Selected Topics in Enveloping Algebras

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Avant-Propos

On beginning research in Mathematics, or indeed in any Science, the novice is faced with the daunting task of finding a result not already submerged in the immense ocean of knowledge stretched out before him. In many ways the greatest gift that a scientific director can bestow upon his student is the specification of a problem to be tackled. This frees him from the harrowing thought that there might be nothing new to discover in whatever line he chooses to pursue. Yet open questions come with a price; they simply would not have remained unresolved if their solution could be found by following some well-trodden path. Again unlike textbook problems encountered in undergraduate experience they possess no guarantee that their solution is accessible or even exists.

Despite what would seem a hopeless task to accomplish original work, the journals abound with new discoveries. Even if one discounts all but the tiniest fraction as being modest variations on a well-known theme, one must still conclude that an extraordinary quantity of new material is being added to the scientific patrimony year by year. How this is achieved may puzzle the onlooker and despite popular conception the answer is seldom provided by Rodin's thinker reflecting quietly in isolation.

Our would-be researcher must not only find accessible problems to tackle; but must also acquire the strength of mind and the motivation to embark on what could be a fruitless and all-consuming search. Again the successful resolution of what at first appeared to be a significant question may in retrospect seem no more than a mere play on words. How indeed does one assess the significance of a mathematical discovery. This can be a task no less easy nor more objective than judging a poem or a painting.

In these lectures we present a number of recent results in the theory of enveloping algebras. They are chosen to illustrate the many different facets of the subject without requiring an extensive technical virtuosity to digest. A primary aim is to show how new ways of presenting such classical concepts as a group or a representation can lead to novel and more efficient proofs of old results as well as inspiring entirely new ideas. Their value is assessed in the overall way we perceive the subject, which of course deepens with time, and how they in turn modify our understanding of it. Besides this there is an undeniable thrill in resolving a question of evident complexity which may bring to light more efficient computational schemes and resolve other previously intractable issues. Perhaps however the most important message we wish to convey through the presentation of these selected topics is that in order to discover original and worthwhile results it is necessary to formulate and to adopt a certain outlook towards one's chosen field as well as to the mathematical endeavour as a whole.

1. A Hopf Algebra Approach to Algebraic Groups and Enveloping Algebras.

1.1. Symmetries and groups.

1.1.1. The notion of a group is so simple, elegant and powerful that it is difficult to believe that it could possibly be just a human invention. Indeed groups come from symmetries and symmetries are found in nature. The builders of the Alhamra Palace in Grenada glorified its walls and ceilings not with human images which were forbidden; but through a pictorial presentation of the 19 two-dimensional crystallographic groups. Rubrick's cube enticed many with the dazzling complexity of groups. Mathematicians on the other hand view this incredibly complex variety of patterns through the apparently banal concept of a group acting on a set. We examine this deceptively simple construction in the next section.

1.1.2. Let X be a set and $S(X)$ the set of all bijections of X. There is a natural product on $S(X)$ obtained by composition of bijections defined as follows. Given $g, h \in S(X)$ define $gh \in S(X)$ through

$$(*) \qquad (gh)x = g(hx), \quad for\ all \ \ x \in X.$$

This may be viewed as an associativity condition and so rather naturally the product in $S(X)$ is associative. Since the trivial map e sending each $x \in X$ to itself is again a bijection, it follows that $S(X)$ possesses a neutral element. Again every bijection has a left and right inverse. Combined these three conditions on the product in $S(X)$ exactly describe the axioms of a group. Moreover given a group G and a set X, then the pair (G, X) satisfying $(*)$ is called a G-set and we say that G acts on X. When X is a finite set possessing say n elements, we denote $S(X)$ by S_n. This is again a finite set indeed of cardinality $n!$ It constitutes the primary example of a finite group.

1.1.3. The richness of group theory stems partly from the ease with which one may find subgroups. This can also be understood from the construction of 1.1.2. Indeed given a group G acting on a set X, a subgroup H of G obtains as a stabilizer of any $x \in X$. For the natural action of $S(X)$ on X, this is not so interesting since $Stab_{S(X)}\ x = S(X - \{x\})$ which is a group of the same type. However we can observe that a group acts on itself by conjugation, that is by the rule $(g, h) \mapsto ghg^{-1}$. Then the set $\{Stab_{S(X)}\ g : g \in S(X)\}$ provides a rich variety of subgroups of $S(X)$.

1.1.4. An entirely new phenomenon arises when we try to consider the action of a group on a Cartesian product. In a natural fashion the direct product $G \times G$ acts on the Cartesian product $X \times Y$ of G sets $X,\ Y$ through the rule $((g, h), (x, y)) \mapsto (gx, hy)$. However viewing G as the diagonal copy in $G \times G$ we can also obtain an action of G on $X \times Y$. This is made possible by the fact that the diagonal map $\Delta : g \mapsto (g, g)$ of G to $G \times G$ is compatible with the product in G and in $G \times G$, that is to say one has $\Delta(gh) = \Delta(g)\Delta(h)$.

1.1.5. We may conclude that the notion of a symmetry, or G-set, leads to a multitude of maps (ultimately five in all). These are the product $\mu : G \times G \to G$, the coproduct $\Delta : G \to G \times G$, the antipode $\sigma : g \mapsto g^{-1}$, the identity map η corresponding to the neutral element of G and its dual the coidentity ε. They satisfy a number of quite restrictive axioms.

1.2. Hopf algebras.

1.2.1. From the above viewpoint one is naturally led to ask if there can be other sets with such maps which are nevertheless different from groups. This is rather optimistic since already the compatibility of the product and coproducts implies severe restrictions. However if we replace sets by vector spaces then additional flexibility is introduced and this leads to the fruitful notion of a Hopf algebra a brief review of which is given below.

1.2.2. Fix a base field K which will be assumed commutative. A Hopf algebra H is a k-vector space together with a collection of five linear maps $(\mu, \Delta, \eta, \varepsilon, \sigma)$. (On any one of these a subscript may be used to designate the Hopf algebra to which they are assigned.) They satisfy in particular the following conditions. The product $\mu : H \otimes H \to H$ is required to be associative. This can be expressed through the commutativity of an appropriate diagram. The coproduct $\Delta : H \to H \otimes H$ is required to be coassociative, a condition which is expressed by reversing the arrows in the commutative diagram defining the associativity of μ. The desired compatibility of μ and Δ is expressed through the commutativity of the diagram

$$
\begin{tikzcd}
 & H \arrow[dr, "\Delta"] & \\
H \otimes H \arrow[ur, "\mu"] \arrow[d, "\Delta \times \Delta"'] & & H \otimes H \\
H \otimes H \otimes H \otimes H \arrow[rr, "\tau"] & & H \otimes H \otimes H \otimes H \arrow[u, "\mu \times \mu"']
\end{tikzcd}
$$

where τ denotes the exchange of the entries of the second and third factors of H. This diagram is unchanged on reversing arrows and so one immediately deduces that not only is the coproduct a morphism (of algebras); but that the product is also a comorphism (of coalgebras). The identity $\eta : k \rightarrow H$ is a morphism and comorphism such that $\eta(1)$ is a left and right identity for H, conditions which can be expressed by the commutativity of appropriate diagrams. The coidentity $\varepsilon : H \rightarrow k$ is a morphism and comorphism satisfying the conditions that the above diagrams, with arrows reversed, are commutative. Finally the antipode $\sigma : H \rightarrow H$ satisfies the condition that the diagram

$$
\begin{tikzcd}
 & H \otimes H \arrow[rr, "\sigma \otimes 1d"] & & H \otimes H \arrow[dr, "\mu"] & \\
H \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] \arrow[rr, "\varepsilon"] & & k \arrow[rr, "\eta"] & & H \\
 & H \otimes H \arrow[rr, "1d \otimes \sigma"] & & H \otimes H \arrow[ur, "\mu"'] &
\end{tikzcd}
$$

is commutative. It is unique if it exists and is automatically an antimorphism and anticomorphism. It is designed to recover the analogue of conjugation.

1.2.3. A fundamental question which now arises is how to construct Hopf algebras. We have of course the example of a group algebra kG and there is also the closely related notion of an enveloping algebra $U(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$, the definition of which we will come to later. Here one may note that the coproduct is fundamentally the same for all group algebras (resp. enveloping algebras). For example in the first case it is the linear map of kG to $kG \otimes kG$ extending the diagonal embedding $g \mapsto (g, g)$. It is immensely difficult to construct further examples. The trouble is that we have no analogue of a G-set which provided such a rich variety of groups. Again it is quite unclear how to describe the Hopf subalgebras of a given Hopf algebra. There are barely no examples of finite dimensional semisimple Hopf algebras besides group algebras and their duals. Eventually we shall introduce an important family of Hopf algebras known as quantized enveloping algebras. These arise in particular as deformations of enveloping algebras. At first it seemed that their study was merely an exercise in handle turning; but it soon became clear that they were deep new objects which could shed a valuable light on the enveloping algebras themselves.

1.2.4. Hopf algebras have some astonishing features which could hardly be guessed at from their formal definition. Perhaps the most important of these is the notion of the Hopf dual. If H is a finite dimensional Hopf algebra then its dual H^* is again a Hopf algebra in which notably the roles of product and coproduct are interchanged. However if H is infinite dimensional, then a difficulty arises from the fact that the natural injection $H^* \otimes H^* \hookrightarrow (H \otimes H)^*$ is not surjective. Consequently the dual map to the product, namely $\mu^* : H^* \to (H \otimes H)^*$ fails to be a coproduct on H^*. At first sight this may seem to be a crippling disadvantage. However it is possible to define a subspace $H^\star$ of H^*, called the Hopf dual, which does acquire the structure of Hopf algebra by restriction of the dual maps. In general there is a price to pay since $H^\star$ may be too small to separate the elements of H.

1.3. Algebraic groups.

1.3.1. One of the first great successes of Hopf algebras came with the theory of algebraic groups. Roughly speaking an algebraic group is a group in which multiplication is a morphism of algebraic varieties. This notion can get along very well without the trappings of Hopf algebra formalism and indeed this was so in the beginnings of the subject. However some simple questions already illustrate the value of this formalism. For example one can ask how the two algebraic structure of group multiplication and morphism of algebraic varieties are related or are even compatible. The answer is dazzlingly simple when put in the Hopf algebra context. One is the product and the other the dual of the coproduct, their compatibility being the commutativity of the first diagram in 1.2.2.

1.3.2. Following the above one may ask if the Hopf dual of an arbitrary group algebra kG provides G with the structure of an algebraic group. Here two difficulties arise. First $(kG)^*$ need not be separating, second it need not be finitely generated. If only the latter fails, it may still be possible to select finitely generated Hopf subalgebras of $(kG)^*$ which are separating. However the choice need not be unique and this places into a natural context the previously known difficulty that the group structure on G need not determine G as an algebraic group.

1.3.3. Perhaps the most striking advantage of the Hopf algebra approach to algebraic groups is that it allows one to pass to the Lie algebra $\mathfrak{g}$ without resorting to analysis. This is achieved by taking the second Hopf dual $(kG)^{\star\star}$ in which the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ resides. However there is a price to pay, namely $\mathfrak{g}$ need not be finite dimensional. This arises when $(kG)^\star$ is not finitely generated. Again one may not recover $\mathfrak{g}$ from $U(\mathfrak{g})$; but only a bigger Lie algebra, called the algebraic hull of $\mathfrak{g}$, which contains $\mathfrak{g}$ as an ideal. The correspondence between Lie algebras and algebraic groups is not so clear as it is for Lie groups. From the point of view of analysis this is a result of the possibility that a differential equation may have an analytic but not an algebraic solution.

1.3.4. Obviously we would like to benefit from the insight gained from algebraic groups to study quantized enveloping algebras, just as non-commutative algebra is inspired by algebraic geometry. It turns out that quantized enveloping algebras have a much tighter structure and can even exhibit the remarkable property of being essentially self-dual. Such can only be true for commutative group algebras of finite groups. Here already a challenging and completely open problem is to describe finite dimensional self-dual Hopf algebras.

1.4. Representations.

1.4.1. Perhaps the most remarkable aspects of Hopf algebras derives from their representation theory. In this let A be an associative k-algebra. Passing to modern terminology define an A module to be a vector space admitting an action of A, that is to say there exists a linear map $\lambda : A \otimes M \to M$ such that the diagram

$$\begin{tikzcd}
& A \otimes M \arrow[dr, "\lambda"] & \\
A \otimes A \otimes M \arrow[ur, "\mu \otimes 1d"] \arrow[dr, "1d \otimes \lambda"'] & & M \\
& A \otimes M \arrow[ur, "\lambda"'] &
\end{tikzcd}$$

is commutative. If $M = A$ and $\lambda = \mu$, this condition is just that A is associative. In addition we require the identity of A to act like the identity on M.

1.4.2. An assocative algebra may have few interesting modules. For example if V is a finite dimensional vector space and $A = End\ V$, then every A module is a direct sum of copies of V. This situation changes drastically when A is a Hopf algebra H. Indeed given H modules M, N their tensor product $M \otimes N$ is naturally an $H \otimes H$ module and hence an H module through the coproduct. In general, $M \otimes N$ is very different to M or N and furthermore its decomposition yields a rich variety of further H modules.

1.4.3. A particularly striking example of the above comes when we consider the set of invertible elements in $End\ V$ as a group through multiplication. Here the linear relations in $End\ V$ are forgotten. The resulting group, called the general linear group and denoted by $GL(V)$, admits a whole host of new modules obtained from V by decomposition of its repeated tensor powers. It is perhaps significant that these representations are of great importance in particle physics. Moreover tensor product has the physical interpretation of combining particles. Then the decomposition of the tensor product describes how particles can be combined to give new ones according to well-defined rules established by the mathematician.

1.4.4. In order to progress in the theory of Hopf algebras we need to overcome a minor technical difficulty introduced by the coproduct. First of all $\Delta(h) : h \in H$, is in general a finite sum of elements of the form $h' \otimes h'' : h', h'' \in H$. Worse still such a presentation is determined only up to linear transformation. This latter difficulty is not so serious in that all the transformations on H and on its tensor powers which will be considered are linear. However we do have to remember that sums are involved and at the same time we must try to avoid cumbersome notation. Here we adopt Sweedler's well-known notation writing for example

$$\Delta(h) = \sum h_{(1)} \otimes h_{(2)} \tag{$*$}$$

for such a sum. We may sometimes wish to drop the summation sign and even the parentheses on suffixes.

1.4.5. Coassociativity of the coproduct means that $(1d \otimes \Delta)\Delta = (\Delta \otimes 1d)\Delta$. This common map will be denoted by Δ^2. Similarly Δ^m denotes the m-th fold iterate of Δ mapping H to the $(m+1)$-th fold tensor power $H^{\otimes(m+1)}$ of H. In calculating $(1d \otimes \Delta)\Delta$ on $h \in H$ we replace $h_{(2)}$ occurring in 1.4.4($*$) by $h_{(2)} \otimes h_{(3)}$ and in this manner we obtain

$$\Delta^2(h) = \sum h_{(1)} \otimes h_{(2)} \otimes h_{(3)}.$$

Since this must also represent $(\Delta \otimes 1d)\Delta$ we see that applying $(\Delta \otimes 1d)$ to 1.4.4($*$) we must not only replace $h_{(1)}$ by $h_{(1)} \otimes h_{(2)}$; but also replace the former $h_{(2)}$ by $h_{(3)}$. In this terminology one may observe that the coidentity ε satisfies

$$\sum h_{(1)}\varepsilon(h_{(2)}) = \sum \varepsilon(h_{(1)})h_{(2)} = h. \tag{$*$}$$

Again the antipode satisfies

$$\sum h_{(1)}\sigma(h_{(2)}) = \sum \sigma(h_{(1)})h_{(2)} = \varepsilon(h). \tag{$**$}$$

1.5. Lie algebras and their enveloping algebras.

1.5.1. To understand the formal definition of a Lie algebra $\mathfrak{g}$ one should first realize that it is in essence an infinitesimal manifestation of a "continuous" group G. Group multiplication becomes addition at the infinitesimal level, so we should at least expect $\mathfrak{g}$ to be a vector space. However this is insufficient to take account of a possible noncommutativity of G. We need also the infinitesimal version of the commutator of two group elements and this becomes the Lie product. Let us avoid the formal definitions for the moment and just remark that its enveloping algebra $U(\mathfrak{g})$ is the associative algebra with identity 1 generated by $\mathfrak{g}$ in which the commutator $xy - yx$ of elements $x, y \in g$ is given by their Lie product. Moreover the coproduct Δ, coidentity ε and antipode σ on

$U(\mathfrak{g})$ are defined at the level of $g \oplus k$ and are then just the infinitesimal analogues of the corresponding objects for G. Indeed

$$\Delta(1) = 1 \otimes 1,\ \Delta(x) = x \otimes 1 + 1 \otimes x;\ \ \varepsilon(1) = 1,\ \varepsilon(x) = 0;\ \ \sigma(1) = 1,\ \sigma(x) = -x,\ \ \forall\, x \in \mathfrak{g}.$$

The first two are extended as algebra morphisms to $U(\mathfrak{g})$ and σ as an algebra antimorphism. In this we need to observe the compatibility of the coproduct with the commutator which is expressed through the formula

$$\Delta(x)\Delta(y) - \Delta(y)\Delta(x) = (xy - yx) \otimes 1 + 1 \otimes (xy - yx) = \Delta(xy - yx).$$

1.5.2. Let V be a finite dimensional vector space. One may consider $End\ V$ as a Lie algebra by taking the Lie product to be the commutator of elements in this associative algebra. It is denoted by $\mathfrak{gl}(V)$ and is the Lie algebra of $GL(V)$ in the sense of algebraic groups. Just as in the case of $GL(V)$ the relations in $\mathfrak{gl}(V)$ do not include all of those in $End\ V$. Moreover the coproduct allows one consider the tensor powers of V as a module over $\mathfrak{gl}(V)$. A crucial result in the decomposition of this tensor product is the subject of our first selected topic.

1.6. Schur's centralizer theorem (after D. Fischman).

In the following the base field k is assumed to be of characteristic zero. Let $SL(V)$ be the subgroup of elements of $GL(V)$ of determinant 1. Then $GL(V)$ is a direct product of $SL(V)$ and the mulplicative group $k^* := k - \{0\}$.

1.6.1. Let V be a finite dimensional vector space. For any positive integer m let $V^{\otimes m}$ denote the m-th tensor power of V. The symmetric group S_m acts naturally on $V^{\otimes m}$ by permuting the entries. One may observe that this action commutes with the diagonal action of $GL(V)$. Consequently the image of $kGL(V)$ in $End\ V^{\otimes m}$ is contained in $End_{S_m} V^{\otimes m}$.

Schur's commutator theorem asserts that this inclusion is an equality. It implies that the isotypical decomposition of $V^{\otimes m}$ as an S_m module coincides with its isotypical decomposition as a $GL(V)$ module and consequently the simple $GL(V)$ modules occurring in $V^{\otimes m}$ are classified by simple S_m modules. Moreover V is a faithful $SL(V)$ module from which one may further conclude that every simple continuous finite dimensional $SL(V)$ module occurs in some tensor power of V, which are hence classified by the simple

$S_m : m \in \mathbb{N}^+$, modules up to possible repetitions. For example the trivial $SL(V)$ module corresponds to the trivial S_m module for all $m \in \mathbb{N}^+$.

1.6.2. Schur's centralizer theorem may be expressed in the following equivalent form. Let ρ_m be the homomorphism of $U(\mathfrak{gl}(V))$ into $End\ V^{\otimes m}$ defined by the action of $U(\mathfrak{gl}(V))$ on $V^{\otimes m}$. We denote ρ_1 simply by ρ. Observe that

$$\rho_m(b) = \rho^{\otimes m}(\Delta^{m-1}(b)), \quad for\ all \ \ b \in U(\mathfrak{g}).$$

Theorem. $$Im(U(\mathfrak{gl}(V))) = End_{S_m} V^{\otimes m}.$$

We shall give a proof of this fact due to D. Fischman. It is rather more difficult than Schur's equivalent result and the reason for making this extra effort will be revealed later. Set $\mathfrak{g} = \mathfrak{gl}(V)$. Given $a_1, a_2 \in End\ V$ we denote by $a_1 a_2$ their product in $End\ V$. However we may also consider $a_1 a_2$ as elements of g and we denote their product in $U(g)$ by $a_1.a_2$. This distinction is particularly important when it comes to calculating the coproduct. For example

$$\begin{aligned}\Delta(-a_1a_2 + a_1.a_2) &= -(a_1a_2 \otimes 1 + 1 \otimes a_1a_2) + (a_1 \otimes 1 + 1 \otimes a_1)(a_2 \otimes 1 + 1 \otimes a_2) \\ &= \ a_1 \otimes a_2 + a_2 \otimes a_1 \ ,\end{aligned}$$

which verifies the assertion of the theorem for $m = 2$.

1.6.3. The element

$$e_m := \frac{1}{m!} \sum_{s \in S_m} s$$

is the projector onto the trivial S_m module. Consequently

$$End_{S_m} V^{\otimes m} = e_m(End\ V^{\otimes m}).$$

The inclusion $\rho_m(U(\mathfrak{g})) \subset End_{S_m} V^{\otimes m}$ is trivial. We prove the opposite inclusion by induction on m. For $m = 1$ it follows from the definition of $\mathfrak{gl}(V)$. In particular every element $x \in End\ V$ can be viewed as an element of $\mathfrak{g}$. Then if we let $(1, j) : j = 1, 2, \cdots, m$ be the permutation which exchanges the first and j-th entries in $V^{\otimes m}$, it follows from the definition of the coproduct that

$$(*) \qquad \sum_{j=1}^{m} (1, j)(x \otimes 1^{\otimes m-1}) = \Delta^{m-1}(x).$$

Identify $End\ V^{\otimes m}$ with $(End\ V^{\otimes m})$. For all $a_1, a_2, \cdots, a_m \in End\ V$ we have

$$e_m(a_1 \otimes a_2 \otimes \cdots \otimes a_m) = \frac{1}{m} \sum_{j=1}^{m} (1,j)(a_1 \otimes e_{m-1}(a_2 \otimes \cdots \otimes a_m).$$

By the induction hypothesis there exists $b \in U(\mathfrak{g})$ such that $e_{m-1}(a_2 \otimes \cdots \otimes a_m) = \rho_{m-1}(b) = \rho^{\otimes m-1}\Delta^{m-2}(b)$. From the relations in 1.4.5 we obtain

$$\begin{aligned} 1 \otimes \Delta^{m-2}(b) &= \sum \varepsilon(b_{(1)}) \otimes b_{(2)} \otimes \cdots \otimes b_{(m)} \\ &= \sum (\sigma(b_{(1)}) \otimes 1^{\otimes(m-1)})\Delta^{m-1}(b_{(2)}). \end{aligned}$$

Since $\rho(U(\mathfrak{g}))$ is S_m invariant this and $(*)$ gives

$$\begin{aligned} me_m(a_1 \otimes a_2 \otimes \cdots \otimes a_m) &= \sum_{j=1}^{m} (1,j)[(a_1\rho(\sigma(b_{(1)})) \otimes 1^{\otimes m-1})(\rho^{\otimes m}\Delta^{m-1}(b_{(2)}))] \\ &= \Delta^{m-1}(a_1\rho(\sigma(b_{(1)})))\rho^{\otimes m}(\Delta^{m-1}(b_{(2)})) \\ &= \rho_m(a_1\rho(\sigma(b_{(1)})))\rho_m(b_{(2)}) \\ &= \rho_m(a_1\rho(\sigma(b_{(1)}))) \cdot b_{(2)}) \in \rho_m(U(\mathfrak{g})) \end{aligned}$$

as required. □

1.6.3. A pleasant fact is that the above calculation is valid for $\mathbb{Z}_2$ graded versions of $\mathfrak{gl}(V)$, that is to say for super Lie algebras obtained when V and then $End\ V$ is given a $\mathbb{Z}_2$ graded structure. With an appropriate modification of the action of S_m on $V^{\otimes m}$, the relation $(*)$ of 1.6.2 still holds. Then the image of the enveloping algebra of the correponding super Lie algebra is the centalizer in $End\ V^{\otimes m}$ of this modified S_m action. On the other hand Schur's easy proof using $GL(V)$ fails to extend because one does not have an appropriate "super" analogue of a Lie group.

1.6.4. Schur's centralizer theorem implies that $End_{U(\mathfrak{g})}V^{\otimes m}$ is the image of kS_m in $End\ V^{\otimes m}$. Hence that the images both kS_m and of $U(\mathfrak{g})$ are their own double centralizers in $End\ V^{\otimes m}$. For this reason it is usually called Schur's double centralizer theorem.

1.6.5. Schur's original centralizer theorem also goes through for infinite fields of positive characteristic. A unified proof of such results has recently been given by S. Koenig, I. H. Slungaard and C. Xi. This uses rather more sophisticated methods from representation theory. One calls $End_{S_m}V^{\otimes m}$, a Schur algebra.

2. A Royal Route to Lie Algebras.

2.1. First Beginnings.

2.1.1. Let A be an associative algebra. Given $a, b \in A$, their commutator $[a, b] := ab - ba$ defines a new product on A. This is not associative; but it is antisymmetric and satisfies the Jacobi identity, that is one has

$$\text{(i)} \qquad [a, b] + [b, a] = 0$$

$$\text{(ii)} \qquad [a, [b, c]] + [c, [a, b]] + [b, [c, a]] = 0.$$

One defines *Lie A* to be A as a vector space equipped with this new product.

2.1.2. One defines a Lie algebra to be a vector space with a Lie product being a linear map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying (i), (ii) of 2.1.1. In particular *Lie A* is a Lie algebra and for example in our previous notation *Lie End* $V = \mathfrak{gl}(V)$.

A representation ρ of a Lie algebra $\mathfrak{g}$ is a Lie algebra morphism of $\mathfrak{g}$ into $\mathfrak{gl}(V)$ for some vector space V. Then ρ defines an action of $\mathfrak{g}$ on V satisfying some easily discernible rules and makes V a $\mathfrak{g}$ module.

2.1.3. At first sight the construction of 2.1.1 may seem entirely superfluous and so it should come as a considerable surprise that it stands at the gateway of an immensely rich theroy. The most mundane explanation of this miracle is that the passage from A to *Lie A* adroitly discards relations in A which constricts the flexibility of A. A more informed explanation is that it comes from the possibility of introducing a coproduct - a procedure we already encountered in 1.5.1. This has the following consequence. Let M be an A module. It is clear that M acquires the structure of *Lie A* module. However whereas the tensor powers of M admit no obvious A module structure, they do admit a *Lie A* module structure through the coproduct. Therein lies the main interest of the above construction. A basic question which immediately comes to mind is whether all *Lie A* modules are obtained by decomposing tensor powers of A modules. To expect such a sweeping result strains optimism to the extreme; yet this viewpoint provides an underlying theme for the study of representations of Lie algebras and will be the subject of more than one of our "selecta".

2.1.4. Let $\mathfrak{g}$ be a Lie algebra. The enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is roughly speaking the universal object containing $\mathfrak{g}$ in which the Lie product on $\mathfrak{g}$ becomes the commutator in

$U(\mathfrak{g})$. More precisely consider the set of all pairs (φ, A), where A is an associative algebra and φ a Lie algebra morphism of $\mathfrak{g}$ into $Lie\ A$, for example a pair $(\rho, End\ V)$ defining a representation of $\mathfrak{g}$ on V. The (universal) enveloping algebra $(\iota, U(\mathfrak{g}))$ is such a pair with the property that given any other pair (φ, A) there exist a unique associative algebra morphism $\theta : U(\mathfrak{g}) \to A$ making the diagram

$$\begin{tikzcd} & U(\mathfrak{g}) \arrow[d, "\theta"] \\ \mathfrak{g} \arrow[ur, "\iota"] \arrow[dr, "\varphi"'] & \\ & A \end{tikzcd}$$

commutative. A standard argument shows that the universal pair $(\iota, U(\mathfrak{g}))$ is unique up to isomorphism.

2.1.5. Let V be a vector space. The tensor algebra $T(V)$ of V is the direct sum $k \oplus V \oplus V \otimes V \oplus \cdots$, with a product given by linearly extending the maps $(u_1 \otimes u_2 \otimes \cdots \otimes u_m) \otimes (v_1 \otimes v_2 \otimes \cdots \otimes v_n) \mapsto (u_1 \otimes u_2 \otimes \cdots \otimes v_n)$ of $V^{\otimes m} \otimes V^{\otimes n}$ into $V^{\otimes (m+n)}$. Let $\{v_i : i \in \mathcal{I}\}$ be a basis of V. Then $T(V)$ is the free associative algebra by $v_i : i \in \mathcal{I}$.

The universal pair $(\iota, U(\mathfrak{g}))$ satisfies the condition

$$\iota[a, b] - (\iota(a)\iota(b) - \iota(b)\iota(a)) = 0,$$

It follows easily that $U(\mathfrak{g})$ is just the quotient of $T(\mathfrak{g})$ by the two sided-ideal that these relations generate and ι the composition of the natural embedding $\mathfrak{g} \hookrightarrow T(\mathfrak{g})$ composed with the canonical projection $T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$. It is a fact; but not an obvious one that ι is injective. More precisely the Poincaré–Birkhoff-Witt theorem (PBW) asserts that if $\mathfrak{g}$ admits a linearly ordered basis $\{x_i : i \in \mathcal{I}\}$, then the standard monomials

$$x_{i_1}^{k_1} x_{i_2}^{k_2} \cdots x_{i_m}^{k_m} : k_1, k_2, \cdots, k_m \in \mathbb{N},\ i_1 < i_2 < \cdots < i_m,$$

form a basis for $U(\mathfrak{g})$.

The universality of $(\iota, U(\mathfrak{g}))$ implies that there is an equivalence between (the categories of) $\mathfrak{g}$ modules and $U(\mathfrak{g})$ modules. One further checks that $U(\mathfrak{g})$ is a Hopf algebra with coproduct Δ; coidentity ε and antipode defined as in 1.5.1.

2.2. Filtered and graded Algebras.

2.2.1. Let A be an algebra (not necessarily associative). An (increasing) filtration $\mathcal{F}$ of A is a family of increasing subspaces $\mathcal{F}^i A : i \in \mathbb{Z}$, satisfying

(i) $\displaystyle\bigcap_{i\in\mathbb{Z}} \mathcal{F}^i A = 0,$

(ii) $\displaystyle\bigcup_{i\in\mathbb{Z}} \mathcal{F}^i A = A,$

(iii) $(\mathcal{F}^i A)(\mathcal{F}^j A) \subset A.$

Observe that (iii) uses the semigroup structure on $\mathbb{Z}$. It may be convenient to introduce other semigroups. One may also define decreasing filtrations with analogous properties. An element $a \in \mathcal{F}^i A \setminus \mathcal{F}^{i-1} A$ is said to have degree i. An increasing filtration is said to be bounded from below if $\mathcal{F}^{-j} A = 0$, for j sufficently large. Such a condition is often essential for the success of an induction argument based on degree.

Filtrations most naturally arise from systems of generators. Thus the (canonical) filtration of $U(\mathfrak{g})$ obtains by taking $\mathcal{F}^i U(\mathfrak{g})$ to be the space spanned by all monomials of length $\leq i$ in the elements of $\mathfrak{g}$.

2.2.2. Let A be an algebra (not necessarily associative). A gradation of $\mathcal{G}_i A$ is a family of subspaces $\mathcal{G}_i A : i \in \mathbb{Z}$ of A such that

(i) $\displaystyle\bigoplus_{i\in\mathbb{Z}} \mathcal{G}_i A = A$

(ii) $(\mathcal{G}_i A)(\mathcal{G}_j A) \subset \mathcal{G}_{i+j} A.$

Gradations are generally more difficult to find than filtrations. For example $T(\mathfrak{g})$ admits a gradation defined by i-th tensor powers ; but this gradation does not pass to $U(\mathfrak{g})$ except if $\mathfrak{g}$ is commutative, that is to say unless all Lie products vanish.

2.2.3. Let $(A, \mathcal{F})$ be a filtered algebra. The associated graded algebra $gr_{\mathcal{F}} A$ is defined to be

$$gr_{\mathcal{F}} A = \bigoplus_{i\in\mathbb{Z}} \mathcal{F}^i A / \mathcal{F}^{i-1} A$$

with multiplication $(f, g) \mapsto fg \ mod \ \mathcal{F}^{i+j-1} A$, for all $f \in \mathcal{F}^i A, g \in \mathcal{F}^j A$. It may be viewed as an approximation of A and can be conveniently used to describe and to prove certain results concerning A. For example, let $\mathfrak{g}$ be a Lie algebra. Then the symmetric algebra $S(\mathfrak{g})$ of $\mathfrak{g}$ is defined to be the quotient of $T(\mathfrak{g})$ in which all commutators of elements of $\mathfrak{g}$ are set equal to zero. It is immediate that the identity map on $\mathfrak{g}$ extends to an algebra surjection of $S(\mathfrak{g})$ onto $gr_{\mathcal{F}}U(\mathfrak{g})$. The PBW theorem is equivalent to the assertion that this map is an isomorphism. That $S(\mathfrak{g})$ is a domain implies that so is $U(\mathfrak{g})$. Again suppose $\mathfrak{g}$ is finite dimensional. Then $S(\mathfrak{g})$ is noetherian and an induction argument on degree implies that $U(\mathfrak{g})$ is left and right noetherian.

2.2.4. It might be thought that the passage to the associated graded algebra is merely a book-keeping device enabling one to give a slick proof of obvious results (for example that $U(\mathfrak{g})$ is a domain). Nevetheless there are some extremely deep and difficult results relating to the passage, more than one of which will be treated in our "selecta". For the moment we just observe a beautiful corollary of Schur's theorem established in 1.6. First of all assume that *char* $k = 0$. Let V be a vector space. It is easy to identify $S(V)$ with a subalgebra of $T(V)$ using the projectors in 1.6.3. However it is much more convenient to identify $S(V)$ with the sum of all tensor powers of elements of V. To prove the equivalence of these two presentations is a tricky combinatorial exercise. A more convincing and simpler argument (which I learnt from J. Bernstein) is the following. First of all in any such computation only finitely many element of V are involved so we can assume that V is finite dimensional. Now each symmetric power $V^{\circledS m}$ is just the isotypical component of $V^{\otimes m}$ with respect to the trivial S_m module and moreover it occurs with multiplicity one. Thus by 1.6.2 it is a simple $U(\mathfrak{gl}(V))$ module. On the other hand the space of m tensor powers of elements of V is clearly a non-zero submodule which must coincide with $V^{\circledS m}$ by simplicity.

2.3. Graded Lie Algebras.

2.3.1. The structure theory of finite dimensional Lie algebras is fairly well understood. Familiarity with this theory is useful; but by no means essential for an understanding of the present material. The reason is that in many ways the most important finite dimensional Lie algebras are those which are simple and their structure is particularly well-elucidated and usefully extended in the context of graded Lie algebras. Again just as every finite group is a subgroup of some S_m, it is also true that every finite dimensional Lie algebra is a subalgebra of *End* V for some finite dimensional vector space V.

2.3.2. Let $\mathfrak{g}$ be a graded Lie algebra. Simplifying slightly the notation of 2.2.2 we shall write

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

recalling that by hypothesis

$$[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}, \forall\, i, j \in \mathbb{Z}.$$

This concept is particularly useful when $\mathfrak{g}$ is infinite dimensional and yet $dim\ \mathfrak{g}_i < \infty$, for all $i \in \mathbb{Z}$.

Observe that $\mathfrak{g}_0$ and

$$\mathfrak{g}_{\pm} := \bigoplus_{i>0} \mathfrak{g}_{\pm i}$$

are Lie subalgebras of $\mathfrak{g}$. Moreover we can write

$$\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_-$$

which is called a triangular decomposition of $\mathfrak{g}$. The PBW theorem gives a vector space isomorphism

$$U(\mathfrak{g}_+) \otimes U(\mathfrak{g}_0) \otimes U(\mathfrak{g}_-) \xrightarrow{\sim} U(\mathfrak{g}),$$

defined through multiplication in $U(\mathfrak{g})$. Triangular decomposition is useful because in general $\mathfrak{g}_{\pm}, \mathfrak{g}_0$ have simpler structures then $\mathfrak{g}$ itself. It may also give rise to the construction of highest weight modules which will be a subject of further discussion.

2.3.3. R. Moody and particularly V. Kac introduced and made an intensive study of graded Lie algebras which are generated by their "local" part $\mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. A local Lie algebra is a triple $(\mathfrak{g}_1, \mathfrak{g}_0, \mathfrak{g}_{-1})$ consisting of a Lie algebra $\mathfrak{g}_0$ and $\mathfrak{g}_{\pm 1}$ modules $\mathfrak{g}_0$ equipped with a $\mathfrak{g}_0$ module map

$$\psi : \mathfrak{g}_1 \otimes \mathfrak{g}_{-1} \to \mathfrak{g}_0.$$

These structures recover the commutation relations of $\mathfrak{g}$ which make sense in $\mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. Conversely a straightforward tensor product construction shows that there is a universal graded Lie algebra with a given local part. Of more interest however is the minimal Lie algebra with a given local part. It is defined as the quotient of the universal object by its unique maximal ideal $\mathfrak{r}$ having null intersection with the subspace $\mathfrak{g}_1 \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$. This ideal depends in a very subtle fashion on the nature of the local part.

2.3.4. Astonishingly the above construction not only recovers all finite dimensional simple Lie algebras; but also gives a rich variety of new algebras. We shall be particularly concerned with those algebras; known as Kac-Moody algebras in which the local part is given by an integer-valued $\ell \times \ell$ matrix $A = \{a_{i,j}\}$ in the following way.

Set $I = \{1, 2, \cdots, \ell\}$. Define a "realization" of A to be a triple consisting of a vector space $\mathfrak{h}$, a set of simple roots $\pi = \{\alpha_i : i \in I\} \subset \mathfrak{h}^*$ and a set of simple coroots $\pi^\vee = \{\alpha_i^\vee : i \in I\} \subset \mathfrak{h}$ such that $\alpha_i^\vee(\alpha_j) = a_{i,j}$. We shall require the elements of π and of $\pi^\vee$ to be linearly independant. This is easily arranged by taking $\mathfrak{h}$ to have dimension $2\ell - rk\ A$. Then one defines a local Lie algebra $(\mathfrak{g}_1, \mathfrak{g}_0, \mathfrak{g}_{-1})$ by taking $\mathfrak{g}_0 = \mathfrak{h}$ with the trivial Lie product and giving $\mathfrak{g}_1$ (resp. $\mathfrak{g}_{-1}$) a basis $\{x_i : i \in I\}$ (resp. $y_i : i \in I$) satisfying

$$[h, x_i] = \alpha_i(h)x_i,\ [h, y_i] = -\alpha_i(h)y_i,\ \forall\ h \in \mathfrak{h},\ i \in I.$$

Finally ψ is defined through

$$\psi(x_i, y_j) = \delta_{i,j}\alpha_i^\vee\ ,$$

where δ is the Kronecker delta. It is easy to that ψ is indeed an $\mathfrak{h}$ module map. One denotes by $\mathfrak{g}_A$ the minimal Lie algebra with local part given by A in the above fashion. The relation ideal $\mathfrak{r}$ in the universal graded Lie algebra depends in a complicated fashion on A. Its description in general is an open problem.

2.3.5. One considers more specifically the case when A above has diagonal entries all equal to 2 and negative off-diagonal entries satisfying in addition the requirement that $a_{i,j} = 0 \Longleftrightarrow a_{j,i} = 0$. The resulting minimal algebra $\mathfrak{g}$ is called a Kac-Moody algebra. Such algebras include all finite dimensional simple Lie algebras and their affinizations and will be of particular interest to us. They have a particularly well-developed theory. Outside the Kac-Moody case little is known. However our second selected topic studies some representation theory of a graded Lie algebra belonging to this vast and as yet barely explored field.

2.4. Uniformly bounded modules over a Lie algebra of Weyl type (after Y. Su).

In what follows we assume the base field k algebraically closed of characteristic zero.

2.4.1. The Weyl algebra A is the associative algebra with identity and generators $t, d/dt$ satisfying the relation

$$(d/dt)t - td/dt = 1.$$

An easy verification shows that the multiplicative subset S generated by t is Ore in A and we set $B = S^{-1}A$. One calls $\mathfrak{w} = Lie\ B$ a Lie algbera of Weyl type.

Set $h = td/dt$. It is clear that $\mathfrak{w}$ is a direct sum of its h weight subspaces under adjoint action and that this decomposition provides a gradation of $\mathfrak{w}$. Indeed if we set $H = k[h]$, then the weight subspace $\mathfrak{w}_i$ of weight $i \in \mathbb{Z}$, identifies with the vector subspace t^iH of B.

2.4.2. A $\mathfrak{w}$ module is said to be an h weight module if it is a direct sum of its h weight spaces and to be admissible if each of its weight subspaces are finite dimensional. For example under adjont action $\mathfrak{w}$ is a weight module but it is not admissible. An admissible module is said to be uniform if these dimensions are uniformly bounded. Finally with respect to the triangular decomposition $\mathfrak{w} = \mathfrak{w}_+ \oplus \mathfrak{w} \oplus \mathfrak{w}_-$ (defined as in 2.3.2) we say a module is of highest weight if it has a generator v which is an eigenvector of the subalgebra $\mathfrak{w}_+ \oplus \mathfrak{w}_0$.

2.4.3. The standard B module $M = k[t, t^{-1}]$, in which t, t^{-1} act by multiplication and d/dt acts by differentiation, viewed as a $\mathfrak{w}$ module is uniform. Su shows that M and its translates under $Aut\ \mathfrak{w}$ exhausts all simple uniform $\mathfrak{w}$ modules. Although this result is a little disappointing, his reduction is rather interesting and a good exercise in a little explored area of Lie algebra theory which we present below.

2.4.4. Let V be a weight module. For each $\alpha \in k$ we may define

$$V(\alpha) = \bigoplus_{i \in \mathbb{Z}} V_{\alpha+i}.$$

Since $\mathfrak{w}$ is $\mathbb{Z}$ graded, it follows that $V(\alpha)$ is a submodule of V and then that V is a direct sum of such modules. It what follows we shall assume that $V = V(\alpha)$ for some $\alpha \in k$.

2.4.5. For all $n \in \mathbb{N}$, one has $[t, h^n] = tp_n(h)$ for some polynomial of degree $n-1$. It follows that $ad\ t$ is a locally nilpotent derivation of $\mathfrak{w}$ and

$$(*) \qquad [t, \mathfrak{w}_i] = \mathfrak{w}_{i+1}, \ \forall\ i \in \mathbb{Z}.$$

Similar assertions hold for t^{-1}.

Proposition. *Let V be an admissible $\mathfrak{w}$ module. Suppose $0 \neq v \in V$ satisfies $tv = 0$ (resp. $t^{-1}v = 0$). Then the cyclic module generated by V contains a highest (resp. lowest) weight submodule.*

Proof. Both parts are similar and we consider just the first. Without loss of generality we can assume that v has weight $\alpha \in k$ and that it generates V. Since V_α is finite dimensional, there exists a finite dimensional subspace $\mathfrak{s}_0 \subset \mathfrak{w}_0$ such that $\mathfrak{s}_0 v = \mathfrak{w}_0 v$. Since $ad\ t$ is locally nilpotent there exists an integer $m > 0$ such that $(ad\ t)^{m+1}\mathfrak{s} = 0$. Then for all $r > m$ one has

$$(*) \qquad \mathfrak{w}_r v = ((adt)^r \mathfrak{w}_0)v = t^r \mathfrak{w}_0 v = t^r \mathfrak{s}_0 v = ((ad\ t)^r \mathfrak{s}_0)v = 0.$$

Set $\mathfrak{s}_+ = \sum_{i=1}^{m} \gamma_i$, and let $U(\mathfrak{s}_+)$ denote the subspace of $U(\mathfrak{w}_+)$ spanned by the standard monomials from a basis $x_i : i \in I$ for $\mathfrak{s}$ formed from weight vectors. Then the previous result and triangular decomposition gives

$$(**) \qquad V = U(\mathbf{w})v = U(\mathbf{w}_0 + \mathbf{w}_-)U(\mathfrak{s}_+)v = U(\mathfrak{s}_+)U(\mathbf{w}_0 + \mathbf{w}_-)v.$$

Set $V^m = \sum_{i=0}^{m} V_{\alpha+i}$, and define V^∞ similarly. Clearly $V^\infty \supset U(\mathfrak{w}_+)V^m$. The oppositive inclusion is proved by induction on the degree of $w \in V^\infty$. It is clear if $w \in V_{\alpha+i} : 0 \leq i \leq m$. Take $i > m$. By $(**)$ we may write $w = \sum u_j v_j$, for some $u_j \in \mathfrak{w}_j$ and $v_j \in V_{\alpha+i-j}$ with $0 < j \leq m$. Then $v_j \in U(\mathfrak{w}_+)V^m$, by the induction hypothesis, so $w \in U(\mathfrak{w}_+)V^m$, as required.

For each $i \in I$ we can write $x_i = u_i v$, for some $u_i \in U(\mathfrak{w})$. In view of $(*)$ we can find an integer $s > 0$ such that $\mathfrak{w}_r x_i \subset [\mathfrak{w}_r, u_i]v + u_i \mathfrak{w}_r v = 0$, for all $r > s$. Set

$s = max\{s_i : i \in I\}$. Then $\mathfrak{w}_r V^\infty = \mathfrak{w}_r U(\mathfrak{w}_+)V^m = 0$, for all $r > s$. Since $\mathfrak{w}_{i-1}V_{\alpha+j} = [t^{-1}, \mathfrak{w}_i]V_{\alpha+j} \subset t^{-1}\mathfrak{w}_i V_{\alpha+j} + \mathfrak{w}_i V_{\alpha+j-1}$, the former relation translates to give

$$(***) \qquad \mathfrak{w}_+ V_{\alpha+r} = 0, \quad for\ all \quad r > s.$$

Take $w \in V_{\alpha+i}$. By $(**)$ we may write $w = \sum u_j v_j$, for some $u_j \in \mathfrak{w}_j$ and $v \in V_{\alpha+i-j}$ with $0 < j \leq m$. Hence $V_{\alpha+r} = 0$, for all $r \geq s+m$.

Finally choose r maximal such that $V_{\alpha+r} = 0$. Since $V_{\alpha+r}$ is finite dimensional, $\mathfrak{w}_0$ is commutative and k is algebraically closed we can find $u \in V_{\alpha+r}$ which is a $\mathfrak{w}_0$ eigenvector. Then $U(\mathfrak{w})u$ is a highest weight submodule of V. □

2.4.6. Lemma. *Let V be a simple admissible highest weight module. If t^{-1} does not act injectively on V, then V is the trivial module.*

Proof. Otherwise by 2.4.5, V is both a highest and a lowest weight module. Consequently $\mathfrak{w}_s \subset Ann_{U(\mathfrak{w})}V$, for s sufficiently large. Yet $[t^{-1}, \mathfrak{w}_s] \subset \mathfrak{w}_{s-1}$ and so this inclusion obtains for all $s \in \mathbb{Z}$, as required. □

2.4.7. Lemma. *A simple non-trivial highest weight module V cannot be uniform.*

Proof. Suppose V has highest weight α. The injectivity of t^{-1} implies that the function $i \mapsto dim\ V_{\alpha-i}$ is increasing. If V is uniform, its maximum is attained at some $m \in \mathbb{N}$. Then $t^{-1}V_{\alpha-m-i} = V_{\alpha-m-i-1}$, for all $i > 0$. Since t^{m+1} and t^{-1} commute, we conclude that $t^{m+1} \subset Ann_{U(\mathbf{w})}V$. Commutation with d/dt implies that $t \in Ann_{U(\mathfrak{w})}V$. Yet $\mathfrak{w} = [t, \mathfrak{w}]$ and so V must be the trivial module. □

2.4.8. Lemma. *An indecomposable uniform $\mathfrak{w}$ module V has finite length.*

Proof. Indecomposability implies that V is some $V(\alpha)$. If t does not act injectively on V, then by 2.4.5, V admits a highest weight submodule M. The sum of the maximal submodules of a highest weight module is again maximal and so M admits a (unique) simple quotient which must be the trivial module by 2.4.7. A similar assertion holds if t_{-1} does not act injectively. Admissibility of V implies that in any decreasing sequence $V = V^0 \supset V^1 \supset V^2 \supset \cdots$, of submodules only finitely many factors can be the trivial module and we can assume that the remaining quotients have no trivial factors. Consider any such quotient Q. In any submodule of Q all weight spaces (with weights in $\alpha + \mathbb{Z}$) must have the same dimension. Admissibility implies that Q is of finite length. Hence so is V. □

2.4.9. Set $z := [d/dt, t]$, which is central in $\mathfrak{w}$. Let V be an admissible weight module. Since z preserves weight subspaces it acts finitely on V. In particular if V is indecomposable it acts by a scalar which we denote by c. Of course in a $\mathfrak{w}$ module coming from a B module, z would act by 1. Thus one is lead to ask if there is an automorphism of $\mathfrak{w}$ taking z to cz. Here is convenient to write d/dt simply as s. Then $t^m s^n : m \in \mathbb{Z}, n \in \mathbb{N}$ is a basis for $\mathfrak{w}$ and in terms of this basis the commutator of $x, y \in \mathfrak{w}$ is given by the Moyal bracket defined on the (commutative) Laurent polynomial ring $P = k[t, t^{-1}, s]$ by

$$(*) \qquad \{x, y\}_m = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{\partial^n x}{\partial s^n} \frac{\partial^x y}{\partial t^n} - \frac{\partial^m y}{\partial s^n} \frac{\partial^n x}{\partial t^n} \right).$$

Proposition. *Suppose $\sigma \in Aut\ \mathfrak{w}$ is defined by $\sigma(t^m) = ct^m, \sigma(s^n) = c^{-n+1} s^n$, for all $m \in \mathbb{Z}, n \in \mathbb{N}$. Then $c = \pm 1$. In the case $c = -1$ one has*

$$(**) \qquad \sigma(t^m s^n) = (-1)^{n-1}(t^m s^n + [s^n, t^m]),\ \forall\ m \in \mathbb{Z},\ n \in \mathbb{N}.$$

Conversely σ defined by $(**)$ belongs to $Aut\ \mathfrak{w}$.

