, \dots, m+n\}$.

5.4.3. Consider the case $D(m, n)$. Using (i), (ii) and the fact that $2\xi_{m+n} \in \Delta_{+,0}$, we conclude

$$\begin{aligned} \forall j \ 2x_j \in \mathbb{Z}; \quad \forall i \neq j \ x_i \pm x_j \in \mathbb{Z} \setminus \{0\}; \\ x_i - x_j \geq j - i \text{ for } i < j \leq m \text{ or } m < i < j; \\ x_j > 0 \text{ for } j > m; \quad x_j + x_i > 0 \text{ for } i < j \leq m. \end{aligned}$$

Recall that, if $-(\xi_p + \xi_q) \in \Pi \cap \Delta_1$, then $\Pi' := s_{-\xi_p - \xi_q}(\Pi)$ contains $\xi_p + \xi_q$. Let us assume that $\xi_p + \xi_q \in \Pi \cap \Delta_1, p < q$, that is $x_p + x_q = 1, p \leq m < q$. If $p < m$, then $x_p + x_q > \pm(x_m + x_q)$ (because $x_p \pm x_m, 2x_q > 0$), so $x_p + x_q > |x_m + x_q| \geq 1$, a contradiction. Hence $p = m$, in particular,

$$(5) \quad \pm(\xi_p + \xi_q) \in \Pi \cap \Delta_1, p < q \implies p = m.$$

Since $2x_q \in \mathbb{Z}_{>0}$, $x_p + x_q = 1$ and $x_q \neq x_m$, one has $x_m \leq 0$. Therefore the assumption implies

$$x_m + x_q = 1 \ \& \ q > m \ \& \ x_m \leq 0.$$

Since $\xi_{m-1} - \xi_m \in \Delta^+$ there exists a simple root of the form $\pm\xi_s - \xi_m$ ($s \neq m$).

First, consider the case when $\xi_s - \xi_m \in \Pi$, that is $x_s - x_m = 1$. Since $x_m \leq 0$, one has $x_s + x_m \leq x_s - x_m = 1$ so $x_s + x_m < 0$, that is $-(\xi_s + \xi_m) \in \Delta_+$. Therefore $s > m$ and $x_q + x_s = 2$, because $x_q + x_m = x_s - x_m = 1$. Since $x_{m+n-i} \geq 1/2 + i$ for $i < n$, we conclude that either $x_q = 3/2, x_m = -1/2, x_s = 1/2$, which contradicts to $x_m + x_s \neq 0$, or $q = s = m+n, x_{m+n} = 1, x_m = 0$. Hence $\xi_s - \xi_m \in \Pi$ implies $q = m+n, x_{m+n} = 1, x_m = 0$ and $\xi_{m+n} \pm \xi_m \in \Pi$.

Now consider the case when $-\xi_s - \xi_m \in \Pi$. Then $s > m$ and $x_m + x_q = -x_m - x_s = 1$ so $x_q - x_s = 2$. Since $x_q - x_{q+i} \geq i$ one has $s = q + 1$ or $s = q + 2$. If $s = q + 2$, then we have $2 = x_q - x_{q+2} = (x_q - x_{q+1}) + (x_{q+1} - x_{q+2})$ that is $x_q - x_{q+1} = 1$ so $x_m + x_{q+1} = 0$, a contradiction. Hence $s = q + 1$ that is $-x_m - x_{q+1} = 1$. For $i < m$ one has $0 < x_i + x_m = -x_{q+1} - 1 + x_i$ so $1 < x_i - x_{q+1}$. Therefore for $i < m, t \geq q + 1$ one has $x_i + x_t > x_i - x_t > 1$, so the roots $\pm \xi_i \pm \xi_t$ are not simple. Assume that $m \geq n$ and (S, Π) is an admissible pair. Then for each $m < t \leq m + n$ there exists $i_t < m$ such that one of the roots $\pm \xi_t \pm \xi_{i_t}$ is simple (and lies in S) and $i_t \neq i_p$ for $t \neq p$. By above, $i_{q+1} = m$ and there is no suitable i_t for $t > q + 1$. Hence $q + 1 = m + n$ and $(-\xi_m - \xi_{m+n}) \in S$, that is $\xi_m + \xi_{m+n} \in s_{-\xi_m - \xi_{m+n}} S$.

5.4.4. We conclude that if (S, Π) is an admissible pair for $\mathfrak{g} = D(m, n)$, then one of the following possibilities hold: either all elements of S are of the form $\xi_i - \xi_j$, or all elements of S except one are of the form $\xi_i - \xi_j$, and this exceptional one is β , where

- (1) $\beta = \xi_m + \xi_{m+n}$ and $\xi_{m+n} - \xi_m \in \Pi$;
- (2) $m < n$ and $\beta = \xi_m + \xi_s, m < s < m + n, -\xi_{s+1} - \xi_m \in \Pi$;
- (3) $\beta := -(\xi_m + \xi_s)$, the pair $(s_\beta S, s_\beta \Pi)$ is one of those described in (1)-(2).

