DENOMINATOR IDENTITY FOR AFFINE LIE SUPERALGEBRAS
WITH ZERO DUAL COXETER NUMBER

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Abstract. We prove a denominator identity for non-twisted affine Lie superalgebras with zero dual Coxeter number.

0. Introduction

0.1. Let \( g \) be a complex finite-dimensional contragredient Lie superalgebra. These algebras were classified by V. Kac in [K1] and the list (excluding Lie algebras) consists of four series: \( A(m|n), B(m|n), C(m), D(m|n) \) and the exceptional algebras \( D(2,1,a), F(4), G(3) \). The finite-dimensional contragredient Lie superalgebras with zero Killing form (or, equivalently, with dual Coxeter number equal to zero) are \( A(n|n), D(n|n + 1) \) and \( D(2,1,a) \).

Denote by \( \Delta^+ \) (resp., \( \Delta^- \)) the set of positive even (resp., odd) roots of \( g \). The Weyl denominator \( R \) and the affine Weyl denominator \( \hat{R} \) are given by the following formulas

\[
R = \frac{R_0}{R_1}, \quad \hat{R} = \frac{\hat{R}_0}{\hat{R}_1},
\]

where

\[
R_0 := \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha}), \quad \hat{R}_0 := R_0 \cdot \prod_{k=1}^{\infty} (1 - q^k)^{\text{rank} g} \prod_{\alpha \in \Delta^0} (1 - q^k e^{-\alpha}),
\]

\[
R_1 := \prod_{\alpha \in \Delta^-} (1 + e^{-\alpha}), \quad \hat{R}_1 := R_1 \cdot \prod_{k=1}^{\infty} \prod_{\alpha \in \Delta^1} (1 + q^k e^{-\alpha}).
\]

Let \( \hat{g} \) be the non-twisted affinization of \( g \), \( \hat{h} \) be the Cartan subalgebra of \( \hat{g} \) and \( \hat{\Delta} \) be the set of positive roots of \( \hat{g} \). The affine Weyl denominator is the Weyl denominator of \( \hat{g} \). Let \( \hat{\rho} \in \hat{h} \) be such that \( 2(\hat{\rho}, \alpha) = (\alpha, \alpha) \) for each simple root \( \alpha \in \hat{\Delta} \).

If \( g \) has a non-zero Killing form, the affine denominator identity, stated in [KW] and proven in [KW],[G2], takes the form

\[
\hat{R} e^\hat{\rho} = \sum_{w \in T'} w(Re^\rho),
\]

where \( T' \) is the affine translation group corresponding to the “largest” root subsystem of \( \Delta_0 \) (see Section 1.2.1 below). The affine denominator identity for strange Lie superalgebras \( Q(n) \), which are not contragredient, was stated in [KW] and proven in [Z].

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Suppose \( \mathfrak{g} \) has zero dual Coxeter number, that is \( \mathfrak{g} \) is \( \mathfrak{A}(n|n), \mathfrak{D}(n|n + 1) \) or \( \mathfrak{D}(2, 1, a) \). In this case, \( \hat{\rho} = \frac{1}{2}(\sum_{\alpha \in \Delta_{+0}} \alpha - \sum_{\alpha \in \Delta_{+1}} \alpha) \). In this paper we will prove the following formulas

\[
\begin{align*}
\hat{Re}^{\beta} \cdot f(q, e^{\text{str}}) &= \sum_{w \in T'} w(Re^{\hat{\beta}}) \quad \text{for } \mathfrak{A}(n|n), \\
\hat{Re}^{\beta} \cdot f(q) &= \sum_{w \in T'} w(Re^{\hat{\beta}}) \quad \text{for } \mathfrak{D}(n + 1|n), \mathfrak{D}(2, 1, a),
\end{align*}
\]

where \( T' \) is the affine translation group corresponding to the “smallest” root subsystem of \( \Delta_0 \) (see (1.2) below) and \( f(q, e^{\text{str}}), f(q) \) are given by the formulas (3) below. The affine denominator identity for \( \mathfrak{gl}(2|2) \) was stated by V. Kac and M. Wakimoto in [KW] and proven in [G3] (the proof in [G3] is different from the proof presented below).

In order to write down \( f(q) \), we introduce the following infinite products after [DK]: for a parameter \( q \) and a formal variable \( x \) we set

\[
(1 + x)_{q}^{\infty} := \prod_{k=0}^{\infty}(1 + q^{k}x), \quad \text{and} \quad (1 - x)_{q}^{\infty} := \prod_{k=0}^{\infty}(1 - q^{k}x).
\]

These infinite products converge for any \( x \in \mathbb{C} \) if the parameter \( q \) is a real number \( 0 < q < 1 \). In particular, they are well defined for \( 0 < q = 1 \) and \( (1 \pm q)^{\infty} := \prod_{n=1}^{\infty}(1 \pm q^{n}) \).

For \( \mathfrak{A}(n|n) = \mathfrak{gl}(n|n) \) denote by \( \text{str} \) the restriction of the supertrace to the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \) (thus \( \text{str} \in \mathfrak{h}^{*} \)). One has

\[
\begin{align*}
f(q, e^{\text{str}}) &= \frac{(1-q(-1)^{n}e^{\text{str}})^{-}\cdot(1-q(-1)^{n}e^{-\text{str}})^{-2}}{(1-q)^{-2}} \quad \text{for } \mathfrak{gl}(n|n), \\
f(q) &= \left((1 - q)^{\infty}\right)^{-1} \quad \text{for } \mathfrak{D}(n + 1|n).
\end{align*}
\]

As it was pointed by P. Etingof, the terms \( f(q, e^{\text{str}}), f(q) \) can be interpreted using “degenerate” cases \( n = 1 \); for example, for \( \mathfrak{gl}(1|1) \) we obtain the formula

\[
\hat{Re}^{\beta} = \frac{((1 - q)^{\infty})^{2}}{(1 + q e^{\text{str}})^{\infty} \cdot (1 + q e^{-\text{str}})^{\infty}} Re^{\hat{\beta}},
\]

which is trivial since \( \mathfrak{gl}(1|1) \) has the only positive root \( \beta = \text{str} \), which is odd.

Since \( \mathfrak{sl}(n|n) = \{ a \in \mathfrak{gl}(n|n) | \text{str}(a) = 0 \} \) and \( \text{rank } \mathfrak{sl}(n|n) = 2n - 1 = \text{rank } \mathfrak{gl}(n|n) - 1 \), one has

\[
f(q) = \begin{cases} 
(1 - q)^{\infty} & \text{for } \mathfrak{sl}(2n|2n), \\
\frac{(1 - q)^{\infty}}{(1 - q)^{2}} & \text{for } \mathfrak{sl}(2n + 1|2n + 1).
\end{cases}
\]

The root datum of \( \mathfrak{D}(2, 1, a) \) is the same as the root datum of \( \mathfrak{D}(2|1) \) so the affine denominator identity for \( \mathfrak{D}(2, 1, a) \) is the same as the affine denominator identity for \( \mathfrak{D}(2|1) \).
As it is shown in [KW], the evaluation of the affine denominator identity for \( \mathfrak{gl}(2|2) \) (i.e., \( (2|2) \) for \( A(1|1) \)) gives the following Jacobi identity \([J]\):

\[
\square(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{(j+1)k} j^3 q^{jk},
\]

where \( \square(q) = \sum_{j \in \mathbb{Z}} q^{j^2} \) and thus the coefficient of \( q^m \) in the power series expansion of \( \square(q)^8 \) is the number of representation of a given integer as a sum of 8 squares (taking into the account the order of summands).