Proof. By $(*)$ one has

$$[s^2, t^n] = 2nt^{n-1}s + n(n-1)t^{n-2}.$$

From this relation one checks that

$$\sigma(t^m s^n) = c^{-n+1} s(exp\left(\frac{1-c}{2}\right)\Delta)t^m s^n,\ \forall\ m \in \mathbb{Z},\ n \in \mathbb{N},$$

where $\Delta = \partial^2/\partial t \partial s$ and the right hand side is computed in P. Then the cubic relation

$$[s^3, t^3] = 9t^2 s^2 + 18ts + 6z,$$

forces $c^2 = 1$. When $c = -1$, it follows from $(*)$ that σ is given by the expression in $(**)$.

Conversely one checks for all $x \in \mathfrak{w}$ that

$$(***) \qquad [\sigma(t^2), \sigma(x)] = \sigma[t^2, x],\ [\sigma(s^2), \sigma(x)] = \sigma[s^2, x].$$

Both hold for all $c \in k$. The second relation follows of course from the first part. Since σ is the identity on generators, it implies that σ is involutive. From this a mere manipulation of $(**)$ gives

$$\sigma[t^m, s^n] = [\sigma(t^m), \sigma(s^n)], \ \forall \ m \in \mathbb{Z}, n \in \mathbb{N}.$$

Combining this with the first of the relations $(***)$ and using the Jacobi identity establishes by induction on r that

$$[\sigma(t^m), \sigma(t^r s^n)] = \sigma[t^m, t^r s^n], \ \forall m \in \mathbb{Z}, \ r, n \in \mathbb{N}.$$

To show that this is also valid for negative r, observe that by $(*)$, $(**)$ it can be expressed as an identity in polynomials of degrees bounded by the degree of the exponent of s, that is n. Combined with the second of the relations in $(* * *)$ a similar manipulation gives the required result. □

2.4.10. The main classification result is the following

Theorem.

(i) *Let V be an admissible $\mathfrak{w}$ module in which t acts bijectively and z by scalars. Then V is a finite direct sum of simple uniform modules.*

(ii) *Let V be a simple uniform $\mathfrak{w}$ module distinct from the trivial module. Then V is isomorphic to the standard B module $k[t, t^{-1}]$, up to $Aut\ \mathfrak{w}$.*

Proof. Choose $\alpha \in k$ such that $V_\alpha \neq 0$ and let n be the constant dimension of $V_{\alpha+i} : i \in \mathbb{Z}$. Given $x \in \mathfrak{w}$ which an *ad* h eigenvector of eigenvalue j there exist $n \times n$ matrices X_i such that

$$xV_{\alpha+i} = X_i V_{\alpha+i+j}, \quad for\ all \quad i \in \mathbb{Z},$$

and we say that x acts by X_i. By 2.4.9, z acts by c. Assume $c \neq 0$. Then through the injectivity of t and appropriate choices of bases we can assume that t acts by c. From the relation $[s, t] = z$ it follows that s acts by $i + C$, for some $n \times n$ matrix C. We show that successively that every $x \in \mathfrak{w}$ acts by some polynomial in i and C.

Take $m \in \mathbb{N}$. From the relation $[s, t^m] = mt^{m-1}$, it follows that t^m acts by c. From $[t^2 s, t] = t^2$, it follows that $t^2 s$ acts by $i + D$ for some D. Then from $[t^2 d, t^{-1}] = -z$, it follows that t^{-1} acts by c and so do all the powers of t. From $[s^2, t] = 2s$, s^2 acts by $c^{-1}(i(i-1)+2iC)+E$, for some E. From $[s^2,t^2]=4ts+2z$, it follows that ts acts by $i+C+(1-c)/2$. From the relation $[t^2s^2,t]=2t^2s$, it follows that t^2s^2 acts by $c^{-1}(i(i-1)+2iD)+F$, for some F. Since ts and t^2s^2 commute we conclude that D, F commute with C. Then from $[t^2s,s]=-2ts$, it follows that $D=C+(1-c)$. From the relation $[t^2s^2,s]=-2ts^2$, it follows that ts^2, acts by $c^{-1}(i(i-1)+(i-1)C+iD+CD)$. Finally from $[s,ts^2]=s^2$, it follows that $E=C^2-C$. Since $s^2,\ ts^2,\ t^{\pm 1}$ form a system of generators for $\mathfrak{w}$ the required assertion follows.

Since $h=ts$ acts by $(\alpha+i)$, it follows that C is a multiple of the identity. Consequently V is a direct sum of uniform modules with moreover $n=1$. This concludes the proof of (i). Now assume V is simple. Then C is a scalar, precisely $\alpha-(1-c)/2$. Through the automorphism of $\mathfrak{w}$ which comes from that of B and which translates s by a scalar, we can assume $C=0$. It easily follows that s^n acts by $c^{-n+1}i!/(i-n)!$ for all $n\in\mathbb{N}$, whilst we recall that t^m acts by c for all $m\in\mathbb{Z}$. It is clear that the existence of such representation implies that there exists $\sigma\in Aut\ \mathfrak{w}$ satisfying the hypothesis of 2.4.9. This forces $c=\pm 1$. The case $c=0$ leads to a contradiction by a similar and slightly easier argument. (It just happens to be convenient to assume that t acts by c when $c\neq 0$.) □

Remark. It is clear that admissible *Lie A* modules in which t acts bijectively have a similar description.

3. Demazure Modules and Bruhat Order.

3.1. A Historical Glimpse.

3.1.1. There are many glorious chapters in the theory of semisimple Lie algebras. Selecting out just a few involves some painful choices. Yet however one procedes it is difficult to avoid the Weyl group whose delicate combinatorial structure lies at the heart of the theory. This is particularly true of the character formula for a finite dimensional simple module V over a semisimple Lie algebra $\mathfrak{g}$. For $\mathfrak{g}=\mathfrak{sl}(n)$ the result is due to I. Schur and is already rather striking. It led to the magnificent character formula of H. Weyl in the general case. Later I. N. Bernstein, I. M. Gelfand and S. I. Gelfand (whom we refer collectively to as BGG) found a resolution for such modules bringing a better understanding of this fomula. Their more efficient proof was adapted by V. Kac to extend the result to symmetrizable Kac-Moody algebras. As special cases he found remarkable infinite product formulae of classical number theory due to Fermat, Gauss and Jacobi.

3.1.2. The Weyl character formula took on a new meaning as a result of the work of M. Demazure. Let $\mathfrak{g}$ be a semisimple Lie algebra. As in 2.3 we view $\mathfrak{g}$ as a $\mathbb{Z}$ graded Lie algebra so having a triangular decomposition and write $\mathfrak{g}_0 = \mathfrak{h}$, $\mathfrak{g}_\pm = \mathfrak{n}^\pm$ setting $\mathfrak{b} = \mathfrak{n}^+ \oplus \mathfrak{h}$. In the language of algebraic groups, $\mathfrak{b}$ is a Borel subalgebra, that is to say a maximal solvable subalgebra. Moreover all such Borel subalgberas arise through such a $\mathbb{Z}$ grading of $\mathfrak{g}$. Demazure (as well as BGG) considered certain $U(\mathfrak{b})$ submodules of V in order to study Schubert subvarieties of the flag variety. Demazure proposed a character formula for such modules though unfortunately he relied on a false assertion of D.-N. Verma for its proof. Nevertheless he found a startling reformulation of the Weyl character formula which is also computationally rather useful.

3.1.3. It turned out to be immensely difficult to establish the Demazure character formula. A first attempt was an elementary proof based on computing $\mathfrak{n}$-homologies; but this only works for very dominant weights. (Even in this case a precise formula for the $\mathfrak{n}$-homologies remains a difficult open question.) The first proof valid for all dominant weights was given by H.H. Andersen through a translation principle based on the Steinberg module. A further proof was obtained by O. Mathieu using a deep result of Ramathan asserting that Schubert varieties are Frobenius split. Both these methods involve positive characteristic. M. Kashiwara found an "elementary" proof involving quantum groups; but this is of immense combinatorial difficulty. Finally P. Littelmann found a proof based on crystal graphs and quantum Frobenius splitting which will be detailed in one of our later "selecta". Both these latter methods work over any symmetrizable Kac-Moody algebra and avoid positive characteristic.

3.1.4. For the present we shall just consider a much simpler problem studied by BGG. This concerns the inclusion relations between Demazure modules which turn out to be exactly given by the Bruhat order on the Weyl group. Then we consider certain generalizations of Demazure modules and determine the inclusion relations between them. This applies to the study of primitive ideals over the quantum group. Here the motivation had been to describe the graph of links on the primitive spectrum. Yet for the moment this remains an open problem.

3.2. The Weyl Group and its Combinatorics.

3.2.1. Recall (2.3.5) the notion of a Kac-Moody algebra $\mathfrak{g}_A$ given by a Cartan matrix A. For each $\alpha \in \pi$, let $s_\alpha \in Aut\ \mathfrak{h}^*$ be the reflection defined by

$$s_\alpha \lambda = \lambda - \alpha^\vee(\lambda)\alpha, \ \forall\ \lambda \in \mathfrak{h}^*.$$

The Weyl group W_A (or simply, W) is the subgroup of $Aut\ \mathfrak{h}^*$ generated by the set $S := \{s_\alpha : \alpha \in \pi\}$. When $\alpha = \alpha_i : i \in I$, we shall write s_α simply as s_i.

Although this will not be needed for the present we remark that

$$\mathfrak{g}_A \ \ is\ semisimple \iff A \ \ is\ positive\ definite \iff W_A \ \ is\ a\ finite\ group.$$

3.2.2. Below we review some combinatorial properties of the Weyl group. Proofs not given may be found for example in [, A.1].

3.2.3. Let $\mathfrak{g}_A$ (or simply, $\mathfrak{g}$) be a Kac-Moody algebra. For all $\alpha \in \mathfrak{h}^*$, set

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,\Big|\, [h, x] = \alpha(h)x, \ \forall\ h \in \mathfrak{h}\}.$$

One calls $\alpha \in \mathfrak{h}^*$, a root of $\mathfrak{g}$ if $\mathfrak{g}_\alpha \neq 0$ and $\mathfrak{g}_\alpha$ is called the root subspace corresponding to α. Let Δ denote the set of non-zero roots of $\mathfrak{g}$. From triangular decomposition it easily follows that

$$\Delta = \Delta^+ \amalg \Delta^-, \ \ where \ \ \Delta^\pm = \pm\mathbb{N}\pi \cap \Delta.$$

A key fact (due V. Kac) is that Δ is W stable. This requires the conditions on A imposed in 2.3.5. Although the statement is a purely combinatorial one, the proof requires some Lie algebra theory. As an easy corollary it follows that for all $\alpha \in \pi$, the set $\Delta^+ \setminus \{\alpha\}$ is s_α stable. This innocent looking statement is in fact responsible for a drastic simplification in the structure of W. Define $\Delta_{re} := W\pi$. It is called the set of real roots. Set $\Delta_{re}^\pm = \Delta_{re} \cap \Delta^\pm$.

3.2.4. Take $w \in W$. We can write $w = s_{i_1} s_{i_2} \cdot \cdots \cdot s_{i_n} : i_j \in I$. When n takes its minimal value such an expression (which need not be unique) is called a reduced decomposition and n the reduced length $\ell(w)$ of w. In terms of a reduced decomposition we set $w_j = s_{i_j} \cdot \cdots \cdot s_{i_n}, w_{n+1} = e$ and $\beta_j = w_{j+1}^{-1}\alpha_{i_j} : j = 1, 2, \cdots, n$. Set $S(w) = \{\alpha \in \Delta^+ \,\Big|\, w\alpha \in$

$\Delta^-\} = \Delta^+ \cap w^{-1}\Delta^-$. It is clear that the β_j are natural candidates for the elements of $S(w)$. Precisely one has the

Lemma.

(i) $S(w) = \{\beta_j\}_{j=1}^n$,

(ii) $\ell(w) = |S(w)|$,

(iii) *The map* $w \mapsto S(w)$ *is injective.*

Proof. See [, A.1.1], for example.

Remark. Note in particular that $S(w) \subset \Delta^+_{re}$.

3.2.5. For all $\gamma \in \Delta^+_{re}$, we may write $\gamma = w\alpha$ for some $\alpha \in \pi$, and we set $\gamma^\vee = w\alpha^\vee$. (It is a tricky exercise to show that $\gamma^\vee$ is independent of choices.) Then s_γ is defined as a reflection and coincides with $ws_\alpha w^{-1}$. From 3.2.4 one obtains

$$\ell(s_\gamma w) < \ell(w) \Longleftrightarrow \gamma \in S(w^{-1}).$$

Set $w' = s_\gamma w$. One may note that the pair $w,\ w'$ determines γ. When $\ell(s_\gamma w) = \ell(w) - 1$, we write $w' \xleftarrow{\gamma} w$, or simply $w' \leftarrow w$. The Bruhat order $\leq$ on W is defined by setting $w' \leq w$ if $w' = w$, or if there exists a chain $w' = w_1 \leftarrow w \leftarrow \cdots \leftarrow w_n = w$. The following result is due to *BGG*.

Proposition. *Take* $w, w' \in W$, *with* $w \geq w'$ *and* $\alpha \in \pi$. *Then*

(i) $w \leq e \Longleftrightarrow w = e$,

(ii) $\ell(s_\alpha w) < \ell(w) \Longrightarrow w \geq s_\alpha w$,

(iii) *Either* $s_\alpha w \geq s_\alpha w'$ *or* $w \geq s_\alpha w'$,

(iv) *Either* $s_\alpha w \geq s_\alpha w'$ *or* $s_\alpha w \geq w'$,

(v) *The Bruhat order is the unique order relation on* W *satisfying (i)-(iv).*

Proof. See [, A.1.7] for example.

3.2.6. For all $\alpha, \beta \in \pi$ distinct set $m_{\alpha,\beta} = \alpha^\vee(\beta)\beta^\vee(\alpha)$, which is an integer ≥ 0. One may check that $(s_\alpha s_\beta)^{m_{\alpha,\beta}} = (s_\beta s_\alpha)^{m_{\alpha,\beta}}$, if $m_{\alpha,\beta} = 0, 2, 3$ and $s_\alpha s_\beta s_\alpha = s_\beta s_\alpha s_\beta$, if $m_{\alpha,\beta} = 1$. These are called the Coxeter relations for W. The following result is due to R. Steinberg.

Lemma.

(i) *Any two reduced decompositions of $w \in W$ can be transformed into one another using the Coxeter relations.*

(ii) *W is the group with generators $s_i : i \in I$ satisfying the identities $s_i^2 = 1$ and the Coxeter relations.*

Proof. See [, A.1.9] for example.

3.2.7. In view of 3.2.6 one may define a new associative multiplication $*$ on W by making the generators idempotent and preserving the Coxeter relations. We call this new structure the Hecke monoid. It is a degeneration of the Hecke algebra which is itself a deformation of the group algebra of W. The latter arose from the theory of Chevalley groups and its multiplication corresponds to that obtained by multiplying certain double cosets.

3.2.8. The work of D.A. Kazhdan and G. Lusztig [ ] showed that the Hecke algebra was of fundamental importance in the representation theory of $\mathfrak{g}$ particularly with respect to the decomposition of Verma modules. There are also several manifestations of the Hecke monoid some of which go back even further. These include the multiplication of one dimensional subgroups corresponding to a simple root, the multiplication of almost minimal primitive ideals of $U(\mathfrak{g}) : \mathfrak{g}$ semisimple and the multiplication of the Enright functors. Of particular importance here is the Hecke monoid obtained from Demazure operators. This is described below.

3.2.9. Set $P(\pi) = \{\lambda \in \mathfrak{h}^* \bigm| \alpha^\vee(\lambda) \in \mathbb{Z},\ \forall\ \alpha \in \pi\}$. It is called the lattice of integral weights of $\mathfrak{h}^*$ and in particular is an additive subgroup. It is convenient to consider $P(\pi)$ as a multiplicative group by defining elements $e^\lambda : \lambda \in P(\pi)$ with multiplication $e^\lambda e^\mu = e^{\lambda+\mu}$, for all $\lambda, \mu \in P(\pi)$. Let $\mathbb{Z}e^{P(\pi)}$ denote the corresponding group ring. Choose $\rho \in \mathfrak{h}^*$ to satisfy $\alpha^\vee(\rho) = 1$, for all $\alpha \in \pi$ and define the translated action of W on $\mathfrak{h}^*$ through $\lambda \mapsto w.\lambda := w(\lambda+\rho) - \rho$, for all $\lambda \in P(\pi)$. For all $\alpha \in \pi,\ \lambda \in P(\pi)$ set

$$D_\alpha e^\lambda = (1 - s_\alpha)^{-1}\ (e^\lambda - e^{s_\alpha.\lambda})$$

which is again an element of $\mathbb{Z}e^{P(\pi)}$. Remarkably the $D_\alpha : \alpha \in \pi$, generate the Hecke monoid. In particular D_w is defined for all $w \in W$.

3.2.10. For all $\alpha \in \pi,\ y \in W$, set $s_\alpha \circ y = s_\alpha(s_{\alpha^*} y)$. If $x = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$, is a reduced decomposition, then $x = s_{\alpha_1} * s_{\alpha_2} * \cdots s_{\alpha_n}$, and so $x * y = s_{\alpha_1} * (s_{\alpha_2} * \cdots (s_{\alpha_n} * y) \cdots)$, by the associativity of $*$. Following this we set $x \circ y := (s_{\alpha_1} \circ (s_{\alpha_2} \circ \cdots (s_{\alpha_n} \circ y) \cdots)$. However this latter multiplication is not associative.

Lemma. *For all* $x, y, w \in W$ *one has*

(i) $w^* x \geq y \Longleftrightarrow x \geq w^* \circ y.$

(ii) $w^* \circ y = e \Longleftrightarrow y \leq x.$

Proof. Observe that for Bruhat order $s_\alpha * x = \max\{x, s_\alpha x\}, s_\alpha * y = min\{y, s_\alpha y\}$. Then $x \geq s_\alpha * y$ implies $s_\alpha * x \geq y$, by 3.2.5 (iii), whilst the converse holds by 3.2.5 (iv). Then induction on $\ell(w)$ gives (i). By (i), $x^{-1} * y = e$ implies $y \leq x * e = x$ which in turn implies $x^{-1} * y \leq x^{-1} * x = e$.

3.2.11. For all $\lambda \in \mathfrak{h}^*$, set $W_\lambda := \{w \in W \,\big|\, w\lambda = \lambda\}$. Set $P^+(\pi) = \{\lambda \in P(\pi) \,\big|\, \alpha^\vee(\lambda) \geq 0, \ \forall\ \alpha \in \pi\}$. It is called the set of dominant weights.

Lemma. *For all* $\lambda \in P^+(\pi)$ *one has*

(i) $W_\lambda = < s_\alpha : \alpha^\vee(\lambda) = 0 >,$

(ii) *For each* $w \in W$*, there is a unique element* $w^o \in wW_\lambda$ *of minimal length.*

(iii) $w^o x = w^o * x, \quad for\ all \quad w \in W_\lambda.$

(iv) *For all* $y \in W$*, the maps* $w \mapsto y * w$ *and* $w \mapsto y * w$ *factor to* W/W_λ*.*

Proof. For (i) see [, A.1.1 (vii)], for example. For (ii) set $\pi' = \{\alpha \in \pi \,\big|\, \alpha^\vee(\lambda) = 0\}$ and $\Delta^+_{\pi'} = \Delta \cap \mathbb{N}\pi'$. Since $\ell(w)$ is bounded from below we can choose $w^o \in wW_\lambda$ such that $\ell(w^o s_\alpha) > \ell(w^o)$, for all $\alpha \in \pi$. By 3.2.4 (i) this condition is equivalent to $\Delta^+_{\pi'} \cap S(w^o) = \emptyset$. We may write $w = w^o x$, for some $x \in W_\lambda$. Since W_λ leaves $\Delta^+ \setminus \Delta^+_{\pi'}$ stable it follows that $S(w^o) = S(w) \cap (\Delta^+ \setminus \Delta^+_{\pi'})$. Thus by 3.2.4 (ii), w^o is an element of minimal length in its right W_λ coset proving existence. Moreover

$$S(w) = x^{-1} S(w^o) \amalg S(x).$$

So if $\ell(w)$ is also minimal then $x = e$ and the above formula combined with 3.2.4(iii) proves uniqueness. It also gives $\ell(w) = \ell(w^o) + \ell(x)$ and so $w^o x = w^{o*} x$. Since $x \in W_\lambda$ is arbitray, this is (iii). For the first part of (iv) observe that by associativity $y_* w = y_*(w^o_* x) = (y_* w^o)_* x \in (y_* w^o) W_\lambda$. The second part follows from the first and induction on $\ell(y)$.

3.3. Demazure Modules.

3.3.1. The theory of highest weight modules for Kac-Moody algebras (and their quantum analogues) is a particularly rich chapter in mathematics being the subject of a vast number of papers and texts. We shall start with an extremely brief summary of the salient points of the theory.

3.3.2. For each $\lambda \in \mathfrak{h}^*$ there is a unique up to isomorphism simple highest weight module $V(\lambda)$ of highest weight λ. By definition $V(\lambda)$ is generated by a highest weight vector v_λ of weight λ, that is to say a non-zero element $v_\lambda \in V(\lambda)$ satisfying $\mathfrak{n}^+ v_\lambda = 0$ and $hv_\lambda = \lambda(h)v_\lambda$, for all $h \in \mathfrak{h}$.

For each $\mu \in h^*$, set $V(\lambda)_\mu := \{v \in V(\lambda) \,\big|\, hv = \mu(h)v, \ \forall\ h \in \mathfrak{h}\}$. One calls μ a weight of $V(\lambda)$ if $V(\lambda)_\mu \neq 0$ and $V(\lambda)_\mu$ the corresponding weight subspace. Via the triangular decomposition of $\mathfrak{g}$ it follows that these weight spaces are finite dimensional, that is to say $V(\lambda)$ is admissible. Then one defines the formal character of $V(\lambda)$ by

$$ch\ V(\lambda) := \sum_{\mu \in \mathfrak{h}^*} dim\ V(\lambda)_\mu e^\mu.$$

In general the calculation of *ch* $V(\lambda)$ is of tremendous difficulty. A spectacular achievement of the 1980's was the proof of the Kazhdan-Lusztig conjectures which for $\mathfrak{g}$ semisimple described the form of *ch* $V(\lambda)$ in terms of Hecke algebra combinatorics. These results have been partially extended for $\mathfrak{g}$ affine.

As noted in 3.1.1 when $\lambda \in P^+(\pi)$ the formula for *ch* $V(\lambda)$ is due to Weyl for $\mathfrak{g}$ semisimple and to Kac for $\mathfrak{g}$ symmetrizable Kac-Moody. The general Kac-Moody case is still an open problem, though important progess was made by O. Mathieu [ ] and S. Kumar [ ] using ideas from the Demazure modules. The particular simplication which obtains for $\lambda \in P^+(\pi)$, is that *ch* $V(\lambda)$ is W invariant. This and some general theory involving Verma modules nearly determines *ch* $V(\lambda)$. Moreover using the Hochschild-Serre spectral sequence with respect to codimension one ideals of $\mathfrak{n}^-$ one can even nearly determine $H_*(\mathfrak{n}^-, V(\lambda))$. The extra ingredient required can be either the finite dimensionality of $\mathfrak{n}^-$, which ensures the eventual vanishing of these homology spaces, that is to say for $i > dim\ \mathfrak{n}^-$, or the Casimir invariant constructed by Kac for the case when A is symmetrizable.

3.3.3. In what follows we shall assume $\lambda \in P^+(\pi)$ and we shall only need that *ch* $V(\lambda)$ is W invariant. This is proved as follows. For each $\alpha \in \pi$, set $\mathfrak{r}_\alpha := \mathfrak{g}_\alpha + \mathfrak{h} + g_{-\alpha}$ which is a reductive Lie algebra with semisimple part $\mathfrak{s}_\alpha$ isomophic to $\mathfrak{sl}(2)$. The condition that $\lambda \in P^+(\pi)$ and the representation theory of $\mathfrak{sl}(2)$ ensures that $V(\lambda)$ is a direct sum of finite dimensional $\mathfrak{r}_\alpha$ modules each of which have a formal character invariant under s_α. Thus $s_\alpha ch\ V(\lambda) = ch\ V(\lambda)$, for all $\alpha \in \pi$.

A particular consequence of W invariance is that for each $w \in W$ there is a unique up to scalars non-zero element $v_{w\lambda} \in V(\lambda)$ of weight $w\lambda$. One defines the Demazure module to be

$$V_w(\lambda) := U(\mathfrak{b})v_{w\lambda}.$$

We remark that the Demazure character formula asserts that

$$ch\ V(\lambda) = D_w e^\lambda.$$

3.3.4. For all $\alpha \in \pi$ choose $x_{\pm\alpha} \in \mathfrak{g}_{\pm\alpha}$ non-zero and set $U_\alpha^\pm = k[x_{\pm\alpha}]$. In the notation of 2.3.4 we have $x_\alpha = x_i,\ x_{-\alpha} = y_i$ if $\alpha = \alpha_i$. Set $U_\alpha = U(\mathfrak{r}_\alpha)$ and $U^\pm = U(\mathfrak{n}^\pm)$. It is clear that $V_w(\lambda) = U^+ v_{w\lambda}$. From the relations in 2.3.4 one easily checks the

Lemma. *Let M be a $U(\mathfrak{h})$ stable submodule of $V(\lambda)$. Then for all $\alpha, \beta \in \pi$ one has*

$$U_\alpha^+ U_\beta^- M = U_\beta^- U_\alpha^+ M.$$

3.3.5. Take $\lambda \in P^+(\pi)$.

Lemma. *Suppose $w \in W,\ \alpha \in \pi$ satisfy $\ell(s_\alpha w) < \ell(w)$. Then*

$$V_w(\lambda) = U_\alpha V_{s_\alpha w}(\lambda) = U_\alpha^- V_{s_\alpha w}(\lambda).$$

Proof. See [, 4.4.3] for example. □

Remarks. Take $w \in W$ and let $w = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_n}$, be a reduced decomposition. Then

$$V_w(\lambda) = U_{\alpha_1}^- U_{\alpha_2}^- \cdots\cdots U_{\alpha_n}^- v.$$

This is a second presentation of a Demazure module, which already occurred in his work. It is of crucial importance in the crystal theory. Moreover since the common annihilator of the $v_\lambda : \lambda \in P^+(\pi)$ is zero, it follows that

$$U_w^- := U_{\alpha_1}^- \cdots\cdots U_{\alpha_n}^-,$$

is independent of reduced decomposition. A similar result holds when - is replaced by + and so U_w^+ is also defined. Notice that these give further realizations of the Hecke monoid.

3.3.6 A crucial and not quite trivial fact is the following.

Lemma. *Assume* $y \in W$, $\alpha \in \pi$ *satisfy* $\ell(s_\alpha y) < \ell(y)$. *Let* N *be an* $U(\mathfrak{h})$ *submodule of* $V(\lambda)$ *satisfying* $v_{y\lambda} \in U_\alpha N$. *Then* $v_{s_\alpha y\lambda} \in U_\alpha^+ N$ *and* $v_{y\lambda} \in U_\alpha^- N$.

Proof. Both parts are similar and we establish just the first. Under the hypothesis $x_{-\alpha} v_{y\lambda} = 0$. Thus $M := U_\alpha^+ v_{y\lambda}$ is a simple $\mathfrak{r}_\alpha$ submodule of $U_\alpha N$. By complete irreducibility there exists a surjection p of $\mathfrak{r}_\alpha$ modules of $U_\alpha N$ onto M. Certainly $p(U_\alpha^+ N) \neq 0$ and is U_α^+ stable and so it contains the unique up to scalars highest weight vector $v_{s_\alpha y\lambda} \in M$. Yet the $s_\alpha y\lambda$ weight subspace of $V(\lambda)$ and hence of $U_\alpha N$ is one dimensional. Consequently $v_{s_\alpha y\lambda} \in U_\alpha^+ N$. □

3.3.7. Corollary. *Given* $\alpha \in \pi$, $w, w' \in W$ *such that* $\ell(s_\alpha w) < \ell(w)$ *and* $\ell(s_\alpha w') < (w')$, *then*

$$V_w(\lambda) \supset V_{w'}(\lambda) \textit{ implies } V_{s_\alpha w}(\lambda) \supset V_{s_\alpha w}(\lambda).$$

Proof. Take $y = w'$, $U_\alpha^+ N = V_{s_\alpha w}(\lambda)$ in 3.3.6. □

3.3.8. Set $P^{++}(\pi) = P^+(\pi) + \rho$. Call $\lambda \in \mathfrak{h}^*$ regular if $w\lambda = w'\lambda$ implies $w = w'$. We remark that $P^{++}(\pi)$ is just the set of regular elements of $P^+(\pi)$. Now from 3.2.5, 3.3.5 and 3.3.7 we obtain

Proposition. *Take* $\lambda \in P^{++}(\pi)$. *Then for all* $w, w' \in W$, *one has*

$$w' \leq w \Longleftrightarrow V_{w'}(\lambda) \subset V_w(\lambda).$$

Remark. For $\mathfrak{g}$ semisimple, this result is due to *BGG*. Their proof avoided the use of 3.3.7 by using the existence of the unique longest element $w_0 \in W$. If one includes the property that $w > w_0$ implies $w = w_0$, then (iii) and (iv) of 3.2.5 become equivalent.

3.4. Generalized Demazure Modules.

3.4.1. Given $w, y \in W$ we consider subspaces $U_w^- v_{y\lambda}$ of $V(\lambda)$, which we call generalized Demazure modules. Our aim is to determine the inclusion relations between them in a manner generalizing 3.3.8.

3.4.2. Take $\lambda \in P^+(\pi)$ and let $\geq$ denote the induced Bruhat order on W/W_λ. Observe that by 3.2.11 (iv) the assertions of 3.2.10 pass to the induced order. Given $yW_\lambda \leq zW_\lambda$, set

$$[yW_\lambda, zW_\lambda] = \{x \in W/W_\lambda \,\Big|\, yW_\lambda \leq xW_\lambda \leq zW_\lambda\}.$$

Let $< \, , >$ denote the contravariant form on $V(\lambda)$. It is non-degenerate and respects weight space decomposition. Thus in particular

$$v_{z\lambda} \in U_w^- v_{y\lambda} \iff < v_{z\lambda}, U_w^- v_{y\lambda} > \neq 0 \iff < U_{w^{-1}}^+ v_{z\lambda}, v_{y\lambda} > \neq 0 \iff \; v_{y\lambda} \in U_{w^{-1}}^+ v_{z\lambda}.$$

This and similar identities are used without further mention. Let $\geq$ denote the order relation on $P(\pi)$ defined by $\mu \geq \nu \iff \mu - \nu \in \mathbb{N}\pi$.

Proposition. *For all* $\lambda \in P^+(\pi),\ w \in W;\ y, z \in W/W_\lambda$ *one has*

$$v_{z\lambda} \in U_w^- v_{y\lambda} \iff zW_\lambda \in [yW_\lambda, w * yW_\lambda].$$

Proof. Let y^o be the unique element of minimal length in yW_λ. The proof is by induction on $(\ell(y^o), \ell(w)) \in \mathbb{N}^2$, given the lexicographic order. If $\ell(w) = 0$, the assertion is trivial. Thus we may fix $\alpha \in \pi$, such that $s_\alpha w < w$.

Suppose $z\lambda \geq s_\alpha z\lambda$. This implies $\alpha^\vee(z\lambda) \geq 0$. Then $zW_\lambda \leq s_\alpha zW_\lambda$ by 3.2.4 (i) and so $s_\alpha o zW_\lambda = zW_\lambda$. We obtain

$$\begin{aligned} v_{z\lambda} U_w^- v_{y\lambda} &\iff < U_\alpha^+ v_{z\lambda},\ U_{s_\alpha w}^- v_{y\lambda} > \neq 0 \\ &\iff < v_{z\lambda}, U_{s_\alpha w}^- v_{y\lambda} > \neq 0, \quad \textit{since} \;\; x_\alpha v_{z\lambda} = 0, \\ &\iff v_{z\lambda} \in U_{s_\alpha w}^- v_{y\lambda}, \\ &\iff zW_\lambda \in [yW_\lambda, (s_\alpha w) * yW_\lambda], \quad \textit{by the induction hypothesis}, \\ &\iff zW_\lambda \in [yW_\lambda, w * yW_\lambda], \end{aligned}$$

since by 3.2.10 (i), we have $s_{\alpha^o} zW_\lambda = zW_\lambda \leq (s_\alpha w) * yW_\lambda \iff zW_\lambda \leq s_\alpha * ((s_\alpha w) * yW_\lambda) = (s_\alpha * s_\alpha w) * yW = w * yW_\lambda$.

From now on we assume $z\lambda < s_\alpha z\lambda$. Then $zW_\lambda > s_\alpha zW_\lambda$ and so $s_\alpha \circ zW_\lambda = s_\alpha zW_\lambda$.

Suppose $s_\alpha y\lambda \leq y\lambda$. Then $x_\alpha v_{y\lambda} = 0$ and so $U_\alpha^+ U_{s_\alpha w}^- v_{y\lambda} = U_{s_\alpha w}^- v_{y\lambda}$, by 3.3.4. In particular $U_w^- v_{y\lambda}$ is U_α stable. Then

$$
\begin{aligned}
v_{z\lambda} \in U_w^- v_{y\lambda} &\iff v_{s_\alpha z\lambda} \in U_{s_\alpha w}^- v_{y\lambda}, \quad \textit{by } 3.3.6, \\
&\iff s_\alpha z W_\lambda \in [yW_\lambda, (s_\alpha w) * yW_\lambda], \quad \textit{by the induction hypothesis}, \\
&\iff zW_\lambda \in [yW_\lambda, w * yW_\lambda],
\end{aligned}
$$

since $s_\alpha z W_\lambda = s_\alpha \circ y W_\lambda \leq (s_\alpha w) * yW_\lambda \iff yW_\lambda \leq s_\alpha * (s_\alpha w) * yW_\lambda = w * yW_\lambda$, by 3.2.10 (i), whilst $s_\alpha z W_\lambda \geq yW_\lambda$ implies $zW_\lambda > yW_\lambda$, trivially and by 3.2.5 (iv) the relations $zW_\lambda \geq yW_\lambda$ implies $s_\alpha z W_\lambda > min\{yW_\lambda, s_\alpha y W_\lambda\} = yW_\lambda$.

From now on we assume $s_\alpha y\lambda > y\lambda$, which implies that $s_\alpha y W_\lambda < yW_\lambda$.

Let us first establish $\Longrightarrow$. Suppose $zW_\lambda \geq s_\alpha y W_\lambda$. Then by 3.2.5 (iii) one has $zW_\lambda = \max\{zW_\lambda, s_\alpha z W_\lambda\} \geq yW_\lambda$. Thus it suffices to show that $zW_\lambda \in [s_\alpha y W_\lambda, w * yW_\lambda]$.

If $s_\alpha w^{-1} < w^{-1}$, then $v_{s_\alpha y\lambda} \in U_\alpha^+ v_{y\lambda} \subset U_\alpha^+ U_{w^{-1}}^+ v_{z\lambda} = U_{w^{-1}}^+ v_{z\lambda}$. Hence $zW_\lambda \in [s_\alpha y W_\lambda, w * s_\alpha y W_\lambda]$, by the induction hypothesis. Yet under the hypotheses $w * s_\alpha y W_\lambda = (ws_\alpha) * s_\alpha * (s_\alpha y) W_\lambda = (ws_\alpha) * s_\alpha * yW_\lambda = w * yW_\lambda$, which gives the required result.

If $s_\alpha w^{-1} > w^{-1}$, then $v_{s_\alpha y\lambda} \in U_\alpha^+ v_{y\lambda} \subset U_{(ws_\alpha)^{-1}}^+ v_{z\lambda}$ and so $zW_\lambda \in [s_\alpha y W_\lambda, ws_\alpha * s_\alpha y W_\lambda]$, by the induction hypothesis. Yet as above we have $(ws_\alpha) * s_\alpha y W_\lambda = w * yW_\lambda$.

It remains to establish $\Longleftarrow$. Observe that $zW_\lambda, w * xW_\lambda, yW_\lambda$ all decrease on left multiplication by s_α and so by 3.2.5 (iii), (iv) the hypothesis implies $s_\alpha z W_\lambda \in [s_\alpha y W_\lambda, s_\alpha w * yW_\lambda]$. Noting that the second factor on the right hand side equals $s_\alpha w * s_\alpha * s_\alpha y W_\lambda$, so the induction hypothesis implies that $v_{s_\alpha y\lambda} \in U_{s_\alpha * (w^{-1} s_\alpha)}^+ v_{s_\alpha z\lambda}$. Then since $x_\alpha v_{s_\alpha y\lambda} = 0$, we obtain $v_{y\lambda} \in U_\alpha^- v_{s_\alpha y\lambda} \subset U_\alpha^- U_{s_\alpha * (w^{-1} s_\alpha)}^+ v_{s_\alpha z\lambda}$. Applying the second part of 3.3.6 we obtain $v_{y\lambda} \in U_\alpha^- U_{w^{-1} s_\alpha}^+ v_{s_\alpha z\lambda} = U_{w^{-1} s_\alpha}^+ U_\alpha^- v_{s_\alpha z\lambda} = U_{w^{-1} s_\alpha}^+ U_\alpha^+ v_{z\lambda} = U_{w^{-1}}^+ v_{z\lambda}$. Using the contravariant form this gives the required assertion.

4. Invariants: Chevalley's Theorem and its Generalization.

4.1. Prehistory.

4.1.1. Invariants have always held a special significance for observers of the everchanging face of Nature. The most ancient is certainly that of planetary orbit with its remarkable long-term consistency. Such motion takes place in three-dimensional space yet is confined to a plane. This means that the vector (of angular momentum) orthogonal to that plane is an invariant. Added to this Kepler made two remarkable observations. First the orbits are elipses, second equal areas (of the segment joining two points in the orbit to the focus) are swept out in equal times. It was the spectacular achievement of Newtonian mechanics based on an inverse-square law attraction which gave a rational explanation to such behaviour. In this the vector joining a focus to the lowest point in the orbit defines a further invariant commonly attributed to Hamilton. Moreover one may recognize that planetary orbits are governed by a six dimensional "symmetry group" defined by these invariants.

4.1.2. In quantum mechanics there is a quite different consequence to the presence of a symmetry group. Orbits no longer exist and one may only observe patterns in the different states which the system may adopt. It is the representations of the symmetry group which carry the observable consequences. For atoms, which embody the quantum analogue of planetary motion, the group in question is $SO(4, \mathbb{R})$. To study its finite dimensional representations we may turn to its Lie algebra $\mathfrak{so}(4, \mathbb{R})$. Irreducibles are classified by the scalar action of the centre of its enveloping algebra. Their values become the new invariants of the quantum system.

4.1.3. From the above it should be no surprise that a fundamental problem is to determine the centre $Z(\mathfrak{g})$ of the enveloping algebra of a Lie algebra $\mathfrak{g}$, particularly for $\mathfrak{g}$ semisimple. Initially this was achieved by ad hoc methods. The quadratic Casimir invariant written down first for $\mathfrak{so}(3)$ and later generalized to symmetrizable Kac-Moody algebras, was obtained in this fashion. In higher degree more sophistication is needed. For $\mathfrak{g} = \mathfrak{gl}(V)$ the trace skillfully manipulated is sufficient. Indeed consider the induced action of $\mathfrak{gl}(V)$ on Λ^*V. The map to the trace in each degree gives n invariants elements of $S(\mathfrak{g}^*)$. Using the Killing form and symmetrization one obtains elements of $Z(\mathfrak{g})$ which one may further show generate it as a polynomial algebra.

4.1.4. An effective theory of invariants seems always to invoke the very representations which they are supposed to classify. Thus it is the family of scalar values which the finite irreducible representations take on $Z(\mathfrak{g})$ which determine the structure of the latter. This paradoxical reversal is a common theme in the modern theory of invariants. Moreover the passage to the associated graded algebra, namely $S(\mathfrak{g})$ recovers the classical invariants derived from geometry and mechanics and indeed in a more efficient fashion. This important role of representation theory will be the subject of several "selecta" including the present one. It will be seen that a detailed analysis of certain representations can lead to solutions of problems in geometry not a priori connected to representations of a non-commutative algebra.

4.2. Chevalley's Theorem.

In the remainder of this chapter the base field k is assumed to be algebraically closed of characteristic zero.

4.2.1. Let G be a connected, simply-connected semisimple algebraic group and $\mathfrak{g}$ its Lie algebra. Recall that G acts on $\mathfrak{g}$ and hence on $S(\mathfrak{g})$. Our problem is to determine the set $S(\mathfrak{g})^G$ of G invariant elements. Chevellay provided the following extraordinarily elegant solution. Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{g}$. Consider $S(\mathfrak{g})$ (resp. $S(\mathfrak{h})$) as the algebra of polynomial functions on $\mathfrak{g}^*$ (resp. $\mathfrak{h}^*$). Then restriction of functions provides a homomorphism ψ_0 of $S(\mathfrak{g})$ onto $S(\mathfrak{h})$. Let H be the connected algebraic subgroup of G with Lie algebra $\mathfrak{h}$ and recall that the Weyl group W identifies $N_G(H)/H$. In particular W acts on $S(\mathfrak{h})$ and the image under ψ_0 of a G invariant function is W invariant.

Theorem (Chevalley). *The restriction of ψ_0 to $S(\mathfrak{g})^G$ induces an isomorphism of $S(\mathfrak{g})^G$ onto $S(\mathfrak{h})^W$.*

4.2.2. When $G = GL(n)$, it is not hard to verify the above assertion by hand using for example the explicit description of the invariants given in 4.1.3. Such computations become increasing difficult as we pass through the different series of simple Lie algebras. However notice that it already involves some representation theory. Pushing the use of the trace much further Chevalley considered for each $n \in \mathbb{N}$, and each dominant weight $\lambda \in P^+(\pi)$ the functions on $\mathfrak{g}$ defined by

$$a \mapsto tr(a^n,\ V(\lambda)),$$

where in the right hand side a^n is viewed as an element of $U(\mathfrak{g})$ and $V(\lambda)$ denotes the finite dimensional simple $\mathfrak{g}$ module with highest weight λ. It is clear that any such function belongs to $S(\mathfrak{g}^*)$ (which we may identify with $S(\mathfrak{g})$ through the Killing form) and is G invariant. The key non-trivial contribution from representation theory is the use of the Weyl character formula to determine the right hand side when $a \in \mathfrak{h}$. What we have to know is that $ch\ V(\lambda)$ is a union of W orbit sums and that as λ runs over $P^+(\pi)$ that all orbit sums are obtained. This can be expressed more precisely as follows. Consider the ordering on $P^+(\pi)$ defined by $\lambda \geq \mu$ given $\lambda - \mu \in \mathbb{N}\pi$. With repect to the induced ordering on orbits, $ch\ V(\lambda)$ can be expressed as sums over $W\lambda$ and over smaller orbits. This follows just from the W invariance of $ch\ V(\lambda)$. Since W is a finite group we conclude that every invariant element of $S(\mathfrak{h})$ obtains in the above fashion. This proves surjectivity. Injectivity just follows from $P^+(\pi)$ being Zariski dense in $\mathfrak{h}^*$.

4.2.3. In retrospect it seems quite amazing that it took at least 35 years to consider the following "obvious" generalization of the Chevalley theorem. Namely repeat the same process with two copies of $\mathfrak{g}$ (and of $\mathfrak{h}$) but take invariants with respect to the diagonal copy of G (resp. W). It is rather trivial that the restriction of the corresponding map fails to be injective; but one can still ask if it is surjective. This question was posed by N. Wallach in view of applications to the study of invariant differential operators on $\mathfrak{g}$. Again it is not difficult to compute some cases by hand; but for the exceptional groups Wallach announced that we would have to await the next generation of computers. Here we give a solution to this problem using representation theory. Not surprisingly we need more than just the W invariance of $ch\ V(\lambda)$. Indeed we need some information on the decomposition of tensor products of such modules. In principle this could be obtained from the Weyl character formula; but to do so one must overcome some formidable combinatorics.

4.2.4. One can ask if it possible to further generalize Chevalley's theorem. Indeed the diagonal copy of G is just a reductive subgroup of $G \times G$. As we shall see reductivity is rather essential. However this still leaves a lot of choice. Indeed let $\mathfrak{r}$ be a reductive algebraic subalgebra of $\mathfrak{g}$ and let R be the connected algebraic subgroup of G with Lie algebra $\mathfrak{r}$. Consider the image of $S(\mathfrak{g})^R$ in $S(\mathfrak{h})$ defined as above by restricting ψ_0. Already by Chevalley's theorem this image contains $S(\mathfrak{h})^W$ and one may ask if there is a subgroup W_o of W such that the image is exactly $S(\mathfrak{h}^{W_o})$. The natural answer would be in terms of the Weyl group W_R corresponding to R. However this might not act on $\mathfrak{h}$ and some intelligent choices will have to be made.

4.2.5. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{r}$. It may be completed to a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Let T (resp. H) be the connected algebraic group with Lie algebra $\mathfrak{t}$ (resp. $\mathfrak{h}$). Recalling say [, 2.4.1, 2.4.2], one has $T = R \cap H$ because $\mathfrak{t} = \mathfrak{h} \cap \mathfrak{r}$ by construction. Then $N_R(H) \subset N_R(T)$. We should like to have equality. In this case $W_R = N_R(T)/T$ identifies with a subgroup of the full Weyl group $W = N_G(H)/H$. In particular $N_R(T)$ stabilizes $\mathfrak{h}$.

We may write $\mathfrak{r} = \mathfrak{s} \times \mathfrak{z}$, where $\mathfrak{s}$ is a Levi factor and $\mathfrak{z}$ is the centre of $\mathfrak{r}$. It turns that there is no loss of generality in assuming that $\mathfrak{z}$ is as large as possible. Precisely this means the following. Observe that $\mathfrak{g}^{\mathfrak{s}}$ is a reductive subalgebra of $\mathfrak{g}$ containing $\mathfrak{z}$. Since $\mathfrak{z}$ is commutative and acts by semisimple endomorphisms on $\mathfrak{g}^{\mathfrak{s}}$, it is contained in a Cartan subalgebra of the latter. We assume that $\mathfrak{z}$ is a Cartan subalgebra of $\mathfrak{g}^{\mathfrak{s}}$. Now return to $\mathfrak{t}$ and $\mathfrak{h}$ defined as above. We have $\mathfrak{h} \subset \mathfrak{g}^{\mathfrak{t}}$. We assume that equality holds. This may not be true in general; but still many cases are left, for example this holds when $\mathfrak{r}$ is the Lie algebra corresponding to a maximal compact subgroup of a real form of G. Indeed one has the

Lemma. *Let $\mathfrak{k}$ be the fixed points of an involution φ of $\mathfrak{g}$ and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{k}$. Then $\mathfrak{g}^{\mathfrak{t}}$ is a Cartan subalgebra of $\mathfrak{g}$.*

Proof. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, be the $\mathbb{Z}_2$ grading of $\mathfrak{g}$ defined by φ. Since $\mathfrak{k}^{\mathfrak{t}} = \mathfrak{t}$, one has $[\mathfrak{g}^{\mathfrak{t}}, \mathfrak{g}^{\mathfrak{t}}] \subset [\mathfrak{p}^{\mathfrak{t}}, \mathfrak{p}^{\mathfrak{t}}] \subset \mathfrak{k}^{\mathfrak{t}} = \mathfrak{t}$. Thus $\mathfrak{g}^{\mathfrak{t}}$ is both nilpotent and reductive, hence a Cartan subalgebra of $\mathfrak{g}$. □

4.2.6. Under the assumption that $h = \mathfrak{g}^{\mathfrak{t}}$, it is trivial that $N_R(T)$ stabilizes $\mathfrak{h}$ and so lies in $N_R(H)$. We may now ask if the restriction of ψ_0 to $S(\mathfrak{g})^R$ has image $S(\mathfrak{h})^{W_R}$.

In general the above question has a negative answer. However we shall obtain a representation theory criterion for which it does hold. This is expressed in the way $V(\lambda)$ decomposes when considered as an $\mathfrak{r}$ module. One needs to compare this decomposition with the decomposition of $W\lambda$ into W_R orbits. For each W_R orbit we need a component of $V(\lambda)$ which is roughly speaking supported on that orbit. The precise requirement is formulated in 4.5.2. That this holds in the diagonal case was a conjecture whose initial form had been proposed by Parthasarathy, Ranga Rao and Varadarajan and was put into more precise forms by Kostant and by Verma. We require Verma's formulaton of this conjecture. It was established by Kumar using Frobenius splitting. One may give a proof using crystals and this will be a subject of a further "selectum".

