On π' set $i = -w_0'\big|_{\pi'}$ and suitably extend i to π.

Then Π is the set of $\langle ij \rangle$ orbits in π.

Set $\Pi_1 = \{\Gamma \in \Pi / \, j\,\Gamma \neq \Gamma\}$.

$\Pi_2 = \Pi \setminus \Pi_1$.

Degrees For simplicity we just consider type A

Set $n=|\pi|$. Assign (in type A_n) to the i^{th} simple root the integer $\min\{i, n+1-i\}$.

Similarly assign integers >0 to the roots of π'.

For each $\alpha \in \pi$, let $d_\alpha^{\pi'}$ be the sum of the above two integers assigned to α.

For all $\Gamma \subset \Pi$, set

$$d_\Gamma = \sum_{\alpha \in \Gamma} d_\alpha^{\pi'}.$$

Definition and theorem (FJ2,3)

(i) If $\Gamma \in \Pi_2$, then $d_\Gamma^f = d_\Gamma$

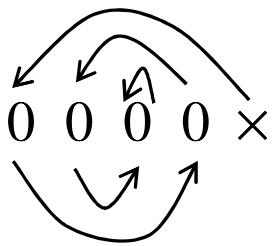
(ii) If Γ, j$\Gamma \in \Pi_1$, then $\{d_\Gamma^f, d_{j\Gamma}^f\} = \{d_\Gamma, d_\Gamma + 1\}$

(iii) In the (present) type A case the d_Γ^f are the true degrees.

Example The big parabolic in $sl(6)$.

Notation 0 means $\alpha \in \pi'$, $\times$ means $\alpha \notin \pi'$

$$\begin{array}{lcll} 1\,2\,3\,2\,1 & & & 2\,4\,5\,3\,1 \\ 0\,0\,0\,0\,\times & \Rightarrow & d_\alpha & 0\,0\,0\,0\,\times \\ 1\,2\,2\,1 & & & \end{array}$$



$0\ 0\ 0\ 0\ \times \quad \Rightarrow \quad \Gamma = \pi$ and $d_\Gamma = 15 = |\Delta^+|$.

Description of the pair *h, y*

Entries in the matrices below signify *h*-eigenvalues of the corresponding matrix unit.

Parabolic exponents are underlined and entries of *S* are encircled.

In the above case

$$
\left(\begin{array}{ccccc|c}
0 & 11 & 12 & 13 & \underline{14} & 10 \\
-11 & 0 & 1 & 2 & 3 & \enclose{circle}{-1} \\
-12 & \enclose{circle}{-1} & 0 & 1 & 2 & -2 \\
-13 & -2 & \enclose{circle}{-1} & 0 & 1 & -3 \\
-14 & -3 & -2 & \enclose{circle}{-1} & 0 & -4 \\
\hline
 & & & & & 0
\end{array}\right)
$$

<u>Example</u> The Borel in *sl(7).*

$$\begin{array}{cccccccc} & & 1 & 2 & 3 & 3 & 2 & 1 \\ d_\alpha & : & \times & \times & \times & \times & \times & \times \end{array}$$

All orbits are in Π_1, and are singletons.

$$\left\{ d_\Gamma^f \right\} = \{1, 2, 2, 3, 3, 4\}$$

In the above case

$$\begin{pmatrix} 0 & \underline{1} & \underline{2} & \underline{3} & \underline{2} & \underline{1} & \underline{0} \\ & 0 & 1 & 2 & 1 & 0 & \circled{-1} \\ & & 0 & 1 & 0 & \circled{-1} & -2 \\ & & & 0 & \circled{-1} & -2 & -3 \\ & & & & 0 & -1 & -2 \\ & & & & & 0 & -1 \\ & & & & & & 0 \end{pmatrix}$$

The rule of the rooks.

Suppose the Levi factor r_Π of p_Π has m blocks. Then there are m^2-1 non - empty upper right hand corner rectangles lying in p_Π whose sides are obtained by extending the sides of the blocks.

These above rectangles are r_Π modules with a natural quotient action of p^-_{Π}.

Consider an element of S as a rook in the corresponding square.

In every rectangle the rooks must cover all the squares not occupied by a parabolic exponent or by a diagonal entry.

Since $S\big|_{h_\Pi}$ is a basis for h^*_Π the rooks need not cover each other.

Integrality

It seems that all the eigenvalues of h should be integer (as consequence of y being regular). For $p = g$ this is expressed by the regular nilpotent orbit being even.

Integrality is quite restrictive; but difficult to express.

The Symmetric Case $j(\pi') = \pi'$

This case is amenable to induction.

One adds one interior block or two interior blocks of the same size in the middle of the previously determined matrix of integers.

