Eigenslices for Parabolic Actions

To Michel on his 62^{nd} Birthday

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June 2005

Notation.

g semisimple, π simple roots, $p = p_{\pi'} : \pi' \subset \pi$ parabolic

Underlying theme. The (semi) invariant algebra $Sy(p)$ should exhibit the same good properties as the invariant algebra $Y(g)$.

Conjecture 1. $Sy(p)$ is polynomial

Theorem (FJ2). Conjecture 1 is true in most cases. For example, all parabolics in type A or C.

Question. What are the degrees of the invariants?

The degrees are known for $p = g$ (Chevalley-Kostant).

The degrees are known for $p = b$ (Prep. thm. 1977.)

The recipe in the two cases is very different.

In our work (FJ1) the generators are parametrized by a set Π of orbits in π under a finite group action depending on π'

To each $\Gamma \in \Pi$ we assign a "false degree" d^f_Γ.

In principle, the generators have degrees $\varepsilon^{\pi'}_\Gamma d^f_\Gamma : \varepsilon^{\pi'}_\Gamma \in \{\frac{1}{2}, 1\}$.

Theorem (FJ2). When $\varepsilon_\Gamma^{\pi'} = 1, \forall\ \Gamma \in \Pi$, then $Sy(p)$ is polynomial and the generators have degrees $d_\Gamma^f : \Gamma \in \Pi$.

When $p = b$, the generators have degree $\varepsilon_\Gamma^{\emptyset} d_\Gamma^f : \Gamma \in \Pi$.

The above "principle" fails when $p = g$, yet the degrees of the generators have the same sum as $\sum_{\Gamma\in\Pi} d_\Gamma^f$.

Since p is algebraic, it admits a unique subalgebra p_Π containing p' such that $Sy(p) = Y(p_\Pi)$.

For any finite dimensional Lie algebra a, let index a be the codimension of a coadjoint orbit of maximal dimension.

Set $c(a) = \frac{1}{2}(\dim a + \text{index } a)$.

Theorem (FJ3)

$$c(p) = c(p_\Pi) = \sum_{\Gamma \in \Pi} d_\Gamma^f$$

If a is algebraic and $Sy(a) = Y(a)$, then $GK\dim\ Sy(a) = \text{index}\, a$ (Chevalley-Dixmier).

Let $A \subset S(a)$ be Poisson commutative, then $GK\dim A \leq c(a)$.

The above bound can always be reached (S.T. Sadetov).

For g semisimple, the above bound can be reached by "shift of argument". The resulting algebra is polynomial and maximal Poisson commutative.

Identify p_Π with $(p_\Pi^-)^*$ through the Killing form. Let h_Π be the Cartan subalgebra of p_Π. In FJ3, h_Π is computed explicitly. Call $y \in p_\Pi$ regular if codim $[p_\Pi^-, y] =$ index p_Π

Set $\mathcal{N} = \mathcal{V}(S(p_\pi^-)Y(p_\pi^-)_+)$- the nil fibre.

Remark. $\mathcal{N}_{reg}$ can be empty

Conjecture 2. Suppose $\mathcal{N}_{reg} \neq \emptyset$. Then there exists $h \in h_{\Pi}, y \in (p_{\Pi})_{reg}$ such that $[h, y] = -y$.

Choose an h stable complement V to $[p_{\Pi}^{-}, y]$ in p_{π}. The eigenvalues of h on V shall be called the parabolic exponents $e_{\Gamma}^{\pi'} : \Gamma \in \Pi$.

Conjecture 3. $Y(p_{\Pi}^{-})$ is polynomial with generators having degrees $e_{\Gamma}^{\pi'} + 1 : \Gamma \in \Pi$.

Corollary (to conjectures 2,3). Restriction of functions gives an isomorphism of algebras

$$Y(p_{\sqcap}^{-}) \xrightarrow{\sim} R[y+V]$$

Remark

$P_{\sqcap}^{-}(y+V) \subset (p_{\sqcap})_{reg}$, but the inclusion may be strict.

Conjecture 4. $\dim(p_{\sqcap})_{\geq 0} = c(p_{\sqcap})$

Remark.

Conjecture 4 is equivalent to $[x,y]=0 : x \in (p_{\sqcap}^{-})_{>0} \Rightarrow x = 0$.

Remark. $\mathcal{S}_0 := (p_\Pi)_0 \cap (p_\Pi)_{reg}$ is usually empty.

Conjecture 5. Suppose $\mathcal{S}_0 \neq \emptyset$. Then there exists $z \in \mathcal{S}$ for which the z shift $T_z(p_\Pi^-)$ of $Y(p_\Pi^-)$ satisfies

$$T_z(p_\Pi^-) \stackrel{\sim}{\longrightarrow} R[y + (p_\Pi)_{\geq 0}]$$

by restriction of functions.

Corollary $T_z(p_\Pi^-)$ is maximal Poisson commutative and, of course, polynomial on $c(p_\Pi^-)$ generators.

Results

Conjectures 1,2,3,4 are true if rank $g \leq 2$.

Set $j = -w_0|_\pi,\ w_0$ being the longest element in W

Conjectures 2,3,4 are true in type A given $j(\pi') = \pi'$. In particular if $\pi' = \emptyset$. They are also true in type A up to rank ≤ 5.

Conjecture 5 is true in type A if $\pi' = \emptyset$.

The case of G_2. Set $\pi = \{\alpha, \beta\}, \pi' = \{\alpha\}, \alpha$ short.
One can easily construct two algebraically independant elements of $Y(p^-_\Pi)$, namely the lowest root vector x and a second invariant x' of degree ≤ 4 by modifying the Casimir of the Levi factor for p^-_Π.
Let Y be the subalgebra they generate.

Set $h = \alpha^\vee,\ y = x_{\alpha+\beta}$. Then $[h, y] = -y$ and $[p^-_\Pi, y]$ is complemented in p_Π by $V := kx_{3\alpha+2\beta} \bigoplus kx_{3\alpha+\beta}$.
Thus the parabolic exponents are $0, 3$.

Conclusion. The restriction map gives an isomorphism $Y \xrightarrow{\sim} R[y + V]$.
Moreover $\deg x' = 4$. As a consequence $Y = Y(p^-_\Pi)$.

By contrast, the false degrees are $2, 3$.

Some examples in type A.

We can assume h to be π' dominant.

In all cases we found we could choose y in the form

$$y = \sum_{\alpha \in S} x_\alpha$$

where $S \subset \triangle^+ \cup \triangle_{\pi'}$ such that $S|_{h_\Pi}$ is the basis for h^*_Π.

Call S the support of y