4.3. The Harish-Chandra Isomorphism.

4.3.1. Let $\mathfrak{g}$ be a Kac-Moody algebra. With respect to a triangular decomposition (2.3.2) set $\mathfrak{b} = \mathfrak{g}_+ + \mathfrak{h}$. For all $\lambda \in \mathfrak{h}^*$ define the one dimensional $\mathfrak{b}$ module $k_\lambda := kv_\lambda$, where v_λ

is annihilated by $\mathfrak{g}_+$ and satisfies $hv_\lambda = \lambda(h)v_\lambda$, for all $h \in \mathfrak{h}$. The Verma module $M(\lambda)$ with highest weight λ is the induced module

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} k_\lambda \ .$$

It is clear that $M(\lambda)$ is a free rank one $U(\mathfrak{g}_-)$ module. One checks as in 2.4.8 that $M(\lambda)$ admits a unique maximal submodule. The quotient $V(\lambda)$ is the unique simple highest weight module with highest weight λ.

4.3.2. BGG introduced the following category of $U(\mathfrak{g})$ modules known as the $\mathcal{O}$ category. An object M of $\mathcal{O}$ is a $U(\mathfrak{g})$ module which satisfies

(i) M is a direct sum of its $\mathfrak{h}$ weight subspaces $M_\lambda : \lambda \in \mathfrak{h}^*$.

(ii) $dim\ M_\lambda < \infty$, for all $\lambda \in \mathfrak{h}^*$, that is M is admissible.

(iii) The set $\Omega(M)$ of weights of M lies in some cone $\mu - \mathbb{N}\pi : \mu \in \mathfrak{h}^*$.

It is clear that $\mathcal{O}$ is an abelian category and that $M(\lambda) \in Ob\mathcal{O}$. Furthermore $\{V(\lambda) : \lambda \in \mathfrak{h}^*\}$ is the set of simple objects of $\mathcal{O}$. For each $M \in Ob\mathcal{O}$ one defines its formal character through

$$ch\ M(\lambda) = \sum_{\mu \in \mathfrak{h}^*} dim\ M_\mu e^\mu .$$

Since $M(\lambda)$ is isomorphic to $U(\mathfrak{g}_-)$ as a $U(\mathfrak{h})$ module, up to translation by λ, it follows easily that

$$ch\ M(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{dim\ \mathfrak{g}_{-\alpha}}} .$$

Thus the calculation of the $ch\ V(\lambda) : \lambda \in \mathfrak{h}^*$ is reduced to determining the transition matrix (which is clearly triangular) relating it to the $ch\ M(\lambda) : \lambda \in \mathfrak{h}^*$. For $\mathfrak{g}$ semisimple the importance of this transition matrix was recognized already in the study of Prim $U(\mathfrak{g})$ and moreover it was found to be mysteriously related to the decomposition of the regular representation of W. This latter result found a more precise formulation due to Kazhdan-Lusztig (whom we abbreviate as KL) who introduced a combinatorially defined transition matrix between appropriate bases of the corresponding Hecke algebra and conjectured that these transition matrices were the same. This gave rise to a whole host of similar conjectures coming generally under the heading of KL theory, many of which have now been proved.

4.3.3. Let P the projection onto $U(\mathfrak{h})$ defined by the direct sum decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (\mathfrak{g}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{g}_+)$. Identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$ and the latter with the space of polynomial functions on $\mathfrak{h}^*$. Let σ be a Chevalley antiautomorphism. For each $\lambda \in \mathfrak{h}^*$ one may define a symmetric bilinear form on $U(\mathfrak{g})$ through $< a, b > = P(\sigma(a)b)(\lambda)$. It passes to $M(\lambda)$ and satisfies $< \sigma(a)m, n > = < m, an >$, for all $m, n \in M(\lambda),\ a \in U(\mathfrak{g})$. It is called the contravariant form on $M(\lambda)$. It passes to a non-degenerate form on $V(\lambda)$.

4.3.4. There are several deep results concerning the contravariant form. Already because it is non-degenerate on $V(\lambda)$ it identifies the latter with its graded dual. This gave KL the idea to define the basis in the Hecke algebra corrsponding to $V(\lambda)$ to consist of self-dual elements. Secondly for $\mathfrak{g}$ semisimple and for $\lambda \in P^+(\pi)$, the contravariant form is positive definite in the following sense. Let $\mathfrak{g}_{\mathbb{R}}$ be the $\mathbb{R}$ linear subspace of $\mathfrak{g}$ spanned by the Chevalley basis and set $V(\lambda)_{\mathbb{R}} = U(\mathfrak{g}_{\mathbb{R}})\bar{v}_\lambda$, where $\bar{v}_\lambda$ denotes the image of v_λ in $V(\lambda)$. Then the restriction of $<, >$ to $V(\lambda)_{\mathbb{R}}$ is positive definite. (This holds equally well over $\mathbb{Q}$.) It reflects the fact that $\mathfrak{g}$ is the Lie algebra of a compact real form of G and the corresponding representation of G is unitary. However it also has a purely algebraic proof which only involves the Casimir invariant and hence passes [ ] to the case when $\mathfrak{g}$ is symmetrizable Kac-Moody.

Thirdly one may calculate the determinants of the restriction of the form for each weight subspace of $M(\lambda)$. Here λ is viewed as a parameter. The calculation is due to Shapovalov [ ]. Using the Janzten filtration which is determined by "powers of zeros" of the form one may determine [ ] the simple factors which can occur in a given Verma module. Unfortunately this does not determine their multiplicities. Nevertheless the truth of a conjecture of Jantzen [ ] describing the behaviour of the filtration under Verma module embeddings results in exactly [ ] the multiplicities predicted by KL theory. Unfortunately at present the heavy machinery needed to prove this conjecture is even more forbidding than that used to prove the KL conjectures themselves! Still Jantzen's arguments are rather powerful and have been used to obtain the form [ ] of the parabolic analogues of these determinants as well as extending the Shapovalov result to the symmetrizable Kac-Moody case [ ].

4.3.6. Adopt the notation of 3.3.4. Then $[x_\alpha, x_{-\beta}] = 0$, for all $\alpha, \beta \in \pi$ distinct.

Otherwise $[x_\alpha, x_{-\alpha}] = h_\alpha \in \mathfrak{h}$, interation of which gives

$$[x_\alpha, x_{-\alpha}^{n+1}] = (n+1)x_{-\alpha}^n(h_\alpha - \frac{1}{2}n(\alpha,\alpha)), \quad for\ all \quad n \in \mathbb{N}.$$

Recall the translated action (3.2.9) of W on $\mathfrak{h}^*$.

Lemma. *Take $\lambda \in \mathfrak{h}^*$ and $\alpha \in \pi$ such that $\alpha^\vee(\lambda) \in \mathbb{N}$. There exists an embedding $M(s_\alpha.\lambda) \hookrightarrow M(\lambda)$ of Verma modules.*

Proof. Set $n = \alpha^\vee(\lambda)$. From the above relations it follows that $x_{-\alpha}^{n+1}v_\lambda$ is a weight vector annihilated by $\mathfrak{g}_+$. It hence generates a highest weight submodule of $M(\lambda)$ which being free over $U(\mathfrak{g}_-)$ must again be a Verma module and has highest weight $\lambda - (n+1)\alpha = s_\alpha.\lambda$. □

Remark. This provides a first step in computing the Shapovalov determinants.

4.3.7. Assume $\mathfrak{g}$ semisimple. The restriction ψ of P to $U(\mathfrak{g})$ is an algebra morphism called the Harish-Chandra homomorphism. The following result is due to Harish-Chandra. It extends the Chevalley theorem to $U(\mathfrak{g})$, where it can be applied to representation theory (see 4.3.8).

Theorem. *The restriction of ψ to $Z(\mathfrak{g})$ is an isomorphism onto $S(\mathfrak{h})^{W_o}$.*

Proof. Observe that $P(a)(\lambda) = < v_\lambda, av_\lambda >$, for all $a \in U(\mathfrak{g})$. It follows that $z \in Z(\mathfrak{g})$ acts by the scalar $\psi(z)(\lambda)$ on $M(\lambda)$. By 4.5.6 one has

$$\psi(z)(\lambda) = \psi(z)(s_\alpha.\lambda),\ \forall\ \lambda \in P^+(\pi),\ \alpha \in \pi.$$

Since $P^+(\pi)$ is Zariski dense in $\mathfrak{h}^*$, it follows that $\psi(z) \in S(\mathfrak{h})^{W_o}$. The proof is completed using 4.2.1 and the canonical filtration. Alternatively one may directly use the Letzter map described below. □

Remark 1. This result goes through for the "simply-connected" quantized enveloping algebra using the quantum analogue of the Letzter map [ ].

Remark 2. One cannot expect an easy analogue of this result for $\mathfrak{g}$ Kac-Moody. Already V. Chari [ ] has shown that $U(\mathfrak{g})$ is generated by the centre of $\mathfrak{g}$ for $\mathfrak{g}$ affine. One needs to complete $U(\mathfrak{g})$ to obtain the Casimir invariant constructed by Kac. This does not act on all modules; but does act on those in the $\mathcal{O}$ category. Higher degree invariants are more obscure [ ]. For the quantized enveloping algebra invariants may be obtained though completion [ ]. For $\mathfrak{g}$ affine one may show that they can be made to act in a natural way on (a suitable subcategory of) the $\mathcal{O}$ category [ ].

4.3.8. Suppose $\mathfrak{g}$ is semisimple. Even in this case an object of $\mathcal{O}$ is not necessarily of finite length. Nevertheless one has the

Proposition. *Suppose $M \in Ob\mathcal{O}$ is finitely generated. Then M has finite length.*

Proof. With loss of generality we can assume the generators of M to be weight vectors. Let a be such a generator. Since the action of $Z(\mathfrak{g})$ preserves weight and M is admissible, it follows that $Z(\mathfrak{g})a$ is finite dimensional. Then the common annihilator of the generators of M over $Z(\mathfrak{g})$, which is just $Ann_{Z(\mathfrak{g})}M$, has finite codimension. Thus M has a finite filtration by $U(\mathfrak{g})$ submodules with quotients admitting a scalar action of $Z(\mathfrak{g})$ and these are finitely generated since $U(\mathfrak{g})$ is noetherian. Consequently we may assume $Z(\mathfrak{g})$ to act by scalars on M.

The noetherianity of $U(\mathfrak{g})$ implies that every submodule of M admits a maximal submodule. Through this one may construct a decreasing chain of submodules with simple quotients. Since M is admissible there can only be finite repetitions of a given simple factor. Yet the simple objects of $\mathcal{O}$ are the $V(\lambda)$. By 4.3.7 there can be at most $|W| < \infty$, simples admitting a given scalar action of the centre. Consequently M has finite length. □

Remark. Noetherianity can be avoided through [ ].

4.4. The Letzter map.

4.4.1. The conception of the Letzter map was in part the fortuitous consequence of a secretarial error. Funding for joint work with G. Letzter had been turned down by the BSF; but their secretary had written to me saying that it had been approved !

The essence of the construction was a contribution of Letzter to the first of our many successful collaborations. It had originally been aimed at the construction of the centre of the quantized enveloping algebra. However the idea is rather simple and flexible applying to "reductive" systems whether they be classical, quantum or super. It replaces "group averaging" used to obtain invariants in the theory of finite groups.

4.4.2. Let $\mathfrak{r}$ be a reductive Lie algebra, V a finite dimensional $U(\mathfrak{r})$ module and P its annihilator.

Lemma. $P^2 = P$.

Proof. Write $\mathfrak{r} = \mathfrak{s} \times \mathfrak{z}$ as in 4.2.5. Since $\mathfrak{z}$ must act by scalars we can assume $\mathfrak{r} = \mathfrak{s}$ without loss of generality. Set $U = U(\mathfrak{s})$.

Multiplication in U gives a surjective map of $P \times U/P$ onto P/P^2. Since U is right noetherian, P has a finite number n of generators as right U module and so $dim\ P/P^2 \leq n\ dim\ U/P < \infty$. Consequently U/P^2 is a finite dimensional left U module. Since $\mathfrak{s}$ is semisimple, it is a direct sum of simple U modules. Thus its annihilator P^2 must be semiprime and hence equal to P. □

Remark. It is generally false that the annihilator of a simple highest weight $U(\mathfrak{g})$ module V is idempotent. However if one considers its image in $U(\mathfrak{g})/U(\mathfrak{g})Ann_{Z(\mathfrak{g})}V$, then this is often the case. Indeed there are no known counterexamples.

4.4.3. We may now define the Letzter map. Define $\mathfrak{r}$ as in 4.3.5, so that $W_R = N_R(H)/T$. Let $U(\mathfrak{r})_+$ denote the augmentation ideal of $U(\mathfrak{r})$, that is to say the annihilator of the trivial module. Take $a \in S(\mathfrak{h})$ of degree n. Then $(ad\ U(\mathfrak{r}))a$ belongs to the subspace of polynomials on $\mathfrak{g}^*$ of degree $\leq n$ and is hence finite dimensional. Since $\mathfrak{r}$ is reductive, it is also a direct sum of simple $U(\mathfrak{r})$ modules. Now either $(ad\ U(\mathfrak{r})_+)a$ equals $(ad\ U(\mathfrak{r}))a$ or is of codimension one. In the latter case it has a complement which is the trivial $U(\mathfrak{r})$ module. On the other hand $(ad\, U(\mathfrak{r})_+)a$ is also a direct sum of finite dimensional $\mathfrak{r}$ modules in which the trivial $\mathfrak{r}$ module cannot appear because $U(\mathfrak{r})_+$ is idempotent. We conclude that there is a unique R invariant element $\theta_0^R(a)$ of $(ad\ U(\mathfrak{r}))a$ such that $\theta_0^R(a) - a \in (ad\ U(\mathfrak{r})_+)a$.

The uniqueness of $\theta_0^R(a)$ immediately implies the linearity of the map $a \mapsto \theta_0^R(a)$. Given $w \in W_R$, choose a representative $n_w \in N_R(H)$. Then $\theta_0^R(a) - n_w(a) = n_w(\theta_0^R(a) - a) \in n_w(ad\ U(\mathfrak{r})_+)a = (ad\ U(\mathfrak{r})_+)n_w(a)$, since $\mathfrak{r}$ is stable under conjugation by R. Uniqueness then implies that $\theta_0^R(n_w(a)) = \theta_0^R(a)$. Consequently θ_0^R factors to a linear map of $S(\mathfrak{h})^{W_R}$ onto $((ad\ U(\mathfrak{r}))S(\mathfrak{h}))^R$. It is called the Letzter map.

Consider the restriction ψ_0^R of ψ_0 to $((ad\ U(\mathfrak{r}))S(\mathfrak{h}))^R$. We eventually want to show that ψ_0^R is an isomorphism onto $S(\mathfrak{h})^{W_R}$. We claim that it is enough to show that both ψ_0^R and θ_0^R are injective. Indeed if this holds their composition $\psi_0^R\theta_0^R$ is also injective. Yet both maps preserve degree and so for each $n \in \mathbb{N}$, their composition restricts to an injection of the subspace of W_R invariant polynomials on $\mathfrak{h}^*$ of degree $\leq n$. Yet these subspaces are finite dimensional, so the composition is an isomorphism forcing the surjectivity of ψ_0^R.

In general the above isomorphism cannot be the identity map. Indeed, whereas ψ_0^R is an algebra map, θ_0^R is only linear.

4.4.4. Let us first examine the possible injectivity of ψ_0^R.

The pairing $S_n(g) \times S_n(\mathfrak{g}^*) \to k$ defined by linearity and

$$(*) \qquad < x_1 x_2 \cdot \cdots \cdot x_n, \xi_1 \xi_2 \cdot \cdots \cdot \xi_n > = \prod_{i=1}^{n} \xi_i(x_i), \ \forall \ \xi_i \in \mathfrak{g}^*, x_i \in \mathfrak{g},$$

identifies $S(\mathfrak{g}^*)$ with the graded dual of $S(\mathfrak{g})$. Given $a \in S_\alpha(\mathfrak{g})$, $\nu \in \mathfrak{g}^*$, define $a(\nu) \in k$, by evaluation. Then $a(\nu) = < a, \nu^n >$.

Lemma. *With respect to the above pairing, the ideal I of definition of the closure $\overline{R\mathfrak{h}^*}$ of $R\mathfrak{h}^*$ in $S(\mathfrak{g})$ is $((ad\ U(\mathfrak{r}))S(\mathfrak{h}^*))^\perp$.*

Proof. Take $a \in S_n(\mathfrak{g})$. By the above formula we have

$$a \in I \Longleftrightarrow < a, (r\nu)^n > = 0, \ \forall \ r \in R, \nu \in \mathfrak{h}^* \Longleftrightarrow < a, (ad\ U(\mathfrak{r}))S_n(\mathfrak{h}^*) > = 0,$$

which extended by linearity gives the required assertion. □

4.4.5. By 4.4.4, the injectivity of ψ_0^R is reduced to showing that

$$(*) \qquad ((ad\ U(\mathfrak{r}))S(\mathfrak{h}^*))^\perp \cap (ad\ U(\mathfrak{r}))S(\mathfrak{h}) = 0.$$

This certainly requires $\mathfrak{r}$ to be reductive, for example it fails when $\mathfrak{r}$ is a Borel subalgebra. For k the complex field, the idea of the proof is to construct an $ad\ U(\mathfrak{r})$ invariant positive definite hermitian form on $S(\mathfrak{g})$ for which $(ad\ U(\mathfrak{r}))S(\mathfrak{h})$ is stable under complex conjugation.

Let V be a k vector space and recall that we are assuming k to be of characteristic zero. We say that $V_\mathbb{Q}$ is a $\mathbb{Q}$-lattice for V if $V_\mathbb{Q}$ is a $\mathbb{Q}$ subspace of V and the canonical map $V_\mathbb{Q} \otimes_\mathbb{Q} k \to V$ is bijective. Such a lattice is just obtained by taking the $\mathbb{Q}$ linear span of a k basis of V.

Let $\mathfrak{r}_\mathbb{Q}$ be the $\mathbb{Q}$-lattice for $\mathfrak{r}$ obtained from a Chevalley basis. Since $\mathfrak{r}$ is reductive in $\mathfrak{g}$, the latter is a direct sum of simple highest weight $U(\mathfrak{r})$ modules under adjoint action.

Fix a highest weight vector v in each direct summand V. Set $V_{\mathbb{Q}} = U(\mathfrak{r}_{\mathbb{Q}})v$. Their direct sum is a $\mathbb{Q}$-lattice $\mathfrak{g}_{\mathbb{Q}}$ for $\mathfrak{g}$.

Since $\mathfrak{r}$ is reductive and V is a finite dimensional simple $\mathfrak{r}$ module one may define (4.3.4) a contravariant form $< \, , \, >$ on V. A key fact is that the contravariant form $< \, , \, >$ on $V_{\mathbb{Q}}$ is positive definite. Taking direct sums we obtain a positive definite contravariant form on $\mathfrak{g}_{\mathbb{Q}}$. It extends to a positive definite contravariant form on each homogeneous subspace $S_n(\mathfrak{g}_{\mathbb{Q}})$ by a formula similar to 4.4.4 $(*)$ and hence to such a form on $S(\mathfrak{g}_{\mathbb{Q}})$ which is $\mathbb{Q}$-lattice for $S(\mathfrak{g})$. Extend this form by linearity to a non-degenerate form on $S(\mathfrak{g})$. This restricts to a non-degenerate form on $\mathfrak{g}$ and so defines an isomorphism γ of $\mathfrak{g}$ onto $\mathfrak{g}^*$ which by contravariance is a $U(\mathfrak{r})$ module map.

Since $\mathfrak{h} = \mathfrak{g}^{\mathfrak{t}}$ it follows that $\gamma(\mathfrak{h}) = \mathfrak{h}^*$ and that $\mathfrak{h}_{\mathbb{Q}} := \mathfrak{g}_{\mathbb{Q}}^{\mathfrak{t}_{\mathbb{Q}}}$ is a $\mathbb{Q}$-lattice for $\mathfrak{h}$. Then $(ad\, U(\mathfrak{r}_{\mathbb{Q}}))S(\mathfrak{h}_{\mathbb{Q}})$ is a $\mathbb{Q}$-lattice for $(ad\, U(\mathfrak{r}))S(\mathfrak{h})$ lying in $S(\mathfrak{g}_{\mathbb{Q}})$. Then $< \, , \, >$ is positive definite on $(ad\, U(\mathfrak{r}_{\mathbb{Q}}))S(\mathfrak{h}_{\mathbb{Q}})$ and so is non-degenerate on $(ad\, U(\mathfrak{r}))S(\mathfrak{h})$. Applying γ gives $(*)$ and so proves the

Proposition. *The restriction* ψ_0^R *of* ψ_0 *to* $((ad\, U(\mathfrak{r}))S(\mathfrak{h}))^R$ *is injective.*

4.5. A criterion for the injectivity of the Letzter map.

4.5.1. In this section it is convenient to assume that $k = \mathbb{C}$. This involves no loss of generality. Indeed through a Chevalley basis for $\mathfrak{g}$ we may assume that $\mathfrak{g}$ is defined over $\mathbb{Q}$. Then $\mathfrak{r}$ being a finite dimensional subspace of $\mathfrak{g}$ will be defined over a finite extension of $\mathbb{Q}$ and it remains to extend scalars to $\mathbb{C}$.

4.5.2. In order to establish the injectivity of the Letzter map we need to pass to a non-commutative context. First an exactly similar construction to 4.4.3 gives a linear map $\theta^R : U(\mathfrak{h})^{W_R} \to ((ad\, U(\mathfrak{r}))U(\mathfrak{h}))^R$. Let $s : S(\mathfrak{g}) \to U(\mathfrak{g})$ be the symmetrization map. It is a linear isomorphism of vector spaces commuting with the adjoint action. It follows easily that the diagram

$$\begin{array}{ccc} S(\mathfrak{h})^{W_R} & \xrightarrow{\theta_0^R} & ((ad\, U(\mathfrak{r}))S(\mathfrak{h}))^R \\ s \wr \downarrow & & s \wr \downarrow \\ U(\mathfrak{h})^{W_R} & \xrightarrow{\theta^R} & ((ad\, U(\mathfrak{r}))U(\mathfrak{h}))^R \end{array}$$

is commutative. Thus the injectivity of θ^R and of θ_0^R are equivalent.

4.5.3. We formulate a separation property which will give a criterion for θ^R to be injective. Fix $\lambda \in \mathfrak{h}^*$ and let $\overline{V(\lambda)}$ denote the subspace spanned by the weight vectors of $V(\lambda)$ whose weights do not lie in $W\lambda$. View W_R as a subgroup of W and write $W\lambda$ as a disjoint union $\coprod \Omega_i$ of W_R orbits.

Let Δ (resp. Δ_R) be the set of non-zero roots of $\mathfrak{g}$ (resp. $\mathfrak{r}$). Since $\mathfrak{t}$ may be strictly contained in $\mathfrak{h}$, the roots of $\mathfrak{r}$ will be restrictions of roots of $\mathfrak{g}$ and moreover different roots may have the same restriction. Nevertheless the standard construction gives a positive system Δ^+ for Δ such that $\Delta_R^+ := \Delta^+ \cap \Delta_R$ is a positive system for Δ_R. Root vectors of $\mathfrak{r}$ will be linear combinations of root vectors for $\mathfrak{g}$ corresponding to roots having the same restriction.

Again distinct elements of a given W_R orbit may coincide on $\mathfrak{t}$. Thus we decompose each Ω_i into disjoint subsets $\Omega_{i,s}$ corresponding to elements which coincide on $\mathfrak{t}$. Let $\Omega_{i,1}$ denote the subset corresponding to the unique weight $\mu_i \in \mathfrak{t}^*$ in the W_R orbit Ω_i, dominant with respect to Δ_R^+. If $\nu_i \in \Omega_{i,1}$, then $\mu_i = \nu_{i|_{\mathfrak{t}}}$. Set $W^i = Stab_W \mu_i$ and $W_R^i = W^i \cap W_R$. Observe that $\Omega_{i,1} = W_R^i \nu_i$.

For each $w \in W$, choose scalars so that $v_{w\lambda} = wv_\lambda$. We make the following separation hypothesis.

($\mathcal{S}_R$). *For each i there exists a $U(\mathfrak{r})$ direct summand V_i of $V(\lambda)$, whose highest weight vector can be written as $v_{i,1} = v'_{i,1} + v''_{i,1}$ where $v''_{i,1} \in \overline{V(\lambda)}$ and*

$$v'_{i,1} = \sum_{y \in W_R^i} c_y v_{y\lambda} : c_y \in \mathbb{C}$$

is non-zero.

One may remark that $\mathbb{C}v_{i,1}$ is a one dimensional W_R^i module. Let χ^i be the linear character it defines. Then $c_y = \chi^i(y^{-1})$, $\forall\ y \in W_R^i$, up to an overall scalar. On the other hand the $wv_{i,1} : w \in W^i/W_R^i$ run over highest weight vectors of direct summands of $V(\lambda)$ isomorphic to V_i. Thus the separation hypothesis can require multiplicities in the decomposition of $V(\lambda)$.

From the action of W_R it is immediate that the extreme vectors $v'_{i,s}$ of V_i (those having weights in $\Omega_{i,s}$) can be written in a similar form.

Recall that $V(\lambda)$ admits a positive definite hermitian form $< \, , \, >$. It is convenient to assume that $< v_{i,s}, v_{i,s} >= 1$. Observe that

$$(*) \qquad < v'_{i,s}, v''_{i,s} > = 0,$$

by the orthogonality of different $\mathfrak{h}$ weight subspaces.

4.5.4. Proposition. *Suppose $\mathcal{S}_R$ holds. Then θ^R is injective.*

Proof. Take V to be defined by $\mathcal{S}_R$ and let $P_i \in Hom_{U(\mathfrak{r})}(V(\lambda), V_i)$ be a projection onto V_i . Then $a \in ker\ \theta^R$ implies $a \in (ad\ U(\mathfrak{r})_+)a$ and so $tr(P_i a P_i) = tr(P_i a) = tr(ad\ U(\mathfrak{r})_+(P_i a)) = \varepsilon(U(\mathfrak{r})_+) tr(P_i a) = 0$, where ε is the coidentity. (The penultimate identity follows from the invariance of the trace. This needs modification for quantized enveloping algebras which are not co-commutative.)

To show that $a = 0$, it is enough to show that a vanishes on the Zariski dense subset $P(\pi)$. This is proved by a two-fold induction with respect to the order relation on $P^+(\pi)$ defined as before by $\mu \geq \nu$ if $\mu - \nu \in \mathbb{N}\pi$, and the corresponding order relation on the dominant weights of $\mathfrak{t}$. We can assume that a vanishes on every W orbit in the weights of $V(\lambda)$ excepting $W\lambda$, since the dominant element of such an orbit is strictly smaller than λ. Thus $\overline{aV(\lambda)} = 0$. Again we can assume that a vanishes on the weights of V_i excepting $W_R \mu_i$. (These weights may be restrictions of weights of $V(\lambda)$ lying in $W\lambda$ and so are weights of some other V_j whose highest weight will be strictly smaller than μ_i. An example is given in Remark 2 of 4.5.6.) Then a vanishes on the non-extremal weight vectors of V_i. Hence

$$\begin{aligned} 0 = tr(P_i a P_i) &= \sum_s < v_{i,s}, a v_{i,s} >, \quad \text{by the last remark,} \\ &= a(\mu_i) \sum_s < v_{i,s}, v'_{i,s} >, \quad \text{since } a \text{ is } W \text{ invariant and } a\overline{V(\lambda)} = 0, \\ &= a(\mu_i) \sum_s < v'_{i,s}, v'_{i,s} >, \quad \text{by 4.5.3}(*). \end{aligned}$$

Since the $v'_{i,s}$ are non-vanishing and the form is positive definite, the above sum is strictly positive and so $a(\mu_i) = 0$, as required. □

4.5.5. Combining 4.4.5, 4.5.3 and the remarks in 4.4.3 we obtain

Theorem. *Suppose $\mathcal{S}_R$ holds. Then ψ_0^R is an isomorphism of $((ad\ U(\mathfrak{r}))S(\mathfrak{h}))^R$ onto $S(\mathfrak{h})^{W_R}$.*

4.5.6. Of course the separation hypothesis holds trivially when $\mathfrak{r} = \mathfrak{g}$. Moreover $(ad\ U(\mathfrak{g}))S(\mathfrak{h}) = S(\mathfrak{h})$. (This may be checked by a direct computation or by 4.4.5 $(*)$ noting [ ] that $G\mathfrak{h}^*$ is dense in $\mathfrak{g}^*$). Then Chevalley's theorem obtains from 4.2.1.

In general it is extremely hard to verify the separation theorem for a given choice of $\mathfrak{r}$ and indeed as noted in the example below it may fail. When $\mathfrak{r}$ is the diagonal copy of $\mathfrak{g}$ in $\mathfrak{g} \times \mathfrak{g}$, it results from the following theorem of S. Kumar. Take $\lambda, \mu \in P^+(\pi)$. Then $V(\lambda) \otimes V(\mu)$ is a simple finite dimensional $\mathfrak{g} \times \mathfrak{g}$ module and all such modules are so obtained. Observe that the restriction of a weight $(\lambda', \mu') \in \mathfrak{h}^* \times \mathfrak{h}^*$ to $\mathfrak{t}$ (which identifies with the diagonal copy of $\mathfrak{h}$) is just the sum $\lambda' + \mu'$. Again W_R identifies with the diagonal copy of W. The W orbits in $W\lambda \times W\mu$ are generated by the $\lambda + w\mu : w \in W$.

Given $w \in W$, choose $y \in W$ such that $\nu := y\lambda + yw\mu$ is dominant with respect to Δ^+. Observe that elements w, w' lying in the same $W_\lambda \setminus W/W_\mu$ double coset give the same dominant element.

Theorem.

(i) $U(\mathfrak{g})(v_\lambda \otimes v_{w\mu})$ *contains a unique copy of* $V(\nu)$.

(ii) *As* w *runs over* $W_\lambda \setminus W/W_\mu$, *the simple modules defined in (i) above form a direct sum* S.

Remark 1. We shall give a proof of this result based on Littelmann's crystal theory.

Remark 2. Assume $\lambda, \mu \in P^+(\pi) \cap \mathbb{N}\pi$. It can easily happen that $y\lambda + w\mu = 0$, for some $y, w \in W$. Yet under these assumptions the zero weight space belongs to every component $V(\nu)$ of the tensor product $V(\lambda) \otimes V(\mu)$. This is why a two-fold induction is needed in 4.5.4.

4.5.7. Take $\mathfrak{g}$ to be $\mathfrak{g} \times \mathfrak{g}$ and $\mathfrak{r}$ to be its diagonal copy. One may identify $\mathfrak{r}$ as the fixed points under the involution φ : sending (x, y) to (y, x). Thus ψ_0^R is injective by 4.4.5.

Theorem. *For the above pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$ the separation hypothesis holds. In particular the corresponding Chevalley map is surjective.*

Proof. Take $\lambda, \mu, \nu \in P^+(\pi)$ and $y, w \in W$ as in 4.5.6 and set $M = \overline{V(\lambda) \otimes V(\mu)}$, $W^\nu =$

$\{w \in W \mid \nu \in W(\lambda + w\mu)\}$, $W_\nu = Stab_W \nu$. Let S^ν denote the isotypical component of S of type ν.

It is clear that W^ν is a union of $W_\lambda \mid W/W_\mu$ double cosets and we choose representatives $w_i \in W^\nu$ and $y_i \in W$ such that $\nu = y_i\lambda + y_i w_i \mu$. Let v_ν be a highest weight vector of a simple $V \subset S^\nu$. Then $\mathbb{C}v_\nu$ is W_ν module and we let χ be its character. Then, *mod* M, we can express v_ν as a linear combination of terms of the form

$$\sum_{z \in W_\nu} \chi(z^{-1})(v_{zy_i\lambda} \otimes v_{zy_iw_i\mu}).$$

Since S^ν is a direct sum of $|W_\lambda \setminus W^\nu/W_\mu|$ copies of $V(\nu)$ we may choose linear combinations of highest weight vectors so that either there is always just one such term for each such highest weight vector or we can find a copy V whose the highest weight vector is zero *mod* M. In the first case the separation hypothesis holds. In the second case the action of W implies that every extreme vector of V lies in M.

Let $<\ ,\ >$ be the contravariant form on $V(\lambda) \otimes V(\mu)$. We claim that if the second case holds then

$$(*) \qquad < V, v_{y\lambda} \otimes v_{yw\mu} > = 0, \quad for\ all \quad y \in W, w \in W^\nu.$$

It is enough to prove this orthogonality for each weight vector of V. For extremal vectors of V it holds since each susch vector lies in M. Again for $y \in W, w \in W^\nu$ the $y\lambda + yw\mu$ are the extremal weights of V so it holds for the remaining weight vectors of V by the orthogonality of $\mathfrak{h}$ weight vectors. Recalling 4.5.6 (i) it follows from $(*)$ that V is orthogonal to the generators of S^ν and hence is orthogonal to S^ν. Then $V \subset S^\perp \cap S = 0$, since the form is positive definite. This contradiction proves the separation hypothesis. The last part follows from 4.5.5 (and the injectivity of ψ_0^R). □

4.5.8. Take $g = \mathfrak{sl}(3)$ and write $\pi = (\alpha, \beta)$. Let $\mathfrak{r}$ be a copy of $\mathfrak{sl}(2)$ generated by $x := x_\alpha + x_\beta$, $y := x_{-\alpha} + x_{-\beta}$. Let V be one of the fundamental $\mathfrak{sl}(3)$ modules. It is three dimensional and its weights lie in a single W orbit. Since $|W_R| = 2$, this cannot remain a single W_R orbit. On the other hand V restricted to $\mathfrak{r}$ remains simple. Hence the separation condition fails. One may also check that ψ_0^R is not surjective. In this case the image is the space of elements of $S(\mathfrak{h})$ invariant under W_R and the Dynkin automorphism interchanging α, β.

5. Admissible Weight Modules.

5.1. Finiteness in the Infinite.

5.1.1. An unwritten principle and a fervent belief amongst some is that a mathematical theory only transcends the elementary when finiteness is discarded. Yet perhaps because the human condition is finite, few theories are truly infinite and even the most adventurous are attached by heavy anchors to the finite. Thus we prefer to consider infinite groups which are finite dimensional vector spaces, infinite dimensional Lie algebras which are graded by finite dimensional subspaces and infinite dimensional gradations which are obtained by finite generation.

5.1.2. Concerning modules over a Lie algebra $\mathfrak{g}$, finiteness is often introduced through the notion of "admissibility". This was already a key element for our second "selectum" and indeed figures prominently in any discussion concerning highest weight modules. In general terms we admit infinite dimensional (simple) $\mathfrak{g}$ modules; but require them to break into (infinitely many) isomorphism classes of finite dimensional submodules with respect to a subalgebra $\mathfrak{k}$ of $\mathfrak{g}$. A crucial issue is the size of $\mathfrak{k}$ relative to $\mathfrak{g}$. If it is too big then admissibility may become too restrictive. If it is too small then no serious classification can be undertaken. Inspiration from the Harish-Chandra theory of unitary representations deemed that $\mathfrak{k}$ should be roughly half of $\mathfrak{g}$, specifically the fixed points of an involution. For $\mathfrak{g}$ semisimple such modules, known as Harish-Chandra modules, have been intensively studied and admit an immensely rich structure theory. By contrast admissibility with respect to fixed subalgebras of other finite automorphism groups has been universally ignored.

5.1.3. Let $\mathfrak{g}$ be a Kac-Moody algebra and $\mathfrak{h}$ a Cartan subalgebra. Admissibility with respect to $\mathfrak{h}$ would seem to be hopelessly out of line with the point of view proposed in 5.1.2. Indeed *dim* $\mathfrak{h}$ is only finite and even in the semisimple case its value is more close to the square root of *dim* $\mathfrak{g}$. Of course highest weight modules, which are certainly of great importance, do fit within this framework. However in this case we have to remember that they are locally finite with respect to $\mathfrak{b}$ (as defined in 4.3.1) which is again roughly half of $\mathfrak{g}$. Of course $\mathfrak{b}$ is not reductive and such modules have only a filtration whose quotients are finite dimensional $\mathfrak{b}$ modules. It is perhaps also pertinent to recall that their serious study came long after the study of Harish-Chandra modules and that they only gained full respectibility through Casselman-Jacquet functor relating them to Harish-Chandra modules.

5.1.4. Should one study modules which are genuinely $\mathfrak{h}$ admissible? After all nothing is forbidden in mathematics, though travellers to inappropriate foreign lands may be "disadvised" from impolitic adventure. Yet taking this particular route one discovers a fascinating dichotomy. Either dimensions are bounded and the modules must remain small, or admissibility holds with respect to a larger subalgebra. Indeed this we already discovered in our second "selectum". In that case there was a disappointing paucity of truly $\mathfrak{h}$ admissible modules. Here we shall study the semisimple case in which one finds a small but rich variety of examples. In one of our further "selecta" we study "integrable" modules for $\mathfrak{g}$ affine. By the very definition of these modules they are locally finite under the action of the finite dimensional semisimple subalgebras given by a proper subset of the Dynkin diagram. Here a crucial issue is whether they are admissible with respect to the Cartan subalgebra. Not surprisingly such admissible modules are not much bigger than finite dimensional modules simple over a maximal Levi factor.

5.2. Complete subsets of a root system.

In the remainder of this chapter the base field $\mathfrak{k}$ is assumed to be algebraically closed of characteristic zero.

5.2.1. Let $\mathfrak{g}$ be a Kac-Moody algebra and adopt the notation of 4.3.2. A $U(\mathfrak{g})$ module is said to be $\mathfrak{h}$ admissible if (i), (ii) of 4.3.2 are satisfied. We denote the category of $\mathfrak{h}$ admisible modules by $\mathcal{F}_{\mathfrak{g}}$, or simply $\mathcal{F}$. It is clearly an abelian category containing $\mathcal{O}$ as a subcategory. If $\mathfrak{g}$ is semisimple one is able to give a fairly good description of such modules, the basic results being due to S. L. Fernando. Little is known in the general case though of course $\mathfrak{g}$ itself is $\mathfrak{h}$ admissible. Already this fails for tensor products of $\mathfrak{g}$ with itself unless $dim\ \mathfrak{g} < \infty$, though in the affine case such tensor products have again admissible simple quotients.

From now on we assume $\mathfrak{g}$ semisimple.

5.2.2. A subset $S \subset \Delta$ is said to be closed if $\alpha, \beta \in S,\ \alpha + \beta \in \Delta$ implies $\alpha + \beta \in S$. A subset S is said to be coclosed if $\Delta \setminus S$ is closed. If both hold S is said to be complete. The novelty in Fernando's approach was to realize the importance of this concept and to classify complete subsets. Here it is interesting to remark that the notion of a complete subset of Δ^+ is much older. Moreover it had been known for some time that every such subset is some $S(w)$ - notation 3.2.4; yet previous to Fernando no one thought to carry out the analogous computation for Δ!

5.2.3. Let M be a $U(\mathfrak{g})$ module. An element $x \in \mathfrak{g}$ is said to be injective (resp. locally nilpotent) on M if for all $m \in M$ non-zero one has $xm = 0$ (resp. there exists $n \in \mathbb{N}^+$ such that $x^n m = 0$). Set $S_M = \{\alpha \in \Delta \,\big|\, x_\alpha$ is locally nilpotent on $M\}$, $T_M = \{\alpha \in \Delta \,\big|\, x_\alpha$ is injective on $M\}$.

Lemma. *Assume* $M \in Ob\mathcal{F}$ *is simple.*

(i) $S_M \coprod T_M = \Delta$,

(ii) S_M *is complete.*

Proof. It is convenient to omit the subscript M. Then $S \cap T = \emptyset$, since the conditions on S and T are mutually exclusive. Suppose $\alpha \in T$. Then there exists a non-zero $m \in M$ such that $x_\alpha m = 0$. By simplicity $M = U(\mathfrak{g})m$. Since $ad\ x_\alpha$ is locally nilpotent on $U(\mathfrak{g})$, it follows that $\alpha \in S$. Hence (i).

Suppose $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$. If $(\alpha, \alpha) \geq (\beta, \beta)$ then $2\alpha + \beta \notin \Delta$. Choose $m \in M$ non-zero such that $x_\alpha m = 0$. Choose $n \in \mathbb{N}$ such that $x_\beta^n m = 0$. Applying $(ad\ x_\alpha)^n$ to this expresson gives $x_{\alpha+\beta}^n m = 0$ and so $\alpha + \beta \in S$. It remains to show that T is closed. Fix $\lambda \in \Omega(M)$; $\alpha, \beta \in T$ and suppose $\gamma := \alpha + \beta \in S$. Then $dim\ M_{\lambda+n(\alpha+\beta)} \geq dim\ M_\lambda$, for all $n \in \mathbb{N}$, by the injectivity of x_α, x_β. Yet M belongs to the $\mathcal{O}$ category with respect to the reductive Lie algebra $kx_\gamma + kx_{-\gamma} + \mathfrak{h}$. By 4.3.8 every such module is a direct sum of modules of finite length. Moreover for all $n \in \mathbb{N}$ and $\mu \in \mathfrak{h}^*$ for which $\gamma^\vee(\mu) \notin -\mathbb{N}^+$, every simple $V \in Ob\mathcal{O}$ satisfies $dim\ V_{\mu+n\gamma} \leq dim\ V_\mu$, and this inequality becomes strict for n sufficiently large. This leads to a contradiction with the first inequality, so proving (ii).

5.2.4. Finding complete subsets of Δ (resp. Δ^+) is related to the problem of decomposing $\mathfrak{g}$ (resp. $\mathfrak{n}^+$) into complementary subalgebras. Recall that $\mathfrak{p}$ is said to be a parabolic subalgebra of $\mathfrak{g}$ if it contains a Borel subalgebra $\mathfrak{b}$. Any Borel can be conjugated into a standard Borel defined by the presentation of $\mathfrak{g}$ as a Kac-Moody algebra. We may therefore write

$$(*) \qquad \mathfrak{p} = \mathfrak{h} \oplus \bigoplus_{\alpha \in S} \mathfrak{g}_\alpha$$

for some $S \subset \Delta$ which is closed, contains Δ^+ and hence satisfies $S \cup -S = \Delta$. Moreover $\mathfrak{p}$ has a Levi decomposition $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{r}$, where $\mathfrak{m}$ is the nilradical of $\mathfrak{p}$ and where $\mathfrak{r}$ is a Levi factor which can be assumed to contain $\mathfrak{h}$. The roots of $\mathfrak{r}$ exactly those belonging to the set $S^L := S \cap -S$. Consequently $-(\Delta \setminus S)$ is the set of roots of $\mathfrak{m}$ which is hence closed. Thus S is complete.

We call S a parabolic subset of Δ if S is closed and $S \cup -S = \Delta$.

Lemma. *Suppose S is a parabolic subset of Δ. Then S can be conjugated under W to contain Δ^+. In particular S is complete and $S \setminus S^L$ is closed. Moreover $S^L = \mathbb{Z}\pi_1 \cap \Delta$ for some subset π_1 of π.*

Proof. Take a conjugate of S so that $|S \cap \Delta^+|$ is maximal. If $\alpha \in \pi,\ \alpha \notin S$, then $-\alpha \in S$ and so $s_\alpha S \cap \Delta^+ = \{\alpha\} \coprod s_\alpha(S \cap \Delta^+ \setminus \{\alpha\}) = \{\alpha\} \coprod s_\alpha(S \cap \Delta^+)$, which contradicts the assumed maximality, proving the first part. Set $\pi_1 = \pi \cap S^L$. Then $S^L \cap \Delta^+ \supset \mathbb{N}\pi_1 \cap \Delta^+$. If this inclusion is strict the complement contains a root γ such that $\gamma - \alpha$ is not a root for all $\alpha \in \pi_1$. Suppose $\gamma - \alpha$ is a root for some $\alpha \in \pi \setminus \pi_1$ Then $\alpha, \gamma - \alpha \in \Delta^+ \subset S$. Yet $-\gamma \in S$, so then $-\alpha = -\gamma + \gamma - \alpha \in S$ implying the contradiction $\alpha \in \pi_1$. Hence $(\gamma, \alpha) \leq 0$ for all $\alpha \in \pi$, forcing the contradiction $(\gamma, \gamma) \leq 0$. □

5.2.5. If S is a parabolic subset of Δ, we call S^L its Levi factor. Given root subsystems $\Delta_1 = \mathbb{Z}\pi_1 \cap \Delta,\ \Delta_2 = \mathbb{Z}\pi_2 \cap \Delta$ with $\pi_1, \pi_2 \subset \pi$, we say that Δ_1, Δ_2 are disjoint if π_1, π_2 are disjoint subsets of π having no nearest neighbours in the Dynkin diagam. This last condition is equivalent to $\Delta_1 \cup \Delta_2 = \mathbb{Z}(\pi_1 \cup \pi_2) \cap \Delta$. Suppose that Δ_1, Δ_2 are disjoint and set $\Delta_i^\pm = \Delta_i \cap \Delta^\pm : i = 1, 2$. Then $S := \Delta_1 \coprod (\Delta^+ \setminus (\Delta_1^+ \cup \Delta_2^+))$ and its complement $T := \Delta_2 \coprod (\Delta^- \setminus (\Delta_1^- \cup \Delta_2^-))$ are closed, hence complete subsets of Δ. The following lemma shows that every complete subset obtains in this fashion.

Lemma. *Suppose S is a complete subset and set $T = \Delta \setminus S$. Then*

(i) $S \setminus S^L = -(T \setminus T^L)$,

(ii) $P := S \coprod T^L$ *is a parabolic subset of* Δ,

(iii) $-(S \coprod T^L) = S^L \coprod T$,

(iv) $S^L \coprod T^L = P^L$,

(v) $S^L,\ T^L$ *are disjoint.*

Proof. (i) is clear. For (ii) choose $\alpha, \beta \in P$. If $\alpha + \beta \in \Delta \setminus P$, we may assume $\alpha \in S$ and $\beta \in T^L$. Then $\alpha + \beta, -\beta \in T$, so $\alpha \in T \cap S = \emptyset$. This contradiction proves that P is closed. Moreover $-P \supset -(S \setminus S^L) = T \setminus T^L$ by (i) and so $P \cup -P = \Delta$. Again $-P = -S \coprod T^L = -(S \setminus S^L) \coprod T^L \coprod S^L = T \coprod S^L$ by (i). Hence (iii). (iv) is clear. Finally suppose $\alpha \in S^L, \beta \in T^L$ and $\alpha + \beta$ is a root. Then it is non-zero and by the symmetry we can assume $\alpha + \beta \in S$, which gives the contradiction $\beta = (\alpha + \beta) - \alpha \in S \cap T = \emptyset$. Hence (v). □

5.3. Gelfand-Kirillov Dimension.

5.3.1. Gelfand-Kirillov (or simply GK) dimension measures growth rate. A number of significant results can be proved using this relatively simple concept. Those we need at present are reviewed below. In a further "selectum" we shall place these results in the wider context of dimension theory. Our present aim is mainly to show that any finitely generated module in $\mathcal{F}_{\mathfrak{g}}$ has finite length, though we do get a little distracted.

5.3.2. Let A be an associative algebra with identity and M a non-zero A module. For simplicity we assume that A is generated by a finite dimensional subspace V which we can assume contains the identity. Then the $V^n : n \in \mathbb{N}$, are finite dimensional, increasing subspaces and their union is A. Again for simplicity we assume that M is generated over A by a finite dimensional subspace W and we set $M^n = V^n W$, which are again finite dimensional, increasing and having union M.

The Gelfand-Kirillov dimension $d_A(M)$, or simply $d(M)$ is defined by

$$d_A(M) = \lim\ \sup \left(\frac{log\ dim\ M^n}{log\ n} \right).$$

One easily checks that it is independent of choices of generators. We denote $d_A(A)$ simply as $d(A)$. Observe that $d_A(M) < 1$ exactly when M is finite dimensional and indeed equals 0 in this case. In general $d(A)$ need not be an integer. One sets $d_A(M) = -\infty$, if $M = 0$.

5.3.3. Many results on GK dimension are fairly easy to prove; but one can also fall into a number of traps. For example if N is a finitely generated submodule of M, then $d_A(M) \geq max\{d_A(M/N), d_A(N)\}$. On the other hand equality can fail! When equality holds (relative to an abelian category of finitely generated A modules) we say that d_A

is partitive. If in addition for any decreasing sequence $M = M_0 \supset M_1 \cdots$, of finitely generated A modules, $d_A(M_i/M_{i+1}) = d_A(M)$ at most finitely many times, we say that d_A is finitely partitive.

Again consider the tensor product $A \otimes B$ of associative algebras $A,\ B$ respectively generated over finite dimensional subspaces $V,\ W$. Then $U = V \otimes 1 + 1 \otimes W$ generates $A \otimes B$. Set $a_n = log\ dim\ U^n/log\ n,\ b_n = log\ dim\ V^n/log\ n,\ c_n = log\ dim\ W^n/log\ n$. Then $a_n \leq b_n + c_n \leq a_{2n}$, since $U^n \subset \sum_{m=0}^{n} V^{n-m} \otimes W^m \subset V^n \otimes W^n \subset U^{2n}$. From the inequality

$$inf\ lim\ b_n + inf\ lim\ c_n \leq sup\ lim(b_n + c_n) \leq sup\ lim\ b_n + sup\ lim\ c_n,$$

we conclude that $d(A \otimes B) \leq d(A) + d(B)$ with equality if $inf\ lim = sup\ lim$ on these sequences. Yet in general equality can fail.