The set of parabolic exponents and V itself simply increase in size.

A corner rectangle which is a square of say size m contains $(m-1)$ rooks not covering the intersection of the first column and some t^{th} row (t being computed inductively).

In that place $m-1$ is placed.

The corresponding minor is an invariant of degree m (in $Y(p^-_\Pi)$).

Suppose our matrix has size $2m$.

Suppose we add one interior block of size $n \geq 1$.

The entries of the interior block are defined by $h(\alpha_i) = 1$: $i = m + 1, \ldots, m + n - 1$.

Since by FJ3 one has $\varpi_m - \varpi_{m+n} \in h_\Pi$ we can choose the $(m+1, m+n+1)$ entry to be -1.

As a consequence the new parabolic exponents appear on the t^{th} row and above the added block.

When two interior blocks of size n are added the solution is more complex.

The cases $n = 1, 2$ are special though easy. Assume $n > 2$.

The entries of the added block are

$\boxed{-1}$

$$
\begin{pmatrix}
0 & \underline{2m+n+1} & \underline{2m+n+2} & & & \\
 & 0 & 1 & & & \\
 & \boxed{-1} & 0 & 1 & & \\
 & & \boxed{-1} & \ddots & \ddots & \\
 & & & & & 1 \\
 & & & & \boxed{-1} & 0
\end{pmatrix}
\quad \overbrace{}^{n-2}
\quad
\begin{pmatrix}
\underline{m+n-1} & & & & \\
 & -(m+1) & & & \\
 & & \boxed{-1} & & \\
 & & & \boxed{-1} & \\
 & & & & \ddots \\
-(m+n) & & & & \boxed{-1}
\end{pmatrix}
\quad \overbrace{}^{n-2} \quad \boxed{-1}
$$

$$
\begin{pmatrix}
0 & 1 & & & & \\
\boxed{-1} & 0 & m+1 & & & \\
 & & 0 & 1 & & \\
 & & & 0 & \ddots & \\
 & & & & & 1 \\
 & & & & & 0
\end{pmatrix}
\quad \overbrace{}^{n-3}
$$

The support of y should increase by $2n-1$.

Suppose $m > 1$. We must add two further -1 entries.

Since $\varpi_m - \varpi_{m+2n} \in h_\Pi$ one may choose -1 to be at the $(m+2, m+2n+1)$ place. This puts -1 on the $(t, m+n+2)$ place.

Finally the remaining additional parabolic exponent lies at $(t, m+2)$ and at $(m+2n+1, m+n+2)$.

Observe that
$[x_{t,m+n+2}, x_{m+n+2,m+2}] = x_{t,m+2}$,
$[x_{m+n+2,m+2}, x_{m+2,m+2n+1}] = x_{m+n+2,m+2+1}$

Of course $x_{m+n+2,m+2} \in p^-_\Pi$, but it cannot be used twice, so we should take say $x_{t,m+2} \in V$.

Example. The case of $sl(3)^4$ in $sl(12)$.

$$
\begin{array}{ccc|ccc|ccc|ccc}
0 & \underline{4} & \underline{5} & -7 & \underline{3} & 4 & -2 & \enclose{circle}{-1} & 3 & \underline{2} & 3 & 4 \\
-4 & 0 & 1 & -11 & -1 & 0 & -6 & -5 & -1 & -2 & \enclose{circle}{-1} & 0 \\
-5 & \enclose{circle}{-1} & 0 & -12 & -2 & -1 & -7 & -6 & -2 & -3 & -2 & \enclose{circle}{-1} \\
 & & & 0 & \underline{10} & \underline{11} & \underline{5} & 6 & 10 & 9 & 10 & 11 \\
 & & & -10 & 0 & 1 & -5 & -4 & 0 & \enclose{circle}{-1} & 0 & 1 \\
 & & & -11 & \enclose{circle}{-1} & 0 & -6 & -5 & \enclose{circle}{-1} & -2 & -1 & 0 \\
 & & & & & & 0 & 1 & 5 & 4 & 5 & 6 \\
 & & & & & & \enclose{circle}{-1} & 0 & 4 & 3 & 4 & 5 \\
 & & & & & & -5 & -4 & 0 & -1 & 0 & 1 \\
 & & & & & & & & & 0 & 1 & 2 \\
 & & & & & & & & & \enclose{circle}{-1} & 0 & 1 \\
 & & & & & & & & & -2 & -1 & 0
\end{array}
$$

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[FJ1] F. Fauquant-Millet et A. Joseph, “Sur les semi-invariants d’une sous-algèbre parabolique d’une algèbre enveloppante quantifieé”, Transf. Groups 6, no.2, 125-142, 2001.

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