0.2. In order to define \( T' \) for \( A(n|n), D(n + 1|n) \) we present the set of even roots in the form \( \Delta_0 = \Delta' \coprod \Delta'' \), where

\[
\Delta' \cong \Delta'' = A_{n-1} \quad \text{for} \quad A(n-1|n-1) = \mathfrak{gl}(n|n),
\]

\[
\Delta' = C_n, \quad \Delta'' = D_{n+1} \quad \text{for} \quad D(n+1|n) + 1.
\]

Let \( W' \) be the Weyl group of \( \Delta' \) and \( \hat{W}' \) be the corresponding affine Weyl group. Then \( \hat{W}' = W' \ltimes T' \), where \( T' \) is a translation group, see [K2], Chapter VI. Notice that for \( D(n+1|n) \) the rank of root system \( \Delta' \) is smaller than the rank of \( \Delta'' \); by contrast, for Lie superalgebras with non-zero Killing form, the lattice \( T' \) in \([1]\) corresponds to the root system \( \Delta' \), whose rank is not smaller than the rank of \( \Delta'' \) (one has \( \Delta_0 = \Delta' \coprod \Delta'' \) as before). It is not possible to change \( T' \) to \( T'' \) in Identity \([1]\) and in Identity \([2]\) for \( D(n+1|n) \), since the sum \( \sum_{w \in T''} w(Re^{\rho}) \) is not well defined if \( \Delta' \not\cong \Delta'' \) (see Remark 2.1.4).

We prove Identity \([2]\) and outline a similar proof for Identity \([1]\). The key point is Proposition 2.3.2 where it is shown that for any complex finite-dimensional contragredient Lie superalgebra, the expansion of \( Y := R^{-1} e^{-\hat{\rho}} \sum_{w \in T'} w(Re^{\hat{\rho}}) \) contains only \( \hat{W} \)-invariant elements. This implies that \( Y = f(q) \) for \( \mathfrak{g} \neq \mathfrak{gl}(n|n) \) and \( Y = f(q, e^{-\text{str}}) \) for \( \mathfrak{gl}(n|n) \). We determine \( f(q) \) for \( D(n+1|n) \) and \( f(q, e^{\text{str}}) \) for \( \mathfrak{gl}(n|n) \) using suitable evaluations. For other finite-dimensional contragredient simple Lie superalgebras the equality \( f(q) = 1 \) can be obtained in two steps: first, using the Casimir operator and the fact that the dual Coxeter number is non-zero, we show that \( f(q) \) is scalar; then one deduces that this scalar is equal to 1 from the denominator identity for \( \mathfrak{g} \) (this is done in [G2]).

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1. Preliminary

One readily sees (for instance, [G2, 1.5]) that \( Re^{\hat{\rho}} \) and \( \hat{R}e^{\hat{\rho}} \) do not depend on the choice of set of positive roots \( \Delta_+ \) so it is enough to establish the identity for one choice of \( \Delta_+ \). Similarly, it is enough to establish the identity for one choice of \( A_{n-1} \) for \( \mathfrak{gl}(n|n) \). In Section 1.1 we describe our choice of the set of of positive roots for \( \mathfrak{gl}(n|n), D(n + 1|n) \).
In Section 1.2 we introduce notation for affine Lie superalgebra \( \hat{g} \). In Section 1.3 we introduce the algebra \( \mathcal{R} \) of formal power series in which we expand \( R \) and \( \hat{R} \).

1.1. Root systems. Let \( g \) be \( gl(n|n) \) or \( D(n|n+1) \) and let \( h \) be its Cartan subalgebra. We fix the following sets of simple roots:

\[
\begin{align*}
\Pi &= \{\varepsilon_i - \delta_j\}_{1 \leq i < j \leq n} \text{ for } gl(n|n), \\
\Pi &= \{\varepsilon_i - \delta_j\}_{1 \leq i < j \leq n} \text{ for } D(n+1|n).
\end{align*}
\]

We fix a non-degenerate symmetric invariant bilinear form on \( g \) and denote by \((-,-)\) the induced non-degenerate symmetric bilinear form on \( h^* \); we normalize the form in such a way that \(- (\varepsilon_i, \varepsilon_j) = (\delta_i, \delta_j) = \delta_{ij}\); notice that \( \{\varepsilon_i, \delta_i\}_{1 \leq i \leq n} \) (resp., \( \{\varepsilon_j, \delta_i\}_{1 \leq i \leq n, 1 \leq j \leq n+1} \)) is an orthogonal basis of \( h^* \) for \( gl(n|n) \) (resp., for \( D(n+1|n) \)).

For this choice one has

\[
\begin{align*}
\Delta_{0+} &= \{\varepsilon_i - \varepsilon_j\}_{1 \leq i < j \leq n} \bigcup \{\delta_i - \delta_j\}_{1 \leq i < j \leq n} \text{ for } gl(n|n), \\
\Delta_1^+ &= \{\varepsilon_i - \delta_j\}_{1 \leq i \leq j \leq n} \bigcup \{\delta_i - \varepsilon_j\}_{1 \leq i \leq j \leq n} \text{ for } gl(n|n), \\
\Delta_0^+ &= \{\varepsilon_i \pm \varepsilon_j\}_{1 \leq i < j \leq n+1} \bigcup \{\delta_i \pm \delta_j\}_{1 \leq i \leq j \leq n} \bigcup \{2\delta_i\}_{1 \leq i \leq n} \text{ for } D(n+1|n), \\
\Delta_1^+ &= \{\varepsilon_i - \delta_j\}_{1 \leq i \leq j \leq n} \bigcup \{\delta_i \pm \delta_j\}_{1 \leq i \leq j \leq n+1} \bigcup \{\delta_i + \varepsilon_j\}_{1 \leq i \leq j \leq n+1} \text{ for } D(n+1|n).
\end{align*}
\]

For \( D(n+1|n) \) one has \( \rho = 0 \) for \( D(n+1|n) \). For \( gl(n|n) \) one has \( \text{str} = \sum_{i=1}^{n}(\varepsilon_i - \delta_i) \) and \( \rho = -\frac{1}{2}\text{str} \).

Recall that \( sl(n|n) = \{a \in gl(n|n) | \text{str}(a) = 0\} \) and so \( h^* \) for \( sl(n|n) \) is the quotient of \( h^* \) for \( gl(n|n) \) by \( \mathbb{C}\text{str} \).

By above, \( \Delta_0^* \) is the union of two irreducible root systems, and we write \( \Delta_0^* = \Delta^* \bigcup \Delta' \), where \( \Delta^* \) lies in the span of \( \varepsilon_i \)s and \( \Delta' \) lies in the span of \( \delta_i \)s (this notation is compatible with notations in Section 1.2).

1.2. Non-twisted affinization. Let \( \hat{g} = n_- \oplus h \oplus n_+ \) be any complex finite-dimensional contragredient Lie superalgebra with a fixed triangular decomposition, and let \( \Delta_+ \) be its set of positive roots. Let \( \hat{g} \) be the affinization of \( g \) and let \( \hat{h} \) be its Cartan subalgebra, see [K2], Chapter VI. Recall that \( g = [g, g] \oplus \mathbb{C}D \) for some \( D \in \hat{h} \). Let \( \hat{\Delta} = \hat{\Delta}_0 \bigcup \hat{\Delta}_1 \) be the set of roots of \( \hat{g} \). We set

\[
\hat{\Delta}^+ = \Delta^+ \bigcup (\cup_{k=1}^{\infty} \{\alpha + k\delta | \alpha \in \Delta\}) \bigcup (\cup_{k=1}^{\infty} \{k\delta\}),
\]

where \( \delta \) is the minimal imaginary root. Let \( W^* \) (resp., \( \hat{W}^* \)) be the Weyl group of \( \Delta_0^* \) (resp., \( \hat{\Delta}_0^* \)). One has \( (h^*)^W = \mathbb{C}\delta \) for \( g \neq gl(n|n) \) and \( (\hat{h}^*)^{\hat{W}} = \mathbb{C}\delta \oplus \mathbb{C}\text{str} \) for \( g = gl(n|n) \).

We extend the non-degenerate symmetric invariant bilinear form from \( g \) to \( \hat{g} \) and denote by \((-,-)\) the induced non-degenerate symmetric bilinear form on \( h^* \) (the above-mentioned form on \( h^* \) is induced by this form on \( h^* \)). For \( A \subset h^* \) we set \( A^+ = \{\mu \in h^* | \forall \nu \in A (\mu, \nu) = 0\} \).
1.2.1. In Section 1.2 we introduced the root systems $\Delta', \Delta''$ for $\mathfrak{g} = \mathfrak{gl}(n|n), D(n + 1|n)$. For $\mathfrak{g} \neq \mathfrak{gl}(n|n), D(n + 1|n), D(2,1,a)$ the Killing form $\kappa$ is non-zero; in this case, we introduce $\Delta', \Delta''$ by the formulas: $\Delta' := \{\alpha|\kappa(\alpha, \alpha) > 0\}$, $\Delta'' := \{\alpha|\kappa(\alpha, \alpha) < 0\}$. One has $\Delta_0 = \Delta' \coprod \Delta''$ and $\Delta'' = \emptyset$ if $\Delta_0$ is irreducible. Let $W'$ (resp., $W''$) be the Weyl group of $\Delta'$ (resp., $\Delta''$). One has $W = W' \times W''$.