5.3.4. The following property is known as ideal invariance. Take A and M as before with A noetherian (and M finitely generated).

Lemma. *Let I be a two-sided ideal of A. Then $I \otimes_A M$ is a finitely generated left A module and $d_A(I \otimes_A M) \leq d_A(M)$.*

Proof. By noetherianity we may choose a finite dimensional subspace F of I such that $I = AF = FA$. Choose $V,\ W$ as in 5.3.2. It is clear that there exists integers $s, t > 0$, such that $VF \subset FV^s$ and $FV \subset V^tF$, successive use of which gives $V^mF \subset FV^{ms}$ and $FV^m \subset V^{mt}F$, for all integer $m \geq 0$. For subspaces $G \subset I,\ H \subset M$, let $G \otimes_A H$ denote the image of $G \otimes H$ in $I \otimes_A M$. Then $F \otimes_A V^mW = FV^m \otimes_A W \subset V^{mt}F \otimes_A W$, which shows that $I \otimes_A M$ is finitely generated. Again $V^mF \otimes_A W \subset FV^{ms} \otimes_A W = F \otimes_A V^{ms}W$ and so has dimension $\leq dim\ F\ dim\ V^{ms}W$, proving the second part. □

5.3.5. Retain the notation of 5.3.2 and recall 2.2. It is clear the subspaces V^n define a filtration $\mathcal{F}$ of A. In particular if $\mathfrak{g}$ is a finite dimensional Lie algebra, then $U(\mathfrak{g})$ is generated by $V = k + \mathfrak{g}$ and the resulting filtration is just the canonical one. Again the M^n define a filtration $\mathcal{F}$ of M making it a filtered module for the filtered algebra A. Then $gr_{\mathcal{F}}M$ is a finitely generated graded $gr_{\mathcal{F}}A$ module and

$$(*) \qquad d_A(M) = d_{gr_{\mathcal{F}}A}(gr_{\mathcal{F}}M).$$

This result becomes useful when $gr_{\mathcal{F}}A$ is commutative. Indeed let A be a commutative and associative finitely generated algebra graded by finite dimensional suspaces $\mathcal{G}_nA : n \in \mathbb{N}$, with $\mathcal{G}_0A = k$ and M an A module having gradation $\mathcal{G}_nM = (\mathcal{G}_nA)W$ with W finite dimensional. Then the function $n \mapsto dim\ \mathcal{G}_nM$, for all n sufficiently large, is given by a polynomial (the Hilbert-Samuel polynomial). This polynomial is zero exactly when M is finite dimensional. Otherwise there exist positive integers d, e such that

$$dim\ \mathcal{G}_nM = \frac{en^{d-1}}{(d-1)!}\ + O(n^{d-2}) : n >> 0.$$

Returning to the original pair $A,\ M$ one concludes that $d_A(M) = d$, which is hence an integer. More significantly under the above hypotheses A is noetherian. Then any submodule N of M is finitely generated and by taking the induced filtrations on N and M/N one shows that d_A is partitive and (using the integrality of e) that it is even finitely partitive.

Assume A noetherian. Call an A module M homogeneous (with respect to d_A) if $d_A(M) = d_A(N)$ and critical if $d_A(M/N) < d_A(M)$, for every non-zero submodule N of M. Partitivity implies that a critical module is homogeneous and that every non-zero submodule of a critical module is again critical. Finite partitivity implies that every non-zero submodule admits a non-zero critical submodule. Likewise M/N admits a critical submodule. Taking inverse images this construction gives an increasing sequence $0 = M_0 \subsetneq M \subsetneq \cdots$, of submodules of M with critical successive quotients. Since M is finitely generated this sequence must terminate after finitely many steps in M. One calls this argument noetherian induction.

The assumption that $gr_{\mathcal{F}}A$ is commutative (and finitely generated) means that it can be interpreted as the algebra of regular functions on some affine algebraic variety X. We define the associated variety $\mathcal{V}(M)$ of M as the set of zeros of $gr\ Ann\ W$ in X. This is independent of the choice of generating subspace W by an observation of J. Bernstein. From $(*)$ it is immediate that $d_A(M) = dim\ \mathcal{V}(M)$. (Dimension theory gives a number of equivalent ways of defining the dimension of an algebraic variety $\mathcal{V}$. In particular it may be given by the GK dimension of the algebra of regular functions on $\mathcal{V}$.) The associated variety $\mathcal{V}(Ann\ M)$ of $Ann\ M$ is defined to be $\mathcal{V}(A/Ann\ M)$. The inclusion $Ann\ W \supset Ann\ M$ trivially implies $\mathcal{V}(Ann\ M) \supset \mathcal{V}(M)$. If A is the enveloping algebra of a finite dimensional

Lie algebra $\mathfrak{g}$, then X above may be identified with $\mathfrak{g}^*$. If in addition $\mathfrak{g}$ is the Lie algebra of connected algebraic group G, then $\mathcal{V}(Ann\ M)$ is G stable.

The assumption that $gr_{\mathcal{F}}A$ is commutative implies that it can be equipped with a Lie bracket, called the Poisson bracket, coming from the Lie structure on Lie A. This is defined on homogeneous elements $\bar{a} \in gr_r A,\ \bar{b} \in gr_s A$, as follows and extended by linearity. Choose inverse images $a \in \mathcal{F}^r A,\ b \in \mathcal{F}^s A$ and observe that the above commutativity implies that $ab - ba$ lies in $\mathcal{F}^{r+s-1}A$. Then the Poisson bracket $\{\bar{a}, \bar{b}\}$ is defined to be the image of $ab - ba$ in $gr_{r+s-1}A$.

One calls an ideal I of $gr_{\mathcal{F}}A$, involutive if it is closed under Poisson bracket, that is if $\{I,\ I\} \subset I$. For example if A is a left ideal of A, then $gr_{\mathcal{F}}I$ is involutive. One calls a closed subvariety of X (as above) involutive if its ideal of definition is involutive. If I is semiprime one easily checks that the minimal primes over I are also involutive. Thus the irreducible components of an involutive variety are involutive. However the radical of an involutive ideal need not be involutive. A fundamental conjecture of V. Guillemin, D. Quillen and S. Sternberg [ ] was that the radical of $gr_{\mathcal{F}}I$ is involutive for any left ideal of I. This was proved by them in a special case using a trace argument and generalized by M. Kashiwara, T. Kawai and M. Sato to algebras of differential operators through microlocal analysis. The general case was settled by O. Gabber by purely algebraic means. It can be expressed as follows.

Theorem. *Let A be an assocative algebra with filtration $\mathcal{F}$. If $gr_{\mathcal{F}}A$ is commutative and finitely generated then the radical of $gr_{\mathcal{F}}I$ is involutive for any left (or right) ideal of A.*

Remark. Perhaps not surprisingly the converse fails [ ]. Thus an involutive variety need not be the associated variety of a finitely generated A module. Nevertheless a very significant problem is to construct such modules when they do exist. This is related to the passage from classical to quantum mechanics and is often referred to as quantization.

5.3.6. The concepts introduced in 5.3.5 together with the appropriate proofs will be examined in more detail in later “selecta”. In the present section we just examine some consequences needed in the present chapter.

Lemma. *Let A be a noetherian algebra for which d_A is partitive and M be a critical A module. Then $P := Ann_A M$ is a prime ideal.*

Proof. Let $I,\ J$ be two sided ideals of A such that $IJ \subset P$. We must show that either

$I \subset P$, or $J \subset P$. Observe that multiplication gives a surjective map $I \otimes_A M/JM \to IM$ and so $d(IM) \leq d(I \otimes_A M/JM) \leq d(M/JM)$ by ideal invariance. Critically forces either $IM = 0$ or $JM = 0$, as required. □

5.3.7. Recall the Weyl algebra A defined in 2.4.1 and its standard module $M = k[t]$. One easily checks that $d(A) = 2$ and $d_A(M) = 1$. J. Bernstein remarked that any non-zero A module N being infinite dimensional, must satisfy $d_A(N) \geq 1$. Now take $f \in M$ non-zero. The action on A extends to the localization $M[f^{-1}]$ in which f is inverted. By the finite partitivity of d_A, it follows that any finitely generated submodule of $M[f^{-1}]$ has finite length. An inspired choice of filtration on $M[f^{-1}]$ actually showed that $M[f^{-1}]$ itself has finite length and so is already finitely generated! This means that for i sufficiently large there exists $\partial \in A$ such that $\partial f^{-i} = f^{-(i+1)}$. By a little extra trick Bernstein was thus able to construct a differential operator $\partial \in A(z)$ such that $\partial f^z = f^{z-1}$. This result had the following remarkable application to the analysis. Assume k is the real field and let I be an interval on which f is positive with zeros on the boundary. Then $f^z : Re\ z > 0$, defines a distribution on I and the existence of the above differential operator implies through integration by parts that f^z extends to a meromorphic distribution on $\mathbb{C}$.

Not surprisingly the sheer simplicity of Bernstein's ideas meant that the above analysis extends immediately to the n variable case. How can one "steal his sweet and honeyed sentences" to satisfy other appetites? For the moment we shall just consider the Bernstein inequality $d_A(M) \geq n$, which holds for any non-zero module over the n variable Weyl algebra. In a later "selectum" we consider a harder problem which requires the Bernstein filtration.

5.3.8. The Bernstein inequality was first proved by induction on n. However it really should be viewed as an infinite dimensional version of the fact that any subalgebra A of $End\ V$ satisfies $dim\ A \leq (dim V)^2$ which taking the log and the limit becomes $d(A) \leq 2d_A(V)$ and a proof in this sense was given in [ ]. To go further one first needs the following observations.

Let V be a finite dimensional k-vector space. One calls a Lie subalgebra $\mathfrak{g}$ of $\mathfrak{gl}(V)$ algebraic if it is the Lie algebra of an algebraic subgroup of $GL(V)$. One calls $\mathfrak{g}$ ad-algebraic when V is the adjoint representation of $\mathfrak{g}$. Such a Lie algebra admits a basis $\{x_i\}_{i=1}^r$ such that each $ad_{\mathfrak{g}} x_i$ is either semisimple with integer eigenvalues or is nilpotent. (From this fact one may classify all such Lie algebras and these take the form $\mathfrak{s} + \mathfrak{z} + \mathfrak{n}$, where $\mathfrak{s}$ is semisimple, $\mathfrak{n}$ is the nilradical, $\mathfrak{z}$ is commutative and $ad_{\mathfrak{g}}\mathfrak{z}$ acts by zero on $\mathfrak{s}$ and admits a basis of semisimple endomorphisms of $\mathfrak{n}$ with integer eigenvalues.)

Assume $\mathfrak{g}$ ad-algebraic and choose a basis as above. For each $m \in \mathbb{N}$ extend $ad_{\mathfrak{g}}x$ to a derivation $ad_m x$ of the m-th filtration subspace $U^m(\mathfrak{g})$ of the canonical filtration of $U(\mathfrak{g})$.

Lemma. *There exists an integer $s > 0$ such that for all $x \in \mathfrak{g}$ and all $m, n \in \mathbb{N}$ one may write $(ad_m x)^n$ as a linear combination of the monomials $(ad_m x_1)^{n_1}(ad_m x_2)^{n} \cdots\cdots (ad_m x_r)^{n_r}$ with $\sum n_i \leq ms$.*

Proof. By the PBW theorem it is enough that the characteristic polynomial of $ad_m x_i$ have degree $\leq ms_i$ and then to choose $s = r(max\ s_i)$. For $ad_{\mathfrak{g}}x_i$ nilpotent take s_i to be its nilpotence degree. For $ad_{\mathfrak{g}}x_i$ semisimple, let $t_1, t_2, \cdots\cdots t_u$ be its (integer) eigenvalues and set $s_i = 2(max|t_j| + 1)$. Then the eigenvalues of $ad_m x_i$ lie in the set $\{\sum m_j t_j : m_j \in \mathbb{N}, \sum m_j \leq m\}$ which has cardinality $\leq 2m(max|t_j|) + 1 \leq ms_i$. Hence the assertion. □

5.3.9. Choose s as above. Let M be a $U(\mathfrak{g})$ module with generating subspace W and set $M^m = U^m(\mathfrak{g})W$, for all $m \in \mathbb{N}$.

Lemma. *For all $m \in \mathbb{N}$, $a \in U^m(\mathfrak{g})$, the relation $aM^{ms} = 0$ implies $aM = 0$.*

Proof. For all $\{n_i\} \in \mathbb{N}^r$ with $\sum n_i \leq ms$, the hypothesis implies that $a(x_r^{n_r} x_{r-1}^{n_{r-1}} \cdots x_1^{n_1})W = 0$ and so $((ad_m x_1)^{n_1}(ad_m x_2)^{n_r} \cdots (ad_m x_r)^{n_r} a)M = 0$. (This neat trick was a contribution of O. Gabber). By the previous lemma $((ad_m x)^n a)W = 0$, for all $x \in \mathfrak{g}$ and all $n \in \mathbb{N}$. This in turn gives $a(x^n W) = 0$ and so $aM = a(U(\mathfrak{g})W) = 0$.

5.3.10. With these preliminaries we obtain the following generalization of Bernstein's inequaltity.

Proposition. *Let $\mathfrak{g}$ be an ad-algebraic Lie algebra and M a finitely generated $U(\mathfrak{g})$ module. Then $d(U(\mathfrak{g})/Ann\ M) \leq 2d_{U(\mathfrak{g})}(M)$.*

Proof. Choose s as above. Given $a \in U^m(\mathfrak{g})$, one has $aM^{ms} \subset M^{m(s+1)}$. The resulting map $U^m(\mathfrak{g}) \to Hom(M^{ms}, M^{m(s+1)})$ has kernel $U^m(\mathfrak{g}) \cap Ann\ M$, by 5.3.9. Hence $dim(U^m(\mathfrak{g})/U^m(\mathfrak{g}) \cap Ann\ M) \leq dim\ M^{ms} dim\ M^{m(s+1)}$, from which the assertion follows.

Remarks. The inequality fails when $\mathfrak{g}$ is not ad-algebraic. One may give an alternative though less elementary proof based on Theorem 5.3.5. This uses in addition that G orbits in $\mathfrak{g}^*$ are symplectic varieties and any involutive subvariety of a symplectic subvariety has coisotropic tangent spaces and so must be of at least half the dimension. Here $\mathfrak{g}$ must be an ad-algebraic to be the Lie algebra of G.

5.3.11. Assume $\mathfrak{g}$ semisimple in particular ad-algebraic. Let $\mathcal{M}$ be an abelian category of $U(\mathfrak{g})$ modules. We say that the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts finitely on M if $Ann_{Z(\mathfrak{g})}M$ has finite codimension. One can ask if there is a natural condition on $\mathcal{M}$ which ensures that the objects of $\mathcal{M}$ on which $Z(\mathfrak{g})$ acts finitely have finite length. For example this holds by 4.3.8 in the category $\tilde{\mathcal{O}}$ of modules in which $\mathfrak{b}$ acts locally finitely and have finite dimensional (generalized) $\mathfrak{h}$ weight subspaces. Ideally this should also be true which respect to any subalgebra having roughly half the size of $\mathfrak{g}$. For example consider the category $\mathcal{H}$ of $\mathfrak{g}\times\mathfrak{g}$ modules having a locally finite action of the diagonal copy of $\mathfrak{g}$ with finite dimensional isotypical components. A similar argument to 4.3.8 involving the classification of the simple objects in $\mathcal{H}$ implies finite length when $Z(\mathfrak{g})$ acts finitely. Alternatively this results from the known [ ] equivalence between the categories $\tilde{\mathcal{O}}$ and $\mathcal{H}$. From this fact we shall prove finiteness of length for much more general categories. Specifically we need the following

Lemma. *Let $\chi \in Max\ Z(\mathfrak{g})$ and set $U = U(\mathfrak{g})/U(\mathfrak{g})\chi$. Then U has finite length as a $U(\mathfrak{g})$ bimodule. In particular for any $P \in Spec\ U$, there exists a unique ideal Q minimal over P. Moreover Q is idempotent.*

Proof. Clearly $Z(\mathfrak{g})$ acts by scalars on U whilst $ad\ \mathfrak{g}$ acts locally finitely. Let V be a simple finite dimensional $\mathfrak{g}$ module. Use of the Weyl character formula in particular shows that V occurs $dim\ V_0$ times in U. Thus $U \in Ob\mathcal{H}$ and so has finite length. The last part follows from the primeness of P. □

5.3.12. Let $\mathcal{M}$ be an abelian category of finitely generated $U(\mathfrak{g})$ modules. We say that $\mathcal{M}$ is holonomic if $d(U(\mathfrak{g})/Ann\ M) = 2d_{U(\mathfrak{g})}(M)$, for all $M \in Ob\mathcal{M}$.

Theorem. *Assume $\mathcal{M}$ holonomic. Then every $M \in Ob\mathcal{M}$ admitting a finite action of $Z(\mathfrak{g})$, has finite length.*

Proof. We can assume that $Z(\mathfrak{g})$ acts by scalars on M and since $d_{U(\mathfrak{g})}$ is finitely partitive that M is $d_{U(\mathfrak{g})}$-critical. By noetherian induction it is enough to show that M admits a simple submodule. Set $U = U(\mathfrak{g})/U(\mathfrak{g})Ann_{Z(\mathfrak{g})}M$. By 5.3.6 the annihilator of M in U is a prime ideal P. Take Q as in 5.3.11 and set $N = QM$ which is non-zero. Let us show that N is simple. If not N admits a proper submodule of N_0. Then $d(N/N_0) < d(N)$ since N is critical. Consequently $d(U/Ann\ N/N_0) = 2d(N/N_0) < 2d(N) = 2d(M) = d(U/Ann\ M)$

and so $Ann\ N/N_0 \supsetneq P$. Thus $Q(N/N_0) = 0$, that is $QN \subset N_0$. Yet $QN = Q^2M = QM = N$, which gives the required contradiction.

5.3.13. O. Gabber made the following suggestion as to how to obtain many holonomic categories. Let $\mathcal{N}$ denote the variety of zeros in $\mathfrak{g}^*$ of the augmentation ideal $S(\mathfrak{g})^G_+$ of $S(\mathfrak{g})^G$. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form on $\mathfrak{g}$. Then $\mathcal{N}$ is stable by scalar multiplication and is the set of ad-nilpotent elements of $\mathfrak{g}$. One calls $\mathcal{N}$ the nilpotent cone of $\mathfrak{g}$. It is a finite union of G orbits, called nilpotent orbits. For any nilpotent orbit $\mathcal{O}$ one has the remarkable equality $dim\ \mathcal{O} = 2dim(\mathfrak{n}^+ \cap \mathcal{O})$, due to Grothendieck. By finiteness it follows that for any G stable subvariety $\mathcal{W} \subset \mathcal{N}$ one has $dim\ \mathcal{W} = 2dim(\mathfrak{n}^+ \cap \mathcal{W})$. Moreover every ad-nilpotent element of $\mathfrak{b}$ lies in $\mathfrak{n}^+$ and so $\mathfrak{b} \cap \mathcal{W} = \mathfrak{n}^+ \cap \mathcal{W}$. Set $\mathfrak{b}^- = \mathfrak{h} + \mathfrak{n}^-$.

Proposition. *A category $\mathcal{M}$ of finitely generated $U(\mathfrak{g})$ modules admitting a finite action of $Z(\mathfrak{g})$ and for which $\mathcal{V}(M) \cap \mathfrak{n}^- = 0$, for all $M \in Ob\mathcal{M}$, is holonomic.*

Proof. Take $M \in Ob\mathcal{M}$. We can assume $Z(\mathfrak{g})$ acts by scalars on M. Then $\mathcal{W} := \mathcal{V}(Ann\ M)$ is contained in $\mathcal{N}$ and contains $\mathcal{V}(M)$. Choose $h \in \mathfrak{h}$ such that $\alpha(h)$ is a strictly positive integer for all $\alpha \in \Delta^+$. Let I be the ideal of definition of $\mathcal{V}(M)$.

Define $x_\alpha : \alpha \in \Delta$ to have degree $\alpha(h)$ and the elements of $\mathfrak{h}$ to have degree zero. Let $\mathcal{F}^i S(\mathfrak{g})$ be the span of the monomials of degree $\leq i$. Identify $gr_{\mathcal{F}} S(\mathfrak{g})$ with $S(\mathfrak{g})$ and let $\mathcal{V}_0$ be the zero variety of $gr_{\mathcal{F}} I$. From the remarks preceding the lemma it is enough to show that

$$(*) \qquad \mathcal{V}_0 \subset \mathfrak{b} \cap \mathcal{W} \ \ \textit{and} \ \ dim\ \mathcal{V}_0 = dim\ \mathcal{V}(M).$$

Indeed $d(U(\mathfrak{g})/Ann\ M) = dim\ \mathcal{W} = 2dim(\mathfrak{n}^+ \cap \mathcal{W}) = 2dim(\mathfrak{b} \cap \mathcal{W}) \geq 2dim\ \mathcal{V}(M) = 2d(M)$, by $(*)$, whilst the opposite inequality follows from 5.3.10.

Since $\mathcal{W}$ is a closed subvariety and a union of G orbits, its ideal of definition is *ad h* invariant and so satisfies $gr_{\mathcal{F}} J = J$. Then $I \supset J$ implies $gr_{\mathcal{F}} I \supset J$ and so $\mathcal{V}_0 \subset \mathcal{W}$. The hypothesis $\mathcal{V}(M) \cap \mathfrak{n}_- = \{0\}$ is equivalent to $I + S(\mathfrak{g})\mathfrak{b}^- = S(\mathfrak{g})_+$. Take $\alpha \in \Delta^+$. Since all three subspaces are spanned by homogeneous polynomials we can find $y \in \mathfrak{b}^-$, such that $x_\alpha + y \in I$. By the construction $x_\alpha = gr_{\mathcal{F}}(x_\alpha + y) \in gr_{\mathcal{F}} I$ and so $\mathfrak{n}_+ \subset gr_{\mathcal{F}} I$, which implies $\mathcal{V}_0 \subset (n^+)^\perp = \mathfrak{b}$. Finally $d(S(\mathfrak{g})/I) = d(S(\mathfrak{g})/gr_{\mathcal{F}} I)$, and so $dim\ \mathcal{V}(M) = dim\ \mathcal{V}_0$.

Remark. Geometrically $\mathcal{V}_0$ is a deformation of $\mathcal{V}(M)$. More precisely define $\varphi : k^* \to End\ \mathfrak{g}$ by extending linearly the map $\varphi(t)x = t^m x$, for all $x \in \mathfrak{g}$, satisfying $[h, x] = mx$.

Set $\mathcal{S} = \{(t, \varphi(t)y) : t \neq 0, \ y \in \mathcal{V}(M)\}$. Then $\mathcal{V}_0$ identifies with the zero fibre of $\bar{\mathcal{S}}$ under projection onto the second factor.

5.3.14. Let $\mathfrak{k}$ be a subalgebra of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{k} + \mathfrak{b}$. Up to conjugation, this holds if $\mathfrak{k}$ is the set of fixed points of an involution of $\mathfrak{g}$. Such a subalgebra is reductive. One calls M a Harish-Chandra module for $(\mathfrak{g}, \mathfrak{k})$ if it is direct sum of simple finite dimensional $U(\mathfrak{k})$ modules and is admissible, that is each $\mathfrak{k}$ isotypical component has finite multiplicity, equivalently is finite dimensional.

Corollary. *Let $\mathcal{H}$ be a category of finitely generated $U(\mathfrak{g})$ modules such that for all $M \in Ob\mathcal{H}$, one has*

(i) $\quad Ann_{Z(\mathfrak{g})}M \quad$ *has finite codimension.*

(ii) $\quad dim\ U(\mathfrak{k})m < \infty, \quad$ *for all* $\quad m \in M$.

Then $\mathcal{H}$ is holonomic. In particular each $M \in Ob\mathcal{H}$ has finite length.

Proof. Given $M \in Ob\mathcal{H}$, one has $\mathcal{V}(M) \subset \mathfrak{k}^{\perp}$. Yet $\mathfrak{k}^{\perp} \cap (\mathfrak{b}^-)^{\perp} = 0$ and $(\mathfrak{b}^-)^{\perp} = \mathfrak{n}^-$, so $\mathcal{H}$ satisfies the hypothesis of 5.3.13. The last assertion follows from 5.3.12.

Remarks. For (i) to hold it is enough that M be Harish-Chandra (the proof being as in 5.3.15 below). Of course in 5.3.11 we already used finiteness of length for Harish-Chandra modules corresponding to the pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$. The proof of 5.3.11 also goes through for the category of Harish-Chandra modules with respect to any pair $(\mathfrak{g}, \mathfrak{k})$ if one can show that there are only finitely many simples with a given annihilator in $Z(\mathfrak{g})$. This can be rather arduous. A further proof can be given using the Casselman-Jacquet functor relating such a category to $\tilde{\mathcal{O}}$. A difficult point here is to prove the injectivity of this functor. In a further "selectum" we deduce such a result from a separation theorem of O. Gabber. A further application of 5.3.14 is to Whittaker modules considered in particular by B. Kostant. By definition such modules are locally finite action with respect to $\mathfrak{n}^-$.

5.3.15. The impatient reader may note with some sarcasm that after fourteen subsections, proving that finitely generated $\mathfrak{h}$ admissible modules to have finite length seems nowhere in sight. A weak excuse is that by the time $\mathcal{F}$ arrived on the scene all the above theory was common knowledge, so the result below became a formality.

Proposition. *The subcategory of $\mathcal{F}$ of finitely generated modules is holonomic. In particular any finitely generated module $M \in Ob\mathcal{F}$ has finite length.*

Proof. Let W be a finite dimensional subspace generating M. We can assume that W is $\mathfrak{h}$ stable and hence a direct sum of its weight subspaces. By admissibility we may assume that W is a finite direct sum of weight subspaces of M. Since $Z(\mathfrak{g})$ preserves the weight subspaces of M it leaves W invariant. Since W is finite dimensional its annihilator in $Z(\mathfrak{g})$ has finite codimension. Yet this is also the annihilator of M in $Z(\mathfrak{g})$, that is $Z(\mathfrak{g})$ acts finitely on M. We may assume that $Z(\mathfrak{g})$ acts by scalars on M.

Let $\mathcal{V}$ be an irreducible component of $\mathcal{V}(M)$ and I the ideal of definition of $\mathcal{V}$. Set $P = \{\alpha \in \Delta \,\big|\, x_\alpha \in I\}$ and recall 5.2.4. We claim that P is a parabolic subset of Δ. By 5.3.5 it is closed. Now observe that W is a finite dimensional $U(\mathfrak{g})$ module. Thus for each $\alpha \in \Delta^+$, there exists a polynomial p satisfying $p(x_\alpha x_{-\alpha})W = 0$ and so $x_\alpha x_{-\alpha}$ belongs to the radical of $gr_{\mathcal{F}} Ann_{U(\mathfrak{g})} W$. Thus $P \cup -P = \Delta$. By 5.2.4 we can assume that $P \supset \Delta^+$. Thus $\mathcal{V} \subset \mathfrak{b}$. The proof is then completed by the argument following 5.3.13(∗). □

5.4. Torsion-free $\mathfrak{h}$-admissible modules.

5.4.1. Take $M \in Ob\mathcal{F}$ and recall 5.2.3. One calls M torsion-free if $T_M = \Delta$ and torsion if $S_M = \Delta$. Torsion modules are more commonly known as integrable modules. Any such module (for $\mathfrak{g}$ semisimple) is finite dimensional. Here we reduce the study of simple objects in $\mathcal{F}$ to that of torsion-free simples and then classify the latter.

5.4.2. Take M as above. By 5.2.3 (ii) and 5.2.5 there is a parabolic subalgebra $\mathfrak{p}$ with a Levi factor $\mathfrak{r}$ containing $\mathfrak{h}$ whose roots form the set $S^L \coprod T^L$, and with nilradical $\mathfrak{m}$ whose roots form the set $S \setminus S^L$.

Lemma. *M admits a simple $\mathfrak{p}$ submodule.*

Proof. Recall that $\mathfrak{m}$ is nilpotent and that its root subspaces act locally nilpotently on M. An easy induction on the central descending series of $\mathfrak{m}$ shows that $M' = \{x \in M \,\big|\, \mathfrak{m}x = 0\}$ is non-zero. Since $\mathfrak{m}$ is an ideal of $\mathfrak{p}$, M' is a $\mathfrak{p}$ submodule of M. As an $\mathfrak{r}$ module it belongs to $\mathcal{F}_{\mathfrak{r}}$ and so by 5.3.15 has non-zero simple $\mathfrak{r}$ submodule on which $\mathfrak{m}$ acts by zero.

5.4.3. Let N be defined by the conclusion of 5.4.2. It is a simple $\mathfrak{r}$ module. On the other hand $\mathfrak{r}$ is the sum of two reductive subalgebras whose semisimple parts $\mathfrak{s}_1, \mathfrak{s}_2$ with roots respectively defined by S^L and T^L. By 5.2.5(v) these two algebras commute with one another. This leads to the question posed below.

Let A and B be associative algebras. Give the tensor product $A \otimes B$ the obvious product structure $(a \otimes b)(a' \otimes b') := aa' \otimes bb'$. Let N_1 (resp. N_2) be a simple A (resp. B) module and give $N_1 \otimes N_2$ the obvious $A \otimes B$ module structure $(a \otimes b)(n_1 \otimes n_2) := (an_1 \otimes bn_2)$. The Jacobson density theorem implies that $N_1 \otimes N_2$ is simple.

One can ask if the converse to above holds, that is if N is a simple $A \otimes B$ module, then does there exist a simple A module N_1 and a simple B module N_2 such that N is isomorphic to $N_1 \otimes N_2$? The answer is of course a resounding no! For example take $A,\ B$ to be the Weyl algebra defined in 2.4.1. For rather easy reasons any simple A module is holonomic. Had a decomposition of the above type been possible this would also hold for $A \otimes B$. One might still think (and many did!) that every simple $A \otimes B$ is holonomic. T.J. Stafford [ ] constructed a counterexample and even one that which is the quotient of a principal ideal. In general non-holonomic simple modules remain rather mysterious.

Despite the above one has

Lemma. *Let N be a simple $A \otimes B$ module. Assume that N admits a simple A submodule N_1 and that $Hom_A(N_1, N)$ admits a simple B submodule N_2. Then N is isomorphic to $N_1 \otimes N_2$.*

Proof. Frobenius reciprocity gives an isomorphism

$$Hom_{A \otimes B}(N_1 \otimes N_2, N) \xrightarrow{\sim} Hom_B(N_2, Hom_A(N_1, N)).$$

Yet the right hand side is non-zero by the definition of N_2. It follows that we have an non-zero map $N_1 \otimes N_2 \to N$ which is an isomorphism since both modules are simple.

5.4.4. Combining the above one obtains the following reduction theorem.

Theorem. *Let $M \in Ob\mathcal{F}_{\mathfrak{g}}$ be simple. There exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ and a simple $\mathfrak{p}$ module N such that M is the unique simple quotient of $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} N$. Moreover the semisimple part of a Levi factor for $\mathfrak{p}$ can be written as a product of semisimple Lie algebras $\mathfrak{s}_1, \mathfrak{s}_2$ such that $N \cong N_1 \otimes N_2$ with N_1 a simple finite dimensional $\mathfrak{s}_1$ module and $N_2 \in Ob\mathcal{F}_{\mathfrak{s}_2}$ simple and torsion-free.*

Proof. Take $\mathfrak{s}_1, \mathfrak{s}_2$ and N as in 5.4.3. Since $N \in Ob\mathcal{F}_{\mathfrak{h}+\mathfrak{s}_1}$, it admits a simple $\mathfrak{h} + \mathfrak{s}_1$ submodule N_1 by 5.3.15, which is moreover a simple $\mathfrak{s}_1$ module. Again since $\mathfrak{h} + \mathfrak{s}_2$ normalizes $\mathfrak{s}_1$, $Hom_{\mathfrak{s}_1}(N_1, N)$ is a $\mathfrak{h} + \mathfrak{s}_2$ module. Since N_1 is simple it is generated by an $\mathfrak{h}$ weight vector n of weight, say $\mu \in \mathfrak{h}^*$. Then $(h\theta)(n) = (h - \mu(h))\theta(n)$, for all $h \in \mathfrak{h}, \theta \in Hom_{\mathfrak{s}_1}(N_1, N)$. Yet $\theta(n)$ belongs to a finite direct sum of weight subspaces of N of weights, say $\nu_1, \nu_2, \cdots\cdots, \nu_t$. Then

$$\prod_{i=1}^{t}(h + (\mu - \nu_i)(h))\theta,$$

which is an element of $Hom_{\mathfrak{s}_1}(N_1, N)$, vanishes on n and hence on N_1. Thus $Hom_{\mathfrak{s}_1}(N_1, N)$ is a direct sum of it weight subspaces. Moreover if $\mathcal{O}$ has weight $\nu - \mu$ then $\theta(n)$ belongs to N_ν. Consequently their dimensions are bounded by those of N. This shows that $Hom_{\mathfrak{s}_1}(N_1, N) \in Ob\mathcal{F}_{\mathfrak{h}+\mathfrak{s}_2}$ and so by 5.3.15 admits a simple $h + \mathfrak{s}_2$ submodule, which is moreover simple over $\mathfrak{s}_2$. Applying 5.4.3 gives the required assertion.

5.4.5. We wish to determine the set $\mathcal{F}^0_{\mathfrak{g}}$ of torsion-free simples of $\mathcal{F}_{\mathfrak{g}}$. Take $M \in Ob\mathcal{F}_{\mathfrak{g}}$ simple. Let $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_t$ be the decomposition of $\mathfrak{g}$ into simple factors. Exactly as in the last part of the proof of 5.4.3 it follows that $M = M_1 \otimes M \otimes \cdots \otimes M_t$, with $M_i \in Ob\mathcal{F}_{\mathfrak{g}_i}$. Thus we may assume $\mathfrak{g}$ simple and this we do in the remainder of this section.

5.4.6. Recall 5.3.13.

Lemma. *Every non-zero nilpotent orbit has dimension $\geq 2rk\ \mathfrak{g}$. Moreover equality implies that $\mathfrak{g}$ is of type A or type C and that there is just one nilpotent orbit of this dimension.*

Proof. Let $\mathcal{O} \neq \{0\}$ be a nilpotent orbit. Since $\mathcal{O} \cap \mathfrak{n}_+$ is non-empty, it contains a sum y of root vectors with positive roots. One has $dim\ \mathcal{O} = codim\ \mathfrak{g}^y$. Choose $h \in \mathfrak{h}$ which is strictly positive integer on each positive root and whose values separate the positive roots. Define $\varphi : k^* \to End\ \mathfrak{g}$ by extending linearly the map $\varphi(t)x = t^m x$, for all $x \in \mathfrak{g}$, satisfying $[h, x] = mx$. Set $\mathcal{S} = \{(t, \varphi(t)ky) : t \in k^*\}$. Let $\pi : k \times \mathfrak{g} \to \mathfrak{g}$ be the projection onto the second factor. Then $dim\ \mathfrak{g}^z : z \in \pi(\bar{\mathcal{S}})$ takes its minimal value on an open subset of $\pi(\bar{\mathcal{S}})$ which must be the constant value $dim\ \mathfrak{g}^z$ on $\pi(\mathcal{S})$. Yet $\pi(\bar{\mathcal{S}})$ contains a root vector x_α which therefore satisfies $dim\ \mathfrak{g}^{x_\alpha} \geq dim\ \mathfrak{g}^y$. Since there are only finitely many nilpotent orbits either a strict inequality holds or y and x_α are in the same orbit. One checks that $codim\ \mathfrak{g}^{x_\alpha} \geq 2rk\mathfrak{g}$, with equality if and only if g is of type A or C and α is a long root. Since the long roots vectors are all conjugate under G (even under N_G) the result follows.

Remarks. Through the Jacobson-Morozov theorem, Dynkin classified all nilpotent orbits and this result can also be read off from the classification. The above argument also shows that the orbit of the long root vector is the unique nilpotent orbit of minimal non-zero dimension. It is called the minimal orbit and denoted by $\mathcal{O}_0$.

5.4.7. The above reduces drastically the possible size of $\mathcal{F}^o_{\mathfrak{g}}$. Indeed one has the

Lemma. *Suppose $M \in Ob\mathcal{F}^0_{\mathfrak{g}}$. Then*

(i) *$dim\ M_\mu$ is finite and independent of the choice of $\mu \in \Omega(M)$.*

(ii) *$d(M) = rk\ \mathfrak{g}$.*

(iii) *$\mathfrak{g}$ is of type A or C.*

(iv) *$\mathcal{V}(Ann\ M)$ is the closure of $\mathcal{O}_0$.*

Proof. (i) is obvious. Set $Q = \mathbb{Z}\pi$. For each $\mu \in Q$ we may write $\mu = \sum\limits_{\alpha\in\pi} r_\alpha \alpha$ and we set $|\mu| = \sum |r_\alpha|$. For all $r \in \mathbb{N}$, set $Q_r = \{\mu \in Q, \text{ such that } |\mu| \leq r\}$. Set $s = max_{\beta\in\Delta}|\beta|$. Then

$$\mathcal{F}^n U(g) \subset \sum_{|\mu|\leq sn} U(\mathfrak{g})_\mu,\ \forall\ n \in \mathbb{N},$$

for the canonical filtration $\mathcal{F}$ of $U(\mathfrak{g})$. Take $\lambda \in \Omega(M)$. Then with respect to the filtration $\mathcal{F}^n M := \mathcal{F}^n(U(\mathfrak{g}))M_\lambda$, we obtain by (i) that

$$dim(\mathcal{F}^n M) \leq |Q_{sn}| dim\ M_\lambda.$$

Yet $n \mapsto |Q_n|$ is polynomial in n of degree $rk\ \mathfrak{g}$. Consequently $d(M) \leq rk\ \mathfrak{g}$. On the other hand by 5.3.10, we have $2d(M) \geq d(Ann\ M) > 0$, where the second inequality holds because $U(\mathfrak{g})/Ann\ M$ cannot be finite dimensional. Then (ii) -(iv) result from 5.4.6.

6. Primitive Quotients and Rings of Differential Operators.

6.1. The Role of Quantization.

6.1.1. As we have seen in Chapter 2, differential operators provide a rich source of Lie algebras. As a consequence enveloping algebras are closely related to the algebras of differential operators in one, several or sometimes infinitely many variables. Yet in some sense this connection goes even deeper and relates to the "quantization" of Poisson structure which in physical terms corresponds to the passage from Classical to Quantum Mechanics.

6.1.2. Historically the algebraic approach to quantization obtained from the Born-Heisenberg relation. This is just the commutation relation for the generators of the algebra of polynomial differential operators given in 2.4.1. Shortly afterwards P.A.M. Dirac formulated quantization as the replacement of phase space by the algebra of differential operators on just co-ordinate space (alternatively on momentum space). This meant in particular that half the phase space variables should be discarded and as a deep consequence precise simultaneous values of position and momentum could not be observed. At a similar time, physicists recognized that "Wave Mechanics" involved a superposition principle, which in mathematical terms means that one has a natural linear structure on the set of states of the quantum physical system. Consequently given a symmetry, it becomes particularly relevant to study the representations of the symmetry group in the space spanned by the set of "physical states".

6.1.3. The next breakthrough came from the combined work of J. Dixmier, A.A. Kirillov, B. Kostant and J.-M. Souriau. Motivated partly by physics and partly through the aim to determine the unitary representations of a Lie group G (initially a nilpotent group). Their work showed that the orbits of G acting on the dual of its Lie algebra $\mathfrak{g}$ admit a symplectic structure (in other words define a phase space) and moreover that the construction of the corresponding representations should proceed via an action of G on functions of an appropriate Lagrangian subvariety. The latter defines an appropriate selection of one half the variables and the action of G comes from that of $\mathfrak{g}$ which in turn is usually through differential operators.

6.1.4. On the other hand we saw in 5.3.5 that the canonical filtration of an enveloping algebra leads to a Poisson bracket structure on the graded algebra. This Poisson bracket structure restricts on each coadjoint orbit to give the symplectic structure alluded to above. Moreover by Gabber's theorem the associated variety of any module is involutive if not Lagrangian. This return to phase space is often termed "dequantization".

6.1.5. The above considerations precisely formulated can lead to an attractive formalism, known generally as the "orbit method". However it should be recognized from the start that there are some unavoidable difficulties. These arise because not every involutive subvariety is an associated variety of a $U(\mathfrak{g})$ module. Consequently it is far from clear how to construct a representation from a given Lagrangian subvariety.

6.1.6. A first success of the orbit method was obtained through a so-called polarization of the orbit. If G is the corresponding Lie group with Lie algebra $\mathfrak{g}$ and f an element of $\mathfrak{g}^*$, then a polarization of the coadjoint orbit Gf is a Lie subgroup P such that G/P has one half the dimension of Gf and such that f defines a one dimensional representation of P. Then the required representation of G is obtained by induction and in principle, the dimension condition implies irreducibility.

6.1.7. Whether the above actually leads to a (unitary) representation and then the description of all the refinements and technical details needed to attain this particular nirvana is not a subject of the present discussion. Suffice to say it has had wide-ranging success. However it also happens that there are coadjoint orbits (particularly for $\mathfrak{g}$ semisimple) for which no polarisation exists even in the above rudimentary sense. What one may do in this situation is our present concern. One approach which admittedly requires great optimism is to consider the algebra D of differential operators on a selected Lagrangian subvariety L. We then need a miracle, namely that there exists a homomorphism of the enveloping algebra $U(\mathfrak{g})$ into D. When this happens, a module over D, in particular the module of functions on L, restricts to a $U(\mathfrak{g})$ module. Our present selectum is concerned with the presence of a further miracle, namely that the above homomorphism is surjective. This is relatively straightforward when $\mathfrak{g}$ is nilpotent. However in that case a polarization always exists and indeed such a polarization gives the required Lagrangian subvariety. This is pleasant; but not too interesting. When $\mathfrak{g}$ is semisimple, a polarization which is necessarily defined by a parabolic subalgebra, does give such a homomorphism; but this is never surjective (except for the trivial orbit). Rather we shall consider certain other Lagrangian subvarieties of nilpotent orbits as defined through the construction in 5.3.13. We find that the above homomorphism exists and is surjective. Moreover its kernel is a completely prime, primitive ideal whose associated variety is the closure of the original G orbit. Though this is a rather specialized phenomenum, there are quite a few examples and moreover it leads to some particularly interesting mathematics.

6.2. Algebras which are Modules in the $\mathcal{O}_f$ Category.

6.2.1. Let H be a Hopf algebra with coproduct Δ. Given $h \in H$, we write $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$, as in 1.4.5. A Hopf module algebra A is an associative algebra and an H module satisfying the compatibility condition

$$h(ab) = \sum h_{(1)}(a)h_{(2)}(b), \quad \textit{for all} \ \ h \in H, \ a, b \in A.$$

As we shall see this leads to a significant restriction on the muliplicative structure of A especially if A is small compared to H.

6.2.2. In the remainder of this chapter we shall assume the base field k to be algebraically closed of characteristic zero and $\mathfrak{g}$ a semisimple Lie algebra. One may take $H = U(\mathfrak{g})$ in 6.2.1. Recall 4.3.2 and let $\mathcal{O}_f$ denote the subcategory of $\mathcal{O}$ of finitely generated $U(\mathfrak{g})$ modules. An $\mathcal{O}_f$-algebra A is a $U(\mathfrak{g})$ module algebra which as a module belongs to $\mathcal{O}_f$. This is a severe restriction and one is almost able to give the complete classification of such algebras.

6.2.3. Recall 4.3.2(iii). Fix $M \in Ob\mathcal{O}$. For any subset Ω' of $\Omega(M)$ and any positive integer r set

$$r\Omega' := \{\nu_1 + \nu_2 + \cdots + \nu_r \,\big|\, \nu_i \in \Omega'\}.$$

Lemma. *Take $M \in Ob\mathcal{O}$. For every $n \in \mathbb{N}$ and non-zero submodule N of M there exists a finite subset $\Omega' \subset \Omega(M)$ such that*

$$(*) \qquad \Omega(N) \not\subset \bigcup_{r=1}^{\infty} \bigcup_{s=0}^{n(r+1)} \bigcup_{t=r}^{\infty} s\Omega' + t(\Omega(M) \setminus \Omega').$$

Proof. Since M has finite length (4.3.8) it is enough that $(*)$ holds for every simple submodule of M. Fix $h \in \mathfrak{h}$ such $h(\alpha)$ is an integer > 0 for all $\alpha \in \pi$, and set $S = \{h(\lambda) \,\big|\, \lambda \in \Omega(M)\}$. Choose $\mu \in \mathfrak{h}^*$ such that $\Omega(M) \in \mu - \mathbb{N}\pi$. Express k as a $\mathbb{Q}$ vector space direct sum $k = k' \oplus \mathbb{Q}$ and write $h(\mu) = q' + q$, with respect to this decomposition. Then $S \subset q' + q - \mathbb{N}$. Replacing q by $q + u : u \in \mathbb{N}$ sufficiently large, we can assume $q > 0$.

Let $\{N_i\}$ be the finite set of isomorphism classes of simple submodules of M. Let λ_i be the highest weight of N_i. Then we may write $h(\lambda_i) = q' + q - m_i$. Set $m = (2n+1)q + max\{m_i\}$, and $\Omega' := \{\lambda \in \Omega(M) \,\big|\, h(\lambda) = q' + q - i,\ i = 0, 1, \cdots, m\}$. Clearly Ω' is a finite set.

With above choices

$$(*)\ \ \{h(\lambda)\,\big|\,\lambda \in s\Omega' + t(\Omega(M)\backslash\Omega)\} \subset \{(s+t)q' + (s+t)q - (si+tj)\,\big|\, i \in [0, m]\ \ and\ \ j > m\}.$$

Yet $si + tj > tm \geq 2nqt + qt + max\{m_i\} > (s+t)q + max\{m_i\}$, since $s < n(r+1) \leq 2nr \leq 2nt$ in $(*)$. Consequently none of the λ_i can belong to the right hand side. $\square$

6.2.4. Let n be a positive integer and S_n the symmetric group on n elements. For any ring A and any n elements $a_1, a_2, \cdots, a_n$ of A set

$$I_n(a_1, a_2, \cdots, a_n) = \sum_{\sigma \in S_n} (sn\sigma) a_{\sigma(1)} a_{\sigma(2)} \cdots\cdots a_{\sigma(n)}\ \ and\ \ I_n(A) = \{I_n(a_1, a_2, \cdots, a_n)\,\big|\, a_i \in A\}.$$

One says that A satisfies a standard identity if $I_n(A) = 0$, for some $n \in \mathbb{N}^+$. Such rings are generalizations of matrix rings and lie in a general class of so-called polynomial identity (p.i.) rings.

Proposition. *Let A be an $\mathcal{O}_f$ algebra. Then A satisfies a standard identity.*

Proof. Let $\Omega' \subset \Omega(A)$ be the finite set in the conclusion of 6.2.3. By 4.3.2(ii),

$$r := \sum_{\lambda \in \Omega'} dim\ A_\lambda$$

is finite. Since I_n is alternating and multi-linear we may assume in computing $I_{2r}(A)$ that the a_i are weight vectors and not more than r of them have weights in Ω'. Thus

$$\Omega(I_{2r}(A)) \subset \bigcup_{s=0}^{r} s\Omega' + (2r - s)(\Omega(A) \setminus \Omega').$$

On the other hand the diagonal action of $U(\mathfrak{g})$ on the $2n$-th fold tensor product of A defined through the coproduct on $U(\mathfrak{g})$ commutes with the action of S_{2r}, and so $I_{2r}(A)$ is a submodule of A. Through the above and conclusion of 6.2.3 it must therefore vanish. $\square$

6.2.5. In virtue of 6.2.4 we may use the many powerful theorems of p.i. theory to study $\mathcal{O}_f$-algebras.

Let A be any ring. The prime radical $p-rad(A)$ is the intersection of the prime ideals of A. Now if $\mathfrak{g}$ is a Lie algebra acting by derivations on A, then any minimal prime of A is $\mathfrak{g}$ invariant [ ]. (By contrast this is false for a group action. For example a finite group may transitively permute the irreducible components of an algebraic variety.) A nil-ideal of A is a (two-sided) ideal in which every element is nilpotent. The nilradical $n-rad(A)$ of A is its largest nil-ideal. In general one only has $n-rad(A) \supseteq p-rad(A)$; but equality holds for p.i. rings [ ]. Furthermore for if A is also finitely generated, the nilradical is a nilpotent ideal by Braun's theorem [ ]. This leads to the following important result.

Lemma. *Let A be an $\mathcal{O}_f$ algebra. Then the nilradical N of A is nilpotent.*

Proof. Let Ω' be the finite set in the conclusion of 6.2.3. Set

$$K = \bigoplus_{\mu\in\Omega'} A_\mu \quad and \quad L = \bigoplus_{\mu\in\Omega(A)\setminus\Omega'} A_\mu.$$

The first of these subspaces is finite dimensional and generates a subalgebra B of A which is obviously p.i. Clearly $M := N \cap B$ is contained in the nilradical of B and so by Braun's theorem is nilpotent. Choose $n \in \mathbb{N}$ such that $M^{n+1} = 0$.