5.4.5. Consider the case $D(m, n)$, $n \geq m$. Then $\Delta^\# = \{\xi_i \pm \xi_j; 2\xi_i\}_{i=m+1}^{m+n}$. Let (S, Π) be an admissible pair. If $\xi_m + \xi_s \in S$ for $s < m + n$, then, by above, $-(\xi_m + \xi_{s+1}) \in \Pi$ and the pair (S, Π) is equivalent to the pair $((S \setminus \{\xi_m + \xi_s\}) \cup \{-\xi_m - \xi_{s+1}\}, \Pi)$, which is equivalent to the pair $((S \setminus \{\xi_m + \xi_s\}) \cup \{\xi_m + \xi_{s+1}\}, s_{-\xi_m - \xi_{s+1}} \Pi)$. Thus a pair (S, Π) with $\xi_m + \xi_s \in S$ is equivalent to a pair (S', Π') with $\xi_m + \xi_{m+n} \in S$. If $\xi_m + \xi_{m+n} \in S$, then, by above, $\xi_{m+n} - \xi_m \in \Pi$ and the pair (S, Π) is equivalent to the pair $((S \setminus \{\xi_{m+n} + \xi_m\}) \cup \{\xi_{m+n} - \xi_m\}, \Pi)$. We conclude that any pair (S, Π) is equivalent to a pair (S', Π') , where $S' = \{\xi_i - \xi_j\}_{i=1}^m$.

Consider the case $D(m, n)$, $m > n$. Then $\Delta^\# = \{\xi_i \pm \xi_j\}_{i=1}^m$. By above, any pair (S, Π) is equivalent either to a pair (S', Π') , where $S' = \{\xi_i - \xi_j\}_{i=1}^n$, or to a pair (S', Π') , where $S' = \{\xi_i - \xi_j\}_{i=1}^{n-1} \cup \{\xi_{m+n} + \xi_m\}$ and $\xi_{m+n} - \xi_m \in \Pi'$.

5.4.6. Let (S, Π) be an admissible pair. We conclude that any pair (S, Π) is equivalent to a pair (S', Π') , where either $S' = \{\xi_i - \xi_j\}_{i=1}^{\min(m, n)}$, or, for $D(m, n)$, $m > n$ $S' = \{\xi_i - \xi_j\}_{i=1}^{n-1} \cup \{\xi_{m+n} + \xi_m\}$ and $\xi_{m+n} - \xi_m \in \Pi'$.

5.5. Fix a set of simple roots of \mathfrak{g} and construct f_Π as in 5.4. We mark the points x_i on the real line by a 's and b 's in one of the following ways:

- (M) mark x_i by a (resp., by b) if $1 \leq i \leq m$ (resp., if $m < i \leq m + n$);
- (N) mark x_i by b (resp., by a) if $1 \leq i \leq m$ (resp., if $m < i \leq m + n$).

We use the marking (M) if $\Delta^\#$ lies in the span of $\{\xi_i\}_{i=1}^m$ and the marking (N) if $\Delta^\#$ lies in the span of $\{\xi_i\}_{i=m+1}^{m+n}$. Note that in all cases the number of *as* is not smaller than the number of *bs*.

We fix $\Delta^\#$ and an admissible pair (S, Π) such that $S = \{\xi_i - \xi_j\}_{i=1}^{\min(m,n)}$, or for $D(m, n), m > n, S = \{\xi_i - \xi_j\}_{i=1}^{n-1} \cup \{\xi_{m+n} + \xi_m\}$.

If $\xi_i - \xi_j \in S$ (resp., $\xi_i + \xi_j \in S$) we draw a bow \smile (resp., \frown) between the points x_i and x_j . Observe that the points connected by a bow are neighbours and they are marked by different letters (*a* and *b*). We say that a marked point is a vertex if it is a vertex of a bow. Note that the bows do not have common vertices and that all points marked by *b* are vertices.

From now on we consider the diagrams, which are sequence of *as* and *bs* endowed with the bows (we do not care about the values of x_i). For example, for $\mathfrak{g} = A(4, 1)$ and $\Pi = \{\xi_1 - \xi_2; \xi_2 - \xi_6; \xi_6 - \xi_7; \xi_7 - \xi_3; \xi_3 - \xi_4; \xi_4 - \xi_5\}$ we choose $f = \xi_5^* + 2\xi_4^* + 3\xi_3^* + 4\xi_7^* + 5\xi_6^* + 6\xi_2^* + 7\xi_1^*$ and taking $S = \{\xi_2 - \xi_6; \xi_7 - \xi_3\}$ we obtain the diagram $aaa \smile bb \smile aa$; for $\mathfrak{g} = B(2, 2)$ and $\Pi = \{\xi_1 - \xi_3; \xi_3 - \xi_2; \xi_2 - \xi_4; \xi_4\}$ we choose $f = \xi_4^* + 2\xi_2^* + 3\xi_3^* + 4\xi_1^*$ and taking $S = \{\xi_1 - \xi_3; \xi_2 - \xi_4\}$ we obtain the diagram $a \smile ba \smile b$ for the marking (N), and $b \smile ab \smile a$ for the marking (M). Observe that a diagram containing \frown appear only in the case $D(m, n), m > n$, and such a diagram starts from $a \frown b$, because $\xi_{m+n} + \xi_m \in S$ forces $\xi_{m+n} - \xi_m \in \Pi$, which implies $x_m = 0, x_{m+n} = 1, x_i > 0$ for all $i \neq m$.

5.5.1. Let us see how the odd reflections and the operations of the second type, introduced in Lemma 5.3, change our diagrams.