1.2.2. Now that we have introduced the decomposition $\Delta_0 = \Delta' \coprod \Delta''$ for any complex finite-dimensional contragredient Lie superalgebra, we denote by $\hat{\Delta}$ the affine root system $\hat{\Delta}$. Recall that $\hat{W}' = W' \ltimes T'$, where $T'$ is a translation group (see [K2], Chapter VI).

1.2.3. For $N \subset \hat{\mathfrak{h}}^*$ we use the notation $\mathbb{Z}N$ for the set $\sum_{\mu \in N} \mathbb{Z}\mu$. Set

$$Q^+ := \sum_{\mu \in \Delta_+} \mathbb{Z}_{\geq 0}\mu, \quad Q := \mathbb{Z}\Delta, \quad \hat{Q}^+ := \sum_{\mu \in \Delta_+} \mathbb{Z}_{\geq 0}\mu, \quad \hat{Q} := \mathbb{Z}\Delta_+.$$  

We introduce the standard partial order on $\hat{\mathfrak{h}}^*$: $\mu \leq \nu$ if $(\nu - \mu) \in \hat{Q}^+$.

1.3. Algebra $\mathcal{R}$. We are going to use notation of [G2], 1.4, which we recall below. Retain notation of Section 1.2.

1.3.1. Call a $\hat{Q}^+$-cone a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

For a formal sum of the form $Y := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$, $b_\nu \in \mathbb{Q}$ define the support of $Y$ by $\text{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^*| b_\nu \neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu \in \hat{Q}^+} b_\nu e^{\lambda - \nu}$, where $\lambda \in \hat{\mathfrak{h}}^*$, $b_\nu \in \mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y = \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of $\hat{Q}^+$-cones.

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

1.3.2. Action of the Weyl group. For $w \in \hat{W}$ set $w(\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu) := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. By above, $wY \in \mathcal{R}$ iff $w(\text{supp}(Y))$ is a subset of a finite union of $\hat{Q}^+$-cones. For each subgroup $\hat{W}$ of $\hat{W}$ we set $\mathcal{R}_{\hat{W}} := \{Y \in \mathcal{R}| wY \in \mathcal{R} \text{ for each } w \in \hat{W}\}$; notice that $\mathcal{R}_{\hat{W}}$ is a subalgebra of $\mathcal{R}$.

1.3.3. Infinite products. An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q}$, $r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \Delta_+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)},$$
is again an infinite product of the above form, since the set \( w\hat{\Delta}_+ \setminus \hat{\Delta}_+ \) is finite (see for example [G2], Lemma 1.2.8). Hence \( \mathcal{Y} \) is a \( \hat{W} \)-invariant multiplicative subset of \( \hat{R} \).

The elements of \( \mathcal{Y} \) are invertible in \( \mathcal{R} \): using the geometric series we can expand \( Y^{-1} \) (for example, \( (1 - e^\alpha)^{-1} = -e^{-\alpha}(1 - e^{-\alpha})^{-1} = -\sum_{i=1}^{\infty} e^{-i\alpha} \)).

1.3.4. The subalgebra \( \mathcal{R}' \). Denote by \( \mathcal{R}' \) the localization of \( \hat{R} \) by \( \mathcal{Y} \). By above, \( \mathcal{R}' \) is a subalgebra of \( \mathcal{R} \). Observe that \( \mathcal{R}' \not\subset \hat{R} \): for example, \( (1 - e^{-\alpha})^{-1} = \sum_{j=0}^{\infty} e^{-j\alpha} \not\in \hat{R} \). We extend the action of \( \hat{W} \) from \( \hat{R} \) to \( \mathcal{R}' \) by setting \( w(Y^{-1}Y') := (wY)^{-1}(wY') \) for \( Y \in \mathcal{Y} \), \( Y' \in \mathcal{R}' \).

Notice that an infinite product of the form \( Y = \prod_{\nu \in X}(1 + a_\nu e^{-\nu})^{r(\nu)} \), where \( a_\nu \), \( X \) are as above and \( r(\nu) \in \mathbb{Z} \), lies in \( \mathcal{R}' \) and \( wY = \prod_{\nu \in X}(1 + a_\nu e^{-w\nu})^{r(\nu)} \). The support \( \text{supp}(Y) \) has a unique maximal element (with respect to the standard partial order) and this element is given by the formula

\[
\text{max supp}(Y) = - \sum_{\nu \in X \setminus \Delta_+: a_\nu \neq 0} r_\nu \nu.
\]

1.3.5. Let \( \hat{W} \) be a subgroup of \( \hat{W} \). For \( Y \in \mathcal{R}' \) we say that \( Y \) is \( \hat{W} \)-invariant (resp., \( \hat{W} \)-anti-invariant) if \( wY = Y \) (resp., \( wY = \text{sgn}(w)Y \)) for each \( w \in \hat{W} \).

Let \( Y = \sum a_\mu e^\mu \in \mathcal{R}_\hat{W} \) be \( \hat{W} \)-anti-invariant. Then \( a_{w\mu} = (-1)^{\text{sgn}(w)}a_\mu \) for each \( \mu \) and \( w \in \hat{W} \). In particular, \( \mathcal{W} \text{ supp}(Y) = \text{supp}(Y) \), and, moreover, for each \( \mu \in \text{supp}(Y) \) one has \( \text{Stab}_{\hat{W}} \mu \subset \{ w \in \hat{W} | \text{sgn}(w) = 1 \} \). The condition \( Y \in \mathcal{R}_\hat{W} \) is essential: for example, for \( \hat{W} = \{ \text{id}, s_\alpha \} \), the expressions \( Y := e^\alpha - e^{-\alpha} \), \( Y^{-1} = e^{-\alpha}(1 - e^{-2\alpha})^{-1} \) are \( \hat{W} \)-anti-invariant, \( \text{supp}(Y) = \{ \pm \alpha \} \) is \( s_\alpha \)-invariant, but \( \text{supp}(Y^{-1}) = \{ -\alpha, -3\alpha, \ldots \} \) is not \( s_\alpha \)-invariant.

For \( Y \in \mathcal{R}_\hat{W} \) such that each \( \hat{W} \)-orbit in \( \hat{h}^* \) has a finite intersection with \( \text{supp}(Y) \), introduce the sum

\[
\mathcal{F}_\hat{W}(Y) := \sum_{w \in \hat{W}} \text{sgn}(w)wY.
\]

This sum is well defined, but does not always belong to \( \mathcal{R} \). For \( Y = \sum a_\mu e^\mu \) one has \( \mathcal{F}_\hat{W}(Y) = \sum b_\mu e^\mu \), where \( b_\mu = \sum_{w \in \hat{W}} \text{sgn}(w)a_{w\mu} \); in particular, \( b_\mu = \text{sgn}(w)b_{w\mu} \) for each \( w \in \hat{W} \). One has

\[
Y \in \mathcal{R}_\hat{W} \quad \& \quad \mathcal{F}_\hat{W}(Y) \in \mathcal{R} \implies \begin{cases} 
\text{supp}(\mathcal{F}_\hat{W}(Y)) \text{ is } \hat{W}\text{-stable}, \\
\mathcal{F}_\hat{W}(Y) \in \mathcal{R}_\hat{W}; \\
\mathcal{F}_\hat{W}(Y) \text{ is } \hat{W}\text{-anti-invariant}.
\end{cases}
\]

We call a vector \( \lambda \in \hat{h}^* \) \( \hat{W} \)-regular if \( \text{Stab}_{\hat{W}} \lambda = \{ \text{id} \} \), and we say that the orbit \( \hat{W}\lambda \) is \( \hat{W} \)-regular if \( \lambda \) is \( \hat{W} \)-regular (so the orbit consists of \( \hat{W} \)-regular points). If \( \hat{W} \) is an
affine Weyl group, then for any $\lambda \in \hat{h}^*$ the stabilizer $\text{Stab}_W \lambda$ is either trivial or contains a reflection. Thus for $\tilde{W} = \bar{W}'$, $\bar{W}''$ one has

$$Y \in \mathcal{R}_W & \mathcal{F}_{\tilde{W}}(Y) \in \mathcal{R} \implies \text{supp}(\mathcal{F}_{\tilde{W}}(Y)) \text{ is a union of } \tilde{W}\text{-regular orbits.}$$

For $Y \in \mathcal{R}'$ the sum $\sum_{w \in \tilde{W}} \text{sgn}(w)wY$ is not always $\tilde{W}$-anti-invariant: for example, for $\tilde{W} = \{\text{id}, s_\alpha\}$ one has $\sum_{w \in \tilde{W}} \text{sgn}(w)w((1 - e^{-\alpha})^{-1} = (1 - e^{-\alpha})^{-1} - (1 - e^\alpha)^{-1} = 1 + 2e^{-\alpha} + 2e^{-2\alpha} + \ldots$, which is not $\tilde{W}$-anti-invariant.