By the above remarks N is submodule of A and thus so are all its powers. We claim that $N^{n+1} = 0$, which will prove the lemma. For this it is enough to show that any product of $n+1$ weight vectors of N is zero. Any such vector a belongs either to K or to L and the former holds only if a also belongs to M. Thus in any such monomial every sequence of elements in K can be assumed to have length $< n$. It follows that $\Omega(N^{n+1})$ belongs to the right hand side of 6.2.3($*$). Since N^{n+1} is a submodule of A the claim follows from 6.2.3.

6.2.6. Lemma. *Let A be an $\mathcal{O}_f$-algebra.*

(i) *The set P of minimal primes of A is finite.*

(ii) *The centre $Z(A)$ of A is an $\mathcal{O}_f$ algebra.*

Proof. (i). Otherwise $P_1, P_1 \cap P_2, \cdots : P_i \in P$ is an infinite strictly decreasing sequence of $\mathfrak{g}$ submodules of A contradicting its finite length.
(ii). One checks that $Z(A)$ is a submodule of A. Hence $Z(A) \in Ob\mathcal{O}_f$. $\square$

6.3. Semiprime $\mathcal{O}_f$-Algebras.

6.3.1. A ring is said to be semiprime if its nilradical is zero. Here we give a complete classification of semiprime $\mathcal{O}_f$-algebras.

Lemma. *Let A be a semiprime $\mathcal{O}_f$ algebra. The centre $Z(A) \neq 0$ is semiprime.*

Proof. By 6.2.4 and the hypothesis A is a semiprime p.i. ring. Hence $Z(A) \neq 0$ by [ ]. For a commutative algebra nil and prime radicals coincide [, 1.14] so for $Z(A)$ to be semiprime, it is enough to show that any nilpotent element of $Z(A)$ must be zero. Yet such an element would generate a nilpotent ideal of A which is zero by hypothesis. □

6.3.2. Let π' be a subset of π, and let $\mathfrak{p}_{\pi'}$ (or simply $\mathfrak{p}$) denote the corresponding parabolic subalgebra containing the standard Borel $\mathfrak{b}$. Since $\mathfrak{p}$ is a Lie subalgebra it follows that $U(\mathfrak{g})\mathfrak{p}$ is a coideal. Hence $(U(\mathfrak{g})/U(\mathfrak{g})\mathfrak{p})^*$ identifies with a subalgebra of the commutative algebra $U(\mathfrak{g})^*$. On the other hand it is also a $\mathfrak{g}$ module and moreover $\mathfrak{g}$ acts by derivations. Consider the submodule in which $\mathfrak{h}$ acts locally finitely. It is clearly also a subalgebra and also the $\mathcal{O}$-dual $\delta M_{\pi'}(0)$ of the generalized Verma module $M_{\pi'}(0)$ of highest weight 0. It is therefore an $\mathcal{O}_f$ subalgebra. Below it is shown that every completely prime $\mathcal{O}_f$ algebra occurs in this fashion.

6.3.3. For each $\pi' \subset \pi$, set $\Delta_{\pi'} = \Delta \cap \mathbb{Z}\pi'$, $\Delta^{\pm}_{\pi'} = \Delta^{\pm} \cap \Delta_{\pi'}$ and $\mathfrak{r}_{\pi'}, \mathfrak{m}^{\pm}_{\pi'}$ the subspaces of $\mathfrak{g}$ spanned by the corresponding root vectors. Then $\mathfrak{p}^{\pm}_{\pi'} = \mathfrak{r}_{\pi'} + \mathfrak{m}^{\pm}_{\pi'}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{m}^{\pm}_{\pi'}$ and Levi factor $\mathfrak{r}_{\pi'}$. We may sometimes omit the π' subscript and the $+$ superscript on these subalgebras.

Let K denote the Killing form $K(a, b) = tr(ad\ a)(ad\ b)$, $\forall a, b \in \mathfrak{g}$ which we recall is $ad\mathfrak{g}$ invariant and non-degenerate. Recall that ad extends to $U(\mathfrak{g})$ as an algebra homomorphism. In what follows we use a subscript to denote the weight of an element of $U(\mathfrak{g})$. Let $x_\alpha : \alpha \in \Delta, \alpha^\vee : \alpha \in \pi$, be a Chevalley basis for $\mathfrak{g}$ and $\kappa : x_\alpha \mapsto x_{-\alpha}$ the Chevalley antiautomorphism.

Lemma. $\sum_{\alpha\in\pi\setminus\pi'}(ad\ \mathfrak{p}^-)x_{-\alpha} = \mathfrak{m}^-$.

Proof. The inclusion $\subset$ is obvious. If equality fails there exists $\beta \in \Delta^+\setminus\Delta^+_{\pi'}$ minimal and $0 \neq x_\beta \in \mathfrak{m}^+$ such that $K(x_\beta, \sum\limits_{\alpha\in\pi\setminus\pi'} (ad\mathfrak{p}^-)x_{-\alpha}) = 0$. Then $\beta \notin \pi$, by the non-degeneracy of K. Assume $\alpha \in \pi$ is such that $\beta - \alpha \in \Delta^+ \setminus \Delta^+_{\pi'}$. By the invariance of K and the minimality of β we obtain $(ad\ x_{-\alpha})x_\beta = 0$. Through the Chevalley antiautomorphism we obtain $x_{-\beta} \in \mathfrak{m}^-$ such that $(ad\ x_\alpha)x_{-\beta} \in \mathfrak{r}$ for all $\alpha \in \pi$. Applied to the canonical generator v_0 of $M_{\pi'}(0)$, this gives $x_\alpha x_{-\beta} v_0 = 0$, for all $\alpha \in \pi$. Through the Casimir invariant this forces $(\rho - \beta, \rho - \beta) = (\rho, \rho)$, which by [, A.1.3] is impossible. □

6.3.4. In what follows we show that the algebra structure on $\delta M_{\pi'}(0)$ is determined by its $\mathfrak{g}$ module structure assuming that

(i) $\mathfrak{p}$ *acts by derivations on* $\delta M_{\pi'}(0)$.

(ii) *The (one-dimensional) zero weight subspace of* $\delta M_{\pi'}(0)$ *(which is a subalgebra) has no zero divisors.*

Recall that $(\delta M_{\pi'}(0))^{\mathfrak{n}} = (\delta M_{\pi'}(0))_0$ and is one-dimensional, hence commutative. It follows easily from (i) applying $\mathfrak{n}$ and by induction on weights that $\delta M_{\pi'}(0)$ is commutative. Again fix $0 \neq \xi_0 \in \delta M_{\pi'}(0)$ of weight 0. By (ii) we can assume ξ_0 to be idempotent. Then by applying $\mathfrak{n}$ and by induction on weights it follows that ξ_0 acts as an identity 1 on $\delta M_{\pi'}(0)$. A further induction argument shows (given (i)) that (ii) is equivalent to the multiplication in $\delta M_{\pi'}(0)$ being non-trivial.

6.3.5. For each $\alpha \in \pi \setminus \pi'$ there exists a unique up to scalars vector $p_{-\alpha} \in \delta M_{\pi'}(0)$ of weight $-\alpha$. Furthermore $x_\alpha p_{-\alpha}$ is a non-zero scalar, so we may assume that $x_\alpha p_{-\alpha} = K(x_{-\alpha}, x_\alpha)$. Then for all $x_\beta \in \mathfrak{m}^+$, $a_{-(\beta-\alpha)} \in \mathfrak{p}^-$ one has

$$(*) \qquad x_\beta a_{-(\beta-\alpha)} p_{-\alpha} = K((ad\ a_{-(\beta-\alpha)})x_{-\alpha}, x_\beta).$$

Moreover this formula extends to all $a_{-(\beta-\alpha)} \in U(\mathfrak{p}^-)$. Set $M^- = \sum_{\alpha\in\pi\setminus\pi'} U(\mathfrak{p}^-)p_\alpha$ and define $\theta(a_{-(\beta-\alpha)}p_\alpha) := (ad\ a_{-(\beta-\alpha)})x_{-\alpha}$, $\forall \alpha \in \pi \setminus \pi'$, $a_{-(\beta-\alpha)} \in U(\mathfrak{p}^-)$.

Lemma. *The map θ extends linearly to a $U(\mathfrak{p}^-)$ module surjection of M^- onto $\mathfrak{m}^-$ satisfying $K(\theta(p_{-\beta}), x_\beta) = x_\beta p_{-\beta}$, for all $\beta \in \Delta^+ \setminus \Delta^+_{\pi'}$.*

Proof. Suppose $\sum_{\alpha\in\pi\setminus\pi'} a_{-(\beta-\alpha)}p_{-\alpha} = 0$ and set $a_{-\beta} = \sum_{\alpha\in\pi\setminus\pi'}(ad\ a_{-(\beta-\alpha)})x_{-\alpha}$. Then by $(*)$ we obtain $K(a_{-\alpha}, x_\beta) = 0$ and so $a_{-\beta} = 0$, by the non-degeneracy of K. Hence θ is well-defined. It is surjective by 6.3.3. By construction it is a $U(\mathfrak{p}^-)$ module map and by $(*)$ satisfies the required identity. □

6.3.6. Retain the above notation and recall that $\mathfrak{r}$ acts reductively on $\delta M_{\pi'}(0)$. Let V be an $\mathfrak{r}$ stable complement of $Ker\ \theta$ in M^-. By the formula in the conclusion of 6.3.4 we obtain for each $\beta \in \Delta^+ \setminus \Delta^+_{\pi'}$ an element $p_{-\beta} \in V$ satisfying $x_\beta p_{-\beta} = 1$. Trivially $x_{\beta'} p_{-\beta} = 0$, unless $\beta' \leq \beta$. Since $\mathfrak{p}$ and in particular $\mathfrak{m}$ acts by derivations on $\delta M_{\pi'}(0)$ a Taylor lemma [ ] argument shows that the $p_{-\beta} : \beta \in \Delta^+ \setminus \Delta^+_{\pi'}$ are algebraically independent. Thus we obtain an embedding $S(V) \hookrightarrow \delta M_{\pi'}(0)$ of $\mathfrak{r}$ modules which by comparison of formal characters must be an isomorphism. Observe further that $V \cong \mathfrak{m}^-$ as an $\mathfrak{r}$ module by comparison of characters.

6.3.7. Choose V as in 6.3.6. Let V' be an $\mathfrak{r}$ module isomorphic to $\mathfrak{m}^-$ such that the $\mathfrak{r}$ module structure on $S(V')$ extends to $U(\mathfrak{p})$ module isomorphism $S(V') \xrightarrow{\sim} \delta M_{\pi'}(0)$. For example take $V' = V$. A second example is given in 6.3.8, for which $V' + k \subset S(V')$ is also $\mathfrak{p}$ stable and in which the $\mathfrak{p}$ action on $S(V')$ is obtained by extending this linear action by derivation. Let ψ (resp. ψ') be the algebra homomorphism of $U(\mathfrak{p})$ into $End\ S(V)$ (resp. $S(V')$).

Lemma. *There exists an algebra isomorphism φ of $S(V)$ onto $S(V')$ intertwining ψ, ψ', that is $\psi'(y)\varphi(a) = \varphi(\psi(y)a)$, for all $y \in U(\mathfrak{p}),\ a \in S(V)$.*

Proof. Let $\varepsilon : S(V) \to k$ denote the augmentation and define a pairing $U(\mathfrak{m}) \times S(V) \to k$, by $< x, a >= \varepsilon(\psi(x)a)$, for all $x \in U(\mathfrak{m}),\ a \in S(V)$. This implies

$$(*) \qquad < xy, a >=< x, \psi(y)a >, \ \forall x, y \in U(\mathfrak{m}),\ a \in S(V).$$

Clearly $< 1, 1 >= 1$ and $<,>$ is $\mathfrak{r}$ invariant and so in particular respects weight space decomposition. Let $\beta \in \mathbb{N}\pi$ be minimal such that $0 \neq a_{-\beta} \in S(V)$ lies in $ker <,>$. Then $\beta > 0$ and through $(*)$ we obtain $< a_{-\beta-\gamma}, \psi(x_\gamma)a_{-\beta} >= 0$ for all $\gamma \in \Delta^+ \setminus \Delta^+_{\pi'}$, and all $a_{\beta-\gamma} \in U(\mathfrak{m})_{\beta-\gamma}$. Hence the minimality of β gives $a_{-\beta} \in S(V)^{\mathfrak{m}} = k$. This contradicts $\beta > 0$. Finally since $dim\ U(\mathfrak{m})_\beta = dim\ S(V)_{-\beta},\ \forall\ \beta \in \mathbb{N}\pi$, we conclude that $<,>$ is non-degenerate.

A similar result holds for the pairing $U(\mathfrak{m}) \times S(V') \to k$ defined by $< y, b >'= \varepsilon(\psi'(y)b),\ \forall\ y \in S(\mathfrak{m}), b \in S(V')$. Define a map $\varphi : S(V)$ into $S(V')$ by $< x, \varphi(a) >'= < x, a >$, for all $x \in S(\mathfrak{m}),\ a \in S(V)$. By $\mathfrak{r}$ invariance it is an isomorphism of $\mathfrak{r}$ modules. Moreover $< x, \psi'(y)\varphi(a) >'=< xy, \varphi(a) >'=< xy, a >=< x, \psi(y)a >=< x, \varphi(\psi(y)a) >'$, for all $x, y \in S(\mathfrak{m}),\ a \in S(V)$. Through the non-degeneracy of $<,>'$ we obtain $\psi'(y)\varphi(a) = \varphi(\psi(y)a)$.

Clearly $\varphi(1) = 1$. To establish that $\varphi(ab) - \varphi(a)\varphi(b) = 0$, for all $a, b \in S(V)$, it is enough to show that the left hand side is annihilated by the $\psi'(x) : x \in \mathfrak{m}^+$, when x, a, b are all weight vectors. This results from $\psi(x), \psi'(x)$ being derivations and from φ intertwining ψ, ψ', through induction on weights. $\square$

6.3.8. It is useful to observe that the action of $\mathfrak{p}$ on $S(V)$ can be assumed to be linear (inhomogeneous), that is to say we can choose an $\mathfrak{r}$ module V such that $\mathfrak{p}V \subset V \oplus k$. Indeed through the Killing form we may identify $\mathfrak{m}^-$ with $\mathfrak{m}^*$ and then the coadjoint action of $\mathfrak{p}$ on $\mathfrak{m}^*$ defines a linear action of $\mathfrak{p}$ on $\mathfrak{m}^-$. We deform the resulting action on $S(\mathfrak{m}^-)$ (which is not quite what we want) as follows. Let $\mathfrak{z}_{\pi'}$ denote the centre of $\mathfrak{r}_{\pi'}$ and identify $\mathfrak{m}^- \oplus \mathfrak{z}$ with $(\mathfrak{m} \oplus \mathfrak{z})^*$ through the Killing form. This gives $\mathfrak{m}^- \oplus \mathfrak{z}$ a $\mathfrak{p}$ module struture and moreover $x_\beta x_{-\beta}$ is a non-zero element of $\mathfrak{z}$ for all $\beta \in \Delta^+ \setminus \Delta^+_{\pi'}$. Now choose $\lambda \in z^*$ such that $\lambda(x_\beta(x_{-\beta})) \neq 0$, for all $\beta \in \Delta^+ \setminus \Delta^+_{\pi'}$, and let I be the ideal of $S(\mathfrak{m}^- \oplus \mathfrak{z})$ generated by the $h - \lambda(h) : h + \mathfrak{z}$. Let V denote the image of $\mathfrak{m}^-$ under the canonical projection $S(\mathfrak{m}^- \oplus z) \rightarrow S(\mathfrak{m}^- \oplus z)/I =: S$. It is clear that $S(V)$ identifies with S. By construction V is an $\mathfrak{r}$ submodule of S and $\mathfrak{p}V \subset V \oplus k$. Moreover $c_\beta := x_\beta(x_{-\beta})$ is a non-zero scalar for all $\beta \in \Delta^+ \setminus \Delta^+_{\pi'}$ which forces $S(V)^{\mathfrak{m}} = k$. It is also clear that the action of $\mathfrak{p}$ on $V \oplus k$ obtains from the (linear) action of $\mathfrak{p}$ on $\mathfrak{m}^-$ by just adding the derivation $c_\beta \partial/\partial x_{-\beta}$ to $x_\beta : \beta \in \Delta^+ \setminus \Delta^+_{\pi'}$.

Now suppose V' to be the $\mathfrak{r}$ submodule of $\delta M_{\pi'}(0)$ obtained from the construction of 6.3.6. Through 6.3.7 (which only used the $\mathfrak{p}$ module structure of $S(V')$, the $\mathfrak{r}$ module structure of V' and that $S(V')^{\mathfrak{m}} = k$) we obtain an isomorphism $S(V) \cong S(V') \cong \delta M_{\pi'}(0)$, of $U(\mathfrak{p})$ modules. We may summarize this result as follows.

Proposition. *Give the $U(\mathfrak{g})$ module $\delta M_{\pi'}(0)$ a non-trivial multiplication in which $\mathfrak{p}$ acts by derivations. Then there exists an $\mathfrak{r}$ submodule V of $\delta M_{\pi'}(0)$ isomorphic to $\mathfrak{m}^-$ such that*

(i) $\mathfrak{p}V \subset V \oplus k$.

(ii) *The $\mathfrak{p}$ module structure on $V \oplus k$ is uniquely determined by the condition $S(V)^{\mathfrak{m}} = k$ and extends to an isomorphism $\varphi : S(V) \xrightarrow{\sim} \delta M_{\pi'}(0)$ of $U(\mathfrak{p})$ module algebras.*

(iii) *The $U(\mathfrak{p})$ module algebra structure on $S(V)$ extends uniquely to a $U(\mathfrak{g})$ module algebra structure.*

Proof. It remains to verify (iii). It is equivalent to showing that φ commutes with the $\mathfrak{m}^-$ action, that is $x_{-\alpha}\varphi(a_{-\beta}) = \varphi(x_{-\alpha}a_{-\beta}), \forall\, \alpha, \beta \in \Delta^+ \setminus \Delta^+_{\pi'},\ x_{-\alpha} \in \mathfrak{m}^-,\ a_{-\beta} \in S(V)$.

This follows by induction on weights applying $x_\gamma : \gamma \in \pi$ to both sides and using that $S(V)^{\mathfrak{n}}$ reduces to scalars. Finally this unique $\mathfrak{g}$ action on $S(V) = \delta M_{\pi'}(0)$ must be one described in 6.3.2 making it a $U(\mathfrak{g})$ module algebra. $\square$

6.3.9. The above result shows that $\delta M_{\pi'}(0)$ can be given a non-trivial multiplication making it a $U(\mathfrak{g})$ module algebra in only one way. The resulting algebra structure is that described by 6.2.2 and made more precise through 6.3.8. In the following $\delta M_{\pi'}(0)$ denotes this unique $\mathcal{O}_f$-algebra.

Theorem. *Let A be an $\mathcal{O}$-algebra with $A^{\mathfrak{n}}$ one-dimensional and having a non-trivial multiplication. Then there exists $\pi' \subset \pi$ such that $A \cong \delta M_{\pi'}(0)$.*

Proof. It follows exactly as in 6.3.4 that A is commutative, that $A^{\mathfrak{n}} = A_0$ and consists of multiples of an identity for A.

Let δA denote the $\mathcal{O}$-dual of A. Through the contravariant form $\delta A/\mathfrak{n}^- \delta A$ is dual to $A^{\mathfrak{n}}$ and then by weight space decomposition it follows that δA is generated over $U(\mathfrak{n}^-)$ by its zero weight subspace $(\delta A)_0$.

Set $\pi' = \{\alpha \in \pi \mid (\delta A)_{-\alpha} = 0\}$. Then $x_{-\alpha}(\delta A)_0 = 0$, for all $\alpha \in \pi'$ and so by the above and universality of induction, δA is an image of $M_{\pi'}(0)$. Consequently A embeds in $\delta M_{\pi'}(0)$.

Now recall the construction of M^- and V in 6.3.5. By choice of π' it follows that M^- and hence V belongs to A. This forces $A = S(V) \cong \delta M_{\pi'}(0)$, as required. $\square$

6.3.10. Let A be an $\mathcal{O}_f$ algebra. Then *dim* $A^{\mathfrak{n}} < \infty$. Thus if A is an integral domain (and k is algebraically closed) the subalgebra $A^{\mathfrak{n}}$ is necessarily one dimensional. This and 6.3.9 establishes the claim in 6.3.2. For a more general semiprime $\mathcal{O}_f$ algebra we start with the following

Lemma. *Let A be a semiprime $\mathcal{O}_f$ algebra. Then*

$$A = \bigoplus_{i=1}^{m} A_i,$$

where each A_i is a semiprime $\mathcal{O}_f$ subalgebra of A with centre $Z(A_i)$ being completely prime and containing an identity for A_i.

Proof. Set $Z = Z(A)$. By 6.3.1 and the above remarks, Z is a non-zero finite dimensional algebra over k with no non-zero nilpotent elements. Hence Z admits an identity 1 and finitely many pairwise orthogonal idempotents $e_i : i = 1, 2, \cdots, m$ which sum to 1.

Let e be any central idempotent of A. Then for all $x \in \mathfrak{g}$ one has $xe = xe^2 = 2exe$ and $exe = 0$, hence $xe = 0$. Thus the $A_i := Ae_i$, are $\mathcal{O}_f$ subalgebras forming a direct sum in A. To show that this is A itself, it is enough to show that 1 is an identity of A.

Since 1 is a central $\mathfrak{g}$ invariant idempotent its kernel K is a semiprime $\mathcal{O}_f$ subalgebra of A. If $K \neq 0$, then by 6.3.1 its centre $Z(K)$ is a non-zero submodule of A and so $Z(K) \neq 0$. Yet the latter lies in Z on which 1 acts by the identity. This contradiction proves that 1 is an identity for A.

Finally $Z(A_i) = Ze_i = ke_i$. □

6.3.11. Proposition. *Let A be a semiprime $\mathcal{O}_f$ algebra whose centre $Z(A)$ is completely prime. Then there exists $\pi' \subset \pi$ and an isomorphism of $\mathfrak{r}_{\pi'}$ module algebras*

$$(*) \qquad A \xleftarrow{\sim} A^{\mathfrak{m}_{\pi'}} \otimes Z(A),$$

given by multiplication, with $A^{\mathfrak{m}_{\pi'}}$ central simple.

Proof. By 6.2.6 (ii), $Z(A)$ is an $\mathcal{O}_f$-subalgebra of A and so by the hypothesis and 6.3.11 there exists $\pi' \subset \pi$ such that $Z(A)$ is isomorphic to the $\mathcal{O}_f$ algebra $\delta M_{\pi'}(0)$.

Recall the notation of 6.3.6. Since the matrix with entries $x_\beta p_{-\beta'} : \beta, \beta' \in \Delta^+ \setminus \Delta^+_{\pi'}$ is triangular with ones on the diagonal, a Taylor lemma argument [ ] gives $(*)$. In particular A is a free $Z(A)$ module. Since $Z(A)$ is noetherian (even a polynomial algebra) and A is semiprime p.i. it follows by Formanek's theorem [, 5.1.6] that A has finite rank over $Z(A)$ and so *dim* $A^{\mathfrak{m}} < \infty$. Moreover $A^{\mathfrak{m}}$ is semiprime through $(*)$ because A is semiprime and its centre is reduced to scalars by definition of $Z(A)$. Thus $A^{\mathfrak{m}}$ is central, simple. Since $\mathfrak{r}$ normalizes $\mathfrak{m}$, it is $\mathfrak{r}$ module with $\mathfrak{r}$ acting by derivations because already $\mathfrak{g}$ acts by derivations on A. □

6.3.12. Let $A^{\mathfrak{m}_{\pi'}}$ be the $U(\mathfrak{r}_{\pi'})$ module algebra in the conclusion of 6.3.11.

Lemma. *There exists a finite dimensional $\mathfrak{r}_{\pi'}$ module E such that the $\mathfrak{r}_{\pi'}$ module algebra End E (in which $\mathfrak{r}_{\pi'}$ acts diagonally) is isomorphic to $A^{\mathfrak{m}_{\pi'}}$.*

Proof. Recall that $A^{\mathfrak{m}}$ is simple and finite dimensional over an algebraically closed field k. Then by the Artin-Wedderburn theorem [, Thm. 2.1.6] there exists a finite dimensional

k-vector space E such that $A^{\mathfrak{m}} \cong End_k E$ as an algebra.

Take $x \in \mathfrak{r}$. Then x is a derivation of $End_k E$, hence inner by a corollary to the Noether-Skolem theorem [, Proposition following Thm. 4.3.1]. This means that we have an element $\psi(x) \in End\ E$ such that the action of x on $End\ E$ is given by $(x, a) \mapsto \psi(x)a - a\psi(x)$, for all $a \in End\ E$. Since the centre of $End\ E$ reduces to scalars, taking $\psi(x)$ to have trace zero determines $\psi(x)$ uniquely as an element of the Lie subalgebra $\mathfrak{sl}(E)$ of $\mathfrak{gl}(E)$. Uniqueness implies that the map $x \mapsto \psi(x)$ is a Lie algebra map. This makes E an $\mathfrak{r}$ module as required. □

6.3.13. Just for the moment assume $\mathfrak{g}$ to be an arbitrary Lie algebra and let $\mathfrak{a}$ be a Lie subalgebra. For any $\mathfrak{g}$ module V set

$$F_{\mathfrak{a}}(V) := \{v \in V \,\big|\, dim\ U(\mathfrak{a})v < \infty\}.$$

It is called the $\mathfrak{a}$ locally finite part of V. If with respect to adjoint action $F_{\mathfrak{a}}(U(\mathfrak{g})) = U(\mathfrak{g})$, for example if $\mathfrak{g}$ is finite dimensional, then $F_{\mathfrak{a}}(V)$ is a $\mathfrak{g}$ submodule of V.

Suppose now that $\mathfrak{g}$ is semisimple and recall that the Chevalley antiautomorphism κ interchanges $\mathfrak{p}$ and $\mathfrak{p}^-$. If V is a $\mathfrak{p}$ module with finite dimensional weight spaces, we set $\delta V := F_{\mathfrak{h}}((V^\kappa)^*)$, which is a $\mathfrak{p}^-$ module with weight spaces of the same dimension as those of V. If $V \in Ob\mathcal{O}$, then δV is just the $\mathcal{O}$ dual of V.

Now suppose V is a finite dimensional (left) $\mathfrak{p}$ module. Then V^κ is a finite dimensional right $\mathfrak{p}^-$ module and

$$(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V)^\kappa \cong V^\kappa \otimes_{U(\mathfrak{p}^-)} U(\mathfrak{g}).$$

Consequently

$$\begin{aligned}\delta(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V) &= F_{\mathfrak{h}}(Hom_k(V^\kappa \otimes_{U(\mathfrak{p})} U(\mathfrak{g}), k))\\ &= F_{\mathfrak{h}}(Hom_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), \delta V)),\end{aligned}$$

by Frobenius reciprocity.

Let E be a finite dimensional $\mathfrak{r}$ module, for example that occuring in the conclusion of 6.3.12. Then $A := End\ E \otimes \delta M_{\pi'}(0)$ is a prime ring with centre $\delta M_{\pi'}(0)$ and admits a $U(\mathfrak{r})$ module algebra structure through the diagonal action of $\mathfrak{r}$. Extend E to a $U(\mathfrak{p}^-)$ module by letting $\mathfrak{m}^-$ act by zero. Then $\delta(End\ E)$ becomes a $U(\mathfrak{p})$ module (coalgebra).

Proposition. (i) *The action of* $\mathfrak{r}_{\pi'}$ *on* A *extends to a* $\mathfrak{g}$ *action making it an* $\mathcal{O}_f$ *algebra and the resulting module is* $\delta(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \delta(End\ E))$.

(ii) *The subalgebra* $A^{\mathfrak{m}_{\pi'}}$ *identifies with* $End\ E$ *as a* $U(\mathfrak{r}_{\pi'})$ *module.*

Proof. Certainly $\delta(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \delta(End\ E)) \in Ob\mathcal{O}_f$ and by the above is isomorphic to $B := F_{\mathfrak{h}}(Hom_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), End\ E))$. The latter admits a $U(\mathfrak{g})$ module structure through multiplication in $End\ E$ and the coproduct on $U(\mathfrak{g})$. The subalgebra $F_{\mathfrak{h}}(Hom_{U(\mathfrak{p}^-)}(U(\mathfrak{g}), 1d_E))$ of B is just $\delta M_{\pi'}(0)$ as a $U(\mathfrak{g})$ module algebra.

Let $U(\mathfrak{m})_+$ be the augmentation of $U(\mathfrak{m})$. It is a coideal of $U(\mathfrak{m})$ and hence of $U(\mathfrak{g})$. Thus $B_0 := \{\theta \in B \mid \theta(U(\mathfrak{m})_+) = 0\}$ is $U(\mathfrak{r})$ module subalgebra of B and lies in $B^{\mathfrak{m}}$. The $U(\mathfrak{p}^-)$ module map $\theta \mapsto \theta(1)$ restricted to B_0 is trivially injective. Since $U(\mathfrak{g})$ is free over $U(\mathfrak{p}^-)$ its image is $End\ E$. The Taylor lemma argument of 6.3.11 gives an $\mathfrak{r}$ module isomorphism $B^{\mathfrak{m}} \otimes \delta M_{\pi'}(0) \xrightarrow{\sim} B$ and then comparison of formal characters forces $B^{\mathfrak{m}} = B_0 = End\ E$. □

6.3.14. Combining 6.3.10 - 6.3.13 we obtain the

Theorem. (i) *Every semiprime* $\mathcal{O}_f$ *algebra is a finite direct sum of prime* $\mathcal{O}_f$ *algebras.*

(ii) *Take* $\pi' \subset \pi$ *and let* E *be a finite dimensional* $\mathfrak{r}_{\pi'}$ *module. Then* $A := End\ E \otimes \delta M_{\pi'}(0)$ *admits the structure of a prime* $\mathcal{O}_f$ *algebra and every prime* $\mathcal{O}_f$ *algebra obtains in this way.*

(iii) *In (ii) one has* $A^{\mathfrak{g}} = (End\ E)^{\mathfrak{r}_{\pi'}}$.

Proof. It remains to verify (iii). By 6.3.13(ii) it is immediate that $A^{\mathfrak{g}} \subset A^{\mathfrak{p}} = (End\ E)^{\mathfrak{r}}$. For equality recall the construction of the $\mathfrak{g}$ module structure of A in 6.3.13. Since $End\ E$ is given a trivial $\mathfrak{m}^-$ action, it follows that $\delta(End\ E)^{\mathfrak{r}}$ acquires a trivial $\mathfrak{p}$ module structure and so $\delta(U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \delta(End\ E)^{\mathfrak{r}})$ is just $(dim\ E)^{\mathfrak{r}}$ copies of $\delta M_{\pi'}(0)$. The latter has as its socle the trivial $\mathfrak{g}$ module. Hence $dim\ A^{\mathfrak{g}} \geq dim(End\ E)^{\mathfrak{r}}$, forcing the required equality.

□

6.4. Bernstein filtrations and Goldie Rings.

6.4.1. Let M, N be $U(\mathfrak{g})$ modules. Then $Hom(M,N)$ is a $U(\mathfrak{g})$ bimodule and so a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ module through the antipode. However in general it is too big to be of much interest and a fundamental step is to reduce it to something smaller. This is done in the following way. First observe that $Hom(M,N)$ is $U(\mathfrak{g})$ module for the diagonal action $(x,\theta) \mapsto x\theta - \theta x : x \in \mathfrak{g}, \theta \in Hom(M,N)$. Here $x\theta - \theta x)(m) = x\theta(m) - \theta(xm)$, for all

$m \in M$. Now let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$. Then with respect to diagonal action of $\mathfrak{a}$ we set

$$F_{\mathfrak{a}}(M, N) = \{\theta \in Hom(M, N) \,\Big|\, dim\ U(\mathfrak{a})\theta < \infty\}.$$

It is called the $\mathfrak{a}$ locally finite part of $Hom(M, N)$. Since $\mathfrak{g}$ is finite dimensional, it is a $U(\mathfrak{g}) \otimes U(\mathfrak{g})$ submodule of $Hom(M, N)$.

When $M = N$ we shall simply write $F_{\mathfrak{a}}(M, M)$ as $F_{\mathfrak{a}}(M)$. This could lead to confusion with the notation introduced in 6.3.13; but it will generally be clear what is meant.

We shall write $F_{\mathfrak{g}}$ simply as F. The algebra $F(M)$ is particularly interesting and about which much is known especially for M simple and in the $\mathcal{O}$ category. We shall permit ourselves a slight digression to discuss this.

6.4.2. Suppose M is simple. Then by Quillen's lemma the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ acts by scalars on M. It follows that the centre $Z(\mathfrak{g}) \otimes Z(\mathfrak{g})$ of $U(\mathfrak{g}) \otimes U(\mathfrak{g}) \cong U(\mathfrak{g} \times \mathfrak{g})$ acts by scalars on $End\ M$ and hence by scalars on $F(M)$. Since $\mathfrak{g}$ is reductive it follows by the definition of $F(M)$, that it is a direct sum of finite dimensional simple $U(\mathfrak{g})$ submodules. We claim that the number of times $[F(M) : E]$ that a given simple module E occurs in $F(M)$ is finite. Admitting this it follows that $F(M)$ is a Harish-Chandra module (5.3.14) for the pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{k})$, where $\mathfrak{k}$ denotes the diagonal copy of $\mathfrak{g}$ in $\mathfrak{g} \times \mathfrak{g}$. Thus $F(M)$ has finite length by the argument which precedes lemma 5.3.11. In particular $F(M)$ is finitely generated as a $U(\mathfrak{g})$ bimodule and so finitely generated as a left (or as a right) $U(\mathfrak{g})$ module (because the diagonal action is locally finite).

One may remark that the action of $U(\mathfrak{g})$ on M defines an embedding of $U(\mathfrak{g})/Ann\ M$ into $F(M)$. However this is seldom surjective. Indeed $F(M)$ is an example of a Dixmier algebra, namely an overring D of a primitive quotient U of $U(\mathfrak{g})$ which admits a locally finite action of the diagonal copy of $\mathfrak{g}$ and is finitely generated as a left (or right) $U(\mathfrak{g})$ module. Thus D is an image of $E \otimes U$, for some E finite dimensional and so $Z(\mathfrak{g})$ acts by scalars on the right and similarly on the left. Then by 5.3.14, D has finite length as a bimodule and so is a Harish-Chandra module.

Very little is known about the classification of Dixmier algebras, though a few results can be mentioned. For example, a prime Dixmier algebra is always a subalgebra of $F(M)$ for some $M \in Ob\mathcal{O}_f$. A deeper fact is that if $L \in Ob\mathcal{O}$ is simple, then $F(L)$ is very nearly equal to its bimodule socle $Soc\ F(L)$. Indeed the GK dimension (5.3.2) of $F(L)/Soc\ F(L)$

is strictly less then that of $F(L)$. Moreover the simples occuring in $Soc\ F(L)$ can be described. On the other hand rather little is known about the simples occuring in $F(L)$ itself (or indeed in $U(\mathfrak{g})/Ann\ L$).

6.4.3. To prove the claim in 6.4.2 one first observes that by the reductivity of $\mathfrak{g}$ and Frobenius reciprocity

$$(*) \qquad [F(M):E] = dim\ Hom_{U(\mathfrak{g})}(M, E^* \otimes M).$$

Now for M simple and "reasonable" (for example if it belongs to the category $\mathcal{M}$ defined in 5.3.13) one may show that $E^* \otimes M$ has finite length and so the right hand side of $(*)$ is finite by Quillen's lemma. However finite length can fail in general, though of course $E^* \otimes M$ is still finitely generated and moreover has the same GK dimension $d(M)$ as M. Now recall (5.3.2 - 5.3.5) that for any non-zero finitely generated $U(\mathfrak{g})$ module N one may define a filtration $N^n : n \in \mathbb{N}$ compatible with the canonical filtration $U^m(\mathfrak{g})$ on $U(\mathfrak{g})$ such that $n \mapsto dim\ N^n$ is polynomial for n sufficiently large. Moreover there exist integers $d(N) \geq 0, e(N) > 0$ such that

$$dim\ N^n = \frac{e(N)n^{d(N)}}{d(N)!} + O(n^{d(N)-1}) : n >> 0.$$

Here $d(N)$ is just the GK dimension of N. One calls $e(N)$ the multiplicity of N. Both are independent of the finite filtration chosen. (A filtration is finite if $U^m(\mathfrak{g})N^n = N^{m+n}$, for $m, n >> 0$).

One easily checks that $e(E^* \otimes M) = (dim\ E^*)e(M)$. Combined with Quillen's lemma and the simplicity of M this shows that the right hand side of $(*)$ is bounded by $dim\ E^*$.

6.4.4. Take $L \in Ob\mathcal{O}$ simple and denote $F_{\mathfrak{n}^-}(L)$ simply as $A(L)$. It turns out that the surjectivity result concerning differential operators referred to in 6.1.7 can best be viewed in the more general context of when $F(L) = A(L)$. At first sight it may seem extremely surprising that such an equality should hold since $\mathfrak{g}$ is much bigger than $\mathfrak{n}^-$. Yet we shall show that it nearly always holds. A first rather technical step is to show that $A(L)$ is a left Goldie ring. Since it is trivially a prime ring (because L is simple) a deep theorem of Goldie can then be applied.

6.4.5. Bernstein deftly used the notion of multiplication to give a growth rate condition for a module to be finitely generated. We shall turn this into a way of showing that $A(L)$ is Goldie.

6.4.6. Recall 2.2.1. Let A be a k-algebra with an increasing filtration by finite dimensional subspaces $A^n : n \in \mathbb{N}$. Let $d \geq 0, e > 0$ be integers. We say that $\{A^n\}$ is a Bernstein filtration of type (d, e) of A if

$$dim\ A^n \leq \frac{en^d}{d!} + o(n^d),\ \forall\ n \in \mathbb{N}.$$

Now suppose that A is a left $U(\mathfrak{g})$ module. Let ℓ be a positive integer. We say that $\{A^m\}$ is ℓ-compatible with the canonical filtration of $U(\mathfrak{g})$ if $\mathfrak{g}A^n \subset A^{n+\ell}$, for all $n \in \mathbb{N}$.

Lemma. *Assume that A admits an ℓ-compatible Bernstein filtration of type (d, e). Then for any increasing filtration $N_1 \subset N_2 \subset \cdots$, of A by finitely generated left $U(\mathfrak{g})$ modules, one has $d(N_i/N_{i+1}) \leq d$ with equality at most $e\ell^d$ times.*

Proof. Take $N = N_i$ and let N^0 be a finite dimensional subspace over which N is generated. Then $N^n := U^n(\mathfrak{g})N^0$ is a finite filtration of N. Choose t such that $N^0 \subset A^t$. Then $N^n \subset A^{n\ell+t}$ and so

$$dim\ N^n \leq \frac{e}{d!}(n\ell + t)^d + o(n^d).$$

Consequently $d(N) \leq d$ and equality implies that $e(N) \leq e\ell^d$. Yet $e(\cdot)$ is additive on those short exact sequences in which all objects have the same value of $d(\cdot)$. Since $e(N)$ is a positive integer the required conclusion obtains. □

6.4.6. Assume $L \in Ob\mathcal{O}_f$. Then (4.3.8) L has finite length and its simple subquotients are highest weight modules. It follows from the decomposition $U(\mathfrak{g}) = U(\mathfrak{n}^-) \oplus U(\mathfrak{g})\mathfrak{b}$ that L is finitely generated over $U(\mathfrak{n}^-)$ and by the additivity on exact sequences that $d_1 := d(L), e_1 := e(L)$ calculated with respect to $\mathfrak{g}$, are the same as those integers calculated with respect to $\mathfrak{n}^-$. Choose a finite dimensional subspace L^0 of L satisfying $U(\mathfrak{n}^-)L^0 = L$ (which we can assume to be $\mathfrak{h}$ stable) and set $L^n = U^n(\mathfrak{n}^-)L^0$.

Define $\delta \in \mathfrak{h}^*$ such that $(\delta, \alpha) = 1$, for all $\alpha \in \pi$. Let ℓ be a positive integer bounding both the nilpotence degree of $ad_{\mathfrak{g}}\mathfrak{n}^-$ and the set $\{(\alpha, \delta) : \alpha \in \Delta^+\}$. For example $dim\ \mathfrak{g}$ will do.

For each $n \in \mathbb{N}$ set

$$L^{(n)} := \oplus\{L_\mu \,\Big|\, -(\delta, \mu) \leq n\}.$$

These form an increasing sequence of finite dimensional subspaces with union L satisfying $L^{(-t)} = 0$, for some integer t.

Now let A be a subalgebra of $A(L)$ containing the image of $U(\mathfrak{g})$ in $End\ L$. For the moment, view A as a $U(\mathfrak{g})$ module under diagonal action and for each $n \in \mathbb{N}$, set $A^{(n)} = \{a \in A \mid U^{n+1}(\mathfrak{n}^-)a = 0\}$. Then $A^{(n)}$ is $\mathfrak{h}$ stable and these subspaces define a filtration of A.

Now given $a \in A^{(n)}$, then $aL^n = 0$ implies $a = 0$ and so the action of A on L defines an embedding of $A^{(n)}$ into $Hom(L^n, L)$ of $U(\mathfrak{h})$ modules. It follows that $A^{(n)}$ admits a weight space decomposition with finite dimensional weight subspaces. Moreover in the notation of 4.3.2 (iii) one has

$$(*) \qquad \Omega(A^{(n)}) \subset S_n - \mathbb{N}\pi$$

for some finite subset $S_n \subset \mathfrak{h}^*$. One may remark that $A^{(0)} = A^{\mathfrak{n}-}$ and that we can take $S_0 = \{0\}$ if L is cyclic.

It follows from the above that the

$$A^n := \bigoplus_{\{\mu \mid |(\mu,\delta)| \leq n\}} A_\mu^{(n)} : n \in \mathbb{N}$$

also form a filtration of A.

Now consider A as a $U(\mathfrak{g})$ module under left multiplication.

Lemma. *Set $d = 2d_1, e = (\ell+1)^d e_1^2 \binom{2d_1}{d_1}$. Then $\{A^n\}$ is an ℓ-compatible Bernstein filtration of type (d, e).*

Proof. By definition of ℓ one has $\mathfrak{g}A^n \subset A^{n+\ell}$, for all $n \in \mathbb{N}$. Since L^0 is finite dimensional, there exists an integer s such that $L^0 \subset L^{(s)}$. Since $(\alpha, \delta) \geq 1$, for all $\alpha \in \Delta^+$, it follows from the definition of ℓ, s, t that

$$L^{(n-t)} \subset L^n \subset L^{(n\ell+s)}, \forall\ n \in \mathbb{N}.$$

On the other hand if $a \in A^n$, then $aL^{(n\ell+s)} \subset L^{(n(\ell+1)+s}$, whilst $aL^{n'} = 0 : n' \geq n$ implies

$a = 0$. Hence

$$\begin{aligned}
dim\ A^n &\leq dim\ L^{(n\ell+s)} dim\ L^{(n(\ell+1)+s)}, \\
&\leq dim\ L^{n\ell+s+t} dim\ L^{n(\ell+1)+s+t}, \\
&= \frac{e_1^2 (n\ell+s+t)^{d_1} (n(\ell+1)+s+t)^{d_1}}{(d_1!)^2} + o(n^{2d_1}), \\
&\leq \frac{e_1^2 (\ell+1)^{2d_1} n^{2d_1}}{(d_1!)^2} + o(n^{2d_1}), \\
&= \frac{en^d}{d!} + o(n^d),
\end{aligned}$$

as required. □

6.4.7. Recall 5.3.2. Retain the hypotheses and notation of 6.4.6. Set $U = U(\mathfrak{g})/Ann\ L$.

Lemma. *One has $d_U(U) = d_A(A) = d$.*

Proof. Trivially $d_U(U) \leq d_U(A) \leq d_A(A) \leq d$. It remains to show that $2d_1 \leq d_U(U)$. This follows from 5.3.13, though for $L \in Ob\mathcal{O}$ a more elementary proof based on weight space decomposition is available [ ]. □

6.4.8. The above Bernstein-inspired growth estimate gives the following crucial result. We extend $d_U(M)$ to a not necessarily finitely generated U module M by taking the supremum over finitely generated submodules.

Proposition. *Take $L \in Ob\mathcal{O}_f$ and let A be a subalgebra of $A(L)$ containing $U := U(\mathfrak{g})/Ann\ L$. Then any strictly increasing sequence of left ideals $I_1 \subsetneq I_2 \subsetneq \cdots$, of A satisfying $d_U(I_{i+1}/I_i) = d_A(A)$, whenever $I_{i+1} \supsetneq I_i$, is finite.*

Proof. Since U is finitely generated and since $d_U(M)$ takes integer values for any finitely generated U module M, one has $d_U(I_{i+1}/I_i) = d_U(M)$, for some such $M \subset I_{i+1}/I_i$. Thus we can assume the I_i to be successively replaced by finitely generated U modules J_i satisfying $d_U(J_{i+1}/J_i) = d_U(I_{i+1}/I_i) = d_A(A)$. Then the conclusion obtains from 6.4.4 - 6.4.7. □

6.4.9. Recall that a ring A is said to be (left) Goldie if

(i) *A admits no infinite direct sums of left ideals,*

(ii) *Any increasing sequence $I_1 \subset I_2 \cdots$, of left annihilators in A is stationary.*

Here a left annihilator I (of a subset S of A) is just $I := \{a \in A \,\big|\, as = 0, \forall\ s \in S\}$. It is hence a left ideal of A. Thus a left noetherian ring is left Goldie; but the latter condition is significantly weaker. A left ideal I of a ring A is called essential if $I \cap L \neq 0$, for every non-zero left ideal L of A. For example in any prime ring A any non-zero two-sided ideal I of A is essentially. Indeed if L is a non-zero left ideal, one has $I(LA) \subset (I \cap L)A$, so $I \cap L$ cannot be zero by the primeness of A. A crucial result of Goldie is the following.

Theorem. *In any prime left Goldie ring A any essential left ideal contains a non-zero divisor.*

From this one may go on to show that A admits a classical ring of fractions which by the Artin-Wedderburn theorem is a matrix ring $M_n(K)$ over a skew-field K. One calls n (resp. K) the Goldie rank (resp. Goldie field) of A. Remarkably $rk(U(\mathfrak{g})/P) : P \in Prim\ U(\mathfrak{g})$ with $\mathfrak{g}$ semisimple is given by a finite family of polynomials on $\mathfrak{h}^*$ via Duflo's parametrization of $Prim\ U(\mathfrak{g})$ and moreover these polynomials are known. We remark that J.-C. Jantzen has devoted a whole text to the exposition of this work. A natural extension of the Gelfand-Kirillov conjecture asserts that the Goldie field of $U(\mathfrak{g})/P : P \in Prim\ U(\mathfrak{g})$, $\mathfrak{g}$ algebraic is a Weyl skew field of index $n : \frac{1}{2}\ d(U(\mathfrak{g})/P)$, that is to say it is obtained by inverting differential operators on the affine space $\mathbb{A}^n$. This is known for $\mathfrak{g}$ solvable and for $\mathfrak{g} = \mathfrak{sl}(n)$. Otherwise it is an open problem.

6.4.10. Take $L \in Ob\mathcal{O}_f$ and set $U := U(\mathfrak{g})/Ann\ L$. Let A be a subalgebra of $A(L)$ containing U. Assume that L is a simple A module. This makes A a prime ring. Below we show that A is left Goldie through the following key

Lemma. *One has $d_U(Aa) = d_A(A)$, for all $0 \neq a \in A$.*

Proof. Recall that we may write $L = U(\mathfrak{n}^-)L^0$, with L^0 finite dimensional. Simplicity implies that $AaL = L$ and so we choose $H_1 \subset Aa$ finite dimensional such that $H_1L \supset L^0$. Let H be obtained from H_1 through the diagonal action of $\mathfrak{n}^-$. Since this action is locally finite, there exist integers $m, n \geq 0$ such that $H_1 \subset H \subset U^m(\mathfrak{n}^-)H_1U^n(\mathfrak{n}^-)$. We can assume $U^m(\mathfrak{n}^-)H_1 \subset H_1$ without loss of generality. Then since $U^n(\mathfrak{n}^-)$ is finite dimensional,

$$d_U(UH_1) \leq d_U(UH) = sup_{b \in U^n(\mathfrak{n}^-)} d_U(UH_1b) \leq d_U(UH_1),$$

so equality holds throughout. Furthermore $HL = HU(\mathfrak{n}^-)L = U(\mathfrak{n}^-)HL \supset U(\mathfrak{n}^-)L^0 = L$.

Choose $b_i \in A$ such that $\{h_i := b_i a,\ i = 1, 2, \cdots, s\}$ is a basis for H_1. Since $Ab_i a$ is a submodule of Aa we obtain

$$d_U(UH_1) \leq d_U(AH_1) = sup_i\ d_U(Ab_i a) \leq d_U(Aa).$$

Now suppose $d_U(Ua) < d_A(A)$ and set $I_i = Ann_U h_i$. Then $d_U(U/I_i) = d_U(Uh_i) < d_A(A) = d_U(U)$. We claim that this strict inequality implies that I_i is an essential left ideal of U and so $I := \cap I_i \neq 0$. Then $0 = IHL = IL$ in contradiction to L being a faithful U module.