Recall that $\xi_i - \xi_j \in \Delta_+$ iff $x_i > x_j$. For an odd simple root β one has $s_\beta(\Delta_+) = (\Delta_+ \setminus \{\beta\}) \cup \{-\beta\}$, so the order of x_i s for $s_{\xi_p - \xi_q}(\Delta_+)$ is obtained from the order of x_i s for Δ_+ by the interchange of x_p and x_q (if $\xi_p - \xi_q \in \Pi \cap \Delta_1$). Therefore the odd reflection with respect to $\xi_p - \xi_q \in S$ corresponds to the following operation with the diagram: we interchange the vertices (i.e., the marks *a, b*) of the corresponding bow:

$$\dots a \smile b \dots \mapsto \dots b \smile a \dots; \quad \dots b \smile a \dots \mapsto \dots a \smile b \dots$$

If the diagram has a part $a \smile ba$ (resp. $ab \smile a$), where the last (resp. the first) sign *a* is not a vertex, and x_i, x_j, x_k are the corresponding points, then the quadrupole $(S, \Pi), \gamma := \xi_i - \xi_j, \gamma' := \xi_j - \xi_k$ (resp. $\gamma' := \xi_i - \xi_j, \gamma := \xi_j - \xi_k$) satisfies the assumptions of Lemma 5.3. The operation $(S, \Pi) \mapsto (S', \Pi)$, where $S' := (S \setminus \{\gamma\}) \cup \{\gamma'\}$ corresponds to the following operation with our diagram: $a \smile ba \mapsto ab \smile a$ (resp. $ab \smile a \mapsto a \smile ba$). Hence we can perform the operation of the second type if *a, b, a* are neighbouring points, *b* is connected by \smile with one of *as* and another *a* is not a vertex; in this case, we remove the bow and connect *b* with another *a*:

$$\dots ab \smile a \dots \mapsto \dots a \smile ba \dots; \quad \dots a \smile ba \dots \mapsto \dots ab \smile a \dots$$

Since both our operations are involutions, we can consider the orbit of a given diagram with respect to the action of the group generated by these operations. Let us show that all diagrams without \frown lie in the same orbit. Indeed, using the operations $a \smile b \mapsto b \smile a$ and $ab \smile a \mapsto a \smile ba$ we put b to the first place so our new diagram starts from $b \smile a$. Then we do the same with the rest of the diagram and so on. Finally, we obtain the diagram of the form $b \smile ab \smile a \dots b \smile a \dots a$. By the same argument, all the diagram starting from $b \frown a$ lie in the same orbit. Hence, for $\mathfrak{g} \neq D(m, n), m > n$ all diagrams lie in the same orbit, and for $\mathfrak{g} = D(m, n), m > n$ there are two orbits: the diagrams with \frown and the diagrams without \frown .

5.5.2. We claim that if $(S, \Pi), (S, \Pi')$ are admissible pairs, then $\Pi = \Pi'$. It is clear that if the claim is valid for some S , then it is valid for all S' such that (S, Π) is equivalent to (S', Π') . Thus it is enough to verify the claim for any representative of the orbit. Each S determines the diagram and the diagram determines the order of x_i 's (for instance, the diagram $b \smile ab \smile a \dots b \smile aa \dots a$ gives $x_{m+n} < x_m < x_{m+n-1} < x_{m-1} < \dots < x_{m+1} < x_{m-n+1} < x_{m-n} < \dots < x_1$ for the marking (M) and $x_m < x_{m+n} < x_{m-1} < x_{m+n-1} < \dots < x_1 < x_{n+1} < x_n < \dots < x_{m+1}$ for the marking (N)). For $A(m-1, n-1), B(m, n)$ one has $\{x_i\}_{i=1}^{m+n} = \{1, \dots, m+n\}$, by 5.4.2, so each diagram determines Π .

For $D(m, n), m > n$ the above diagram means that $-\xi_{m+n} + \xi_m, -\xi_m + \xi_{m+n-1}, \dots \in \Pi$. Since $2\xi_{m+n} \in \Delta_+$, we conclude that $2\xi_{m+n} \in \Pi$ that is $x_{m+n} = \frac{1}{2}$ and $\{x_i\}_{i=1}^{m+n} = \{\frac{1}{2}, \dots, m+n - \frac{1}{2}\}$. Hence the above diagram determines Π .

For $D(m, n), m \leq n$ the same reasoning shows that the diagram $a \smile bb \smile a \dots b \smile aa \dots a$ determines Π .

It remains to consider the case $D(m, n), m > n$ and the diagram $a \frown bb \smile ab \smile a \dots b \smile aa \dots a$. In this case, $\xi_m + \xi_{m+n} \in S$ so, by above, $x_m = 0$ and $\{x_i\}_{i=1}^{m+n} = \{0, \dots, m+n-1\}$. Thus the diagram determines Π .

5.5.3. Consider the case $\mathfrak{g} \neq D(m, n), m > n$. Fix an admissible pair (S, Π) . By 5.5, any admissible pair (S', Π') is equivalent to an admissible pair (S, Π'') ; by 5.5.2 one has $\Pi = \Pi''$ so (S', Π') is equivalent to (S, Π) .

Consider the case $\mathfrak{g} = D(m, n), m > n$. Fix admissible pairs $(S, \Pi), (S', \Pi')$, where S consists of the roots of the form $\xi_i - \xi_j$ and S' contains $\xi_m + \xi_{m+n}$. Arguing as above, we conclude that any admissible pair (S'', Π'') is equivalent either to (S, Π) or to (S', Π') .

5.6. **Corollary.** (i) If $(S, \Pi), (S, \Pi')$ are admissible pairs, then $\Pi = \Pi'$.