2. Proof

As it is pointed out in Section [1] it is enough to establish the denominator identity for a particular choice of $\Delta_+$ and we do this for the choice described in Section [1.1]. Recall that the group $T'$ was introduced in Section [1.2.2]. The steps of the proof are the following.

1) In Section [2.1] we check that for $g = \mathfrak{gl}(n|n), D(n+1|n)$, the sum $\mathcal{F}_{T'}(Re^\delta)$ is well-defined and belongs to $\mathcal{R}$.

2) In Section [2.2] we prove the inclusions

$$\text{(5)} \quad \text{supp}(\mathcal{F}_{T'}(Re^\delta)), \text{supp}(Re^\delta) \subset U,$$

where

$$\text{(6)} \quad U := \{\mu \in \hat{\rho} - \hat{Q}^+ | (\mu, \mu) = (\hat{\rho}, \hat{\rho})\}$$

for $g = \mathfrak{gl}(n|n)$ and $D(n+1|n)$.

For simple contragredient Lie superalgebras with non-zero Killing form steps (1), (2) are performed in [G2], 2.4.

3) In Section [2.3] we show that for any finite-dimensional simple contragredient Lie superalgebra $g$ the inclusions (5) imply that $\text{supp}(\hat{\mathcal{F}} e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^\delta)) \subset \hat{Q}^W$. As a result, $\hat{\mathcal{F}} e^{-\hat{\rho}}\mathcal{F}_{T'}(Re^\delta)$ takes the form $f(q)$ (resp., $f(q, e^{\text{str}})$) for $g \neq \mathfrak{gl}(n|n)$ (resp., for $\mathfrak{gl}(n|n)$).

4) In Section [2.4] we compute $f(q)$ (resp., $f(q, e^{\text{str}})$) for $D(n+1|n)$ (resp., for $\mathfrak{gl}(n|n)$). This completes the proof of Identity (2).

In Section [2.5] we briefly repeat the arguments of [G2] showing that $f(q) = 1$ for $g \neq \mathfrak{gl}(n|n), D(n+1|n), D(2,1,a)$. This completes the proof of Identity (1).

2.1. Step 1. In this subsection we show that for $g = \mathfrak{gl}(n|n), D(n+1|n)$, the sum $\mathcal{F}_{T'}(Re^\delta)$ is a well-defined element of $\mathcal{R}$. Since $\hat{\rho} = \rho$ is $\bar{W}$-invariant, it is enough to verify that $\mathcal{F}_{T'}(R)$ is a well-defined element of $\mathcal{R}$.

Recall that $T' = \mathbb{Z}\{t_\delta, -t_\delta + 1\}_{i=1}^{n-1}$ for $\mathfrak{gl}(n|n)$ and $T' = \mathbb{Z}\{t_\delta\}_{i=1}^n$ for $D(n+1|n)$, where

$$\text{(7)} \quad t_\mu(\alpha) = \alpha - (\alpha, \mu)\delta \quad \text{for any } \alpha \in \hat{Q}.$$
2.1.1. By Section 1.3.4 one has
\[
\max \text{supp}(w(R)) = \sum_{\alpha \in \Delta_{0+}:w(\beta) < 0} w\alpha - \sum_{\beta \in \Delta_{1+}:w(\beta) > 0} w\beta.
\]

For \( w \in T' \) write \( w = t_\mu \), where \( \mu \in \mathbb{Z}\{\delta_i - \delta_{i+1}\}_{1 \leq i < n} \) for \( \mathfrak{g}(n|n) \) and \( \mu \in \mathbb{Z}\{\delta_i\}_{i=1}^n \) for \( D(n + 1|n) \). From (7) we get
\[
\{\beta \in \Delta_{i+}|w(\beta) < 0\} = \{\beta \in \Delta_{i+}|(\beta, \mu) > 0\} \quad \text{for } i = 0, 1.
\]
We obtain \( \max \text{supp}(t_\mu(R)) = -v(\mu) + (v(\mu), \mu)\delta \), where
\[
v(\mu) := \sum_{\beta \in \Delta_{0+}: (\beta, \mu) > 0} \beta - \sum_{\beta \in \Delta_{1+}: (\beta, \mu) > 0} \beta.
\]

In order to prove that \( \mathcal{F}_{T'}(R) \) is a well-defined element of \( \mathcal{R} \) we verify that
\[(8) \quad (i) \, \forall \mu \, (v(\mu), \mu) \leq 0; \quad (ii) \, \forall N > 0 \, \{\mu| (v(\mu), \mu) \geq -N\} \text{ is finite}.
\]
The condition (ii) ensures that the sum \( \mathcal{F}_{T'}(R) = \sum_\mu t_\mu(R) \) is well-defined and the condition (i) means that for each \( \mu \) one has
\[
\max \text{supp}(t_\mu(R)) = -v(\mu) \leq \sum_{\beta \in \Delta_{i+}} \beta
\]
so \( \text{supp}(\mathcal{F}_{T'}(R)) \subset \sum_{\beta \in \Delta_{i+}} \beta - \hat{Q}^+ \) and thus \( \mathcal{F}_{T'}(R) \in \mathcal{R} \).

2.1.2. Case \( \mathfrak{g}(n|n) \). Recall that \( w \in T' \) has the form \( w = t_\mu \), \( \mu = \sum_{i=1}^n k_i \delta_i \), where the \( k_i \)s are integers and \( \sum_{i=1}^n k_i = 0 \). One has
\[
\{\alpha \in \Delta_{i+0}| (\alpha, \mu) > 0\} := \{\delta_i - \delta_j| i < j, k_i > k_j\},
\]
\[
\{\alpha \in \Delta_{i+1}| (\alpha, \mu) > 0\} := \{\varepsilon_i - \delta_j| k_j < 0, i < j\} \cup \{\delta_i - \varepsilon_j| k_i > 0, i < j\},
\]
where \( 1 \leq i, j \leq n \).

Write \( v(\mu) = v' + v'' \), where \( v' = \sum_{i=1}^n a_i \delta_i \) and \( v'' \) lies in the span of \( \varepsilon_i \)s. By above, for \( k_i > 0 \) one has \( a_i \leq (n-i) - (n-i) = 0 \) and for \( k_j < 0 \) one has \( a_j \geq -(j-1) + j = 1 \). Therefore \( (v(\mu), \mu) = \sum_{i=1}^n a_i k_i \leq \sum_{k_i < 0} k_i \leq 0 \) and the set \( \{\mu| (v(\mu), \mu) \geq -N\} \) is a subset of the set \( \{\mu| \sum_{k_i < 0} k_i \geq -N\} \), which is finite for any \( N \), because \( k_i \)s are integers and \( \sum_{i=1}^n k_i = 0 \). This establishes conditions (8).