To prove the claim, note that U is noetherian, so there are finitely many prime ideals over 0 and then at least one of these, say P, must satisfy $d_U(\bar{U}) = d_U(U)$, where $\bar{U} = U/P$. Consider the image $\bar{I}_i$ of I_i in $\bar{U}$. One has $d_U(\bar{U}/\bar{I}_i) = d_U(U/I_i) < d_U(\bar{U})$. Let $\bar{K}$ be a non-zero left ideal of $\bar{U}$. If $\bar{I}_i \cap \bar{K} = 0$, then $\bar{K} \mapsto \bar{U}/\bar{I}_i$, so $d_U(\bar{K}) < d_U(\bar{U})$. Set $\bar{J} = \bar{K}\bar{U}$, which is an ideal of the prime Goldie ring $\bar{U}$ and so contains a non-zero divisor c. Then $d_U(\bar{J}) \geq d_U(\bar{U}c) = d_U(\bar{U})$. On the other hand U being noetherian means that we can write $\bar{J}$ as a finite sum $\sum \bar{K}b_j : b_j \in \bar{U}$ and then $d_U(\bar{J}) = sup_j d_U(\bar{K}b_j) \leq d_U(\bar{K}) < d_U(\bar{U})$, which is a contradiction. □

6.4.11. Retain the notation and hypotheses of 6.4.10.

Theorem. *A is a left Goldie ring.*

Proof. The existence of an infinite direct sum of left ideals obviously contradicts 6.4.9 and 6.4.10. Let $I_i \subsetneq I_2 \subsetneq \cdots$, be a strictly increasing sequence of left annihilators of A. Then there exists $a \in A$ such that $I_i a = 0$, yet $I_{i+1}a \neq 0$. Then $b \mapsto ba$ induces a surjection of I_{i+1}/I_i onto $I_{i+1}a$. Hence $d_U(I_{i+1}/I_i) \geq d_U(I_{i+1}a) \geq d_A(A)$, by 6.4.10. Then the above sequence is finite by 6.4.9, as required. □

Remarks. One may deduce from this result using the Weyl group and Chevalley anti-automorphism [ ] that $A(L)$ is also right Goldie; but this will not be needed here. It is not known if $A(L)$ is left noetherian.

6.4.12. Let $\hat{\mathcal{O}}$ denote the category of $U(\mathfrak{g})$ modules M satisfying

(i) M is a direct sum of its weight subspaces

(ii) $dim\ U(\mathfrak{n})m < \infty,\ \forall\ m \in M.$

(ii) The weight subspaces of $M^{\mathfrak{n}}$ are finite dimensional.

Clearly $\hat{\mathcal{O}} \supset \mathcal{O} \supset \mathcal{O}_f$.

By the Harish-Chandra isomorphism (4.3.7) every maximal ideal of $Z(\mathfrak{g})$ can be written in the form $Ann_{Z(\mathfrak{g})} M(\mu) : \mu \in \mathfrak{h}^*$ and moreover any two such ideals coincide if and only if the corresponding highest weights lie in the same W. orbit. We write $\hat{\mu} = W.\mu$ and $\chi(\hat{\mu}) = Ann_{Z(\mathfrak{g})} M(\mu)$. For any $U(\mathfrak{g})$ module M, let $M(\hat{\mu})$ be the submodule formed by vectors annihilated by some power of $\chi(\hat{\mu})$. It is called the $Z(\mathfrak{g})$ primary component of M of type $\hat{\mu}$.

Proposition. *Suppose $M \in \hat{\mathcal{O}}$. Then*

(i) *M is a (possibly infinite) direct sum of its $Z(\mathfrak{g})$ primary components.*

(ii) *$M(\hat{\mu}) \in Ob\mathcal{O}_f$, for all $\mu \in \mathfrak{h}^*$.*

(iii) *For all $i \in \mathbb{N}$, $M/\mathfrak{n}^i M$ has finite dimensional weight spaces.*

Proof. (i) Set $M^{(n)} = \{m \in M \,\big|\, U^{n+1}(\mathfrak{n})m = 0\}$. These are $U(\mathfrak{h}) \otimes Z(\mathfrak{g})$ submodules of M whose union is M by property (ii) of 6.4.12. By property (i) each $M^{(n)}$ is a direct sum of $\mathfrak{h}$ weight spaces. Given $m \in M^{(n)}$ we show that $I := Ann_{Z(\mathfrak{g})} m$ has finite codimension by induction on n. By the Chinese remainder theorem this will give (i). We can assume that m is a weight vector of weight μ.

Clearly $M^{(0)} = M^{\mathfrak{n}}$ and by property (i) it is a direct sum of its weight subspaces $M^{\mathfrak{n}}_{\mu} : \mu \in \mathfrak{h}^*$. Moreover $Ann_{Z(\mathfrak{g})} M^{\mathfrak{n}}_{\mu} = \chi(\hat{\mu})$. This proves the assertion for $n = 0$.

The map $\varphi : m \mapsto \{x_\alpha m\}_{\alpha \in \Delta^+}$ is a $Z(\mathfrak{g})$ module homomorphism of $M^{(n+1)}$ into card Δ^+ copies of $M^{(n)}$ with kernel $M^{\mathfrak{n}}$. By the induction hypotheses $J_\alpha := Ann_{Z(\mathfrak{g})} x_\alpha m$ has finite codimension and hence so has $J := \cap J_\alpha$. By construction $J\varphi(m) = 0$ and so $Jm \in ker\ \varphi$. We conclude that $Jm \subset M^{\mathfrak{n}}_{\mu}$ and so $I \supset \chi(\hat{\mu})J$, which again has finite codimension. This completes the proof of (i).

(ii) Since $\chi(\hat{\nu}) M(\hat{\mu})^{\mathfrak{n}}_{\nu} \subset \chi(\hat{\nu}) M^{\mathfrak{n}}_{\nu} = 0$, it follows that

$$M(\hat{\mu})^{\mathfrak{n}} = \bigoplus_{\nu \in W.\mu} M^{\mathfrak{n}}_{\nu}$$

which by property (iii) has some finite dimension, say s. (We remark that although $Z(\mathfrak{g})$ acts by scalars on $U(\mathfrak{g}) M^{\mathfrak{n}}(\hat{\mu})$, the latter may fail to be the whole of $M(\hat{\mu})$).

Let N be a finitely generated submodule of $M(\hat{\mu})$. By properties (i), (ii) we can assume that N is generated by a finite dimensional $U(\mathfrak{b})$ stable subspace. Then through the triangular decomposition of $\mathfrak{g}$ it follows that N has finite dimensional weight spaces. Hence $N \in Ob\mathcal{O}_f$. In particular N has finite length (4.3.8).

Since for any $L \in Ob\mathcal{O}$ simple, one has $dim\ L^{\mathfrak{n}} = 1$, we conclude from the embeddings $(Soc\ N)^{\mathfrak{n}} \mapsto N^{\mathfrak{n}} \mapsto M(\hat{\mu})^{\mathfrak{n}}$, that the length of $Soc\ N$ is bounded by s.

Now $\mathcal{O}_f$ has enough injectives so the N is contained in the injective hull of $Soc\ N$ which again has finite length. Finally let $N_1 \subset N_2 \subset \cdots$, be an increasing sequence of finitely generated submodules of $M(\hat{\mu})$. By the first remark above the increasing sequence $Soc\ N_1 \subset Soc\ N_2 \subset \cdots$, is stationary. Then by the second remark above, so is the original sequence. Hence $M(\hat{\mu}) \in Ob\mathcal{O}_f$. □

(iii) Since $\chi(\hat{\mu})(M/\mathfrak{n}M)_\nu = 0$ if $\nu \in W.\mu$ it follows that the dimension of $(M/\mathfrak{n}M)\mu$ is bounded by the length of $M(\hat{\mu})$ which is finite by (ii). Finally by multiplication we obtain a surjection of $\mathfrak{n}^i/\mathfrak{n}^{i+1} \times M/\mathfrak{n}M$ onto $\mathfrak{n}^i M/\mathfrak{n}^{i+1}M$ on $\mathfrak{h}$ modules. Since the left hand factor has finite dimensional weight spaces so has the right hand factor. This proves (iii). □

6.4.13. Recall the notation of 6.3.13. For any left $U(\mathfrak{b})$ module M consider M^* as a left $U(\mathfrak{b})$ module through the principal antiautomorphism (or antipode) σ.

Lemma. *Let M be a $U(\mathfrak{b})$ module satisfying $dim\ M/\mathfrak{n}^i M < \infty$, for all $i \in \mathbb{N}$. Then*

$$F_{\mathfrak{b}}(M^*) = \varinjlim (M/\mathfrak{n}^i M)^*.$$

Proof. Through the surjection $M \to M/\mathfrak{n}^i M$, we obtain an injection of the finite dimensional $\mathfrak{b}$ module $(M/\mathfrak{n}^i M)^*$ into M^*. Thus the right hand side belongs to $F_{\mathfrak{b}}(M^*)$. Conversely consider $\xi \in F_{\mathfrak{b}}(M^*)$. Set $V = U(\mathfrak{b})\xi$, which is finite dimensional by hypothesis, hence a direct sum of generalized $\mathfrak{h}$ weight subspaces. Since the weights of $\mathfrak{n}$ are all strictly positive one must have $\mathfrak{n}^i V = 0$, for i sufficiently large. In particular $\mathfrak{n}^i\xi = 0$. Yet $(M/\mathfrak{n}^i M)^*$ identifies with the set $\{\eta \in M^* \mid \mathfrak{n}^i\eta = 0\}$ and so contains ξ. □

6.4.14. Proposition. *Suppose $M \in Ob\hat{\mathcal{O}}$ satisfies $\bigcap \mathfrak{n}^i M = 0$. Then M is a direct sum of simple finite dimensional modules each occuring with finite multiplicity.*

Proof. Consider first a $Z(\mathfrak{g})$ primary component N of M. By 6.4.12 one has $N \in Ob\mathcal{O}_f$. In particular the weights of N lie in some cone $\mu - \mathbb{N}\pi$. Consequently the weights of

$F_{\mathfrak{b}}(N^*)$ lie in $-(\mu - \mathbb{N}\pi)$. Since $\mathfrak{n}^-$ has strictly negative weights it follows that for any weight vector $\xi \in F_{\mathfrak{b}}(N^*)$ one has $(\mathfrak{n}^-)^i\xi = 0$, for i sufficiently large. Thus $\mathfrak{n}^-$ acts locally finitely on $F_{\mathfrak{b}}(N^*)$. Since $\mathfrak{g} = \mathfrak{n}^- \oplus b$, this implies that $\mathfrak{g}$ acts locally finitely on $F_{\mathfrak{b}}(N^*)$. Moreover $F_{\mathfrak{b}}(N^*)$ coincides with $\delta(N)^\iota$, where $\iota = \kappa\sigma$. In particular $F_{\mathfrak{b}}(N^*)$ has finite length and its annihilator in $Z(\mathfrak{g})$ is just $\chi(\hat{\nu})$, where $\nu = -\mu$. Since $\mathfrak{g}$ is reductive and the action of $Z(\mathfrak{g})$ separates non-isomorphic simple finite dimensional modules, we conclude that $F_{\mathfrak{b}}(N^*)$ is a finite direct sum of isomorphic simple finite dimensional $U(\mathfrak{g})$ modules.

Since the functor $M \to F_{\mathfrak{b}}(M^*)$ commutes with direct sums we conclude that $F_{\mathfrak{b}}(M^*)$ is a direct sum of simple finite dimensional modules each occuring with finite multiplicity.

Finally the hypothesis $\bigcap \mathfrak{n}^i M = 0$ implies that the canonical map $M \to (\varinjlim(M/\mathfrak{n}^i M)^*)^*$ is injective. This also commutes with direct sums and so by the first part and 6.4.13 the required conclusion is obtained. $\square$

6.4.15. Recall the notation of 4.3.2 (iii). Take $L \in Ob\mathcal{O}_f$. One cannot expect that $A(L) \in Ob\hat{\mathcal{O}}$; but rather $A(L)^\iota \in Ob\hat{\mathcal{O}}$. We shall say that $M \in Ob\hat{\mathcal{O}}$ belongs to $\hat{\mathcal{O}}_S$ if there exists a finite set $S \subset \mathfrak{h}^*$ such that

$$\Omega(M^{\mathfrak{n}}) \subset S + \mathbb{N}\pi.$$

Lemma. *For all $L \in Ob\mathcal{O}_f$ one has $A(L)^\iota \in Ob\hat{\mathcal{O}}_S$.*

Proof. This follows from 6.4.6(∗). $\square$

6.5. The Surjectivity Theorem.

6.5.1. From now on we fix $L \in Ob\mathcal{O}_f$. Eventually we shall assume L simple and then our aim is to show that $F(L) = A(L)$ unless L is obtained from an induced module by translation. We shall often simply write $F = F(L)$ and $A = A(L)$.

6.5.2. Let $\mathfrak{u}$ denote the diagonal copy of $\mathfrak{n}^-$ acting on A. Then for any subspace $S \subset A$ we denote by $\mathfrak{u}(S)$ the linear span of $xs - sz$, $\forall\ x \in \mathfrak{n}^-, s \in S$. In particular $\mathfrak{u}(S) + \mathfrak{u}(S') = \mathfrak{u}(S + S')$ if S' is a second subspace.

Set $G = \bigcap\limits_{i=0} \mathfrak{u}^i A$. Clearly $\mathfrak{u}(G) = G$ and is the unique largest subspace of A with this property.

Lemma. *G is a two-sided ideal of A.*

Proof. Recall 6.4.6. The subspaces

$$A^{(n)} = \{a \in A \,\big|\, U^{n+1}(\mathfrak{u})a = 0\} : n \in \mathbb{N},$$

define a filtration of A. Obviously $A^{(0)} = A^{\mathfrak{u}}$ and $\mathfrak{u}A^{(n+1)} \subset A^{(n)}$. Clearly $A^{\mathfrak{u}}G \subset G$ and assume we have shown that $A^{(n)}G \subset G$. Then $A^{(n+1)}G = A^{(n+1)}\mathfrak{u}(G) \subset \mathfrak{u}(A^{(n+1)}G) + \mathfrak{u}(A^{(n+1)})G \subset \mathfrak{u}(A^{(n+1)}G) + G$. Iteration gives $A^{(n+1)}G \subset \mathfrak{u}^i(A^{(n+1)}G) + G = \mathfrak{u}^i(A^{(n+1)}G + G \subset \mathfrak{u}^i(A)$, for all i and so $A^{(n+1)}G \subset G$. Hence $AG \subset G$. Similarly $GA \subset G$. □

That $A(L)$ is a Goldie ring (6.4.11) has the following crucial consequence. Retain the notation of 6.5.1.

Corollary. (i) *If $G \neq 0$, then $F \cap G \neq 0$.*

(ii) *If $G = 0$, then $A = F$.*

Proof. (i) If $G \neq 0$, then being a two-sided ideal of A it contains a non-zero divisor 6.4.9. This forces $d_A(A/G) \leq d_A(A) - 1 < d_A(A)$. Yet $d_F(F/F \cap G) \leq d_F(A/G) \leq d_A(A/G)$, whilst $d_F(F) = d_A(A)$ by 6.4.7 since $U \subset F$. Combining these three inequalities we obtain $F \cap G \neq 0$.

(ii) results from 6.4.14 and 6.4.15. □

6.5.3. Although we have a well-developed structure theory of $\mathcal{O}_f$-algebras, this does not immediately apply to $A(L)$ since the latter is too big. The necessary cutting down is made by the following observation. Recall the notation of 6.4.15.

Lemma. *Let A be a $U(\mathfrak{g})$ module algebra with $A \in Ob\hat{\mathcal{O}}_S$. Then the subalgebra A_ϕ generated by the $Z(\mathfrak{g})$ primary component $A(\hat{0})$ is an $\mathcal{O}_f$-algebra.*

Proof. One has

$$\Omega(A(\hat{0})) \subset \bigcup_{w\in W} \Omega(L(w\rho - \rho)) \subset -\mathbb{N}\pi.$$

Since $\mathfrak{h}$ acts by derivations on $A(L)$ we conclude that

$$(*) \qquad \Omega(A_\phi) \subset -\mathbb{N}\pi.$$

On the other hand the hypothesis on A gives

$$(**) \qquad \Omega(A_\phi^{\mathfrak{n}}) \subset S + \mathbb{N}\pi,$$

for some $S \subset \mathfrak{h}^*$ finite. Together these force $\Omega(A_\phi^{\mathfrak{n}})$ to be a finite set. Through the Harish-Chandra isomorphism (4.3.7) we conclude that A_ϕ is a finite direct sum of its $Z(\mathfrak{g})$ primary components which by 6.4.12(iii) belongs to $\mathcal{O}_f$. □

Remark. Suppose L is generated over $U(\mathfrak{n}^-)$ by a weight vector v (for example if L is simple). Set $I = Ann_{U(\mathfrak{n}^-)}v$ and $N(I) = \{a \in U(\mathfrak{n}^-) \mid Ia \subset I\}$. Then the map $a \mapsto (bv \mapsto ba\ v)$ of $N(I)$ into $End_{\mathfrak{n}^-}L$ factors to an isomorphism $N(I)/I \xrightarrow{\sim} End_{\mathfrak{n}^-}L = A(L)^{\mathfrak{n}^-}$. In particular $\Omega(A(L)^{\mathfrak{n}^-}) \subset -\mathbb{N}\pi$. Thus for $A = A(L)^\iota$ in the lemma we can take $S = \{0\}$ forcing $\Omega(A_\phi^{\mathfrak{n}}) = \{0\}$. This means to $A_\phi = A(\hat{0})$. Such a result is not a priori obvious since $Z(\mathfrak{g})$ does not act by derivations.

6.5.4. Given $A = A(L)$ we shall write $((A^\iota)_\phi)^\iota$ simply as A_ϕ. If $A = F$, then A_ϕ reduces to scalars. However the converse, which we shall eventually need, can fail. To recover the converse we must replace L by a translate of itself, that is to say a simple submodule L' of some $L \otimes E$ with E finite dimensional. This process is really necessary since (as was pointed out to me by J.T. Stafford) both maps $End\ E \otimes F(M) \to F(E \otimes M)$ and $End\ E \otimes A(M) \to A(E \otimes M)$ are bijective. Consequently if $A(M) = F(M)$, then $A(N) = F(N)$ for every direct summand N of $E \otimes M$. Yet one can easily arrange (by translation to a wall) that N is properly induced but M is not.

The way to find the appropriate translate of L is provided by the following. Let $\mathcal{E}$ denote the set of simple finite dimensional $U(\mathfrak{g})$ modules. We shall say that $V \in Ob\mathcal{O}_f$

admits $E \in \mathcal{E}$ as a non-split submodule if some exact sequence

$$(*) \qquad 0 \longrightarrow E \xrightarrow{\psi} V \longrightarrow V/E \longrightarrow 0$$

does not split. Let $E_\phi \in \mathcal{E}$ denote the trivial module.

Lemma. *Suppose $V \in Ob\mathcal{O}_f$ admits $E \in \mathcal{E}$ as a non-split submodule, then $V \otimes E^*$ admits E_ϕ as a non-split submodule.*

Proof. Choose $\mu \in P^+(\pi)$ such that $E \cong L(\mu)$. Since $\delta M(\mu)$ is the injective hull of $V(\mu)$ in $\mathcal{O}_f$ we obtain a map $V \to \delta M(\mu)$ making the diagram

$$(**) \qquad \begin{tikzcd} 0 \arrow[r] & L(\mu) \arrow[r, "\psi"] \arrow[d] & V \arrow[dl, "\varphi"] \\ & \delta M(\mu) & \end{tikzcd}$$

commutative. Then $V' := Im\varphi \supsetneqq L(\mu)$, otherwise $Ker\ \varphi$ is a complement to $L(\mu)$, that is $(*)$ splits. Moreover $Soc\ V' \subset Soc\ \delta M(\mu) = L(\mu)$. We conclude that $L(\mu)$ cannot occur as a quotient of V', for otherwise it occurs as a subquotient of $\delta M(\mu)/L(\mu)$ which is impossible. (Indeed $\delta M(\mu)/L(\mu)$ admits no finite dimensional subquotients). Finally we can replace V by V'.

Now tensor $(**)$ by E^* and take the primary component corresponding to $\chi(\hat{0})$. By [ ] one has $(\delta M(\mu) \otimes L(\mu)^*)(\hat{0}) \cong \delta M(0)$, which has socle E_ϕ. Thus we obtain a commuting diagram

$$\begin{tikzcd} 0 \arrow[r] & E_\phi \arrow[r, "\psi'"] \arrow[d] & V' \otimes E^* \arrow[dl, "\varphi'"] \\ & \delta M(0) & \end{tikzcd}$$

Suppose $Im\varphi' = Soc\,\delta M(0) = E_\phi$. Then $\varphi' \in Hom_{U(\mathfrak{g})}(V' \otimes E^*, E_\phi) \cong Hom_{U(\mathfrak{g})}(V', L(\mu)) = 0$, which is absurd. Thus $V'' := Im\varphi' \supsetneqq E_\phi$. Since $Soc\ V'' \subset Soc\,\delta M(0) = E_\phi$, it admits E_ϕ as a non-split submodule. Hence so does $V \otimes E^*$. $\square$

6.5.5. A criterion for $M \in Ob\mathcal{O}_f$ to be properly induced (from a parabolic subalgebra) is provided by the following. Suppose $P := \delta M_{\pi'}(0)^\iota$ occurs as a $\mathcal{O}_f$ subalgebra of $A(M)$.

Recall that $(Soc\ \delta M_{\pi'}(0))^\iota$ is a trivial $\mathfrak{g}$ module and consists of multiples of the identity e of P (as an algebra).

Proposition. *Suppose e acts by a scalar on M (necessarily non-zero). Then there exists a $U(\mathfrak{p}_{\pi'})$ module N of M such that the map $x \otimes n \mapsto xn$ is an isomorphism of $Ind\ N := U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\pi'})} N$ onto M.*

Proof. Recall (6.3.6) that P is a polynomial algebra and let P_+ denote its augmentation. It is a $U(\mathfrak{p}_{\pi'})$ submodule of P. Set $N = \{m \in M \,\big|\, pm = 0,\ \forall\ p \in P_+\}$ which is a $U(\mathfrak{p}_{\pi'})$ submodule of M.

Let us further recall that P has polynomial generators p_γ of weight $\gamma \in \Delta^+ \setminus \Delta^+_{\pi'}$. Since $\Omega(M) \subset \mu - \mathbb{N}\pi$ for some $\mu \in \mathfrak{h}^*$, it follows that p_γ act locally nilpotently on M.

The (diagonal) action of $U(\mathfrak{g})$ on $P \subset A(M)$ restricts to an action of $U(\mathfrak{m}^-_{\pi'})$. By weight space considerations this action is locally nilpotent and by 6.3.8 one has $P^{\mathfrak{m}^-_{\pi'}} = ke$. (Of course $\mathfrak{m}, \mathfrak{m}^-$ are interchanged through the present use of ι.) Inverting the matrix $x_{-\gamma}(p_{\gamma'}) : \gamma, \gamma' \in \Delta^+ \setminus \Delta^+_{\pi'}$, which is triangular with e on the diagonal, we obtain the derivations $q_{-\gamma} := e\partial/\partial p_{-\gamma} : \gamma \in \Delta^+ \setminus \Delta^+_{\pi'}$ of P lying in the subalgebra A of $A(M)$ generated by P and the image U^- of $U(\mathfrak{m}^-_{\pi'})$. In particular A is a Weyl algebra with centre ke and of index $|\Delta^+ \setminus \Delta^+_{\pi'}|$.

Let Q denote the subalgebra of A generated by the $q_{-\gamma} : \gamma \in \Delta^+ \setminus \Delta^+_{\pi'}$. Through the definition of N, the nilpotent action of the p_γ on M, the scalar action of e and the Weyl algebra structure of A, a Taylor lemma argument shows that the map $q \otimes n \mapsto qn$ is an isomorphism of $Q \otimes N$ onto M.

In A we have the relations $x_{-\gamma}p_{\gamma'} - p_{\gamma'}x_{-\gamma} = x_{-\gamma}(p_{\gamma'}) \in P$, for all $\gamma, \gamma' \in \Delta^+ \setminus \Delta^+_{\pi'}$ which in particular forces $A = U^-P$ and even an isomorphism $U^- \otimes P \to A$ of vector spaces with $U^- \cong U(\mathfrak{m}^-_{\pi'})$. Combined with our previous observation, this gives an isomorphism $U(\mathfrak{m}^-_{\pi'}) \otimes N \xrightarrow{\sim} M$ of $U(\mathfrak{m}^-_{\pi'})$ modules forcing $M \cong Ind\ N$. □

Remarks. Let M be the image of a Verma module, that is to say a highest weight module. Then $End_{U(\mathfrak{g})}M$ reduces to scalars and the hypothesis of the proposition is satisfied.

The assertion and proof of the proposition are analogous for the corresponding criterion for a module of a finite group to be induced.

Conversely $A(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\pi'})} N)$ admits $(\delta M_{\pi'}(0))^\iota$ as a $U(\mathfrak{g})$ module subalgebra and so the former is distinct from $F(U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\pi'})} N)$.

6.5.6. From now on we assume $L \in Ob\mathcal{O}_f$ simple. Set $A = A(L), F = F(L)$. Suppose $A \neq F$. Then by 6.5.2 one has $F \cap G \neq 0$. It follows that there exists $E \in \mathcal{E}$ which is a direct summand of F occuring in G. Let V be an indecomposable direct summand of G containing E in its socle. From the definition of G the submodule E occurs as a non-split submodule of $V \subset A(L) \subset End\ L$. By 6.5.4 the trivial module E_ϕ occurs as a non-split submodule of $V \otimes E^* \subset End\ L \otimes E^*$. This is not quite what we want however. Rather we must find $E' \in \mathcal{E}$ such that E_ϕ occurs as a non-split submodule of $A(L \otimes E')$. This is achieved below.

Lemma. *Let $M \in Ob\mathcal{O}_f$ be simple with highest weight vector v. Let E be a simple submodule of $F(M)$ with zero weight space E_0. One has $E_0 v \neq 0$.*

Proof. Otherwise Ev is contained in the $\mathfrak{h}$ stable complement M^- of kv in M. Then $M^- \supset U(\mathfrak{n}^-)Ev = EU(\mathfrak{n}^-)v = EM$, forcing EM to be a proper submodule of M_1 hence 0. □

6.5.7. Take $E \in \mathcal{E}$. We claim that there exists $E' \in \mathcal{E}$ such that E occurs $dim\ E_0$ times in $End\ E'$. Actually there is a rather precise formula describing the number of occurrences of E in $Hom(E', E'') : E', E'' \in \mathcal{E}$. Suitably interpreted the result is valid for any symmetrizable Kac-Moody algebra $\mathfrak{g}$. It also has an analogue for the tensor product of crystals. Let us make a short digression to discuss this. Our method of proof will be valid both for modules over $U(\mathfrak{g})$ and its Drinfeld-Jimbo quantization.

6.5.8. Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. For each $\alpha \in \pi$ let $\mathfrak{s}_\alpha$ denote the subalgebra generated by $x_\alpha, x_{-\alpha}$.

Call a $\mathfrak{g}$ module V integrable if

(i) V is a direct sum of its $\mathfrak{h}$ weight subspaces.

(ii) $dim\ U(\mathfrak{s}_\alpha)v < \infty$, for all $\alpha \in \pi, v \in V$.

For example if $\mu \in P^+(\pi)$, then the simple highest weight module $L(\mu)$ is integrable. Moreover in this case $L(\mu)$ can be identified with the quotient $M(\mu)\Big/\sum_{\alpha\in\pi} M(s_\alpha.\mu)$. The proof (see for example,) requires the Casimir invariant whose existence depends on the Cartan matrix being symmetrizable. The general case is an open problem.

Take $v_\mu \in L(\mu)$ of weight μ. Then the above result implies that

$$I(\mu)^- := \sum_{\alpha\in\pi} U(\mathfrak{n}^-) x_{-\alpha}^{\alpha^\vee(\mu)+1}$$

is the annihilator of v_μ in $U(\mathfrak{n}^-)$.

Let $V(\mu)^\#$ denote the graded dual of $V(\mu)$, that is to say $F_{\mathfrak{h}}(V(\mu)^*)$, where $V(\mu)^*$ is viewed as a left $U(\mathfrak{g})$ module via the antipode σ. One has $V(\mu)^\# \cong (\delta V(\mu))^\iota$. One may also describe $V(\mu)^\#$ as the simple lowest weight module with lowest weight μ. Let $v_{-\mu} \in V(\mu)^\#$ be of weight $-\mu$ and set

$$I(\mu)^+ := \sum_{\alpha\in\pi} U(\mathfrak{n}^+) x_{\alpha}^{\alpha^\vee(\mu)+1}$$

which is just the annihilator of $v_{-\mu}$ in $U(\mathfrak{n}^+)$.

Let V be an integrable module. Given $\mu, \nu \in P^+(\pi)$ and $v \in V_{\mu-\nu}$, it follows from property (ii) and $\mathfrak{sl}(2)$ theory that

$$(*) \qquad Ann_{U(\mathfrak{n}^+)} v \supset I(\nu)^+ \Longleftrightarrow Ann_{U(\mathfrak{n}^-)} v \supset I(\mu)^-.$$

Let $V_{\mu-\nu}^{\mu,\nu}$ denote the subspace of $V_{\mu-\nu}$ of vectors satisfying these two equivalent conditions.

Proposition. *Let V be an integrable module of a symmetrizable Kac-Moody algebra $\mathfrak{g}$. Then for all $\mu, \nu \in P^+(\pi)$ there exists an isomorphism $v \mapsto \varphi_v$ of $V_{\mu-\nu}^{\mu,\nu}$ onto $Hom_{U(\mathfrak{g})}(V(\mu)\otimes V(\nu)^\#, V)$ satisfying $\varphi_v(v_\mu \otimes v_{-\nu}) = v$.*

Proof. Take $v \in V_{\mu,\nu}^{\mu-\nu}$. By definition of the latter there exists $\theta \in Hom(V(\nu)^\#, V)$ satisfying $\theta(av_{-\nu}) = av$, for all $a \in U(\mathfrak{n}^+)$. Under diagonal action θ is annihilated by $\mathfrak{n}^+$ and has weight μ. Thus by the universality of the Verma module $M(\mu)$ there exists a $U(\mathfrak{g})$ module map of $M(\mu)$ into $Hom(V(\nu)^\#, V)$ taking the canonical generator $\hat{v}_\mu$ of $M(\mu)$ to θ. Under Frobenius reciprocity this transforms to a map $\hat{\varphi}_v \in Hom_{U(\mathfrak{g})}(M(\mu)\otimes V(\nu)^\#, V)$ defined by $\hat{\varphi}_v(\hat{v}_\mu \otimes v_{-\nu}) = v$. Now $\mathfrak{n}^- v_{-\nu} = 0$, so $\hat{\varphi}_v(a\hat{v}_\mu \otimes v_{-\nu}) = \hat{\varphi}_v(a(\hat{v}_\mu \otimes v_{-\nu})) = av = 0$, if $a \in Ann_{U(\mathfrak{n}^-)} v_\mu$. Set $\overline{M(\mu)} = Ker(M(\mu) \to V(\mu))$. We conclude from the above that $\hat{\varphi}_v$ vanishes on $\overline{M(\mu)} \otimes v_{-\nu}$ and hence (through the action of $U(\mathfrak{n}^+)$) on $\overline{M(\mu)} \otimes V(\nu)^\#$. Consequently $\hat{\varphi}_v$ factors to the required map φ_v. Since $v_\mu \otimes v_{-\nu}$ generates $V(\mu)\otimes V(\nu)\#$, it follows that φ_v is injective. Conversely given $\varphi \in Hom(V(\mu)\otimes V(\nu)^\#, V)$, then $\varphi(v_\mu \otimes v_{-\nu}) \in V_{\mu-\nu}^{\mu,\nu}$ and so φ_v is surjective. □

6.5.9. Retain the above notation. Assume $\mu, \nu \in P^+(\pi)$ and let $I(\mu-\nu)$ denote the ideal of codimension 1 in $U(\mathfrak{h})$ which is the annihilator of a vector of weight $\mu-\nu$.

Corollary. $Ann_{U(\mathfrak{g})}(v_\mu \otimes v_{-\nu}) = U(\mathfrak{g})(I(\mu)^- + I(\nu)^+ + I(\mu-\nu))$.

Proof. The inclusion $\supset$ is obvious. Conversely the quotient of $U(\mathfrak{g})$ by the right hand side is a cyclic module V whose canonical generator v (that is the image of 1) has weight $\mu-\nu$ and satisfies both conditions of 6.5.8($*$). Thus V is integrable and $v \in V_{\mu,\nu}^{\mu-\nu}$. By 6.5.8, we obtain a surjection φ_v of $V(\mu)\otimes V(\gamma)^{\#}$ onto V satisfying $\varphi_v(v_\mu \otimes v_{-\nu}) = v$. This forces the opposite inclusion. $\square$

6.5.10. Let V be an integrable module generated by a vector of weight λ (for example if V is simple. Then one can clearly choose $\mu, \nu \in P^+(\pi)$ sufficiently large so that $\lambda = \mu - \nu$ and $v \in V_{\lambda-\mu}^{\lambda,\mu}$. Then V is an image of $V(\mu)\otimes V(\nu)^{\#}$. This is the best one can say about the classification of simple integrable modules in general.

One calls a weight module admissible if all its weight spaces are finite dimensional. Suppose V is simple and integrable. If $\mathfrak{g}$ is semisimple, then V is highest weight, hence V is admissible. For $\mathfrak{g}$ Kac-Moody or even just affine both conclusions fail. However if $\mathfrak{g}$ is affine and V has a zero weight space, there are some indications that V may be admissible. For example it is enough to show that V occurs as a subquotient of $U(\mathfrak{g})$ viewed as a $\mathfrak{g}$ module under adjoint action. For $\mathfrak{g}$ affine there is a classification theory of simple integrable admissible modules. Finally if V is admissible, then one may define its graded dual δV (as for the $\mathcal{O}$ category) and then 6.5.8 implies an isomorphism

$$(*) \qquad Hom_{U(\mathfrak{g})}(\delta V, Hom(V(\mu), V(\nu))) \xrightarrow{\sim} V_{\mu-\nu}^{\mu,\nu}.$$

For $\mathfrak{g}$ semisimple, this is completely obvious by finite dimensionality and reductivity.

6.5.11. Return from now on to the special case when $\mathfrak{g}$ is semisimple. Then any simple integrable module is finite dimensional and self-dual. Thus by 6.5.10($*$) (or 6.5.8) it follows that the number n_μ of copies of V in $End\ V(\mu)$ is just $dim\ V_0^{\mu,\mu}$. Since $dim\ V_0 < \infty$ one has $n_\mu = dim\ V_0$ for $\mu \in P^+(\pi)$ sufficiently large.

6.5.12. Fix $E \in \mathcal{E}$ and choose $E' \in \mathcal{E}$ such that $dim\ Hom_{U(\mathfrak{g})}(E, End\ E') = dim\ E_0$. Let $\{e_j\}$ be a basis for E_0 and $\{\varphi_i\}$ a basis for $Hom_{U(\mathfrak{g})}(E, End\ E')$. Let v' be a highest weight vector for E'. Then we may write $\varphi_i(e_j)v' = c_{i,j}v'$, for some $c_{i,j} \in k$.

Lemma. $det\ c_{i,j} \neq 0$.

Proof. Otherwise we can choose $\varphi \in Hom_{U(\mathfrak{g})}(E, End\ E')$ such that $\varphi(E_0)v' = 0$. By 6.5.6 this is impossible. $\square$

6.5.13. Return to the notation and hypothesis of 6.5.6. Suppose E occurs $F(L)$. Choose $E' \in \mathcal{E}$ such that E^* occurs with multiplicity $dim\ E_0^*$ in $End\ E'$. Let v (resp. v') be a highest weight vector of L (resp. E'). Let a be a generator of the unique copy of E_ϕ in $E \otimes E^*$.

Proposition. *There exists* $\varphi \in Hom_{U(\mathfrak{g})}(E^*, End\ E')$ *such that* $(1d \otimes \varphi)(a)(v \otimes v') \neq 0$.

Proof. Let $\{b_\xi^j\}$ be a basis for E_ξ and $\{c_{-\xi}^j\}$ a dual basis for $E_{-\xi}^*$. Then we may write

$$a = \sum_{\xi \in \Omega(E)} \sum_j b_\xi^j \otimes c_{-\xi}^j.$$

Consequently

$$(1d \otimes \varphi)(a)(v \otimes v') = \sum_i b_0^j v \otimes \varphi(c_0^j)v'.$$

If the above expression is zero for all $\varphi \in Hom_{U(\mathfrak{g})}(E^*, End\ E')$, then by 6.5.12 we obtain $E_0 v = 0$, which by 6.5.6 is impossible. $\square$

6.5.14. Recall that our aim is to show that if $F(L) \neq A(L)$, then L can be obtained by translation and induction. To be precise this means that there exists a proper subset $\pi' \subset \pi$, a simple highest weight $\mathfrak{r}_{\pi'}$ module L' viewed as a $\mathfrak{p}_{\pi'}$ module through the trivial action of $\mathfrak{m}_{\pi'}^+$, and $E \in \mathcal{E}$ such that L is a submodule of $E \otimes Ind\ L'$. Should $Ind\ L'$ be simple, this also means that it occurs as a submodule of $E^* \otimes L$. In this case we say that L is not rigid.

In general induction does not preserve simplicity; but the following result of Jantzen allows us to choose L' so that $Ind\ L'$ is simple. The proof is much more rudimentary than that of Jantzen, gives much less, but is sufficient for our purposes. It applies to the affine and quantum cases also to which Jantzen's analysis has not yet been extended.

Set ${\pi'}^\perp = \{\mu \in \mathfrak{h}^* \mid \alpha^\vee(\mu) = 0,\ \forall\ \alpha \in \pi'\}$. Let $L_{\pi'}(\lambda) : \lambda \in \mathfrak{h}^*$ denote the simple $\mathfrak{r}_{\pi'}$ module with highest weight λ, viewed as $\mathfrak{p}_{\pi'}$ module as above. For all $\nu \in {\pi'}^\perp$ let k_ν denote the one-dimensional $\mathfrak{p}_{\pi'}$ module of weight ν. Then $L_{\pi'}(\lambda) \otimes k_\nu$ identifies with $L_{\pi'}(\lambda + \nu)$.

Lemma. *For all* $\nu \in \pi' \cap P^+(\pi)$ *sufficiently large,* $Ind\ L_{\pi'}(\lambda - \nu)$ *is simple.*

Proof. It suffices to show that for such ν only the canonical generator $v_{\lambda-\nu}$ of $L_{\pi'}(\lambda-\nu)$ is a highest weight vector. Let $v_{\lambda-\nu-\beta}$ be a second highest weight vector. Then $v_{\lambda-\nu-\beta}$ cannot belong to $L_{\pi'}(\lambda-\nu)$ since the latter is already simple. This forces β to be a non-zero element of $\mathbb{N}(\Delta^+ \setminus \Delta^+_{\pi'})$ modulo $\mathbb{N}\pi'$. Consequently there exists $\alpha \in \pi \setminus \pi'$ such that $(\omega_\alpha, \beta) > 0$ (where ω_α denotes the fundamental weight corresponding to α). On the other hand the Casimisr invariant must act by the same scalar on $v_{\lambda-\nu}$ and on $v_{\lambda-\nu-\beta}$ which gives $(\lambda-\nu+\rho, \lambda-\nu+\rho) = (\lambda-\nu-\beta+\rho, \lambda-\nu-\beta+\rho)$, hence $2(\lambda-\nu+\rho, \beta) = (\beta, \beta) \geq 0$. By taking the coefficient of ω_α in ν to be sufficiently large, we can force the left hand side to be strictly negative giving the required contradiction. □

6.5.15. Retain the notation of 6.5.13; but let us simply use a to denote its image under $1d \otimes \varphi$ with φ as in the conclusion of 6.5.13. Set $M = U(\mathfrak{g})(v \otimes v') \subset T := L \otimes E'$. Then a acts by a non-zero scalar c on M. Moreover if $F \neq A$ we can assume (see 6.5.6) that E occurs a a non-split submodule of $A(L)$ and then $E_\phi = ka$ occurs (6.5.4) as a non-split submodule of $A' := A(T)$ and hence of A'_ϕ, which we recall is a direct summand of A'. Let $\mathcal{O}_f^-$ denote the category of $U(\mathfrak{g})$ modules obtained from $\mathcal{O}_f$ by transport under ι.

Theorem. *Take $L \in Ob\mathcal{O}_f$ simple. If $A(L) \neq F(L)$, then L is not rigid, that is to say there exists a subset $\pi' \subsetneq \pi$, a simple highest weight $\mathfrak{p}_{\pi'}$ module V and $E \in \mathcal{E}$ such that $Ind\ V$ is simple and L is a submodule of $E \otimes Ind\ V$.*

Proof. By 6.5.3, A'_ϕ is an $\mathcal{O}_f^-$ subalgebra of A', hence in a p.i. ring. Let N denote its prime or nilradical, which by 6.2.5 is nilpotent and a $\mathfrak{g}$ submodule. Let $\bar{a}$ denote the image of a in A'_ϕ/N.

By 6.3.14(i), A'_ϕ/N is a finite direct sum of prime $\mathcal{O}_f^-$ algebras $A_i : i = 1, 2, \cdots, s$. Let $e_i \in A_i$ be the projector onto A_i. It is central and $\mathfrak{g}$ invariant (see 6.3.10).

Since N is nilpotent we can choose $t \in \mathbb{N}$ such that $M' := M \cap N^t T \neq 0$, and $M \cap N^{t+1}T = 0$. By construction we obtain an embedding $M' \hookrightarrow N^tT/N^{t+1}T$ of $U(\mathfrak{g})$ modules, whilst the right hand factor is also a A'_ϕ/N module. With respect to this embedding choose $i \in \{1, 2, \cdots, s\}$ such that $e_iM' \neq 0$ and set $a_i = e_i\bar{a}$. Now a acts by the non-zero scalar c on M' and so a_i acts by c on e_iM'. This forces

$$(*) \qquad ka \cap (N + ker\ e_i) = 0.$$

Moreover the image of a in A_i is just a_i.

Now recall that ka is a non-split $\mathfrak{g}$ submodule of A'_ϕ isomorphic to the trivial module. This means that we have an indecomposable submodule I of A'_ϕ such that $ka = Soc\ I \subsetneq I$. By $(*)$ we obtain an embedding $I \hookrightarrow A_i$ of $\mathfrak{g}$ modules. Under this embedding $ka_i = Soc\ I \subsetneq I$.

By 6.3.14(ii), there exists $\pi' \subset \pi$ and a finite dimensional $\mathfrak{r}_{\pi'}$ module E'' such that

$$A_i = End\ E'' \otimes (\delta M_{\pi'}(0))^\iota.$$

Suppose $\pi' = \pi$. Then E'' is a $\mathfrak{g}$ module and $A_i^{\mathfrak{g}} = (End\ E')^{\mathfrak{g}}$ is a direct summand of A_i. This contradicts the fact that ka_i is a non-split submodule of A^i. Hence $\pi' \subsetneq \pi$.

Now consider $M'' := e_i(N^tT/N^{t+1}T)$ which is a $\mathfrak{g}$ module and non-zero since it contains e_iM'. It is an A_i module, hence an $(\delta M_{\pi'}(0))^\iota$ module whose socle is just ke_i, which acts by multiples of the identity on M''. Hence by 6.5.5 there exists a $\mathfrak{p}_{\pi'}$ module S such that $M'' = Ind\ S$. Now $e_iM' \in Ob\mathcal{O}_f$ so has finite length. Let L' be a simple submodule of e_iM' and hence of M''. Of course M'' has finite length since it is a subquotient of T and so S must have finite length since Ind is exact and faithful. Choose a maximal submodule S'' of S such that the image of L' in $Ind\ S$ has null intersection with $Ind\ S''$. Since L' is simple it embeds in any submodule in $Ind\ S'''$ for any submodule S''' of S strictly containing S''. Choosing $V' := S'''/S''$ simple, we conclude that L' embeds in $Ind\ V'$. Now by 6.5.14 we can choose $\nu \in {\pi'}^\perp \cap P^+(\pi)$ such that $L'' := Ind(V' \otimes k_{-\nu})$ is simple. Yet $Ind\, V' = Ind(V' \otimes k_{-\nu} \otimes k_\nu) \hookrightarrow Ind(V' \otimes k_{-\nu} \otimes L(\nu)) \cong L(\nu) \otimes Ind(V' \otimes k_{-\nu}) = L(\nu) \otimes L''$, by [ ]. Moreover $L(\nu) \in \mathcal{E}$, since $\nu \in P^+(\pi)$. We conclude L' embeds in $E''' \otimes L''$, for some $E''' \in \mathcal{E}$. On the other hand L' is a submodule of e_iM' which is in turn a submodule of $T = L \otimes E'$. Through Frobenius reciprocity we obtain an embedding $L \hookrightarrow L' \otimes (E')^* \hookrightarrow L'' \otimes E''' \otimes (E')^*$. Hence the conclusion of the theorem results with $E = E''' \otimes (E')^*$ and $V = S' \otimes k_{-\nu}$. □

Remark. Conjecturally the converse holds. For the moment the best we can do is the following. Let M be a $U(\mathfrak{g})$ module and suppose that there exists $E \in \mathcal{E}, \pi' \subsetneq \pi$ and a $U(\mathfrak{p}_{\pi'})$ module such that $U(\mathfrak{g}) \otimes_{U(\mathfrak{p}_{\pi'})} V$ is a direct summand of $E \otimes M$. Then by 6.5.4 and the remarks in 6,5.5 one has $A(M) \supsetneq F(M)$.

6.6. Associated Varieties and Maximal Ideals.

6.6.1. Take $L \in Ob\mathcal{O}_f$ simple. We should like to turn the conclusion of 6.5.15 into a criterion for $A(L) = F(L)$ which will be a little weaker; but much more useful. This is provided by studying the associated variety $\mathcal{V}(L)$ of L as introduced in 5.3.5. Roughly speaking we show that $A(L)=F(L)$ unless $\mathcal{V}(L)$ is properly induced.

6.6.2. Let us start in a fairly general context. Let $(A,\mathcal{F})$ be a filtered associative algebra in the sense of 2.2.1. In particular for all $i,j\in\mathbb{Z}$ one has $(\mathcal{F}^iA)(\mathcal{F}^jA)\subset\mathcal{F}^{i+j}A$ and this gives

$$\mathcal{A}(\mathcal{F})=\bigoplus_{i\in\mathbb{Z}}\mathcal{F}^iA$$

the structure of a graded algebra. It is called the Rees algebra of the pair $(A,\mathcal{F})$. Its immediate application is to recover the associated graded algebra $gr_{\mathcal{F}}A$ as a quotient. For this we must assume that A admits an identity 1 lying in $\mathcal{F}^oA$. We denote by t the image of 1 under the embedding $\mathcal{F}^oA\hookrightarrow\mathcal{F}'A$ *viewed as an element if* $\mathcal{A}(\mathcal{F})$. Obviously t is central and $gr_{\mathcal{F}}A=\mathcal{A}(\mathcal{F})/\mathcal{A}(\mathcal{F})t$.

6.6.3. Let $(M,\mathcal{F}')$ be a filtered module for the filtered algebra $(A,\mathcal{F})$. Then

$$\mathcal{M}(\mathcal{F}')=\bigoplus_{i\in\mathbb{Z}}\mathcal{F}'^iM$$

is graded module for the Rees algebra $\mathcal{A}(\mathcal{F})$. Observe that $\mathcal{M}(\mathcal{F}')$ being finitely generated over $\mathcal{A}(\mathcal{F})$ exactly means there exists a finite subset $\{m_i\}_{i=1}^r$ of M and integers $s_i: i=1,2,\cdots,r$ such that

$$(*)\qquad \mathcal{F}'^nM=\sum_{i=1}^r(\mathcal{F}'^{n-s_i}A)m_i,\quad for\ all\quad n\in\mathbb{Z}.$$

In particular M itself is finitely generated (by the m_i). Conversely if M is finitely generated, then $(*)$ defines a filtration $\mathcal{F}'$ on M making $\mathcal{M}(\mathcal{F}')$ finitely generated. For the latter it is often convenient to take $s_i=0$, for all i (as in 5.3.5). A filtration satisfying $(*)$ is said to be of finite type (or simply, finite). We shall often simply write $\mathcal{A}(\mathcal{F})$ (resp. $\mathcal{M}(\mathcal{F}')$) as $\mathcal{A}$ (resp. $\mathcal{M}$).