(ii) For $\mathfrak{g} \neq D(m, n), m > n$ there is one equivalence class of the pairs (S, Π) and the left-hand (resp., right-hand) side of (1) is the same for all admissible pairs (S, Π) .

(iii) For $\mathfrak{g} = D(m, n), m > n$ there are two equivalence classes of the pairs (S, Π) . In the first class S consists of the elements of the form $\pm(\xi_i - \xi_j)$ and in the second class S

contains a unique element of the form $\pm(\xi_i + \xi_j)$. The left-hand side (resp., right-hand) of (1) is the same for all admissible pairs (S, Π) belonging to the same class.

6. STEPS (II), (II')

6.1. Assume that

$$(6) \quad \begin{aligned} &\forall \alpha \in \Pi \quad (\alpha, \alpha) \geq 0; \\ &W^\# \text{ is generated by the set of reflections } \{s_\alpha \mid (\alpha, \alpha) > 0\}. \end{aligned}$$

We start from the following lemma.

6.1.1. **Lemma.** *One has*

- (i) $\rho \geq w\rho$ for all $w \in W^\#$;
- (ii) the stabilizer of ρ in $W^\#$ is generated by the set $\{s_\alpha \mid (\alpha, \alpha) > 0, (\alpha, \rho) = 0\}$.

Proof. Since $(\alpha, \rho) = \frac{1}{2}(\alpha, \alpha) \geq 0$ for all $\alpha \in \Pi$, one has $(\beta, \rho) \geq 0$ for all $\beta \in \Delta_+$. Take $w \in W^\#, w \neq \text{id}$. One has $w = w's_\beta$, where $w' \in W^\#, \beta \in \Delta_+$ are such that the length of w is greater than the length of w' and $w'\beta \in \Delta_+$ (see, for example, [J], A.1). One has

$$\rho - w\rho = \rho - w'\rho + \frac{2(\rho, \beta)}{(\beta, \beta)} \cdot (w'\beta).$$

Now (i) follows by induction on the length of w , since $(\rho, \beta) \geq 0, (\beta, \beta) > 0, w'\beta \in \Delta_+$. For (ii), note that $\rho \geq w'\rho$ by (i), thus $w\rho = \rho$ forces $\rho = w'\rho$ and $(\rho, \beta) = 0$. Hence (ii) also follows by induction on the length of w . \square

6.1.2. Retain notation of (3). For $w \in W^\#$ one has $-\varphi(w), |w|\mu \in Q^+$, by definition, and $w\rho \leq \rho$, by Lemma 6.1.1. Therefore $\varphi(w) - |w|\mu + w\rho \leq \rho$ and the equality means that $\mu = 0, w\rho = \rho$ and $\varphi(w) = 0$, that is $wS \subset \Delta_+$. The above inequality gives $\text{supp}(X) \subset (\rho - Q^+)$. Moreover, by above, the coefficient of e^ρ in the expansion of X is equal to

$$\sum_{w \in W^\#: w\rho = \rho, wS \subset \Delta_+} \text{sgn}(w).$$

6.2. **Root systems.** Recall that we consider all choices of Δ_+ with a fixed $\Delta_{+,0}$. From now on we assume that $m \geq n$ (we consider the types $A(m, n), B(m, n), B(n, m), D(m, n), D(n, m)$) and we embed our root systems in the standard lattices spanned by $\{\varepsilon_i, \delta_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ chosen in such a way that $\Delta^\# = \Delta_0 \cap \text{span}\{\varepsilon_i\}_{i=1}^m$. More precisely, for $A(m, n), m \geq n$ we take

$$\Delta_{+,0} = \{\varepsilon_i - \varepsilon_{i'}, \delta_j - \delta_{j'} \mid 1 \leq i < i' \leq m, 1 \leq j < j' \leq n\}, \quad \Delta_1 = \{\pm(\varepsilon_i - \delta_j)\};$$

for other cases we put

$$U' := \{\varepsilon_i \pm \varepsilon_{i'}, \delta_j \pm \delta_{j'} \mid 1 \leq i < i' \leq m, 1 \leq j < j' \leq n\}$$

and then

$$\begin{aligned} \Delta_{+,0} &= U' \cup \{\varepsilon_i\}_{i=1}^m \cup \{2\delta_j\}_{j=1}^n, & \Delta_1 &= \{\pm\varepsilon_i \pm \delta_j, \pm\delta_j\} & \text{for } B(m,n), m > n \\ & & & & \text{and } B(n,n), \Delta^\# = B(n); \\ \Delta_{+,0} &= U' \cup \{2\varepsilon_i\}_{i=1}^m \cup \{\delta_j\}_{j=1}^n, & \Delta_1 &= \{\pm\varepsilon_i \pm \delta_j, \pm\varepsilon_i\} & \text{for } B(n,m), m > n \\ & & & & \text{and } B(n,n), \Delta^\# = C(n); \\ \Delta_{+,0} &= U' \cup \{2\delta_j\}_{j=1}^n, & \Delta_1 &= \{\pm\varepsilon_i \pm \delta_j\} & \text{for } D(m,n), m > n; \\ \Delta_{+,0} &= U' \cup \{2\varepsilon_i\}_{i=1}^m, & \Delta_1 &= \{\pm\varepsilon_i \pm \delta_j\} & \text{for } D(n,m), m \geq n. \end{aligned}$$

We normalize the form $(-, -)$ by the condition $(\varepsilon_i, \varepsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$. One has $\Delta^\# = \Delta_0 \cap \text{span}\{\varepsilon_i\}_{i=1}^m = \{\alpha \in \Delta_0 \mid (\alpha, \alpha) > 0\}$.