2.1.3. Case \( D(n + 1|n) \). Recall that \( w \in T' \) has the form \( w = t_\mu \), \( \mu = \sum k_i \delta_i \), where the \( k_i \)s are integers. One has
\[
\{\alpha \in \Delta_{i+0}| (\alpha, \mu) > 0\} := \{\delta_i - \delta_j| i < j, k_i > k_j\} \cup \{\delta_i + \delta_j| i \neq j, k_i + k_j > 0\} \cup \{2\delta_i| k_i > 0\},
\]
\[
\{\alpha \in \Delta_{i+1}| (\alpha, \mu) > 0\} := \{\varepsilon_s - \delta_j| k_j < 0, s \leq j\} \cup \{\delta_i - \varepsilon_s| k_i > 0, i < s\} \cup \{\delta_i + \varepsilon_s| k_i > 0\},
\]
where \( 1 \leq i, j \leq n \) and \( 1 \leq s \leq n + 1 \).
Write \( v(\mu) = v' + v'' \), where \( v' = \sum_{i=1}^{n} a_i \delta_i \) and \( v'' \) lies in the span of \( \varepsilon_i \)s. By above, for \( k_i > 0 \) one has \( a_i \leq (2n + 1 - i) - (2n + 2 - i) = -1 \) and for \( k_j < 0 \) one has \( a_j \geq -(j - 1) + j = 1 \). Therefore

\[
(v(\mu), \mu) = \sum_{i=1}^{n} a_i k_i \leq - \sum_{k_i > 0} k_i + \sum_{k_j < 0} k_j = - \sum_{i=1}^{n} |k_i| \leq 0
\]

so the set \( \{ \mu | (v(\mu), \mu) \geq -N \} \) is a subset of the set \( \{ \mu | \sum_{i=1}^{n} |k_i| \leq N \} \), which is finite for any \( N \). This establishes conditions (3).

2.1.4. Remark. For \( \mathfrak{gl}(n|n) \) one can interchange \( \Delta' \) and \( \Delta'' \) so the sum \( F_{T'}(R) \) is well-defined. One readily sees that \( F_{T'}(R) \) is not well-defined for \( D(n+1|n) \). For instance, for \( n > 1 \), for each \( k > 0 \) one has \( v(-2k \varepsilon_1) = 0 \) so max supp \((t_{-2k \varepsilon_1} (R)) = 0 \) and the sum \( \sum_{k=1}^{\infty} t_{-2k \varepsilon_1} (R) \) is not well-defined; hence \( F_{T'}(R) \) is not well-defined as well.

2.2. Step 2. By Section 1.3.3, \( \check{R} \) is an invertible element of \( \mathcal{R}' \). From representation theory we know that since \( \check{g} \) admits a Casimir element \( \check{K}_2 \), Chapter II, the character of the trivial \( \check{g} \)-module is a linear combination of the characters of Verma \( \check{g} \)-modules \( M(\lambda) \), where \( \lambda \in -\check{Q} \) are such that \( (\lambda + \check{\rho}, \lambda + \check{\rho}) = (\check{\rho}, \check{\rho}) \). Since the character of \( M(\lambda) \) is equal to \( \check{R}^{-1} e^\lambda \), we obtain

\[
1 = \sum_{\lambda \in Q^-} a_\lambda \check{R}^{-1} e^\lambda,
\]

where \( a_\lambda \in \mathbb{Z} \). This can be rewritten as

\[
\check{R} e^\check{\rho} = \sum_{\lambda \in Q^-} a_\lambda e^\lambda,
\]

that is \( \text{supp}(\check{R}) \subset U \), see (9) for notation.

It remains to verify the inclusion \( \text{supp}(F_{T'}(\check{R} e^\check{\rho})) \subset U \). The denominator identity for \( \mathfrak{g} \) (see [KW], [GH]) takes the form

\[
\check{R} e^\check{\rho} = F_{W''} \left( \frac{e^\rho}{1 + e^{-\beta}} \right),
\]

where \( S := \{ \varepsilon_i - \delta_i \}_{i=1}^{n} \) (the identity for \( \mathfrak{gl}(n|n) \) immediately follows from the identity for \( \mathfrak{sl}(n|n) \)). Since \( \rho = \check{\rho} \) is \( W \)-invariant, this implies

\[
t_\mu(\check{R} e^\check{\rho}) = e^{\check{\rho}} \sum_{w \in W''} \text{sgn}(w) \prod_{\beta \in S} (1 + e^{-t_\mu w \beta})^{-1}.
\]

For each \( t_\mu \in T' \) and \( w \in W'' \) one has

\[
\text{supp}(\prod_{\beta \in S} (1 + e^{-t_\mu w \beta})^{-1}) \subset V, \text{ where } V := \mathbb{Z}\{t_\mu w \beta | \beta \in S \} \cap \check{Q}^-.
\]
Since \((t_\mu w^\beta, t_\mu w^\beta') = (\beta, \beta') = (t_\mu w^\beta, \hat{\rho}) = (\hat{\rho}, \beta) = 0\) for any \(\beta, \beta' \in S\), one has \((V, V) = (V, \hat{\rho}) = 0\). Therefore \(V + \hat{\rho} \subset U\) so \(\text{supp}(t_\mu(Re^{\hat{\rho}})) \subset U\) for each \(\mu\). This establishes the required inclusion \(\text{supp}(F_T(Re^{\hat{\rho}})) \subset U\) and completes the proof of (5).

2.3. Step 3. Let us deduce the inclusion \(\text{supp}(\hat{R}^{-1}e^{\hat{\rho}} \cdot F_T(Re^{\hat{\rho}})) \subset (\hat{Q}^-)^W\) from (5).

2.3.1. Lemma. For any simple finite-dimensional contragredient Lie superalgebra \(g\) the term \(F_T(Re^{\hat{\rho}})\) is a \(W'\)-anti-invariant element of \(R_{\hat{W}'}\).

Proof. In the light of Section 1.3.5, it is enough to present \(F_T(Re^{\hat{\rho}})\) in the form \(F_{\hat{W}'}(Y)\) for some \(Y \in R_{\hat{W}'}\). Let \(R_0', R_0''\) be the Weyl denominators for \(\Delta', \Delta''\) respectively (i.e., \(R_0' = \prod_{\alpha \in \Delta'_+} (1 - e^{-\alpha})\)). Below we will prove the formula

\[
F_T(Re^{\hat{\rho}}) = F_{\hat{W}'}(\frac{R_0''e^{\hat{\rho}}}{R_1}).
\]

By Section 1.3.3 \(R_1^{-1}R_0''e^{\hat{\rho}} \in R_{\hat{W}}\), so the formula establishes the required assertion.

Let us show that the right-hand side of (9) is well-defined. Since \(R_0''\) is \(W'\)-invariant, it is enough to verify that \(F_{\hat{W}'}(e^{\hat{\rho}}R_1^{-1})\) is a well-defined element of \(R\). For \(g \neq gl(n|n), D(n + 1|n)\) this is proven in \([G2], 2.4.1\) (i). Consider the case \(g = gl(n|n), D(n + 1|n)\). Since \(\hat{\rho}\) is \(\hat{W}\)-invariant, it is enough to check that \(F_{\hat{W}'}(R_1^{-1})\) is a well-defined element of \(R\). By Section 1.3.4 for each \(w \in \hat{W}'\) one has

\[
\max \text{supp}(w(R_1^{-1})) = \sum_{\beta \in \Delta_{1+}: w\beta < 0} w\beta.
\]

In particular, \(\text{supp}(w(R_1^{-1})) \subset \hat{Q}^-\), so, if the sum \(F_{\hat{W}'}(R_1^{-1}) = \sum_{w \in \hat{W}'} \text{sgn} w \cdot w(R_1^{-1})\) is well-defined, it lies in \(R\). In order to see that this sum is well-defined let us check that for each \(\nu \in \hat{Q}^-\) the set

\[
X(\nu) := \{w \in \hat{W}'| \sum_{\beta \in \Delta_{1+}: w\beta < 0} w\beta \geq \nu\}
\]

is finite. One has

\[
X(\nu) \subset \{w \in \hat{W}'| \forall \beta \in \Delta_{1+} w\beta \geq \nu\}.
\]