Lemma. *Let $(A,\mathcal{F})$ be a filtered algebra and M an A module. The prime radical of $Ann_{gr_{\mathcal{F}}A}(gr_{\mathcal{F}'}M)$ is independent of the finite filtration $\mathcal{F}'$ on M.*

Proof. Let $\mathcal{F}',\mathcal{F}''$ be filtrations on M of finite type. Considering generators and their degrees one obtains positive integers r,s such that

$$\mathcal{F}''^nM\subset\mathcal{F}'^{n+r}M\subset\mathcal{F}''^{n+r+s}M,\ \forall\, n\in\mathbb{Z}.$$

In terms of the element $t \in \mathcal{A}$ defined previously and setting $\mathcal{M}' = \mathcal{M}(\mathcal{F}'), \mathcal{M}'' = \mathcal{M}(\mathcal{F}'')$ this gives

$$t^{r+s}\mathcal{M}'' \subset t^s\mathcal{M}' \subset \mathcal{M}''.$$

Now $\mathcal{A}/t\mathcal{A} = gr_{\mathcal{F}}A$, whilst $\mathcal{M}'/t\mathcal{M}' = gr_{\mathcal{F}'}M$ and $\mathcal{M}''/t\mathcal{M}'' = gr_{\mathcal{F}''}M$. If say $s = 0,\ r = 1$, then $gr_{\mathcal{F}''}M$ has a filtration with factors $\mathcal{M}''/\mathcal{M}'$ and $\mathcal{M}'/t\mathcal{M}''$, whilst $gr_{\mathcal{F}'}M$ has a filtration with factors $\mathcal{M}'/t\mathcal{M}''$ and $t\mathcal{M}''/t\mathcal{M}' \cong \mathcal{M}''/\mathcal{M}'$. One may reduce to this case and the case $s = 1,\ r = 0$, by taking intermediate filtrations (for example, ${\mathcal{F}'''}^n M := {\mathcal{F}''}^n M + {\mathcal{F}'}^n M$). Thus we may conclude that $gr_{\mathcal{F}'}M,\ gr_{\mathcal{F}''}M$ admit finite filtrations with factors isomorphic up to permutation. Consequently the set of primes containing $Ann_{gr_{\mathcal{F}}A}(gr_{\mathcal{F}'}M)$ and $Ann_{gr_{\mathcal{F}}A}(gr_{\mathcal{F}''}M)$ coincide. This proves the lemma. □

6.6.4. For the moment set $S = gr_{\mathcal{F}}A,\ N' = gr_{\mathcal{F}'}M$. If S is commutative, then the localization S_P of S at $P \in Spec\ S$ is defined and $S \to S_P$ is an exact functor. Moreover (assuming $\mathcal{F}'$ of finite type) N' is finitely generated over S and so the length ℓ_P of $N'_P := S_P \otimes SN$ is finite. The proof of the lemma gives the refinement that $\ell_P(gr_{\mathcal{F}'}M)$ is independent of the finite filtration chosen. This can be of great importance to comparing Goldie rank to geometric data.

6.6.5. Now assume that $gr_{\mathcal{F}}A$ is commutative. In this case for any ideal I of $gr_{\mathcal{F}}A$ the prime and nilradicals of I coincide [, 1.14] and both will be denoted by $\sqrt{I}$. In this notation 6.6.3 expresses the independence of $\sqrt{Ann_{gr_{\mathcal{F}}A} gr_{\mathcal{F}'}M}$ on the finite filtration $\mathcal{F}'$ of M.

Now suppose M is a cyclic A module with cyclic generator m. For the finite filtration ${\mathcal{F}'}^n M := (\mathcal{F}^n A)n$ one has $gr_{\mathcal{F}}Ann_A m = Ann_{gr_{\mathcal{F}}A}(gr_{\mathcal{F}'}m) = Ann_{gr_{\mathcal{F}}A}(gr_{\mathcal{F}'}M)$, since $gr_{\mathcal{F}'}m$ is a generator of $gr_{\mathcal{F}'}M$ over the commutative algebra $gr_{\mathcal{F}}A$. Then by 6.6.3 we conclude that $\sqrt{gr_{\mathcal{F}}Ann_A m}$ is independent of the cyclic generator m.

More generally if M has a finite set of generators $m_1, m_2, \cdots, m_t$, then similarly $\sqrt{\bigcap_{i=1}^t gr_{\mathcal{F}}Ann_A m_i}$ is independent of their choice. However we can immediately say that this equals $\sqrt{gr_{\mathcal{F}}(\bigcap_{i=1}^t Ann_A m_i)}$. For this we need some noetherianity.

From now on we assume $\mathcal{F}^{-1}A = 0$. Below "noetherian" means with respect to left (or right) ideals.

Proposition. $(\mathcal{F}^{-1}A = 0)$. *The following three conditions are equivalent .*

(i) $gr_{\mathcal{F}}A$ *is noetherian.*

(ii) $\mathcal{A}(\mathcal{F})$ *is noetherian for graded ideals.*

(iii) $\mathcal{A}(\mathcal{F})$ *is noetherian.*

Proof. (iii) $\Longrightarrow$ (i) trivially. Set $B = gr_{\mathcal{F}}A$ which we assume noetherian. By Hilbert's basis theorem $B[x]$ is noetherian. Write $\mathcal{A} = \mathcal{A}(\mathcal{F})$ and recall that $B = \mathcal{A}/\mathcal{A}t$. Then the map $\bar{a} \mapsto \bar{a}t^s$ of $(\mathcal{A}/\mathcal{A}t)x^s$ onto $\mathcal{A}t^s/\mathcal{A}t^{s+1}$ extends linearly to a surjection of $B[x]$ onto $C := \oplus\mathcal{A}t^s/\mathcal{A}t^{s+1}$. Hence C is noetherian.

Let I be a graded ideal of $\mathcal{A}$. Then $J := \oplus J_s$, where $J_s = (I \cap \mathcal{A}t^s)/(I \cap \mathcal{A}t^{s+1})$ is a graded ideal of C. Let $\{a_{i,s}\}$ be a finite system of generators of J with $a_{i,s} \in J_s$. Choose $b_{i,s} \in I \cap \mathcal{A}t^s$ homogeneous such that $b_{i,s} - a_{i,s} \in I \cap \mathcal{A}t^{s+1}$ and let I' be the graded ideal they generate. Take $b \in I$ homogeneous of degree n. Then for each $m \in \mathbb{N}$ we may find $a_m \in I'$ homogeneous of degree n such that $b - a_m \in I \cap \mathcal{A}t^m$. Taking $m > n$, forces $b = a_m$. This establishes that (i) $\Longrightarrow$ (ii). That (ii) $\Longrightarrow$ (iii) is just a general fact about $\mathbb{Z}$ graded rings noted below. $\square$

6.6.6. Let $A = \bigoplus_{i\in\mathbb{Z}} A_i$ be a graded ring. Given $a \in A$ we can write $a = \sum_{i=s}^{s+\ell} a_i$, with $a_s, a_{s+\ell} \neq 0$. We call a_s the principal term of a and ℓ the length of a.

Lemma. *A is noetherian if it is noetherian for graded ideals.*

Proof. Let I be an ideal of A. For each integer $r \geq 0$, let $I(r)$ be the graded ideal of A formed from the principal elements of I of length $\leq r$. Clearly $I(0) \subset I(1) \subset \cdots$, which by the hypothesis is stationary. Choose $s \in \mathbb{N}$ such that $I(s) = I(s+t)$, for all $t \in I$. For each $I(t) : t \leq r$ choose a set $\{a_{t,1}, \cdots, a_{t,i_t}\}$ of generators of $I(t)$ and choose $b_{t,s} \in I$ of length $\leq t$ such that $a_{t,s}$ is the principal term of $b_{t,s}$. Induction on length shows that these latter elements generate I. $\square$

6.6.7. Now assume that $gr_{\mathcal{F}}A$ is commutative, finitely generated and that $\mathcal{F}^{-1}A = 0$. This is the case if $A = U(\mathfrak{g}) : \mathfrak{g}$ finite dimensional, given its canonical filtration $\mathcal{F}$. Then $gr_{\mathcal{F}}U(\mathfrak{g})$ identifies with $S(\mathfrak{g})$, which further identifies with the algebra of regular functions on $\mathfrak{g}^*$. More generally we can view $gr_{\mathcal{F}}A$ as the algebra of regular functions on an affine algebraic variety X. Thus for any ideal I of $gr_{\mathcal{F}}A$ we can define its set $v(I)$ of zeros in X.

Under these hypotheses $gr_{\mathcal{F}}A$ is noetherian and by 6.6.5 so is $\mathcal{A}(\mathcal{F})$. This has the following immediate, but important, consequence. Given an A module M with a finite filtration $\mathcal{F}'$, then the filtration ${\mathcal{F}'}^n N := ({\mathcal{F}'}^n M) \cap N$ induced on a submodule N is again finite. This is also trivially true for the filtration ${\mathcal{F}'}^n (M/N) := ({\mathcal{F}'}^n M + N)/N$ induced on the quotient.

Now for each $P \in Spec(gr_{\mathcal{F}} A)$, let $\ell_P(M)$ denote the length of $(gr_{\mathcal{F}} A)_P \otimes_{gr_{\mathcal{F}} A}$ $(gr_{\mathcal{F}'} M)$, which by 6.6.3 is independent of the finite filtration $\mathcal{F}'$ on M.

Lemma. *ℓ_P is additive on exact sequences.*

Proof. Let

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

be an exact sequence of finitely generated A modules. Fix a finite filtration $\mathcal{F}'$ on M_2 and let $\mathcal{F}'$ on M_1 (resp. on M_3) be the induced filtrations. Then

$$0 \to gr_{\mathcal{F}'} M_1 \to gr_{\mathcal{F}'} M_2 \to gr_{\mathcal{F}'} M_3 \to 0$$

is an exact sequence of $gr_{\mathcal{F}}$ modules. Since localization is exact, lengths add as required. □

6.6.8. We now obtain the result announced in 5.3.5.

Corollary. *Let $(A, \mathcal{F})$ be an associative filtered algebra with $\mathcal{F}^{-1} A = 0$ and such that $gr_{\mathcal{F}} A$ is commutative and finitely generated. Let M be a finitely generated A module. Then $\mathcal{V}(M) := \mathcal{V}(gr_{\mathcal{F}} Ann_A W)$ is independent of the finitely dimensional generating subspace W of M chosen.*

Proof. Via 6.6.7 the proof reduces to the case of a cyclic module which was established in 6.6.5. □

6.6.9. Return to the case when $\mathfrak{g}$ is semisimple and recall the notation of 6.5.4.

Lemma. *Let L, L' be simple $U(\mathfrak{g})$ modules. If there exists $E \in \mathcal{E}$ such that L is a submodule of $E \otimes L'$, then $\mathcal{V}(L) = \mathcal{V}(L')$.*

Proof. We claim that $\mathcal{V}(M) = \mathcal{V}(E \otimes M)$ for any finitely generated $U(\mathfrak{g})$ module M. Let $\mathcal{F}$ be the canonical filtration of $U(\mathfrak{g})$. Choose a finite dimensional generating subspace W of M. Then $E \otimes W$ is a finite dimensional generating subspace of $E \otimes M$. Define a finite filtration on $E \otimes M$ (resp. M) through $\mathcal{F}^n(E \otimes M) = \mathcal{F}^n(U(\mathfrak{g}))(E \otimes W) = E \otimes \mathcal{F}^n(U(\mathfrak{g}))W$

(resp. $\mathcal{F}^n M = \mathcal{F}^n(U(\mathfrak{g}))W$). Take $a \in \mathcal{F}^n(U(\mathfrak{g}))$. Through the coproduct we obtain $a(e \otimes w) = e \otimes aw \ mod \ E \otimes \mathcal{F}^{n-1}(U(\mathfrak{g}))W$. We conclude that $gr_{\mathcal{F}}(M \otimes E)$ is isomorphic to $gr_{\mathcal{F}} M \otimes E$, with E given the trivial $S(\mathfrak{g}) = gr_{\mathcal{F}} U(\mathfrak{g})$ module structure (and using the coproduct on $S(\mathfrak{g})$). Thus $Ann_{S(\mathfrak{g})}(gr_{\mathcal{F}}(W \otimes E)) = Ann_{S(\mathfrak{g})} gr_{\mathcal{F}} W$, which establishes the claim.

The embedding $L \mapsto E \otimes L'$ gives (6.6.7) an inclusion $\mathcal{V}(L) \subset \mathcal{V}(E \otimes L') = \mathcal{V}(L')$. Yet by Frobenius reciprocity L' is a submodule of $E^* \otimes L'$ which gives the opposite inclusion.

$\square$

6.6.10. Let M be a $U(\mathfrak{g})$ module. From its definition $\mathcal{V}(M)$ is a closed subvariety of $\mathfrak{g}^*$. However there are a number of much deeper properties of $\mathcal{V}(M)$ which we shall treat in subsequent chapters. For example we have already noted Gabber's theorem (5.3.5) that $\mathcal{V}(M)$ is involutive. A further result of Gabber is that $\mathcal{V}(M)$ is equidimensional if M is $d_{U(\mathfrak{g})}$-homomorphism (definition 5.3.5). This holds for example if M is simple or a primitive quotient $U(\mathfrak{g})/P$ of $U(\mathfrak{g})$ viewed as a left $U(\mathfrak{g})$ module. This latter result eventually implies that $\mathcal{V}(U(\mathfrak{g})/P)$ (which we shall write simply as $\mathcal{V}(P)$) is irreducible and hence the closure nilpotent orbit $\mathcal{O}_P$ (5.3.13). Thus for example if M is simple and holonomic (5.3.12) then $\mathcal{V}(M)$ is the closure of a Lagrangian subvariety of $\mathcal{O}_{Ann\ M}$, equidimensional but *not* necessarily irreducible.

There is a natural definition of an induced orbit (in $\mathfrak{g}^*$) and this plays a major role in the decomposition of $\mathfrak{g}$ into sheets and the subsequent parametrization of nilpotent orbits. This will be discussed in a later chapter. It leads to the notion of an induced (Lagrangian) subvariety of a coadjoint orbit. This goes as follows.

Fix a parabolic subalgebra $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form (so that $\mathfrak{m}^*$ identifies with $\mathfrak{m}^-$ and $\mathfrak{r}^*$ with $\mathfrak{r}$). Let $\mathcal{O}'$ be a coadjoint orbit in $\mathfrak{r}$ and $\mathcal{L}'$ a (Lagrangian) subvariety of $\mathcal{O}'$, that is to say an involutive subvariety of dimension $\frac{1}{2}\ dim\mathcal{O}'$. We shall define $Ind\ \mathcal{L}'$ and $Ind\ \mathcal{O}'$ so that their closures become $\overline{(\mathcal{L}' + \mathfrak{m})}$ and $\overline{G(\mathcal{L}' + \mathfrak{m})} = \overline{G(\mathcal{O}' + \mathfrak{m})}$, respectively (notation 5.3.5). Observe that $\overline{G(\mathcal{O}' + \mathfrak{m})}$ is irreducible, since $\mathcal{O}' + \mathfrak{m}$ is irreducible and G is connected. On the other hand any invariant function on $\mathfrak{g}^*$ takes the same value on $G(\mathcal{O}' + \mathfrak{m})$ as its restriction to $\mathcal{O}'$, which is a scalar. Hence $\overline{G(\mathcal{O}' + \mathfrak{m})}$ is a finite union of G orbits and so admits a unique dense orbit which we shall denote by $Ind\ \mathcal{O}'$. We set $Ind\ \mathcal{L}' = \overline{(\mathcal{L}' + \mathfrak{m})} \cap Ind\ \mathcal{O}'$, which is dense in $(\mathcal{L}' + \mathfrak{m})$. In general $Ind\ \mathcal{L}'$ is different to $\mathcal{L}' + \mathfrak{m}$; but for the moment this subtlety will not be of any concern, since we are only interested in closures.

6.6.11. Suppose $L \in Ob\mathcal{O}_f$. Since $\mathfrak{b}$ acts locally finitely on L and $\mathfrak{b}^\perp$ identifies with $\mathfrak{n}$ through the Killing form, it follows that $\mathcal{V}(L) \subset \mathfrak{n} \cap G\mathcal{V}(L) \subset \mathfrak{n} \cap \mathcal{N}$ (notation 5.3.13). Since $\mathcal{N}$ is a finite union of nilpotent orbits, it follows that any irreducible component $\mathcal{V}$ of $\mathcal{V}(L)$ lies in some $\mathfrak{n} \cap \mathcal{O}$, with $\mathcal{O}$ a nilpotent orbit. Consequently $dim\ \mathcal{V} \leq dim(\mathfrak{n} \cap \mathcal{O}) = \frac{1}{2}\ dim\mathcal{O}$ (c.f. 5.3.13). On the other hand $\mathcal{V}$ is involutive in $\mathcal{O}$ so the opposite equality also holds. One concludes the $\mathcal{V}(L)$ is a finite union of Lagrangian varieties of nilpotent orbits. However even if L is simple it need not be irreducible (equivalently the closure of single irreducible Lagrangian variety).

The irreducible Lagrangian subvarieties of $\mathfrak{n} \cap \mathcal{O}$ are called orbital varieties. Rather a lot is known about them and this will be discussed more fully in a later chapater.

A rather simple condition for an orbital variety to not be properly induced can be described as follows. First for any orbital variety $\mathcal{O}$ there exists a unique maximal subset $\pi' \subset \pi$ such that $\mathcal{V} \subset \mathfrak{m}_{\pi'}$. This definition also makes sense for any union of orbital varieties. Usually π' is easy to compute and is called the τ-invariant $\tau_{\mathcal{V}}$ of $\mathcal{V}$. For example if $L \in Ob\mathcal{O}_f$ is locally finite with respect to some parabolic $\mathfrak{p}_{\pi'}$ then $\mathcal{V}(L) \subset \mathfrak{p}_{\pi'}^\perp = \mathfrak{m}_{\pi'}$ and so $\tau_{\mathcal{V}(L)} \supset \pi'$. Furthermore if π' is maximal with respect to the former property, then equality holds.

Now suppose $\pi' := \tau_{\mathcal{V}}$ is of cocardinality 1 in π. If $\mathcal{V}$ is properly induced, then $\mathcal{V} \supset \mathfrak{m}_{\pi''}$, for some $\pi'' \subsetneq \pi$. This forces $\tau_{\mathcal{V}} = \pi''$ and that $\pi'' = \pi'$. Combining 6.5.15, 6.6.9 and the above we obtain the following

Theorem. *Suppose $L \in Ob\mathcal{O}$ is simple and locally finite with respect to some proper maximal parabolic subalgebra $\mathfrak{p}_{\pi'}$. Then either $\mathcal{V}(L) = \mathfrak{m}_{\pi'}$ or $A(L) = F(L)$.*

6.6.12. A particularly interesting case occurs when $\mathfrak{m}_{\pi'}$ is commutative (which requires $\mathfrak{p}$ to be maximal). The hypothesis that L is locally $\mathfrak{p}$ finite and implies

$$A(L) := F_{\mathfrak{n}^-}(End\ L) = F_{\mathfrak{m}^-_{\pi'}}(End\ L).$$

Moreover this further implies that L is generated over $U(\mathfrak{g})$ by a finite dimensional simple $\mathfrak{p}_{\pi'}$ module V. By universality of induction, it follows that L is a quotient of $Ind\ V$. Now suppose that we can choose V to be one dimensional. Then as a $U(\mathfrak{m}^-)$ module L is isomorphic to $U(\mathfrak{m}^-)/I$ for some ideal I of $U(\mathfrak{m}^-)$. If $\mathfrak{m}$ (and hence $\mathfrak{m}^-$) is commutative, we can identify $U(\mathfrak{m}^-)$ with $S(\mathfrak{m}^-)$ and then I is just the ideal of definition of $\mathcal{V}(L) \subset \mathfrak{m}$. Moreover in this case $A(L)$ becomes exactly the algebra $\mathcal{D}(\mathcal{V}(L))$ of differential operators on $\mathcal{V}(L)$.

On the other hand one can often show in these circumstances that the image of $U(\mathfrak{g})$ in $End\ L$ is exactly $F(L)$. Then we can conclude that $U(\mathfrak{g})/Ann\ L$ is exactly $\mathcal{D}(\mathcal{V}(L))$. This is the miracle referred to in 6.1.1.

6.6.13. There are some significant simplifications to the above theory for maximal ideals.

Theorem. *Suppose $L \in Ob\mathcal{O}$ is simple and $J := Ann\ L$ is a maximal ideal. Then either $A(L) = F(L)$ or L is properly induced.*

Proof. Set $A = A(L)$, $F = F(L)$ and recall (Remark 6.5.3) that $A(\hat{0})^\iota$ is an $\mathcal{O}_f$ subalgebra of A. Let N be the nilradical of $A(\hat{0})$. Since N is $\mathfrak{g}$ stable and L is simple, either $NL = 0$, in which case $N = 0$ or $NL = L$ which contradicts the nilpotence of N. Since $End_{\mathfrak{g}}L$ reduces to scalars so does $A(\hat{0})^{\mathfrak{g}}$ and then by 6.3.10 we conclude that $A(\hat{0})$ is prime.

If $A \supsetneq F$ then (notation 6.5.2) one has $G \neq 0$. Set $U = U(\mathfrak{g})/J$. By 6.4.7 one has $d_U(U) = d_A(A) > d_A(A) - 1 \geq d_A(A/G) \geq d_U(A/G)$, via Goldie's theorem (as in 6.5.2). Consequently $U \cap G \neq 0$. Yet J is maximal, so U is simple as a $U(\mathfrak{g})$ bimodule which forces $U \subset G$. From the definition of G, it follows that the trivial module E_ϕ (which occurs with multiplicity 1 in U) is a non-split submodule of G and hence of $A(\hat{0})$. Then by 6.3.14(ii) and the conclusion of the previous paragraph it follows that E_ϕ occurs as the socle of the $\mathcal{O}_f$ subalgebra $(\delta M_{\pi'}(0))^\iota$ of A for some $\pi' \subsetneq \pi$. Thus L is properly induced by 6.5.5. □

6.6.14. The above result is useful in the study of $Max\ U(\mathfrak{g})$. For example if $\mathfrak{g}$ has only type A factors, then one may go on to show [, 3.6] that any maximal ideal is induced from a primitive ideal of finite codimension in some $U(\mathfrak{r}_{\pi'})$. (An ideal is said to be induced, if it is the annihilator of a module induced from a parabolic subalgebra.)

6.6.15. There is a rather satisfactory classification theory of the primitive spectrum $Prim\ U(\mathfrak{g})$ for $\mathfrak{g}$ semisimple. This is based on two sets of invariants. First the centre $Z(\mathfrak{g})$ of $U(\mathfrak{g})$ which is a polynomial algebra on $dim\ \mathfrak{h}$ generators and second the Goldie ranks of the quotient algebras $U(\mathfrak{g})/P : P \in Prim\ U(\mathfrak{g})$. Remarkably these are described by polynomials which themselves form bases for W modules, whilst $\mathcal{V}(P)$ is a nilpotent orbit closure. The ensuing map from W modules to nilpotent effects a remarkable link with Springer's classification of nilpotent orbits. Despite all this one is very far from a satisfactory description of the subset $Prim^1 U(\mathfrak{g})$ of ideals of Goldie rank 1, equivalently the set $\{P \in Prim\ U(\mathfrak{g}) \,\big|\, U(\mathfrak{g})/P \text{ is a domain}\}$. This is partly due to the rather complex nature of these "Goldie rank" polynomials which are given by the Kazhdan-Lusztig polynomials up to some not completely known scale factors.

Call $P \in Prim\ U(\mathfrak{g})$ rigid if it is not properly induced and let $Prim_r U(\mathfrak{g})$ denote the set of rigid primitive ideals. Although $Prim_r U(\mathfrak{g})$ is neally all $Prim\ U(\mathfrak{g})$, by contrast $Prim_r^1 U(\mathfrak{g}) := Prim^1 U(\mathfrak{g}) \cap Prim_r U(\mathfrak{g})$ is a finite set [ ]. This is a difficult result which is a striking success of the Goldie rank theory mentioned above. Yet even this is a little unsatisfactory. For example by Moeglin's theorem [ ] one has $|Prim_r^1 U(\mathfrak{g})| = 1$, if $\mathfrak{g}$ has only type A factors.

McGovern [ ] has classified $Max^1 U(\mathfrak{g}) := Prim^1 U(\mathfrak{g}) \cap Max\ U(\mathfrak{g})$, for $\mathfrak{g}$ classical. This involves describing the finite set $Max_r^1 U(\mathfrak{g}) := Prim_r^1 U(\mathfrak{g}) \cap Max\ U(\mathfrak{g})$. Here the results mentioned in 6.6.13 and 6.6.14 are useful.

6.6.16. A far reaching question is whether

$$(*) \qquad Max_r^1 U(\mathfrak{g}) = Prim_r^1 U(\mathfrak{g}).$$

This was raised by McGovern in [, 8.3] though it was already implicit in [, 5.8] which even indicated a possible method of proof. This is discussed briefly below.

Take $P \in Prim\ U(\mathfrak{g})$. By Duflo's theorem [ ] $P = Ann\ L(\lambda)$, for some $\lambda \in \mathfrak{h}^*$. Now choose $\mu \in W.\lambda$ dominant, that is satisfying $\alpha^\vee(\mu) \notin -\mathbb{N}^+$, for all $\alpha \in \Delta^+$. The module $M := M(\mu)/PM(\mu)$ is critical (for Gelfand-Kirillov dimension) has a simple socle S and moreover $P = Ann\ M = Ann\ S$. Furthermore as $U(\mathfrak{g})$ bimodules $V := Soc\ F(M) = Soc\ F(S) = Soc(U(\mathfrak{g})/P)$, and is simple. Also $F(M), F(S)$ and $U(\mathfrak{g})/P$ are critical as left $U(\mathfrak{g})$ modules.

Now assume $P \in Prim^1 U(\mathfrak{g})$. For M constructed as above one can ask if it is always true that

$$(**) \qquad A(M) \ \textit{is a simple ring.}$$

Proposition. $(**)$ *imples* $(*)$.

Proof. Set $A = A(M), F = F(M)$ and define G as in 6.5.2. The simplicity of A implies that either $G = A$ or $G = 0$. In the first case the trivial module E_ϕ (which occurs with multiplicity 1 in A because M is a quotient of a Verma module) is a non-split submodule of G and hence of A_ϕ.

Set $B = \{b \in A \,\big|\, bS = 0\}$. Let us show that $B = 0$. Otherwise B is a non-zero $U(\mathfrak{g})$ submodule of A and so $BM \supset S$. By definition of A we can find a finite dimensional $\mathfrak{u}$ (notation 6.5.2) stable subspace H of B such that HM contains the canonical generator s of S. Then $S = U(\mathfrak{n}^-)s \subset U(\mathfrak{n}^-)HM = HU(\mathfrak{n}^-)M = HM$. On the other hand HM is a $U(\mathfrak{n}^-)$ module image of $H \otimes M/S$. This forces (notation)

$$d_{U(\mathfrak{n}^-)}(S) \leq d_{U(\mathfrak{n}^-)}(H \otimes M/S) \leq d_{U(\mathfrak{n}^-)}(M/S).$$

Yet $d_{U(\mathfrak{n}^-)}$ and $d_{U(\mathfrak{g})}$ coincide on the $\mathcal{O}$ category, so this last inequality contradicts the critically of M.

It follows from the above (and the argument in 6.6.13) that the nilradical of A_ϕ is zero. Hence by the previous observation (and as in 6.6.13) we conclude that M is properly induced and then (by the definition) so is P.

Finally suppose $G = 0$. Then $A = F$ by 6.5.2(ii). We claim that $V = Soc\ F$ is an ideal of F. It is clear that this forces $F = V = U(\mathfrak{g})/P$ and so P is maximal establishing the lemma.

As a left $U(\mathfrak{g})$ module F is critical, that is $d_{U(\mathfrak{g})}(F/V) < d_{U(\mathfrak{g})}F$. Let J be the annihilator of F/V considered as a left $U(\mathfrak{g})$ module. Now $F,\ F/V$ are Harish-Chandra modules (for the pair $(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$) and for any such module T viewed as a left $U(\mathfrak{g})$ module one has $d_{U(\mathfrak{g})}(U(\mathfrak{g})/Ann\ T) = d_{U(\mathfrak{g})}T$. (This follows from the fact that T is finitely generated as a right $U(\mathfrak{g})$ module, say by $t_1, t_2, \cdots, t_n$, and then $\bigcap_i Ann_{U(\mathfrak{g})}t_i = Ann_{U(\mathfrak{g})}T$ with respect to the left action. Consequently $d_{U_{(\mathfrak{g})}}(U(\mathfrak{g})/Ann\ T) = max_i\ d_{U(\mathfrak{g})}(U(\mathfrak{g})/Ann_{U(\mathfrak{g})}t_i) = max_i\ d_{U(\mathfrak{g})}(U(\mathfrak{g})t_i) = d_{U(\mathfrak{g})}T$.) Thus our previous inequality and the simplicity of V forces $JF = V$. Then $VF = JFF = JF = V$. Similarly $FV = V$. $\square$

6.6.17. It is worth noting that 6.6.16 admits a partial converse. More precisely suppose $P \in Prim^1 U(\mathfrak{g})$ is induced from a primitive ideal of some $U(\mathfrak{r}_{\pi'})$, that is there exists a simple (highest weight) $U(\mathfrak{p}_{\pi'})$ module L' such that $P = Ann\ L$, where $L = U(\mathfrak{g}) \otimes_{U(\mathfrak{g})} L'$.

By [, 3.5], $A(L)$ is a Weyl algebra over the corresponding algebra $A'(L')$ defined with respect to L' and $\mathfrak{r}_{\pi'}$. Thus the simplicity of $A'(L')$ implies that of $A(L)$. We can assume without loss of generality that $Ann\ L'$ is rigid. Yet unless $\mathfrak{g}$ has only type A factors one cannot say that $Ann\ L' \in Prim^1 U(\mathfrak{r})$. However when it does hold we may conclude from $(*)$ that $Ann\ L' \in Max\ U(\mathfrak{r})$. In this situation we can choose L' (as in 6.6.16) so that $U(\mathfrak{r})/Ann\ L' = F_{\mathfrak{r}}(L')$. The latter equals $A'(L')$ by 6.6.13 and the assumed rigidity of $Ann\ L'$. Thus $A'(L')$ is simple and so is $A(L)$.

6.6.18. A possible method to establish $(**)$ is to show that any ideal of $A(M)$ can be used to translate P to an ideal having a more singular central character. Such translation can only decrease Goldie rank (cf.) and so we can assume that no translation can take place forcing $A(M)$ to be simple.

From the assumption that $P \in Prim^1 U(\mathfrak{g})$ and the definition of M, it follows by [, 7.11] that $A(M)$ is a domain. This has to be used in some non-trivial fashion. Indeed even in type A it is false that any primitive ideal can be translated to either an induced or to a maximal ideal. For example in type A_5 with $\pi = \{\alpha_1, \alpha_2, \cdots, \alpha_5\}$ there exists [ ] a Goldie rank polynomial of the form (identifying $\alpha_i^\vee(\lambda)$ with α_i)

$$\frac{1}{12}\alpha_1\alpha_2\alpha_4\alpha_5(\alpha_1+\alpha_2)(\alpha_4+\alpha_5)(\alpha_1+2\alpha_2+3\alpha_3+2\alpha_2+\alpha_5).$$

The corresponding coherent family (obtained as λ runs $P^+(\pi)$ - see [, 2.4]) of primitive ideals cannot admit a properly induced member. This is because even on the wall $\alpha_3 = 0$, the above expression cannot be expressed as a product of positive roots as would be required by [, 2.7]. Then by 6.6.13 no member of the family can be maximal either. This may also be checked directly. Again the minimal Goldie rank of $U(\mathfrak{g})/P$, for P in the family, is 2 (corresponding to $\alpha_1 = \alpha_2 = \alpha_4 = \alpha_5 = 1,\ \alpha_3 = 0$). Thus $(**)$ fails already at Goldie rank two!

6.6.19. The condition that $A(M)$ is a domain is equivalent to the condition that $A(M)^{\mathfrak{n}^-}$ is a domain. Since M is a quotient of $M(\mu)$, it is in particular a cyclic $U(\mathfrak{n}^-)$ module, hence of the form $U(\mathfrak{n}^-)/I$ for some left ideal I of $U(\mathfrak{n}^-)$. Set $N(I) = \{a \in U(\mathfrak{n}^-) \mid Ia \subset I\}$. Then $A(M)^{\mathfrak{n}^-}$ identifies with $N(I)/I$.

6.6.20. The classification of $Prim^1 U(\mathfrak{g})$ must also involve a systematic construction of enough of its elements. Clues as to how to do this will be given in later chapters. For the moment suppose $L \in Ob\mathcal{O}$ is a cyclic $U(\mathfrak{n}^-)$ module with cyclic generator v. Set $I = Ann_{U(\mathfrak{n}^-)} v$ and suppose further that $gr_{\mathcal{F}} I$ is a prime ideal J of $S(\mathfrak{n}^-)$, with $\mathcal{F}$ the canonical filtration of $U(\mathfrak{n}^-)$. In this case we say that L is a strong quantization of (the orbital variety closure) $\mathcal{V}(J)$. For the induced filtration on the quotient $N(I)/I$ we have $gr_{\mathcal{F}}(N(I)/I) \cong gr_{\mathcal{F}} N(I)/gr_{\mathcal{F}} I \hookrightarrow S(\mathfrak{n}^-)/J$ and so $N(I)/I$ is a domain. Consequently $U(\mathfrak{g})/Ann\ L \hookrightarrow A(L)$ is a domain, that is $Ann\ L \in Prim^1 U(\mathfrak{g})$. The trouble with this method is that it is notoriously difficult to construct enough strong quantizations.

7. Nilpotent Orbits and Sheets.

7.1. An overview.

7.1.1. Let $\mathfrak{g}$ be a finite dimensional Lie algebra. As already noted in Sect. 6.1 the orbit method furnishes the beginnings of a procedure for constructing unitary representations, or primitive ideals, from orbits in $\mathfrak{g}^*$. If $\mathfrak{g}$ is not far from being commutative (for example if $\mathfrak{g}$ is nilpotent) then one does not require an in-depth understanding of these orbits. It is enough to find a polarization $\mathfrak{p}$ of a given element $f \in \mathfrak{g}^*$ and then the corresponding representation does the required job (6.1.6). More precisely it can be given a unitary structure through the construction of a positive definite Hermitian form, or its annihilator is a primitive ideal. With some extra effort this still goes through for $\mathfrak{g}$ solvable; but the theory has many severe difficulties for $\mathfrak{g}$ semisimple the most evident of which is the failure for a polarization to exist.

7.1.2. As can be expected a resolution of the difficulties evoked above must necessarily pass through an in-depth understanding of co-adjoint orbits.

Let us from now on assume that $\mathfrak{g}$ is a semisimple Lie algebra over an algebraically closed field of characteristic zero. Call $s \in \mathfrak{g}$ (resp. $n \in \mathfrak{g}$) semisimple (resp. nilpotent) if $ad\ s$ (resp. $ad\ n$) is a semisimple (resp. nilpotent) derivation of $\mathfrak{g}$. Both properties are preserved under the action of the adjoint group G and so one may speak of semisimple (resp. nilpotent) orbits.

Recall that the Killing form on $\mathfrak{g}$ is G invariant and non-degenerate. This sets up an equivalence between G orbits in $\mathfrak{g}$ and G orbits in $\mathfrak{g}^*$. The latter are just the coadjoint orbits. These have a symplectic structure and so it is considered that one may construct representations of G by a process which mimics the passage from classical to quantum mechanics.

For $\mathfrak{g}$ semisimple every derivation of $\mathfrak{g}$ is inner. From this one may deduce that the Jordan decomposition of an element of $ad\ \mathfrak{g}$ passes to $\mathfrak{g}$. More precisely each $x \in \mathfrak{g}$ can be written uniquely as a sum of a semisimple element s and a nilpotent element n which commute with one another. This fact is a crucial ingredient for decomposing $\mathfrak{g}^*$ into finitely many "sheets", which are the maximal irreducible subvarities of $\mathfrak{g}^*$ consisting of orbits having all the same dimension. One can equally well speak of sheets in $\mathfrak{g}^*/G$ and this is more appropriate for comparison with $Prim\ U(\mathfrak{g})$.

7.1.3. Of the above families of orbits, it is the semisimple ones which are the best behaved. They are closed, polarizable, form a dense open subset in $\mathfrak{g}^*$ and are separated by the G invariant functions on $\mathfrak{g}^*$, that is to say by $Y(\mathfrak{g}) := S(\mathfrak{g})^G$. The nilpotent orbits are the worst behaved. Only if the trivial nilpotent orbit is closed, they may fail to be polarizable, they are finite in number and all lie in the zero variety of the augmentation ideal of $S(\mathfrak{g})^G$. However like the proverbial naughty boys they are the most intriguing.

7.1.4. Let us leap ahead for the moment to inspect the available results on $Prim\ U(\mathfrak{g}) : \mathfrak{g}$ semisimple. The first difficulty is that unlike the solvable case one does not obtain a homeomorphism of $\mathfrak{g}^*/G$ onto $Prim\ U(\mathfrak{g})$. One reason is that $Prim^1 U(\mathfrak{g}) = Prim\ U(\mathfrak{g})$, for $\mathfrak{g}$ solvable, that is to say all primitive ideals in $U(\mathfrak{g})$ are completely prime, whilst for $\mathfrak{g}$ semisimple $Prim^1 U(\mathfrak{g})$ is only a very small part of $Prim\ U(\mathfrak{g})$. Yet this difficulty is not entirely resolved by restricting to $Prim^1 U(\mathfrak{g})$. However $Prim^1 U(\mathfrak{g})$ does decompose into finitely many sheets defined in a manner similar to sheets in $\mathfrak{g}^*/G$. It is not quite true that there is a bijective correspondence between sheets in these two settings, nor is it true that a sheet in $Prim^1 U(\mathfrak{g})$ is homeomorphic to a sheet in $\mathfrak{g}^*/G$. However one can hope to prove that every sheet in $Prim^1 U(\mathfrak{g})$ is homeomorphic to a closed subvariety of a sheet in $\mathfrak{g}^*/G$. A detailed discussion of this open question is beyond the scope of the present volume. However we do discuss two hard questions which must first be resolved. One, how to construct primitive ideals from non-polarizable nilpotent orbits. Second, how to be sure that we have constructed all of $Prim^1 U(\mathfrak{g})$. The second question was already raised in 6.6.16 and on it not much can be said except that it was resolved by C. Moeglin for $\mathfrak{g} \cong \mathfrak{sl}(n)$ by special and rather complicated means. The first question and perhaps also the second must involve an in-depth analysis of nilpotent orbits and sheets in $\mathfrak{g}^*$.

7.1.5. A classification of nilpotent orbits was given by E.B. Dynkin more than forty years ago. A key step is the Jacobson-Morosov theorem which leads quickly to there being at most $3^{rank\ \mathfrak{g}}$ such orbits. Indeed they are classified by assigning 0, 1 or 2 to each simple root. Although it is known which assignments occur, the list is not known to have any clear interpretation. Again the study of sheets leads to the notion of a rigid orbit which is necessarily nilpotent. All other orbits are by definition obtained by induction from a rigid orbit of a proper Levi factor. The list of rigid nilpotent orbits is known (in terms of the Dynkin classification) but does not make any particular sense. From the Dynkin data of a nilpotent orbit one can (implicitly) describe the ideal of definition of its closure; but this has not yet been exploited to construct an element of $Prim^1 U(\mathfrak{g})$.

A second way to classify nilpotent orbits is through the Steinberg map. This is a map from the Weyl group W to the set of nilpotent orbits. It is known to be surjective; but its fibres, called the geometric two-sided cells, have still not been completely determined. From the Steinberg map one can read off orbit closures from a given element of W. This gives information on orbit closure inclusions and ideals of definition.

A third way to classify nilpotent orbits is through the remarkable theory of Springer which assigns to each nilpotent orbit an irreducible representation of W, this correspondence being injective. It has the advantage of admitting a link-up with the Goldie rank theory of $Prim\ U(\mathfrak{g})$. Indeed $Prim\ U(\mathfrak{g})$ decomposes into finitely many families (each family having infinitely many members) and to any such family there is assigned an irreducible representation of W provided by the transformation properties of the Goldie rank polynomials (see 7.3.1). On the other hand the associated variety of a primitive ideal is a nilpotent orbit closure so we have a map φ from $Prim\ U(\mathfrak{g})$ to the set of nilpotent orbits. The fibres are exactly the families mentioned above and moreover the corresponding Weyl group representation is exactly that given by the Springer theory up to tensoring with the sign representation. This is discussed in the chapter on orbital varieties. For the moment we remark that there are two outstanding difficulties with this link-up to primitive ideal theory. First it is not clear that φ is surjective, although enough is known to check this case by case. Secondly it is not clear if a given fibre of φ has non-empty intersection with $Prim^1 U(\mathfrak{g})$, nor what is the cardinality of this intersection.

There is yet a fourth way to describe nilpotent orbits. It was first noticed by Carter and Elkington that their classification is remarkably similar to that of the classification of conjugacy classes in W (except that there can be more of the latter). Of course for

$\mathfrak{g} = \mathfrak{sl}(n)$ this is a classical result, both sets being given by the partitions of n. More recently Kazhdan and Lusztig have constructed (in a rather abstract fashion, but partly inspired by an observation of Kostant) a map from the set of nilpotent orbits to the set of conjugacy classes in W. It is not yet known if this map is injective. It is completely different to the Steinberg map whose fibres are subsets of W but very far from being conjugacy classes. On the other hand this Kazhdan-Lusztig map may well have a link-up with the primitive ideal picture.

7.1.6. We may summarize the above by saying that there are at least four different ways of viewing nilpotent orbits and each of these have their own individual mysteries. Could there be a reader who is not left unmoved by this situation?

7.1.7. Observe that $x \in \mathfrak{g}$ is nilpotent exactly if the characteristic polynomial of $ad\ x$ is monic with all non-leading coefficients being zero. It follows that the set $\mathcal{N}$ of all nilpotent elements is a closed subvariety of $\mathfrak{g}^*$. Again if $x \in \mathcal{N}$, then $cx \in \mathcal{N}$ for all $x \in k$ and so $\mathcal{N}$ is a cone. Finally $\mathcal{N}$ is obviously G stable and so a union of G orbits which are exactly the nilpotent orbits.

7.1.8. Now let $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ be a triangular decomposition. Since the weights of $\mathfrak{n}^+$ are strictly positive, it follows that $\mathfrak{n}^+ \subset \mathcal{N}$. On the other hand via the Jacobson-Morosov theorem one can show that every $x \in \mathcal{N}$ can be conjugated into $\mathfrak{n}^+$. Thus $\mathcal{N} = G\mathfrak{n}^+$ and so is an irreducible variety. In particular $\mathcal{N}$ admits a dense orbit and this has dimension $dim\ \mathfrak{g} - rk\mathfrak{g}$. For more details see 7.2.14.

7.1.9. Recall the definition of $Y(\mathfrak{g})$ and set $Y_+ = Y(\mathfrak{g}) \cap S(\mathfrak{g})_+$. It is the space of invariant homogeneous polynomials of strictly positive degree. Since the coefficients of the characteristic polynomial of $ad\ x : x \in \mathfrak{g}$, besides the leading term, are G invariant polynomials of degree > 0 it follows that if $p(x) = 0$, for all $p \in Y_+$, then $x \in \mathcal{N}$. A hard theorem of Kostant which we shall establish in Chapter 8 asserts that $P := S(\mathfrak{g})Y_+$ is a prime ideal. Since $dim\ \mathcal{N} = dim\ \mathfrak{g} - rk\mathfrak{g}$ and $Y(\mathfrak{g})$ is a polynomial algebra on $\ell := rank\mathfrak{g}$ homogeneous generators $z_1, z_2, \cdots, z_\ell$, it follows from Kostant's theorem that P is exactly the ideal of definition $I(\mathcal{N})$ of $\mathcal{N}$. This means that $\mathcal{N}$ is a complete intersection, more precisely it is the intersection of the hyperplanes $z_i = 0 : i = 1, 2, \cdots, \ell$.

7.1.10. In this chapter we shall describe the classification of sheets in $\mathfrak{g}^*$ resulting from the works of J. Dixmier, H.-P. Kraft and W. Borho. Briefly these are obtained by "induction" starting from a rigid nilpotent orbit of a Levi factor $\mathfrak{r}$. If the rigid orbit is just $\{0\}$ the resulting sheet is a Dixmier sheet. If in addition $\mathfrak{r} = \mathfrak{h}$, then the resulting sheet is called the regular sheet $\mathcal{S}_{reg}$. It contains the unique dense orbit in $\mathcal{N}$. A by-product of Kostant's theorem is a description of $\mathcal{S}_{reg}/G$ as an affine variety. This is described in Chapter 8. One can expect a similar description for any Dixmier sheet. The general case is more subtle and may involve the Kazhdan-Lusztig map.

7.1.11. Kostant's theorem has some deep implications for the representation theory of $\mathfrak{g}$. This stems partly from the following observation. Set $H = (ad\ U(\mathfrak{g}))S(\mathfrak{n}^+)$. From the remarks in 7.1.8 this is exactly the span of the powers of nilpotent elements. It is called Kostant's harmonic space. Through the longest element of the Weyl group which exchanges $\mathfrak{n}^+$ and $\mathfrak{n}^-$, one also has $H = (ad\,U(\mathfrak{g}))S(\mathfrak{n}^-)$. Yet exactly as in 4.4.4, it follows that $I(\overline{G\mathfrak{n}^+}) = I(\mathcal{N})$ is just the orthogonal of $(ad\ U(\mathfrak{g}))S(\mathfrak{n}^-)$ with respect to the bilinear form on $S(\mathfrak{g})$ defined by extending the Killing form. Thus H identifies with space of regular functions on $\mathcal{N}$ and is a graded complement to P. From this one may deduce that the map $H \otimes Y(\mathfrak{g}) \to S(\mathfrak{g})$ defined by multiplication in $S(\mathfrak{g})$, is a linear isomorphism. This means that $S(\mathfrak{g})$ is a free $Y(\mathfrak{g})$ module. Furthermore by restricting to the semisimple orbits and using Frobenius reciprocity one may calculate the multiplicity of any simple $\mathfrak{g}$ module E in H to be the dimension of the zero weight space E_0 of E.

The above result, known as Kostant's separation theorem, lifts to a corresponding assertion for $U(\mathfrak{g})$, which in particular must be a free module over its centre $Z(\mathfrak{g})$. This result is a key ingredient of Duflo's theorem describing $Prim\ U(\mathfrak{g})$ as well as of the Beilinson-Bernstein equivalence of categories which besides other things gives particularly deep information on $Prim\ U(\mathfrak{g})$. This further leads to a description of sheets in $Prim\ U(\mathfrak{g})$ and more precisely the description of $Prim\ U(\mathfrak{g})$ as a topological space. As an example one may remark that the set of minimal primitive ideals (being those generated by a maximal ideal of $Z(\mathfrak{g})$) form a sheet in $Prim\ U(\mathfrak{g})$ homeomorphic to $Max\ Z(\mathfrak{g})$ which in turn is homeomorphic to $\mathcal{S}_{reg}/G$. Again for any ideal I in this sheet $U(\mathfrak{g})/I$ is isomorphic to H as a $\mathfrak{g}$ module.

7.1.12. A further remarkable aspect of the proof of Kostant's theorem is that it describes the degrees of the z_i in terms of data coming from the Jacobson-Morosov theorem applied to the unique dense orbit in $\mathcal{N}$. One would obviously like to similarly probe the corresponding data coming from the remaining nilpotent orbits.

7.2 The Jacobson-Morosov Theorem and its Consequences.

7.2.1. Consider the matrices

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

They form a basis of the Lie algebra $\mathfrak{sl}(2)$ and satisfy

$$[h,x] = 2x, \;\; [x,y] = h, \;\; [h,y] = -2y.$$

Viewed as elements of a semisimple Lie algebra satisfying the above identities one calls (x,h,y) an s-triple. The following result is known as the Jacobson-Morosov theorem though they studied this question rather independently. Our proof follows that of Kostant.

Theorem. *Let $\mathfrak{g}$ be a semisimple Lie algebra over a field k of characteristic zero. Let $0 \neq x \in \mathfrak{g}$ be nilpotent. There exist $h, y \in \mathfrak{g}$ such that (x,h,y) is an s-triple.*

7.2.2. The proof of 7.2.1 results from the following three lemmas. Let V be a finite dimensional k-vector space and $a, b \in End\, V$.

Lemma. *Suppose $[a,[a,b]] = 0$. Then*

(i) $[a,b]$ *is nilpotent.*

(ii) *If a is nilpotent, then ab is also nilpotent.*

Proof. (i) From the identity $[a,b]^{r+1} = [a, b[a,b]^r]$, it follows that $tr[a,b]^{r+1} = 0$, for all $r \geq 0$. Hence (i).

(ii) We may assume k algebraically closed. Set $V_r = ker\ a^r$. Obviously $aV_{r+1} \subset V_r$. From the identity $a^{r+1}b = ba^{r+1} + r^{(r+1)}[a,b]a^r$, it follows that $abV_r \subset V_r$. Thus if ab is not nilpotent, there exists $r \geq 0$, $v \in V_{r+1} \setminus V_r$ and $c \in k$ such that $abv = cv$. The previous identity gives $a^{r+1}bv = ba^{r+1}v + (r+1)[a,b]a^r v$ and so $[a,b]a^r v = (c/r+1)a^r v$, contradicting the nilpotence of $[a,b]$. □

Remark. This needs $char\ k = 0$ because of the trace argument; but not that k is algebraically closed. It also fails if $dim\ V$ is infinite, for example suppose $ab - ba = 1$ and take $V = k[b]$ where b acts by multiplication and a as d/db.