6.3. Choice of (S, Π) . For the case $B(n, n)$ set

$$S := \{\delta_i - \varepsilon_i\}, \quad \Pi := \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \delta_2 - \varepsilon_2, \dots, \delta_n - \varepsilon_n, \varepsilon_n\},$$

(the root ε_n may be even or odd, depending on the choice of $\Delta^\#$). For other cases set

$$S := \{\varepsilon_i - \delta_i\}_{i=1}^n.$$

In order to describe Π introduce

$$P := \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2, \delta_2 - \varepsilon_3, \dots, \varepsilon_n - \delta_n, \delta_n - \varepsilon_{n+1}, \varepsilon_{n+1} - \varepsilon_{n+2}, \dots, \varepsilon_{m-1} - \varepsilon_m\}$$

and set

Δ	$A(m-1, n-1)$	$B(m, n), B(n, m), n > m$	$D(n, m), m > n$	$D(m, n) m > n$
Π	P	$P \cup \{\varepsilon_m\}$	$P \cup \{2\varepsilon_m\}$	$P \cup \{\varepsilon_{m-1} + \varepsilon_m\}$

$$\Pi := \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \varepsilon_2 - \delta_2, \delta_2 - \varepsilon_3, \dots, \varepsilon_n - \delta_n, \varepsilon_n + \delta_n\} \text{ for } D(n, n).$$

Note that the assumptions (6) hold.

6.4. Step (ii): the case $(\varepsilon_m + \delta_n) \notin \Pi$. By 6.1.2, $\text{supp}(X) \subset (\rho - Q^+)$ and the coefficient of e^ρ in the expansion of X is $\sum_{w \in W^\#: w\rho = \rho, wS \subset \Delta_+} \text{sgn}(w)$. Thus it is enough to show that

$$(7) \quad w \in W^\# \text{ s.t. } wS \subset \Delta_+, w\rho = \rho \implies w = \text{id}.$$

6.4.1. Since $(\alpha, \alpha) \geq 0$ for each $\alpha \in \Pi$, one has $(\rho, \beta) = 0$ for $\beta \in \Delta$ iff β is a linear combination of isotropic simple roots.

Consider the case $\mathfrak{g} \neq D(n, n)$ (one has $\rho = 0$ for $D(n, n)$). In this case $(\rho, \beta) = 0$ for $\beta \in \Delta_+^\#$ forces $\beta = \varepsilon_i - \varepsilon_j$ for $i < j \leq \min(m, n+1)$. From Lemma 6.1.1 we conclude

$$(8) \quad \text{for } \mathfrak{g} \neq D(n, n) \quad \text{Stab}_{W^\#} \rho = \begin{cases} S_n, & \text{if } m = n, \\ S_{n+1}, & \text{if } m > n, \end{cases}$$

where $S_k \subset W^\#$ is the symmetric group, consisting of the permutations of $\varepsilon_1, \dots, \varepsilon_k$ (that is for $w \in S_k$ one has $w\varepsilon_i = \varepsilon_{j_i}$ and $j_i = i$ for $i > k$).

6.4.2. Take $w \in \text{Stab}_{W^\#} \rho$ such that $wS \subset \Delta_+$. Let us show that $w = \text{id}$.

Consider the case when $\mathfrak{g} \neq B(n, n), D(n, n)$. Then $S = \{\varepsilon_i - \delta_i\}_{i=1}^n$. Combining (8) and the fact that $\varepsilon_j - \delta_i \in \Delta_+$ iff $j \leq i$, we conclude that $w\varepsilon_i = \varepsilon_{j_i}$ for $j_i \leq i$ if $i \leq n$ and $j_i = i$ for $i > n + 1$. Hence $w = \text{id}$ as required.

For $\mathfrak{g} = D(n, n)$ one has $S = \{\varepsilon_i - \delta_i\}_{i=1}^n$. Since $-\varepsilon_i - \delta_j \notin \Delta_+$ for all i, j , the condition $wS \subset \Delta_+$ forces $w \in S_n$ (see 6.4.1 for notation). Repeating the above argument, we obtain $w = \text{id}$.

For $\mathfrak{g} = B(n, n)$ one has $S = \{\delta_i - \varepsilon_i\}_{i=1}^n$. Combining (8) and the fact that $\delta_i - \varepsilon_j \in \Delta_+$ iff $j \geq i$, we obtain for all $i = 1, \dots, n$ that $w\varepsilon_i = \varepsilon_{j_i}$ for some $j_i \leq i$. Hence $w = \text{id}$ as required. This establishes (7) and (ii) for the case $(\varepsilon_m + \delta_n) \notin \Pi$.