Write \(\nu = -k\delta + \nu', \) where \(k \geq 0, \nu' \in Q\), and write \(w \in X(\nu)\) in the the form \(w = t_\mu y\), where \(t_\mu \in T', y \in W'\). Since \(w\beta = y\beta - (y\beta, \mu)\delta\) for \(\beta \in \Delta_{1+}\), one has \((y\beta, \mu) \geq -k\) for each \(\beta \in \Delta_{1+}\). Since \(\{\varepsilon_i - \delta_i, \delta_i - \varepsilon_i\} \subset \Delta_{1+}\), this gives \(|(\mu, y\delta_i)| \leq k\) for \(i = 1, \ldots, n\). Combining the facts that \(W'\) is a subgroup of signed permutation of \(\{\delta_i\}_{i=1}^n\) and that \((\mu, \delta_i)\) is integral for each \(i\), we conclude that \(X(\nu)\) is finite. Thus \(F_{\hat{W}'}(R_0''/R_0')\) is a well-defined element of \(R\).
Now let us prove the formula (9). Recall that \( \rho = \rho'_0 + \rho''_0 - \rho_1 \), where
\[
\rho'_0 := \sum_{\alpha \in \Delta''_0} \alpha/2, \quad \rho''_0 := \sum_{\alpha \in \Delta''_+} \alpha/2, \quad \rho_1 := \sum_{\beta \in \Delta_1} \beta/2.
\]

The Weyl denominator identity for \( \Delta''_0 \) takes the form
\[
\hat{R}'_0 e^{\rho'_0} = F_{W'}(e^{\rho''_0}).
\]
Since \( R_1 e^{\rho_1} = \prod_{\beta \in \Delta_1} (e^{\beta/2} + e^{-\beta/2}) \) is \( W \)-invariant and \( \hat{R}'_0 e^{\rho'_0} \) is \( W' \)-invariant, we get
\[
Re^\rho = \frac{R_0^\theta e^{\rho''_0}}{R_1 e^{\rho_1}} \cdot F_{W'}(e^{\rho''_0}) = F_{W'}\left(\frac{e^{\rho''_0} R_0^\theta e^{\rho''_0}}{R_1 e^{\rho_1}}\right) = F_{W'}\left(\frac{R_0^\theta e^{\rho}}{R_1}\right).
\]
Using the \( W \)-invariance of \( \hat{\rho} - \rho \), we obtain
\[
F_{T'}(Re^\hat{\rho}) = F_{T'}\left(F_{W'}\left(\frac{R_0^\theta e^{\rho}}{R_1}\right)\right) = F_{W'}\left(\frac{R_0^\theta e^{\rho}}{R_1}\right)
\]
as required. This completes the proof. \( \square \)

2.3.2. Proposition. Let \( \mathfrak{g} \) be a simple finite-dimensional contragredient Lie superalgebra. One has
\[
supp(\hat{R}^{-1}e^\hat{\rho} : F_{T'}(Re^\hat{\rho})) \subseteq (\hat{Q}^-)^W = \hat{Q}^- \cap \hat{Q}^\perp.
\]

Proof. By Section 2.1.1, \( F_{T'}(Re^\hat{\rho}) \in \mathcal{R} \); by Section 1.3.3 \( \hat{R}^{-1} \in \mathcal{R} \) so
\[
Y := \hat{R}^{-1}e^{-\hat{\rho}} : F_{T'}(Re^\hat{\rho}) \in \mathcal{R}.
\]

The affine root system \( \hat{\Delta}' \) is a subsystem of \( \hat{\Delta}_0 \). Set \( \hat{\Delta}'_+ = \hat{\Delta}' \cap \hat{\Delta}_+ \) and let \( \hat{\Pi}' \) be the corresponding set of simple roots. Fix \( \hat{\rho}' \in \hat{\mathfrak{h}}^* \) such that \( 2(\hat{\rho'}, \alpha) = (\alpha, \alpha) \) for each \( \alpha \in \hat{\Pi}' \).

It is easy to see that \( \hat{R}_0 e^{\hat{\rho}'} : \hat{R} e^{\hat{\rho}} \) are \( \hat{W}' \)-anti-invariant elements of \( \mathcal{R}' \) (see, for instance, [G2], 1.5.1). Thus \( \hat{R}_1 e^{\hat{\rho}' - \hat{\rho}} = \hat{R}_0 e^{\hat{\rho}'} : (\hat{R} e^{\hat{\rho}})^{-1} \) is a \( \hat{W}' \)-invariant element of \( \mathcal{R}' \). By Section 1.3.3 \( \hat{R}_1 \in \mathcal{R}_W \) so \( \hat{R}_1 e^{\hat{\rho}' - \hat{\rho}} \) is a \( \hat{W}' \)-invariant element of \( \mathcal{R}_W \). Using Lemma 2.3.1 we get
\[
(10) \quad \hat{R}_0 e^{\hat{\rho}'} Y = \hat{R}_1 e^{\hat{\rho}' - \hat{\rho}} F_{T'}(R) \quad \text{is a } \hat{W}' \text{-anti-invariant element of } \mathcal{R}_{W'}.
\]

Write \( Y = Y_1 + Y_2 \), where \( \text{supp}(Y_1) = \text{supp}(Y) \cap \hat{Q}^\perp \) and \( \text{supp}(Y_2) = \text{supp}(Y) \setminus \hat{Q}^\perp \). Note that \( Y_1, Y_2 \in \mathcal{R} \). Assume that \( Y_2 \neq 0 \). Let \( \mu \) be a maximal element in \( \text{supp}(Y_2) \). One has \( \text{supp}(\hat{R}^{-1}) \subseteq \hat{Q}^- \) and \( \text{supp}(F_{T'}(R)e^{\rho}) \subseteq \hat{\rho} - \hat{Q}^+ \), by Section 1.3.3 and (5) respectively. Thus \( \text{supp}(Y) \subseteq \hat{Q}^- \) and so \( \mu \in \hat{Q}^- \).

Since \( \text{supp}(Y_1) \subseteq \hat{Q}^\perp \), \( Y_1 \) is a \( \hat{W} \)-invariant element of \( \mathcal{R}_W \) so \( \hat{R}_0 e^{\hat{\rho}'} Y_1 \) is a \( \hat{W}' \)-anti-invariant element of \( \mathcal{R}_{W'} \). In the light of (10), the product \( \hat{R}_0 e^{\hat{\rho}'} Y_2 \) is also a \( \hat{W}' \)-anti-invariant element of \( \mathcal{R}_{W'} \). Clearly, \( \hat{\rho}' + \mu \) is a maximal element in the support of \( \hat{R}_0 e^{\hat{\rho}'} Y_2 \).
By Section 2.3.3, this support is the union of $\hat{W}'$-regular orbits (recall that regularity means that each element has the trivial stabilizer in $\hat{W}'$), so $\hat{\rho} + \mu$ is a maximal element in a regular $\hat{W}'$-orbit and thus $\frac{2(\hat{\rho} + \mu)}{(\alpha, \alpha)} \not\in \mathbb{Z}_{\leq 0}$ for each $\alpha \in \hat{H}'$. Since $\mu \in \hat{Q}^{-}$ one has $\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$ for each $\alpha \in \hat{H}'$. Taking into account that $\frac{2(\hat{\rho}, \alpha)}{(\alpha, \alpha)} = 1$ for each $\alpha \in \hat{H}'$, we obtain

$$\forall \alpha \in \hat{H}' \frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}_{\geq 0}. \tag{11}$$

Recall that $\delta = \sum_{\alpha \in \hat{H}'} k_{\alpha} \alpha$ for some $k_{\alpha} \in \mathbb{Z}_{> 0}$ (see [K2], Chapter VI). Since $\mu \in \hat{Q}^{-}$ one has $(\mu, \delta) = 0$. Combining with (11), we get $(\mu, \alpha) = 0$ for each $\alpha \in \hat{H}'$ so $\mu \in (\hat{\Delta})^{\perp}$.

One has

$$(\hat{\Delta})^{\perp} \cap \hat{Q} = (\hat{Q}^{\perp} \cap \hat{Q}) \oplus V,$$

where the restriction of $(-, -)$ to $QV$ is negatively definite; more precisely, one has

| $g$ | $\mathfrak{gl}(n|n)$ | $\mathfrak{gl}(m|n), \ m \neq n$ | $\mathfrak{C}(n)$ | other cases |
|---|---|---|---|---|
| $Q^{\perp} \cap Q$ | $\mathbb{Z}\{\delta, \text{str}\}$ | $\mathbb{Z}\delta$ | $\mathbb{Z}\delta$ | $\mathbb{Z}\delta$ |
| $V$ | $\mathbb{Z}\Delta''$ | $\mathbb{Z}\Delta'' \oplus \mathbb{C}\xi$ | $\mathbb{Z}\Delta'' \oplus \mathbb{C}\xi$ | $\mathbb{Z}\Delta''$ |

For $g = \mathfrak{gl}(m|n), \ m \neq n$ and $g = \mathfrak{C}(n)$ the element $\xi$ is given in [G2], 3.2; one has $(\Delta'', \xi) = 0$. Since $V \subset \hat{Q}$, one has $(V, \hat{Q}^{\perp}) = 0$. Now combining the formulas $\mu \in (\hat{Q}^{\perp} \cap \hat{Q}) \oplus V$, $(\mu, \mu) = 0$ with the fact that $(\nu, \nu) < 0$ for each non-zero $\nu \in V$, we obtain $\mu \in \hat{Q}^{\perp} \cap \hat{Q} = \hat{Q}^{\perp}$, which contradicts to the construction of $Y_{2}$. Hence $Y_{2} = 0$ as required. \qed

2.3.3. Using the table in the proof of Proposition 2.3.2 we obtain the following corollary.

**Corollary.** For $g \neq \mathfrak{gl}(n|n)$ one has $f(q) \cdot \hat{R}^{\hat{\rho}} = \mathcal{F}_{T'}(\hat{R}^{\hat{\rho}})$ for some $f(q) = \sum_{k=0}^{\infty} a_{k} q^{k}$ $(a_{k} \in \mathbb{Z})$. For $g = \mathfrak{gl}(n|n)$ one has $f(q, e^{\text{str}}) \cdot \hat{R}^{\hat{\rho}} = \mathcal{F}_{T}(\hat{R}^{\hat{\rho}})$ for some $f(q, e^{\text{str}}) = \sum_{k=0}^{\infty} \sum_{m=-\infty}^{\infty} a_{k, m} q^{k} e^{\text{str}}$ $(a_{k, m} \in \mathbb{Z})$.

2.4. Step 4 for $g = \mathfrak{gl}(n|n), D(n+1|n)$. In this subsection we complete the proof of the denominator identities (2) by proving the formulas (3). We prove them by taking a suitable evaluation of $\hat{R}^{-1} \sum_{t \in T} t(R)$. By Corollary 2.3.3 $\hat{R}^{-1} \sum_{t \in T} t(R)$ is equal to $f(q)$ for $D(n+1|n)$ and to $f(q, e^{\text{str}})$ for $\mathfrak{gl}(n|n)$. Now we consider $q$ as a real parameter between 0 and 1. We choose the evaluation in such a way that the evaluation of $\hat{R}^{-1} \sum_{t \in T} t(R)$ is equal to the evaluation of $\hat{R}^{-1}R$. As a result, $f(q)$ (resp., $f(q, e^{\text{str}})$) is equal to the evaluation of $\hat{R}^{-1}R$, which can be easily computed.

2.4.1. Case $D(n+1|n)$. Take a complex parameter $x$ and consider the following evaluation: $e^{-\epsilon_{i}} := x^{a_{i}}, \ e^{-\delta_{j}} := -x^{b_{j}}$, where $a_{i}, \ (i = 1, \ldots, n+1), \ b_{j}, \ (j = 1, \ldots, n)$ are integers such that $a_{i} \pm b_{j} \neq 0, a_{i} \pm a_{j} \neq 0, b_{i} \pm b_{j} \neq 0, b_{i} \neq 0$ for all indexes $i, j$. We
denote the evaluation of $R$ (resp., $\hat{R}$) by $R(x)$ (resp., $\hat{R}(x)$). The functions $R(x), \hat{R}(x)$ are meromorphic. One has

$$R(x) = \frac{\prod_{1 \leq i < j \leq n+1} (1 - x^{a_i + a_j}) \cdot \prod_{1 \leq i \leq n} (1 - x^{b_i}) \cdot \prod_{1 \leq i \leq n} (1 - x^{2b_i})}{\prod_{1 \leq i \leq j \leq n} (1 - x^{a_i + a_j}) \cdot \prod_{1 \leq j < i \leq n+1} (1 - x^{a_i + b_j})}.$$

One readily sees that $R(x)$ has a pole at $x = 1$ of order $|\Delta_{1+}| - |\Delta_{0+}| = n$.

One has

$$\frac{\hat{R}(x)}{R(x)} \bigg|_{x=1} = \frac{((1 - q)_{q}^{\infty})^{\dim g_0}}{((1 - q)_{q}^{\infty})^{\dim g_1}} = ((1 - q)_{q}^{\infty})^{\dim g_0 - \dim g_1} = (1 - q)_{q}^{\infty}.$$

In particular, $\hat{R}(x)$ also has a pole of order $n$ at $x = 1$.

The evaluation of $(t\sum_{k_i\delta_i}(R))(x)$ is

$$\frac{\prod_{1 \leq i < j \leq n+1} (1 - x^{a_i + a_j}) \cdot \prod_{1 \leq i \leq n} (1 - q^{-2k_i}x^{2b_i}) \cdot \prod_{1 \leq i \leq n} (1 - q^{-k_i+k_j}x^{b_i+b_j})}{\prod_{1 \leq i \leq j \leq n} (1 - q^{-2k_i}x^{a_i + a_j}) \cdot \prod_{1 \leq j < i \leq n+1} (1 - q^{-k_j}x^{a_i + b_j})}$$

which is a meromorphic function. Let $s$ be the number of zeros among $k_1, \ldots, k_n$. Then at $x = 1$ the order of zero of the numerator is at least $n(n + 1) + s^2$, and the order of zero of the denominator is $2(n + 1)s$. Therefore at $x = 1$ the function $(t\sum_{k_i\delta_i}(R))(x)$ has the pole of order at most $2(n + 1)s - n(n + 1) - s^2 = n + 1 - (n + 1 - s)^2$; in particular, $(t\sum_{k_i\delta_i}(R))(x)$ has the pole of order at most $n$ and it is equal to $n$ iff $n = s$ that is $\sum_{i} k_i\delta_i = 0$ and $(t\sum_{k_i\delta_i}(R))(x) = R(x)$.

We conclude that $(\hat{R}(x))^{-1} \cdot \sum_{t \in T'} \delta_{t} \cdot (t(R))(x)$ is holomorphic at $x = 1$ and its value is equal to zero, and that $(\hat{R}(x))^{-1} \cdot \sum_{t \in T'} (t(R))(x)$ is holomorphic at $x = 1$ and its value is equal to $\frac{R(x)}{R(x)} \bigg|_{x=1}$. In the light of Corollary 2.3.3 we obtain

$$f(q) = \frac{R(x)}{R(x)} \bigg|_{x=1} = ((1 - q)_{q}^{\infty})^{-1}.$$

2.4.2. Case $\mathfrak{g}l(n|n)$. Fix $y > 1$. Take a complex parameter $x$ and consider the following evaluation

$$e^{-\varepsilon_i} := y, e^{-\varepsilon_i} := x^i, \text{ for } i = 2, \ldots, n, e^{-\delta_i} := -x^{-i} \text{ for } i = 1, \ldots, n.$$

The functions $R(x), \hat{R}(x)$ are meromorphic. One has

$$R(x) = \frac{\prod_{1 \leq i \leq n} (1 - yx^{-i}) \cdot \prod_{1 \leq i < j \leq n} (1 - x^{i-j}) \cdot \prod_{1 \leq i \leq n} (1 - x^{i+j})}{\prod_{1 \leq i \leq n} (1 - yx^i) \cdot \prod_{1 \leq i < j \leq n} (1 - x^{i+j}) \cdot \prod_{1 \leq i \leq n} (1 - x^{i+j})}.$$

Therefore the function $R(x)$ has a pole of order $n - 1$ at $x = 1$. 
One has
\[ \frac{\hat{R}(x)}{R(x)} \bigg|_{x=1} = \frac{((1 - q)^{\infty})^{\dim g_0 - 2(n-1)} \cdot (\frac{1}{q})^{\dim g_1 - 2n} \cdot ((1 - qy)^{\infty})^{n-1} \cdot (\frac{1}{q})^{n} \cdot ((1 - qy^{-1})^{\infty})^{n-1}}{((1 - q)^{\infty})^{\dim g_1 - 2n} \cdot (\frac{1}{q})^{n} \cdot ((1 - qy^{-1})^{\infty})^{n}}. \]