6.5. **Step (ii): the case $(\varepsilon_m + \delta_n) \in \Pi$.** Consider the case $D(m, n)$, $m > n$, $(\varepsilon_m + \delta_n) \in \Pi$. We retain notation of 6.2 and choose a new pair (S, Π) : $S = \{\varepsilon_i - \delta_i\}_{i=1}^{n-1} \cup \{\varepsilon_m + \delta_n\}$ and

$$\Pi := \{\varepsilon_1 - \delta_1, \delta_1 - \varepsilon_2, \dots, \delta_{n-1} - \varepsilon_n\} \cup \{\varepsilon_i - \varepsilon_{i+1}\}_{i=n}^{m-2} \cup \{\varepsilon_{m-1} - \delta_n, \delta_n - \varepsilon_m, \delta_n + \varepsilon_m\}.$$

The assumptions (6) are satisfied. By 6.1.2, the coefficient of e^ρ in the expansion of X is equal to $\sum_{w \in A} \text{sgn}(w)$, where $A := \{w \in \text{Stab}_{W^\#} \rho \mid wS \subset \Delta_+\}$. Let us show that $A = \{\text{id}, s_{\varepsilon_{m-1}-\varepsilon_m}, s_{\varepsilon_{m-1}-\varepsilon_m} s_{\varepsilon_{m-1}+\varepsilon_m}\}$; this implies that the coefficient of e^ρ in the expansion of X is equal to 1.

Take $w \in W^\#$ such that $wS \subset \Delta_+$. Note that $-\varepsilon_j - \delta_i \notin \Delta_+$ for all i, j and for $i < n$ one has $\varepsilon_j - \delta_i \in \Delta_+$ iff $j \leq i$. The assumption $wS \subset \Delta_+$ means that $w\varepsilon_i - \delta_i \in \Delta_+$ for all $i < n$ and $w\varepsilon_m + \delta_n \in \Delta_+$. For $i < n$ this gives $w\varepsilon_i = \varepsilon_{j_i}$ for some $j_i \leq i$. Hence $w\varepsilon_i = \varepsilon_i$ for $i = 1, \dots, n-1$. The remaining condition $w\varepsilon_m + \delta_n \in \Delta_+$ means that $w\varepsilon_m = \varepsilon_{j_m}$ or $w\varepsilon_m = -\varepsilon_m$.

For $m = n + 1$ one has $\rho = 0$, so $w \in A$ iff $w \in W^\#$, $wS \subset \Delta_+$. Thus, by above, $w \in A$ iff $w\varepsilon_i = \varepsilon_i$ for $i < n = m - 1$ and $w\varepsilon_m \in \{\pm\varepsilon_m, \varepsilon_{m-1}\}$.

Take $m > n + 1$. The roots $\beta \in \Delta_+^\#$ such that $(\rho, \beta) = 0$ are of the form $\beta = \varepsilon_i - \varepsilon_j$ for $i < j \leq n$ or $\beta = \varepsilon_{m-1} \pm \varepsilon_m$. From Lemma 6.1.1 we conclude that the subgroup $\text{Stab}_{W^\#} \rho$ is a product of S_n defined in 6.4.1 and the group generated by the reflections $s_{\varepsilon_{m-1} \pm \varepsilon_m}$. By above, $w \in A$ iff $w\varepsilon_i = \varepsilon_i$ for $i < m - 1$ and $w\varepsilon_m \in \{\pm\varepsilon_m, \varepsilon_{m-1}\}$.

Since $W^\#$ is the Weyl group of $D(m)$, i.e. the group of signed permutations of $\{\varepsilon_i\}_{i=1}^m$, changing the even number of signs, the set

$$\{w \in W^\# \mid w\varepsilon_i = \varepsilon_i \text{ for } i < m - 1 \ \& \ w\varepsilon_m \in \{\pm\varepsilon_m, \varepsilon_{m-1}\}\}$$

is $\{\text{id}, s_{\varepsilon_{m-1}-\varepsilon_m}, s_{\varepsilon_{m-1}-\varepsilon_m} s_{\varepsilon_{m-1}+\varepsilon_m}\}$. Hence $A = \{\text{id}, s_{\varepsilon_{m-1}-\varepsilon_m}, s_{\varepsilon_{m-1}-\varepsilon_m} s_{\varepsilon_{m-1}+\varepsilon_m}\}$ as required. This establishes (ii) for the case $(\varepsilon_m + \delta_n) \in \Pi$.

6.6. **Step (ii')**. Consider the case $\mathfrak{gl}(n|n)$. One has $\rho = 0$. Set $\xi := \sum_{\beta \in S} \beta = \sum \varepsilon_i - \sum \delta_i$. Let us verify that

$$(9) \quad \mathbb{Q}\xi \cap \text{supp}(Re^\rho - X) = \emptyset.$$

Indeed, it is easy to see that ξ has a unique presentation as a positive linear combination of positive roots:

$$(10) \quad \xi = \sum_{\alpha \in \Delta_+} m_\alpha \alpha, \quad m_\alpha \geq 0 \implies m_\beta = 1 \text{ for } \beta \in S, \quad m_\alpha = 0 \text{ for } \alpha \notin S.$$

This implies that for $s \notin \mathbb{Z}_{\geq 0}$ the coefficients of $e^{-s\xi}$ in $Re^\rho = R$ and in X are equal to zero, and that the coefficient of $e^{-s\xi}$ in R is equal to $(-1)^{sn}$ for $s \in \mathbb{Z}_{\geq 0}$. It remains to show that the coefficient of $e^{-s\xi}$ in X is $(-1)^{sn} = (-1)^{\text{ht}(s\xi)}$. It is enough to verify that $|w|\mu - \varphi(w) = s\xi$ for $w \in W^\#$ implies $w = \text{id}$. Assume that $|w|\mu - \varphi(w) = s\xi$. By definition, $|w|\mu, -\varphi(w) \in Q^+$. From (10) we conclude that $|w|\mu, -\varphi(w) \in \mathbb{Z}_{\geq 0}S$. Recall that $\varphi(w) = \sum_{\beta \in S: w\beta \in \Delta_-} w\beta$. By (10), $-\varphi(w) \in \mathbb{Z}_{\geq 0}S$ implies $(-w\beta) \in S$ for each $\beta \in S$ such that $w\beta \in \Delta_-$. However, $(-w\beta) \notin S$ for any $w \in W^\#, \beta \in S$, because $-w(\varepsilon_i - \delta_i) = \delta_i - w\varepsilon_i \notin S$. Thus $wS \subset \Delta_+$ that is $w\varepsilon_i = \varepsilon_{i_j}$ for $i_j \leq i$. Hence $w = \text{id}$. This establishes (9).