Thus \( \hat{R}(x) \) also has a pole of order \( n - 1 \) at \( x = 1 \). Since \( \dim g_0 = \dim g_1 \) and \( e^{\text{str}} = (-1)^n y^{-1} \) for \( x = 1 \) we obtain
\[ \frac{\hat{R}(x)}{R(x)} \bigg|_{x=1} = \frac{((1 - q)^{\infty})^2}{(1 - q(-1)^n e^{\text{str}})^{\infty} \cdot (1 - q(-1)^{n} e^{-\text{str}})^{\infty}}. \]

One has
\[ (t \sum_{k_i \delta_i}(R))(x, y) = \frac{\prod_{1 \leq i \leq n} (1 - y x^{-i}) \cdot \prod_{1 \leq i < j \leq n} (1 - x^{-i - j}) \cdot \prod_{1 \leq i < j \leq n} (1 - q^{k_j - k_i} x^{-i - j})}{\prod_{1 \leq i \leq n} (1 - q^{k_i} x^{i}) \cdot \prod_{1 \leq i < j \leq n} (1 - q^{k_j} x^j) \cdot \prod_{1 \leq j < i \leq n} (1 - q^{-k_j} x^{-i - j})}. \]

which is a meromorphic function.

Let \( s \) be the number of zeros among \( k_1, \ldots, k_n \). Then at \( x = 1 \) the order of zero of the numerator is at least \( \frac{(n-1)(n-2)+s(s-1)}{2} \), and the order of zero of the denominator is \( (n-1)s \). Therefore at \( x = 1 \) the function \( (t \sum_{k_i \delta_i}(R))(x, y) \) has the pole of order at most \( (n-1)s - \frac{(n-1)(n-2)+s(s-1)}{2} = \frac{3n-s - 2(n-s)^2}{2} \), so the order is at most \( n-1 \) and it is equal to \( n-1 \) iff \( s = n-1, n \). Notice that \( s \neq n-1 \), since \( \sum k_i = 0 \). Therefore the pole has order \( n-1 \) iff \( \sum k_i = 0 \).

We conclude that the function \( (\hat{R}(x))^{-1}(F_T'(R))(x, y) \) is holomorphic at \( x = 1 \) and its value is equal to \( \frac{\hat{R}(x)}{R(x)} \bigg|_{x=1} \). Using Corollary 2.3.3 we obtain
\[ f(q, e^{\text{str}}) = \frac{R(x)}{\hat{R}(x)} \bigg|_{x=1} = \frac{(1 - q(-1)^n e^{\text{str}})^{\infty} \cdot (1 - q(-1)^{n} e^{-\text{str}})^{\infty}}{((1 - q)^{\infty})^{2}}. \]

2.5. **Step 4** for \( \mathfrak{g} \neq \mathfrak{gl}(n|n), D(n+1|n), D(2, 1, a) \). In this case the dual Coxeter number is non-zero. Recall that \( q = e^{-\delta} \). Write \( f(q) = \sum_{k=0}^{\infty} a_k q^k e^{-k\delta} \). Since \( f(q) \cdot \hat{R}^\delta = F_T'(R\hat{\delta}) \), we have
\[ \sum_{k=1}^{\infty} a_k e^{-\delta} \cdot \hat{R}^\delta \cdot F_T'(R\hat{\delta}) = a_0 \hat{R}^\delta. \]

By [5], for any \( \nu \) in the support of the right-hand side, one has \( (\nu, \nu) = (\hat{\rho}, \hat{\rho}) \), and for any \( \nu \) in the support of the left-hand side one has \( (\nu, \nu) = (\hat{\rho}, \hat{\rho}) - 2k(\hat{\delta}, \hat{\rho}) \) for some \( k > 0 \). Since \( (\hat{\rho}, \hat{\delta}) \) is equal to the dual Coxeter number, which is non-zero, we conclude that the intersection of supports is empty. Hence \( f(q) = a_0 \). Since the coefficient of \( e^\delta \) in \( \hat{R}^\delta \) is equal to one, \( a_0 \) is equal to the coefficient of \( e^\delta \) in \( F_T'(R\hat{\delta}) \). As it is shown in [G2], this coefficient is equal to one so \( f(q) = 1 \) as required.
3. Other forms of denominator identity

Recall that denominator identity for a basic Lie superalgebra can be written in the form

\[ \hat{Re}^\rho = \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \]

where \( W^\prime := W' \) for \( \mathfrak{g} \neq D(n+1|n), D(2,1,a) \) and \( W^\prime := W'' \) for \( \mathfrak{g} = D(n+1|n), D(2,1,a) \), and \( S \subset \Pi \) is the maximal isotropic system (see [KW], [G1]). If the dual Coxeter number of \( \mathfrak{g} \) is non-zero the affine denominator identity for \( \mathfrak{g} \) can be written in the form

\[ \hat{Re}^\rho = \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \]

see [KW], [G2]. In this section we will show that for \( \mathfrak{g}(n|n) \) the denominator identity can be written in a similar form:

\[ \hat{Re}^\rho = f(q, e^\text{str}) \cdot \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \]

and that the denominator identities for \( D(n + 1|n) \) cannot be written in a similar form, since the expressions \( \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \) are not well defined.

3.1. Case \( D(n+1|n) \). Let us show that the expressions \( \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \), \( \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \) are not well-defined for \( D(n+1|n) \). Fix \( \Pi \) as in Section 1.1 and recall that \( \rho = 0 \).

We repeat the reasonings of Section 2.1.1. One has

\[ \sum_{\beta \in V(w)} w\beta \in \text{supp}(\prod_{\beta \in \mathfrak{s}} \frac{1}{1 + e^{-w\beta}}) \subset \sum_{\beta \in V_S(w)} w\beta - \hat{Q}^+ \subset \hat{Q}^- \]

where

\[ V_S(w) = \{ \beta \in S | w\beta < 0 \} \]

Therefore \( 1 \in \text{supp}(\prod_{\beta \in \mathfrak{s}} \frac{1}{1 + e^{-w\beta}}) \) iff \( wS \subset \Delta_+ \).

Take \( S = \{ \varepsilon_i - \delta_i \} \); then \( t_\mu S \subset \Delta_+ \) if \( (\varepsilon_i - \delta_i, \mu) < 0 \) for all \( i \) which holds for all \( \mu \in \sum_{i} \mathbb{Z}_{<0} \varepsilon_i \) and all \( \mu \in \sum_{i} \mathbb{Z}_{>0} \delta_i \). Hence the sums \( \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \), \( \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) \) contain infinitely many summands equal to 1 and thus they are not well-defined.

3.2. Case \( \mathfrak{g}(n|n) \). Fix \( \Pi \) as in Section 1.1 then \( S = \{ \varepsilon_i - \delta_i \} \).

In order to deduce the formula (13) from (12) and (2) it is enough to verify that the expression

\[ \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} e^{\rho} (1 + e^{-\beta})) = e^\rho \mathcal{F}_{W^\prime}(\prod_{\beta \in \mathfrak{s}} \frac{1}{1 + e^{-\beta}}) \]
is well-defined (since $\rho = \text{str}/2$ is $\hat{W}$-invariant). As in Section 2.1.1, it amounts to show that

$$X_S(\nu) := \{ w \in \hat{W}' | \sum_{\beta \in V_S(w)} w\beta \geq -\nu \}$$

is finite for any $\nu \in \hat{Q}^+$ (where $V_S(w)$ is defined as in Section 3.1). As in Section 2.1.1, writing $\nu = k\delta + \nu_+$, where $\nu_+ \in \mathbb{Z}\Delta$, we get

$$X_S(\nu) \subset \{ t\mu y | \mu \in T', y \in W' \text{ s.t. } (yS, \mu) \geq -k \}.$$ 

Since $y$ permutes $\delta_i$s, $t\mu y \in X_S(\nu)$ forces $(\delta_i, \mu) \geq -k$ for all $\mu$. Taking into account that $\mu$ lies in the $\mathbb{Z}$-span of $\delta_i$ and $(\mu, \sum_{i=1}^n \delta_i) = 0$, we conclude that $X_S(\nu)$ is finite. This establishes (13).

References


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