7. W -INVARIANCE: STEP (III)

In this section we prove that X defined by the formula (2) is W -skew-invariant for a certain admissible pair (S, Π) ; for the case $D(m, n)$ $m > n$ we prove this for two admissible pairs (S, Π) and (S', Π') , which are representatives of the equivalence classes defined in Sect. 5.

Recall that X is $W^\#$ -skew-invariant and that $\Delta = \Delta^\# \amalg \Delta_2$, that is $W = W^\# \times W_2$.

7.1. **Operator F** . Recall the operator $F : \mathcal{R} \rightarrow \mathcal{R}$ given by the formula $F(Y) := \sum_{w \in W^\#} \text{sgn}(w)wY$. Clearly, $w(F(Y)) = F(wY)$ for $w \in W_2$ and $F(wY) = w(F(Y)) = \text{sgn}(w)F(Y)$ for $w \in W^\#$. In particular, $F(Y) = 0$ if $wY = Y$ for some $w \in W^\#$ with $\text{sgn}(w) = -1$. One has

$$X = F(Y), \quad \text{where } Y := \frac{e^\rho}{\prod_{\beta \in S} (1 + e^{-\beta})}.$$

Suppose that $B \in \mathcal{R}$ is such that $w_2w_1B = B$ for some $w_1 \in W^\#, w_2 \in W_2$, where $\text{sgn}(w_1w_2) = 1$. Then

$$w_2^{-1}F(B) = F(w_2^{-1}B) = F(w_1B) = \text{sgn}(w_1)F(B),$$

that is $w_2F(B) = \text{sgn}(w_2)F(B)$.

As a result, in order to verify W -skew-invariance of $F(B)$ for an arbitrary $B \in \mathcal{R}$, it is enough to show that for each generator y of W_2 there exists $z \in W^\#$ such that $\text{sgn}(yz) = 1$ and $yzB = B$ (we consider y running through a set of generators of W_2).

7.2. Root systems. We retain notation of 6.2 for $\Delta_{0,+}$ and Δ_1 , but, except for $A(m-1, n-1)$ we do not choose the same pairs (S, Π) as in 6.3.

For $A(m-1, n-1)$ we choose S, Π as in 6.3 ($S := \{\varepsilon_i - \delta_i\}$). For other cases we choose $S := \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=0}^{n-1}$ and

$$\begin{aligned} \Pi &:= P \cup \{\varepsilon_m\} && \text{for } B(m, n), B(n, m) \\ \Pi &:= P \cup \{\varepsilon_m + \delta_n\} && \text{for } D(m, n) \ m > n, \\ \Pi &:= P \cup \{2\varepsilon_m\}, && \text{for } D(n, m) \ m \geq n, \end{aligned}$$

where $P := \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{m-n-1} - \varepsilon_{m-n}, \varepsilon_{m-n} - \delta_1, \delta_1 - \varepsilon_{m-n+1}, \varepsilon_{m-n+1} - \delta_2, \dots, \delta_n - \varepsilon_m\}$ for $m > n$, and $P := \{\delta_1 - \varepsilon_1, \varepsilon_1 - \delta_2, \dots, \delta_n - \varepsilon_n\}$ for $m = n$. For $D(m, n), m > n$ case we consider two admissible pairs: (S, Π) and (S', Π) , where $S' := \{\delta_{n-i} - \varepsilon_{m-i}\}_{i=1}^{n-1} \cup \{\delta_n + \varepsilon_m\}$.

Recall that $(\varepsilon_i, \varepsilon_j) = -(\delta_i, \delta_j) = \delta_{ij}$ and notice that $(\alpha, \alpha) \geq 0$ for all $\alpha \in \Pi$.

7.3. S_n -invariance. Let $S_n \subset W_2$ be the group of permutations of $\delta_1, \dots, \delta_n$. In all cases S is of the form $S = \pm\{\delta_i - \varepsilon_{r+i}\}$ for $r = 0$ or $r = m - n$. For $i = 1, \dots, n-1$ one has $(\rho, \delta_i - \delta_{i+1}) = (\rho, \varepsilon_{r+i} - \varepsilon_{r+i+1}) = 0$. Therefore the reflections $s_{\varepsilon_{r+i} - \varepsilon_{r+i+1}}, s_{\delta_i - \delta_{i+1}}$ stabilize ρ . Since $s_{\varepsilon_{r+i} - \varepsilon_{r+i+1}} s_{\delta_i - \delta_{i+1}}$ stabilizes the elements of S , one has $s_{\varepsilon_{r+i} - \varepsilon_{r+i+1}} s_{\delta_i - \delta_{i+1}} Y = Y$ for $i = 1, \dots, n-1$. Using 7.1 we conclude that X is S_n -skew-invariant. In particular, this establishes W -invariance of X for $A(m, n)$ -case.

For $D(m, n), m > n$ case consider the admissible pair (S', Π) . Arguing as above, one sees that $s_{\delta_i - \delta_{i+1}}(X) = -X$ for $i = 1, \dots, n-2$. Since $\delta_{n-1} - \varepsilon_{m-1}, \delta_n + \varepsilon_m \in S'$ the product $w := s_{\varepsilon_{m-1} + \varepsilon_m} s_{\delta_{n-1} - \delta_n}$ stabilizes the elements of S' . Since $(\rho, \delta_i) = (\rho, \varepsilon_{m-n+i}) = 0$ for $i = 1, \dots, n$, w stabilizes ρ . Thus $wY = Y$ and so $s_{\delta_{n-1} - \delta_n} X = -X$, by 7.1. Hence X is S_n -skew-invariant.

7.4. $B(m, n), B(n, m), D(m, n), m > n$ cases. In this case W_2 is the group of signed permutations of $\{\delta_i\}_{i=1}^n$ so it is generated by s_{δ_n} and the elements of S_n . In the light of 7.1 and 7.3, it is enough to verify that $s_{\delta_n} s_{\varepsilon_m} Y = Y$. Set

$$\beta := \delta_n - \varepsilon_m \in S.$$

Consider the cases $B(m, n), B(n, m)$. In this case $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}_{i=1}^m$. Since $\beta, \varepsilon_m \in \Pi$, one has $(\rho, \varepsilon_m) = (\rho, \delta_n) = \frac{1}{2}$ so $s_{\delta_n} s_{\varepsilon_m} \rho = \rho + \beta$. Clearly, $s_{\delta_n} s_{\varepsilon_m}$ stabilizes the elements of $S \setminus \{\beta\}$, and $s_{\delta_n} s_{\varepsilon_m} \beta = -\beta$. As a result, $s_{\delta_n} s_{\varepsilon_m} Y = Y$ as required.

Consider the case $D(m, n), m > n$. In this case $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}$, which change even number of signs. Notice that $s_{\varepsilon_{m-n}} s_{\varepsilon_m} \in W^\#$ and $\text{sgn}(s_{\varepsilon_{m-n}} s_{\varepsilon_m}) = 1$. Set $w := s_{\varepsilon_{m-n}} s_{\varepsilon_m} s_{\delta_n}$. One has $(\rho, \delta_n) = (\rho, \varepsilon_m) = (\rho, \varepsilon_{m-n}) = 0$ so $w\rho = \rho$. Since w stabilizes the elements of $S \setminus \{\beta\}$, one has $wY = e^{-\beta} Y$. We obtain

$$s_{\delta_n} F(Y) = s_{\delta_n} F(s_{\varepsilon_{m-n}} s_{\varepsilon_m} Y) = F(s_{\delta_n} s_{\varepsilon_{m-n}} s_{\varepsilon_m} Y) = F(wY) = F(e^{-\beta} Y)$$

and so

$$(1 + s_{\delta_n})F(Y) = F((1 + e^{-\beta})Y) = F\left(\frac{e^\rho}{\prod_{\beta' \in S \setminus \{\beta\}} (1 + e^{-\beta'})}\right) = 0,$$

where the last equality follows from the fact that the reflection $s_{\varepsilon_{m-n}-\varepsilon_m}$ stabilizes ρ and $S \setminus \{\beta\}$. Hence $(1 + s_{\delta_n})F(Y) = 0$ as required.

For the admissible pair (S', Π) we obtain the required formula $(1 + s_{\delta_n})F(Y) = 0$ along the same lines substituting β by $\delta_n + \varepsilon_m$.

7.5. Case $D(n, m), m \geq n$. In this case $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}$, and W_2 is the group of signed permutations of $\{\delta_i\}_{i=1}^n$, which change even number of signs. Note that the reflection s_{δ_i} does not lie in W_2 , but $s_{\delta_i}\Delta = \Delta$, so s_{δ_i} acts on \mathcal{R} and this action commutes with the operator F . Since W_2 is generated by $s_{\delta_1}s_{\delta_2}$ and the elements of S_n , it is enough to verify that $s_{\delta_1}s_{\delta_2}F(Y) = F(Y)$. Set $\beta_i := \delta_i - \varepsilon_{m-n+i} \in S$. One has $(\rho, \varepsilon_{m-n+i}) = (\rho, \delta_i) = 1$ so $s_{\delta_i}s_{\varepsilon_{m+n-i}}\rho = \rho + 2\beta_i$ that is $s_{\delta_i}s_{\varepsilon_{m+n-i}}Y = e^{\beta_i}Y$. Therefore

$$(1 - s_{\delta_i})F(Y) = F(Y) + F(s_{\delta_i}s_{\varepsilon_{m+n-i}}Y) = F((1 + e^{\beta_i})Y) = F\left(\frac{e^{\rho+\beta_i}}{\prod_{\beta \in S \setminus \{\beta_i\}} (1 + e^{-\beta})}\right) = 0,$$

because $s_{\varepsilon_{m+n-i}} \in W^\#$ stabilizes $\rho + \beta_i$ and the elements of $S \setminus \{\beta_i\}$. Thus $s_{\delta_i}F(Y) = F(Y)$ so $s_{\delta_i}s_{\delta_j}F(Y) = F(Y)$ for any i, j . Hence $X = F(Y)$ is W_2 -skew-invariant.

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