7.2.3. Lemma. *Assume $\mathfrak{g}$ semisimple and $x \in \mathfrak{g}$ nilpotent. Then $x \in Im(adx)^2$.*

Proof. Recall that the Killing form $(y, z) = tr\ ady\ adz$ on $\mathfrak{g}$ is non-degenerate and $\mathfrak{g}$ invariant. Consequently $((adx)^2 y, z) = (y, (adx)^2 z) = 0$, if $z \in Ker(adx)^2$, which shows that $Im(adx)^2 = (Ker(adx)^2)^\perp$. Now set $a = (adx)^2,\ b = ady$. Then $ay = 0$, translates through the adjoint representation to $[a, [a, b]] = 0$. By 7.2.2(ii) we conclude that ab is nilpotent and so $(x, y) = 0$. Thus $x \in (Ker\ a)^\perp = Im\ a = Im(adx)^2$, as required. □

7.2.4. Assume $\mathfrak{g}$ semisimple and $0 \neq x \in \mathfrak{g}$ nilpotent. By 7.2.3, there exists $y' \in \mathfrak{g}$ such that $[x, [x, y']] = -2x$. Set $h = [x, y']$. Then $[h, x] = 2x$. Set $K = Ker\ adx$. Certainly $(ad\ h)K \subset K$. In addition $(adx)(adh + 2)y' = ([adx, adh] + 2adx)y' + (adh)(adx)y' = 0$, that is $z := (adh + 2)y' \in K$. Thus theorem 7.2.1 will follow if we can show that $(adh + 2)$ is invertible on K by taking $y = y' - (adh + 2)^{-1}z$. It thus suffices to prove the following

Lemma. *Assume that* $(adx)^{m+1} = 0$. *Then* $\prod_{n=0}^{m} (adh - n)$ *vanishes on* $Ker\ adx$.

Proof. Set $V_r = Im(adx)^r \cap Ker\ adx$ and $a = adx,\ b = adh,\ c = ady'$. One has

$$[a^{r+1}, c] = \sum_{i=0}^{r} a^i b a^{r-i} = \left(\sum_{i=0}^{r} (b - 2i)\right) a^r = (r + 1)(b - r)a^r.$$

Now take $v \in V_r$. We can write $v = a^r z$, with $a^{r+1} z = 0$. Then

$$(adh - r)v = (b - r)a^r z = a^{r+1}(ady')z + (ady')(a^{r+1} z \in Im(adx)^{r+1}.$$

We conclude that $(adh - r)V_r \subset V_{r+1}$ and so

$$\left(\prod_{n=0}^{m} (adh - n)\right) Ker(adx) \subset V_{m+1} = 0,$$

as required. □

Remark. Let (a, b, c) be an s-triple and V a finite dimensional $\mathfrak{sl}(2)$ module. From $[b, a] = 2a$ it follows that a sets nilpotently on V. From the above calculation it follows that b acts semisimply on $ker\ a$ with integer eigenvalues ≥ 0. One may easily conclude that V is a direct sum of simple highest weight modules with dominant highest weights.

7.2.5. Retain the notation of 7.2.1 and set $u = \binom{1}{0},\ v = \binom{0}{1}$. One has $xu = 0,\ xv = u,\ yu = v, yv = 0,\ hu = u,\ hv = -v$. Through the coproduct this action extends to the polynomial ring $S := k[u, v]$. For each $m \in \mathbb{N}$, let S_m denote the subspace of homogeneous polynomials of degree m. One easily checks that S_m is a simple $\mathfrak{sl}(2)$ submodule of dimension $m+1$ and that the set of h eigenvalues of S_m is exactly $\{m, m-2, \cdots, -m\}$. This constructs for each positive integer m a simple finite dimensional highest weight module of highest eigenvalue m. Using the remarks in 7.2.4 one may check that every simple finite dimensional module occurs in S. In particular the eigenvalues of h on any finite dimensional $\mathfrak{sl}(2)$ module are all integer.

Now let $\mathfrak{s}$ be a subalgebra of $\mathfrak{g}$ defined by an s-triple (x, h, y). For each $i \in \mathbb{Z}$ set

$$(*) \qquad \mathfrak{g}_i = \{a \in \mathfrak{g} \,\big|\, [h, a] = ia\}.$$

By the above remarks

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$$

which gives $\mathfrak{g}$ the structure of a graded Lie algebra. Conversely give a $\mathbb{Z}$-gradation on $\mathfrak{g}$, it follows that the map $a \mapsto ia$ on $\mathfrak{g}_i$ extends linearly to a derivation of $\mathfrak{g}$. Since $\mathfrak{g}$ is semisimple, every derivation is inner so there exists $h \in \mathfrak{g}$ such that $\mathfrak{g}_i$ is given by $(*)$. Moreover the centre of $\mathfrak{g}$ is trivial, so h is unique. Nevertheless it does not follow that every $\mathbb{Z}$-gradation comes from an s-triple. Analyzing those which do gives a classification of nilpotent orbits (due to Dynkin).

7.2.6. Let $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$ be a $\mathbb{Z}$-gradation of a semisimple Lie algebra $\mathfrak{g}$ and define h as in 7.2.5. Then $\mathfrak{g}_0 = \mathfrak{g}^h$ which is hence a reductive subalgebra. Let $\mathfrak{h}$ be a Cartan subalgebra for $\mathfrak{g}_0$ containing h. It is clear that $\mathfrak{h}$ is also a Cartan subalgebra of $\mathfrak{g}$. Again each root subspace $\mathfrak{g}_\alpha : \alpha \in \Delta$, is a fortiori an h eigenspace whose eigenvalue we denote by $\nu(\alpha)$. Set

$$\Delta_0 = \{\alpha \in \Delta \,\big|\, \nu(\alpha) = 0\}$$

which is clearly the set of non-zero roots for the pair $(\mathfrak{g}_0, \mathfrak{h})$. Fix a positive system Δ_0^+ for Δ_0 and set

$$\Delta^+ = \Delta_0^+ \cup \{\alpha \in \Delta \,\big|\, \nu(\alpha) > 0\}$$

which is a positive system for Δ. Let $\pi \subset \Delta^+$ be the corresponding set of simple roots. It is clear that $\nu : \Delta \to \mathbb{Z}$ is an additive function and so determined by its values on π

which are integers ≥ 0. Conversely any assignment of integers ≥ 0 to elements of π extends uniquely to an additive function $\nu : \Delta \to \mathbb{Z}$ and the latter defines a gradation on $\mathfrak{g}$ by

$$\mathfrak{g}_i = \bigoplus_{\alpha \in \nu^{-1}(i)} \mathfrak{g}_\alpha \oplus \delta_{i,0}\mathfrak{h}.$$

Lemma. *Suppose h belongs to an s-triple (x, h, y). Then $\nu(\pi) \subset \{0, 1, 2\}$.*

Proof. Take $\alpha \in \pi$. We can assume $\nu(\alpha) > 0$. By the $\mathfrak{sl}(2)$ theory (for example by the remarks in 7.2.5) one has $[y, \mathfrak{g}_\alpha] \neq 0$. Since $y \in \mathfrak{g}_{-2}$ this commutator lies in $\mathfrak{g}_{\nu(\alpha)-2}$. On the other hand y is a sum of root vectors $x_{-\beta}$ with $\beta \subset \nu^{-1}(2) \subset \Delta^+$, so $-\beta+\alpha \in \Delta^- \cup \{0\}$, in particular $\nu(-\beta + \alpha) \leq 0$. This forces $\nu(\alpha) - 2 \leq 0$, as required. □

7.2.7. Not all maps $\nu : \pi \to \{0, 1, 2\}$ arise from s-triples. Those which do were determined by Dynkin. They form a rather mysterious list. The set consisting of 0, is also a nilpotent orbit. By convention one takes $\nu_0 = 0$ in this case. In what follows we generally mean a non-zero nilpotent orbit.

Let $\mathcal{O} \subset \mathfrak{g}$ be a nilpotent orbit. We may choose $x \in \mathcal{O}$ and embed it in an s-triple (x, h, y). Then we may choose a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ containing h and a positive system Δ^+ for the roots defined by $\mathfrak{h}$. Then the pair $(h, \mathfrak{h})$ gives rise to a function $\nu : \pi \to \{0, 1, 2\}$. It is not difficult to show that ν is independent of choices, using in particular 7.2.11(i). We denote it by $\nu_\mathcal{O}$. Again by examining the possible choices one may check that the map $\mathcal{O} \mapsto \nu_\mathcal{O}$ is injective. Thus a nilpotent orbit $\mathcal{O}$ is determined by its Dynkin map $\nu_\mathcal{O}$. We call the corresponding gradation of $\mathfrak{g}$ the Dynkin gradation associated to $\mathcal{O}$.

One may ask if it is possible to obtain $\mathcal{O}$ from $\nu_\mathcal{O}$ in a constructive fashion. Actually as far as quantizing orbits are concerned it is more interesting to construct the ideal of definition $I(\bar{\mathcal{O}})$ of its closure $\mathcal{O}$. (This determines $\mathcal{O}$ as the unique dense G-orbit in the zero set of $I(\bar{\mathcal{O}})$). Towards this aim set

$$\mathfrak{p} = \bigoplus_{i \geq 0} \mathfrak{g}_i, \ \mathfrak{n}_1^\pm = \bigoplus_{i \geq 1} \mathfrak{g}_{\pm i}, \ \mathfrak{n}_2^+ = \bigoplus_{i \geq 2} \mathfrak{g}_{\pm i}.$$

Then $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with nilradical $\mathfrak{n}_1^+$. Let $P,\ N_1^\pm,\ N_2^\pm$ be the connected subgroups of G with Lie algebras $\mathfrak{p}, \mathfrak{n}_1^\pm, \mathfrak{n}_2^\pm$. It is clear that $\mathfrak{n}_2^+$ is also a $\mathfrak{p}$ module, and hence a P module.

Lemma.

(i) $[\mathfrak{p}, x] = \mathfrak{n}_2^+$,

(ii) $\overline{Px} = \mathfrak{n}_2^+$.

Proof. Property (ii) is equivalent to $(ad\ U(\mathfrak{p}))x = \mathfrak{n}_2^+$ and so is an immediate consequently of (i). From $\mathfrak{sl}(2)$ theory it follows that the map $a \mapsto (adx)a$ of $\mathfrak{g}_i$ into $\mathfrak{g}_{i+2}$ is surjective for all $i \geq 0$. This gives (i). $\square$

7.2.8. The Killing form $(x, y) \mapsto K(x, y) := tr(adx)(ady)$ defines by restriction a non-degenerate pairing $\mathfrak{g}_i \times \mathfrak{g}_{-i} \to k$. Thus through this form $S(\mathfrak{g})$ identifies with the algebra of regular functions on $\mathfrak{g}$ whilst $\mathfrak{n}_2^-$ identifies with the space of linear functions on $\mathfrak{n}_2^+$ and $S(\mathfrak{n}_2^-)$ with the algebra of regular functions on $\mathfrak{n}_2^+$. Let $Q : S(\mathfrak{g}) \to S(\mathfrak{n}_2^-)$ be defined by restriction of functions. (One may recognize Q as the projection onto $S(\mathfrak{n}_2^-)$ with kernel $S(\mathfrak{g})\left(\bigoplus_{-i\leq 1} \mathfrak{g}_i\right)$.). Let φ denote the restriction of Q to $S(\mathfrak{g})^{\mathfrak{n}_1^-}$.

Proposition. *Let $\mathcal{O}$ be the nilpotent orbit defined by a Dynkin gradation and $I(\bar{\mathcal{O}})$, or simply I, the ideal of definition of its closure $\bar{\mathcal{O}}$. Then $ker\ \varphi = I^{\mathfrak{n}_1^-}$.*

Proof. Let $N_1^- P$ denote the image of the morphism $(n, p) \mapsto np$ of $N_1^- \times P$ into G. Since $\mathfrak{n}_1^- + \mathfrak{p} = \mathfrak{g}$ it follows that $\overline{N_1^- P} = G$. Recall that $S(\mathfrak{g})^{\mathfrak{n}_1^-} = S(\mathfrak{g})^{N_1^-}$, since N_1^- is connected and $\mathfrak{n}_1^- = Lie\ N_1^-$. Thus for all $f \in S(\mathfrak{g})^{\mathfrak{n}_1^-}$ we have

$$f(\overline{Gx}) = 0 \iff f(N_1^- Px) = 0 \iff f(Px) = 0 \iff f(\mathfrak{n}_2^+) = 0,$$

where the last step follows from 7.2.7(ii). This translates to give

$$f \in I^{\mathfrak{n}_1^-} \iff f \in ker\ \varphi,$$

as required. $\square$

7.2.9. Define $\Delta^\pm$ as in 7.2.7 and set

$$\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha.$$

Then $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}^-$ is a triangular decomposition.

Let V be a finite dimensional $U(\mathfrak{g})$ module. Obviously $V^{\mathfrak{n}_1^+}$ determines $V^{\mathfrak{n}^+}$. Since $\mathfrak{g}$ is reductive V is a direct sum of simple $U(\mathfrak{g})$ modules V_i and each $V_i^{\mathfrak{n}^+}$ is one dimensional. Thus V is generated as a $U(\mathfrak{g})$ module by $V^{\mathfrak{n}^+}$. (Surprisingly this fails if we just assume $V \in Ob\mathcal{O}$). Analogously V is generated by $V^{\mathfrak{n}^-}$ and a fortiori by $V^{\mathfrak{n}_1^-}$. In particular

Corollary. *One has $I(\bar{\mathcal{O}}) = (ad\ U(\mathfrak{g}))\ker\ \varphi$.*

Remark. This means that $I(\bar{\mathcal{O}})$ can be recovered from the pair $(\mathfrak{n}_1^-, \mathfrak{n}_2^-)$, or equivalently from the pair $(\mathfrak{n}_1^+, \mathfrak{n}_2^+)$. On the other hand either pair obtains from $\nu_{\mathcal{O}}$. This gives a constructive proof that the map $\mathcal{O} \mapsto \nu_{\mathcal{O}}$ is injective.

7.2.10. One cannot expect the map $\varphi : S(\mathfrak{g})^{\mathfrak{n}_1^-} \to S(\mathfrak{n}_2^-)$ to be surjective. However it factors to an injection $\varphi : (S(\mathfrak{g})/I(\bar{\mathcal{O}})^{\mathfrak{n}_1^-} \hookrightarrow S(\mathfrak{n}_2^-)$ of integral domains and so extends to an injection $Fract\ \varphi$ of fields of fractions.

Proposition. *$Fract\ \varphi$ is bijective. Moreover $\mathcal{O}$ is a rational variety of dimension $dim\ \mathfrak{n}_1^- + dim\ \mathfrak{n}_2^-$.*

Proof. We claim that the map $(n, px) \mapsto npx$ of $N_1^- \times Px$ into Gx is injective. Now $N_1^- \cap P$ is a connected subgroup of G with Lie algebra $\mathfrak{n}_1^- \cap \mathfrak{p} = 0$ and is hence reduced to the identity. Set $G^x = Stab_G x$ and $P^x = P \cap G^x$. We show in 7.2.11 below that $P^x = G^x$. From this and the previous observation, the claim follows. Since $\overline{N_1^- P} = G$, the closure in Gx of the image is Gx itself. Now $\overline{Px} = \mathfrak{n}_2^+$ by 7.2.7(ii) and so we have a map $(n, z) \mapsto nz$ of $N_1^- \times \mathfrak{n}_2^+$ into $\overline{Gx}$ which is injective on the open set $N_1^- \times Px$. Since $\mathcal{O} = Gx$ is also open in its closure, this means that $\mathcal{O}$ is birationally isomorphic to $N_1^- \times \mathfrak{n}_2^+$ which is itself affine, indeed $N_1^- \cong \mathfrak{n}_1^-$ because $\mathfrak{n}_1^-$ is nilpotent. Consequently $\mathcal{O}$ is a rational variety. Again the induced map $\mathfrak{n}_2^+$ to the orbit space $N_1^- \setminus \overline{Gx}$ is injective on the open set Px. Thus $\mathfrak{n}_2^+$ is birationally isomorphic to $N_1^- \setminus \overline{Gx}$. Equivalently $Fract\ \varphi$ is an isomorphism. □

7.2.11. Let P be a parabolic subgroup of G. Take $x \in \mathfrak{g}$, set $\mathfrak{p} = Lie\ P$ and

$$G^x = Stab_G x,\ P^x = G^x \cap P,\ \mathfrak{g}^x = \{a \in \mathfrak{g} \mid [x, a] = 0\},\ \mathfrak{p}^x = \mathfrak{p} \cap \mathfrak{g}^x.$$

In general the inclusion $G^x \supset P^x$ is strict. Take $z \in G^x$. Since $N_G(P) = P$, for any parabolic P one has first observation $z \in P^x \iff zPz^{-1} = P$. For this reason one says that P is an invariant parabolic when $G^x = P^x$. Suppose $\mathfrak{g}^x = \mathfrak{p}^x$. Then the connected components of the identity G_0^x, P_0^x coincide. Yet G_0^x is normal in G^x and G^x/G_0^x is finite. Consequently $\{zPz^{-1} \mid z \in G^x\}$ is finite.

Now take $x, \mathfrak{p}, P$ as in 7.2.7. One calls P the Dynkin parabolic of x. Set $\mathfrak{m} = \mathfrak{g}^x \cap \mathfrak{n}_1^+$ and let M be the connected subgroup of G with Lie algebra $\mathfrak{m}$. From $\mathfrak{sl}(2)$ one has $\mathfrak{m} = \mathfrak{g}^x \cap Im(ad\ x)$. Since $Lie\ M = \mathfrak{m} \subset \mathfrak{p} = Lie\ P$, one has $M \subset P$.

Proposition.

(i) *The s-triples containing x form a simple M orbit.*

(ii) $\mathfrak{g}^x = \mathfrak{p}^x$.

(iii) $G^x = P^x$.

Proof. (i). Let (x, h, y), (x', h', y') be two such $\mathfrak{sl}(2)$ triples. Clearly $[x, h - h'] = 0$, and so $h - h' \in \mathfrak{g}^x \cap Im\ adx = \mathfrak{m}$. Thus we may write $h - h' = \sum_{i=1}^{t} c_i x_i$ with $x_i \in \mathfrak{g}_i$. Take j minimal with the property that $c_j \neq 0$. Since $j > 0$, it follows that $ad\ x_j$ is nilpotent and so $m_j := exp\ ad(c_j x_j/2j) \in M$. Moreover $m_j h = h - c_j x_j$. Then $m_j h - h' = \sum_{i>j}^{j} c_i x_i$ and so $h - m_j^{-1} h' \in \sum_{i=j+1}^{t} c_i' x_i$. Replacing h' by $m_j^{-1} h' \in Mh'$ and proceeding by induction we obtain $m \in M$ such that $h' = mh$. Replacing (x, h', y') by its translate under m namely (x, h, my'), we can assume $h = h'$. However in this case $[h, (y - y')] = -2(y - y')$ and $[x, y - y'] = h - h' = 0$ and by $\mathfrak{sl}(2)$ theory $y = y'$. This proves (i).

(ii). Through the action of h one has $\mathfrak{g}^x = \mathfrak{p}^x + (\mathfrak{n}_1^-)^x$. Yet by $\mathfrak{sl}(2)$ theory $(\mathfrak{n}_1^-)^x = 0$.

(iii). Take $z \in G^x$. Then $Lie(zPz^{-1}) = z\mathfrak{p}z^{-1}$. Now $z\mathfrak{p}z^{-1}$ is just the parabolic subalgebra defined by the s-triple $(zxz^{-1}, zhz^{-1}, zyz^{-1}) = (x, zhz^{-1}, zyz^{-1})$. By (i) there exists $m \in M \subset P$ such that $m(zhz^{-1}) = mzh(mz)^{-1} = h$, $m(zyz^{-1}) = y$ which forces $mz\mathfrak{p}(mz)^{-1} = \mathfrak{p}$ and $mzP(mz)^{-1} = P$. Thus $mz \in N_G(P) = P$ and so $z \in P$, as required. $\square$

7.2.12. There is a second way to express the dimension of nilpotent orbit $\mathcal{O}$ through its Dynkin map. Fix an s-triple (x, h, y) and decompose $\mathfrak{g}$ into simple modules with respect to its action. Let n_e (resp. n_0) denote the number of even (resp. odd) dimensional modules which occur. From $\mathfrak{sl}(2)$ theory it follows that $n_0 = dim\ \mathfrak{g}^h$, whilst $n_0 + n_e = dim\ \mathfrak{g}^x$. Thus

$$dim\ Gx = dim\ \mathfrak{g} - n_0 - n_e.$$

From this we may recover the last assertion of 7.2.10. Indeed $n_e = dim\ \mathfrak{n}_1^+ - dim\ \mathfrak{n}_2^+$ whilst $dim\ \mathfrak{g} - n_0 = 2dim\ \mathfrak{n}_1^+$, so $dim\ Gx = dim\ \mathfrak{n}_1^+ + dim\ \mathfrak{n}_2^+$.

7.2.13. Let $\rho^\vee$ denote the half sum of the positive coroots $\alpha^\vee := 2\alpha/(a,a) : \alpha \in \Delta^+$. One may check that $\rho^\vee(\alpha) = 1,\ \forall\alpha \in \pi$. Express $\rho^\vee$ as a sum of the $\alpha^\vee : \alpha \in \pi$ and let c_α denote the coefficient of α. Then

$$x = \sum_{\alpha\in\pi} x_\alpha,\ \ h = 2\rho^\vee,\ \ y = 2(\sum_{\alpha\in\pi} c_\alpha x_{-\alpha})$$

is an s-triple. Express $\beta \in \Delta^+$ as a sum of simple roots and let $o(\beta)$ denote the sum of its coefficients. Clearly $o(\beta) = \rho^\vee(\beta) > 0$. Consequently $\mathfrak{g}^h$ is a Cartan subalgebra of $\mathfrak{g}$. Again the eigenvalues of $ad\ h$ on $\mathfrak{g}$ are all even integers. In the notation of 3.2.12 we conclude that

$$(*) \qquad n_e = 0, \quad n_0 = dim\ \mathfrak{h}.$$

In particular $dim\ Gx = dim\ \mathfrak{g} - dim\ \mathfrak{h}$. By 7.2.12 this is the maximal dimension of any nilpotent orbit. Notice that $\nu_{Gx}(\alpha) = 2,\ \forall\alpha \in \pi$ in this case.

7.2.14. Define a map $\psi : G \times \mathfrak{h} \mapsto \mathfrak{g}$, through $\psi(g,h) = g(h)$. Choose $h_0 \in \mathfrak{h}$ such that $\alpha(h_0) \neq 0, \forall\ \alpha \in \Delta$ (such an element is called regular). The tangent map $d_{h_0}\psi : \mathfrak{g}\otimes\mathfrak{h} \to \mathfrak{g}$ at h_0 is given by $(d_{h_0}\psi)(g,h) = [g,h_0]+h$ and hence has image $[\mathfrak{g},h_0]+\mathfrak{h} = \mathfrak{g}$. Consequently $\overline{G\mathfrak{h}} = \mathfrak{g}$. Of course this is still true for a second Cartan subalgebra $\mathfrak{h}'$. Now $G\mathfrak{h}$ is open in its closure which is irreducible forcing $G\mathfrak{h} \cap G\mathfrak{h}' \neq \emptyset$ and proving that $\mathfrak{h}'$ is conjugate to $\mathfrak{h}$. A combinatorial argument shows that any two positive root systems are conjugated by W. From this and the discussion in 7.2.7 we may conclude that any nilpotent $x \in \mathfrak{g}$ can be conjugated into some (fixed) choice of $\mathfrak{n}^+$. Combined with 7.2.13 and the finiteness of $\mathcal{N}/G$ we obtain the

Lemma.

(i) $G\mathfrak{n}^+ = \mathcal{N}$.

(ii) *$\mathcal{N}$ is an irreducible variety and* $|\mathcal{N}/G| \leq 3^{rk\mathfrak{g}} < \infty$.

(iii) *$\mathcal{N}$ admits a unique dense nilpotent orbit $\mathcal{O}_{reg}$.*

(iv) *In the notation of* 3.2.14 *one has $\mathcal{O}_{reg} = Gx_{reg}$ where*

$$x_{reg} = \sum_{\alpha \in \pi} x_\alpha.$$

(v) *$dim\ \mathcal{N} = dim\ \mathfrak{g} - dim\ \mathfrak{h}$.*

7.2.15. One calls $\mathcal{O}_{reg}$ the regular (or principal) nilpotent orbit. One calls a nilpotent orbit even if $n_e = 0$ in the notation of 3.2.12, equivalently if $dim\ \mathfrak{g}_1 = 0$. One calls a nilpotent orbit distinguished if $dim\ \mathfrak{g}_2 = dim\ \mathfrak{g}_0$. The regular nilpotent orbit is both even and distinguished.

Suppose a nilpotent orbit Gx is not distinguished. Since $\mathfrak{g}_2 = [x, \mathfrak{g}_0]$ by $\mathfrak{sl}(2)$ theory, we may conclude that $\mathfrak{g}_0^x \neq 0$. However $\mathfrak{g}_0^x = \mathfrak{g}_0^y$ again by $\mathfrak{sl}(2)$ theory. Consequently $\mathfrak{g}_0^x$ is a reductive subalgebra $\mathfrak{r}$ of $\mathfrak{g}$ containing x of strictly smaller rank (if $x \neq 0$). Let R be the connected algebraic subgroup of G with Lie algebra $\mathfrak{r}$. By repeating this process if necessary, we can assume that $Rx \subset \mathfrak{r}$ is distinguished. In this fashion the classification of nilpotent orbits is reduced to studying distinguished orbits in Levi factors. For example in type $\mathfrak{sl}(n)$ every distinguished orbit is the regular orbit. From this one easily concludes that every nilpotent orbit takes the form

$$\mathcal{O} = G\left(\sum_{\alpha \in \pi'} x_\alpha\right)$$

for some subset $\pi' \subset \pi$. (This is an expression of the Jordan block decomposition theorem up to ordering blocks by size). Through 7.2.12 we further obtain the semisimple element h of an s-triple containing x. Set $2\rho_{\pi'}^\vee = \sum_{\alpha \in \mathbb{N}\pi' \cap \Delta} \alpha^\vee$. Then $[h, x_\beta] = 2\rho_{\pi'}^\vee(\beta)x_\beta$ for all $\beta \in \Delta$. Let $\nu_{\pi'}$ be the unique dominant element in the W orbit of $2\rho_{\pi'}^\vee$. Then $\nu_{\mathcal{O}}(\alpha) = \nu_{\pi'}(\alpha)$. The determination of $\nu_{\pi'}$ is a purely combinatorial problem (involving W) normally carried out case by case.

7.2.16. The study of distinguished orbits is simplified greatly by the fact that such an orbit is even. At first only a long case by case proof was available; but Jantzen later gave a two-line proof using Richardson's theorem. Let P be a parabolic subgroup of G and set $\mathfrak{p} = Lie\ P$. Let $\mathfrak{m}$ be the nilradical of $\mathfrak{p}$. Richardson's theorem asserts that P has an open dense orbit in $\mathfrak{m}$. It is proved using Bruhat decomposition. It means that there exists $z \in \mathfrak{m}$ such that Pz has dimension $dim\ \mathfrak{m}$. Since $[\mathfrak{p}, z]$ is the tangent space to Pz at z, it follows that $[\mathfrak{p}, z]$ has dimension $dim\ \mathfrak{m}$ and hence equals $\mathfrak{m}$.

Lemma. *Every distinguished orbit is even.*

Proof. We apply the above to the Dynkin parabolic P. Take $z \in \mathfrak{g}$ and let z_i be the component of z in $\mathfrak{g}_i$. Then $[\mathfrak{p}, z] = \mathfrak{m}$ implies that

$$[\mathfrak{g}_0, z_1] = \mathfrak{g}_1,\ \ [\mathfrak{g}_0, z_2] + [\mathfrak{g}_1, z_1] = \mathfrak{g}_2.$$

Consequently

$$[\mathfrak{g}_0, (z_1 + z_2)] + [\mathfrak{g}_1, z_1] = \mathfrak{g}_1 + \mathfrak{g}_2$$

and so

$$dim\ \mathfrak{g}_1 + dim\ \mathfrak{g}_2 \leq dim\ \mathfrak{g}_0 + dim[\mathfrak{g}_1, z_1].$$

Thus $dim\ \mathfrak{g}_0 = dim\ \mathfrak{g}_2$ forces $dim\ \mathfrak{g}_1 = dim[\mathfrak{g}_1, z_1]$. Yet $[z_1, z_1] = 0$, so the latter equality forces $\mathfrak{g}_1 = 0$. $\square$

7.2.17. The result described in 7.2.9 first appeared in my mimeographed lecture notes of a troisieme cycle course on nilpotent orbits given in Paris 6 during 1981. The challenge was to use the pair $\mathfrak{n}_1^+, \mathfrak{n}_2^+$ associated to a nilpotent orbit $\mathcal{O}$ to similarly construct a completely prime, primitive ideal I of $U(\mathfrak{g})$ whose associated variety is $\bar{\mathcal{O}}$.

For any $f \in \mathfrak{g}^*$, the Kirillov-Kostant form B_f on $\mathfrak{g}$ is defined through $B_f(a, b) = f([a, b])$, for all $a, b \in \mathfrak{g}$. A subspace $\mathfrak{m}$ is said to be isotropic for B_f if $B_f(\mathfrak{m}, \mathfrak{m}) = 0$. Set $\mathfrak{g}(f) = \{x \in \mathfrak{g} \,\big|\, f(x) = 0\}$. The maximal dimension that an isotropic subspace can take is $\frac{1}{2}(dim\ \mathfrak{g} + dim\ \mathfrak{g}(f))$. A subalgebra of this dimension which is isotropic is called a polarization of f. In this case $x \mapsto f(x)$ defines a one-dimensional $U(\mathfrak{m})$ module k_f and one can consider the induced module $U(\mathfrak{g}) \otimes_{U(\mathfrak{m})} k_f$, whose annihilator is a completely prime ideal I_f of $U(\mathfrak{g})$. In principal the dimension condition on $\mathfrak{m}$ forces I_f to be primitive. For $\mathfrak{g}$ solvable this is ensured by taking a Vergne polarization. For $\mathfrak{g}$ semisimple it is automatic.

The above construction underlies the Dixmier map $f \mapsto I_f$ of $\mathfrak{g}^*/G$ into $Prim^1 U(\mathfrak{g})$, though one has to "twist" induction to ensure independence of the choice of polarization.

Now take $\mathfrak{g}$ semisimple and identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form (7.2.8). Let (x, h, y) be an s-triple and $\mathfrak{g} = \oplus \mathfrak{g}_i$ be the gradation defined by h as in 7.2.5. By $\mathfrak{sl}(2)$

theory $[y, \mathfrak{g}_1] = \mathfrak{g}_{-1}$. Since K restricts to a non-degenerate pairing $\mathfrak{g}_1 \times \mathfrak{g}_{-1} \to k$ it follows that the Kirillov-Kostant form $B_y(a, b) = K(y, [a, b])$ restricts to a non-degenerate form on $\mathfrak{g}_1$. (In particular $\mathfrak{g}_1$ must be even dimensional; but this already follows from 7.2.12.) Let $\mathfrak{m}^-$ be a maximal isotropic subspace of $\mathfrak{g}_1$ with respect to this restriction and set $\mathfrak{p}^- = \bigoplus_{i \le 0} \mathfrak{g}_i$. Then $\mathfrak{p}^- \oplus \mathfrak{m}^-$ is a maximal isotropic subspace for B_y. Unfortunately we cannot hope to choose $\mathfrak{m}^-$ so that $\mathfrak{p}^- \oplus \mathfrak{m}^-$ is a subalgebra, since we already know that not every $y \in \mathfrak{g}^*$ admits a polarization. More specifically a polarization must be a parabolic subalgebra and this is already excluded if $\mathfrak{p}^-$ is maximal parabolic. This bad situation always occurs for the nilpotent orbit of minimal non-zero dimension if $\mathfrak{g}$ is simple and not of type A. However when $\mathfrak{p}^- \oplus \mathfrak{m}^-$ is a subalgebra, then the corresponding induced module identifies with $S(\mathfrak{n})$, where $\mathfrak{n} = \mathfrak{n}_2^+ \oplus \mathfrak{m}$ and $\mathfrak{m}$ is a complement to $\mathfrak{m}^-$ in $\mathfrak{g}_1$ which we can assume to be isotropic. Notice that $\mathfrak{n}$ is always a subalgebra and $x \mapsto f(x)$ defines a one dimensional $\mathfrak{n}$ module if $\mathfrak{m}$ is isotropic.

That $\mathfrak{n}$ is a subalgebra led C. Moeglin to the following construction. Suppose $I \in Prim^1 U(\mathfrak{g})$ has $\bar{\mathcal{O}}$ as its associated variety. Call $M = U(\mathfrak{g})/(I + \sum\limits_{x \in \mathfrak{n}} U(\mathfrak{g})(x - f(0))$ a generalized Whittaker module for I if $I = Ann\ M$. Then under certain additional hypotheses involving a filtration introduced by Kazhdan, $F_{\mathfrak{g}}(M)$ is a Dixmier algebra in the strong sense defined by Vogan. More recent Premet has used a similar construction to describe the regular functions on $\bar{\mathcal{O}}$.

7.3. The Steinberg Map.

From now on we take G to be the simply-connected cover of the adjoint group. For the most part this change affects nothing - only the detailed structure of the flag variety, which we shall only touch upon. Again some arguments go through in positive characteristic, G being replaced by the corresponding Chevalley group whose construction will be briefly discussed (7.4.1).

7.3.1. The Kostant separation theorem (7.1.11) leads to number of important results on $Prim\ U(\mathfrak{g})$. We shall not go into this (very long) story here; but just recall a few facts. First there exists an $ad\,U(\mathfrak{g})$ invariant subspace $\mathbb{H}$ of $U(\mathfrak{g})$ such that the multiplication map gives an isomorphism $\mathbb{H} \otimes Z(\mathfrak{g}) \xrightarrow{\sim} U(\mathfrak{g})$ of $ad\ U(\mathfrak{g}) - Z(\mathfrak{g})$ bimodules. Furthermore since $\mathfrak{g}$ is reductive, $\mathbb{H}$ is a direct sum of simple finite dimensional modules each module E occuring with multiplicity $dim\ E_0$. These facts, interpreted in terms of principal series modules, lead quickly to Duflo's theorem which describes $Prim\ U(\mathfrak{g})$ in the following fashion. Recall that for each $\lambda \in \mathfrak{h}^*$ there exists a unique, up to isomorphism, simple $U(\mathfrak{g})$ module $L(\lambda)$ with highest weight λ. Set $I(\lambda) = Ann\ L(\lambda)$ which is a primitive ideal. Duflo's theorem asserts that the map $\lambda \mapsto I(\lambda)$ of $\mathfrak{h}^*$ into $Prim\ U(\mathfrak{g})$ is surjective. A crucial question had been to determine the fibres of this map. This has now been completely resolved. The first ingredient was the use of the centre $Z(\mathfrak{g})$, more precisely the map $I(\lambda) \mapsto I(\lambda) \cap Z(\mathfrak{g})$ of $Prim\ U(\mathfrak{g})$ into $Max\ Z(\mathfrak{g})$. The latter is identified with $\mathfrak{h}^*/W$. and then $\lambda \mapsto I(\lambda) \cap Z(\mathfrak{g})$ becomes the map $\lambda \mapsto W.\lambda$. In particular $Prim\ U(\mathfrak{g})$ is sandwiched between $\mathfrak{h}^*$ and $\mathfrak{h}^*/W$.. Its precise positioning depends on the level of regularity and integrality of λ. For λ in general position, the inverse image of $W.\lambda$ in $Prim\ U(\mathfrak{g})$ is a singleton. The other extreme is when $\lambda \in P^+(\pi)$. Restricted to $W.\lambda : \lambda \in P^+(\pi)$ the fibres of the map $\mu \mapsto Ann\ L(\mu)$ define subsets of W called left cells (so-called because of their relation to left ideals in $\mathbb{Q}W$). By the Borho-Jantzen translation principle the left cells are independent of the choice of $\lambda \in P^+(\pi)$. The primitive ideals corresponding to a fixed cell form a so-called coherent family and rather remarkably the function $\mu \mapsto rk(U(\mathfrak{g})/I(\mu))$ restricted to such a family extends to a polynomial on $\mathfrak{h}^*$. These "Goldie rank polynomials" are moreover linearly independent. Up to still unknown scalars they can be described in terms of the Kazhdan-Lusztig polynomials. Similar results hold for arbitrary λ if one replaces W by the subgroup generated by the reflections corresponding to the roots integral on λ.

7.3.2. Besides the above there is also a "sandwich" given in terms of associated varieties. Let $\mathcal{V}(\lambda)$ be the associated variety of $L(\lambda)$ and $\mathcal{W}(\lambda)$ the associated variety of $U(\mathfrak{g})/I(\lambda)$. It turns out that $\mathcal{W}(\lambda)$ is irreducible though this is not easy to prove. Moreover it coincides with the G saturation set of $\mathcal{V}(\lambda)$, namely one has $\mathcal{W}(\lambda) = G\mathcal{V}(\lambda)$. Surprisingly $\mathcal{V}(\lambda)$ need not be irreducible, though by a result of Gabber it is always equidimensional. Determining its components is an interesting open question which we will come to in Chapter 9.

Since $I(\lambda) \cap Z(\mathfrak{g})$ is a maximal ideal, it follows that so is $gr\ I(\lambda) \cap Y(\mathfrak{g})$ and hence the augmentation ideal $Y(\mathfrak{g})_+$. Consequently $\mathcal{W}(\lambda) \subset \mathcal{N}$. Being irreducible and G invariant it must admit a unique dense orbit $\mathcal{O}(\lambda)$ by 7.2.14(ii). On the other hand $\mathcal{V}(\lambda)$ is contained in $\mathfrak{n}^+$ and B invariant (because $L(\lambda)$ is highest weight) as well as being involutive (by Gabber's theorem). The first statement implies that $\mathcal{V}(\lambda) \subset \mathcal{W}(\lambda) \cap \mathfrak{n}^+ = \bar{\mathcal{O}} \cap \mathfrak{n}^+$. Now as we shall see the left variety is equidimensional and its components are closures of Lagrangian subvarities of $\mathcal{O} \cap \mathfrak{n}^+$. We call these orbital varieties and we study them more fully in Chapter 9. Involutivity then forces $\mathcal{V}(\lambda)$ to be a union of some of these components.

We may sum up the above by saying that there is a map $\lambda \mapsto \mathcal{O}(\lambda)$ of $\mathfrak{h}^*$ into $\mathcal{N}/G$ and sandwiched in-between $\mathfrak{h}^*$ and $\mathcal{N}/G$ one finds Lagrangian subvarieties of $\mathcal{O}(\lambda)$.

7.3.4. When the maps defined in 7.3.2 and 7.3.3 are combined one obtains an even more remarkable picture. Assume for simplicity that $\lambda \in P^+(\pi)$. Then restricted to $W.\lambda$ the fibres of the map $\mu \mapsto \mathcal{O}(\mu)$ (where we recall that $\mathcal{O}(\mu)$ is the unique dense orbit in $\mathcal{V}(Ann\ L[\mu]))$ defines subsets of W called two-sided cells. These are disjoint union of left cells and the corresponding Goldie rank polynomials form a basis of an irreducible representation of W. The right cells we obtained by replacing w by w^{-1} whilst the two-sided cells are invariant under this operation.

7.3.5. Obviously we should check with the geometers if such phenomena occur in a purely geometric context. Indeed already a sandwich similar to that of 7.3.3. was already provided by the Steinberg map which we now study.

7.3.6. For each $w \in W$ we consider the G saturation set $G(\mathfrak{n}^+ \cap w(\mathfrak{n}^+))$ of the subspace $\mathfrak{n}^+ \cap w(\mathfrak{n}^+)$. Since G is connected, it is an irreducible G stable subvariety of $\mathcal{N}$. Since $\mathcal{N}/G$ is finite by 7.2.14(ii) it follows that $G(\mathfrak{n}^+ \cap w(\mathfrak{n}^+))$ admits a unique dense nilpotent orbit denoted $St(w)$. The map $w \mapsto St(w)$ of W into $\mathcal{N}/G$ is called the Steinberg map. Its fibres are called the two-sided geometric cells. They have not yet been completely determined. Outside type A they difer slightly from the two-sided cells of W; but remarkably they are determined by the same algorithm using only different initial data. More precisely they two-sided cells are computed from the $|W| \times |W|$ matrix with integer entries given by the values of the Kazhdan-Lusztig polynomials at $q = 1$, whereas the two-sided geometric cells are computed in exactly the same way but using a different matrix with entries given in terms of singularity data involving conormals of the cotangent bundle of the flag variety.

Proposition. *The Steinberg map is surjective.*

Proof. Let Gx be a nilpotent orbit. Let π' be the subset of π defining the Levi factor $\mathfrak{r}_{\pi'}$ in which x generates a distinguished orbit and let ν' be the Dynkin map of this orbit. By 7.2.16 this orbit is even and so there exists $\pi'' \subset \pi'$ such that $\nu'(\alpha) = 2$, if $\alpha \in \pi''$ and $\nu'(\alpha) = 0$, if $\alpha \in \pi' \setminus \pi''$. Set $\Delta_{\pi'} = \Delta \cap \mathbb{Z}\pi'$, $\Delta_{\pi''} = \Delta \cap \mathbb{Z}\pi''$, $\Delta^{\pm}_{\pi'} = \Delta_{\pi'} \cap \Delta^{\pm}$, $\Delta^{\pm}_{\pi'} =$

$\Delta_{\pi''}\cap\Delta^{\pm}$ with $W_{\pi'}, W_{\pi''}$ being the corresponding subgroups of W. Let R' be the connected subgroup of G with Lie algebra $\mathfrak{r}_{\pi'}$ and P'' the Dynkin parabolic of R' corresponding to the distinguished nilpotent orbit $R'x$. Let $\mathfrak{m}''$ be the nilradical of *Lie* P''. By Richardson's theorem $\overline{P''x} = \mathfrak{m}''$ and so $\overline{Gx} = G\mathfrak{m}''$. Thus to prove the proposition it is enough to find $w \in W$ such that $\mathfrak{m}'' = \mathfrak{n}^+ \cap w(\mathfrak{n}^+)$.

By definition

$$\mathfrak{m}'' = \bigoplus_{\beta\in\Delta^+_{\pi'}\setminus\Delta^+_{\pi''}} \mathfrak{g}_\beta.$$

Let w_π (resp. $w_{\pi'}, w_{\pi''}$) be the unique element of W (resp. $W_{\pi'}, W_{\pi''}$) sending Δ^+ (resp. $\Delta^+_{\pi'}, \Delta^+_{\pi'}$) to Δ^- (resp. $\Delta^-_{\pi'}, \Delta^-_{\pi'}$). One has

$$w_\pi w_{\pi'} w_{\pi''}(\Delta^+_{\pi'} \setminus \Delta^+_{\pi''}) \subset w_\pi w_{\pi'}(\Delta^+_{\pi'} \setminus \Delta^+_{\pi''}) \subset w_\pi \Delta^-_{\pi'} \subset \Delta^+. \qquad (*)$$

On the other hand the reduced length of $w := w_\pi w_{\pi'} w_{\pi''} = |\Delta^+| - |\Delta^+_{\pi'} \setminus \Delta^+_{\pi''}|$. Combined with $(*)$ this forces for all $\beta \in \Delta^+$ that

$$w\beta \in \Delta^- \iff \beta \in \Delta^+ \setminus (\Delta^+_{\pi'} \setminus \Delta^+_{\pi''})$$

and so w is the required element. □

7.3.7. Let $\mathcal{O}$ be a nilpotent orbit. By definition an orbital variety (associated to $\mathcal{O}$) is an irreducible component of $\mathfrak{n}^+ \cap \mathcal{O}$.

Proposition. *The irreducible components of* $\mathfrak{n}^+\cap\mathcal{O}$ *are the* $\mathcal{V}(w) := \overline{B(\mathfrak{n}^+ \cap w(\mathfrak{n}^+))}\cap\mathcal{O}$: $w \in St^{-1}(\mathcal{O})$. *Moreover these are Lagrangian subvarieties of* $\mathcal{O}$.

7.3.8. The proof of 7.3.7 depends on Bruhat decomposition and the construction of the Steinberg variety. This will be given in the next section. For the moment we observe that $\mathcal{V} := \{\mathcal{V}(w); w \in \mathbf{W}\}$ is sandwiched between W and $\mathcal{N}/G$. The fibres of the map $w \mapsto \mathcal{V}(w)$ of W onto $\mathcal{V}$ are called (right) geometric cells. They have not been completely determined; but they are known to coincide with the right cells in type A and differ slightly from them in the general case.

The last part of 7.3.7 obtains from the following lemma and a dimension estimate given in 7.5.7(i).

Lemma. *$\mathcal{V}(w)$ is an involutive subvariety of $St(w)$.*

Proof. Identify $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form K. Let I be the ideal of the definition of $\overline{\mathcal{V}(w)}$. We must show that $\{I, I\} \subset I$ with respect to the Poisson bracket on $S(\mathfrak{g})$.

Set $S(w) = \{\beta \in \Delta^+ \big| w\beta \in \Delta^-\}$. Then

$$\mathfrak{n}_w^+ := \mathfrak{n}^+ \cap w(\mathfrak{n}^+) = \bigoplus_{\beta \in \Delta^+ \setminus S(w^{-1})} \mathfrak{g}_\beta$$

Consequently

$$\mathfrak{n}_w^- = \bigoplus_{\beta \in S(w^{-1})} \mathfrak{g}_{-\beta}$$

is the orthogonal of $\mathfrak{n}_w^+$ in $\mathfrak{n}^-$. Since $\overline{\mathcal{V}(w)} = \overline{B\mathfrak{n}_w^+}$, its ideal of definition I is just the largest B stable ideal of $S(\mathfrak{g})$ in $S(\mathfrak{g})(\mathfrak{b} + \mathfrak{n}_w^-) = S(\mathfrak{g})\mathfrak{b} + S(\mathfrak{n}^-)\mathfrak{n}_w^-$. Observe that $S(\mathfrak{g})\mathfrak{b}$ is B stable and identify $S(\mathfrak{n}^-)$ with the B module $S(\mathfrak{g})/S(\mathfrak{g})\mathfrak{b}$. Then I takes the form $S(\mathfrak{g})b + J$, where J is the largest B stable ideal of $S(\mathfrak{n}^-)$ contained in $S(\mathfrak{n}^-)\mathfrak{n}_w^-$. Yet $\mathfrak{n}_w^-$ is a subalgebra of $\mathfrak{n}^-$ and so $\{J, J\} \subset J$. Since $\mathfrak{b}$ is a subalgebra and J is B stable, this gives $\{I, I\} \subset I$, as required. □

7.4. The Steinberg Variety.

7.4.1. The Steinberg variety is of fundamental importance in the study of nilpotent orbits, because it brings out their deep relation with the flag variety and consequently with Bruhat decomposition.

Let B be the connected subgroup of G whose Lie algebra is $\mathfrak{b} := \mathfrak{n}^+ \oplus \mathfrak{h}$. It is a maximal solvable subgroup of G, that is to say a Borel subgroup. One has $N_G(B) = B$ and furthermore any Borel subgroup of G is conjugate to B and so the set $\mathcal{B}$ of all Borel subgroup of G identifies with G/B. The latter is a projective algebraic variety. For $G = GL(V)$, the reduction of B to upper triangular form exactly corresponds to the choice of a flag in V and so G/B coincides with variety of all flags in V. (When $dim\ V = 2$, this is just the projective line $\mathbb{P}^1$.) For this reason one simply calls G/B the flag variety of G for any reductive algebraic group G.

Let H be the connected subgroup of G with Lie algebra $\mathfrak{h}$ and recall that $W = N_G(H)/H$. For each $w \in W$, let $n_w \in N_G(H)$ be a representative of w in $N_G(H)$. Bruhat decomposition asserts that

$$G = \coprod_{w \in W} Bn_wB.$$

Given $w \in W$ we write $n_w B n_w^{-1}$ simply as ${}^wB$. Let Z_w be the G orbit in $\mathcal{B}\times\mathcal{B}$ generated by $(B, {}^w B)$. The Bruhat decomposition implies that every G orbit is of this form. Furthermore the Z_w are locally closed and $\bar{Z}_w = \coprod_{y \leq w} Z_y$, where $\leq$ denotes Bruhat order (3.2.5).

Much of the present analysis is valid for an arbitrary closed field k of positive characteristic, where G is the Chevalley group associated to $\mathfrak{g}$ defined as follows. Let $\{x_\alpha, h_\beta\}_{\alpha\in\Delta,\beta\in\pi}$ be a Chevalley basis for $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{Z}$ the $\mathbb{Z}$ submodule they generate. A key property of the Chevalley basis is that for each $\alpha \in \pi$ the exponential $exp\ ad\ x_\alpha$ defines an automorphism of $\mathfrak{g}_\mathbb{Z}$. (It is obvious that $exp\ ad\ x_\alpha$ restricts to a finite sum on $\mathfrak{g}$; but then some integrality conditions also have to be verified). Then for each $\lambda \in k$ one obtains an automorphism $exp\ \lambda\ ad\ x_\alpha$ of $\mathfrak{g}_k := \mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} k$. These elements form a subgroup U_α of $Aut\ \mathfrak{g}_k$ isomorphic to k which is a copy of $\mathbb{A}^1$, as an algebraic variety. One defines G to be the subgroup of $Aut\ \mathfrak{g}_k$ generated by the $U_\alpha : \alpha \in \Delta$. The subgroup L_α generated by $U_\alpha, U_{-\alpha} : \alpha \in \Delta^+$ is a homomorphic image of $SL(2,k)$. Let $H_\alpha : \alpha \in \Delta^+$ denote the image $SL(2,k)$ in L_α. Then H is defined to the subgroup of $Aut\ \mathfrak{g}_k$ generated by the $H_\alpha : \alpha \in \Delta^+$. It acts by conjugation on each $U_\alpha : \alpha \in \Delta^+$ and hence on the subgroup U they generate. Then $B := HU$ and is a semidirect product.

Call $u \in G$ unipotent if $u - e$ is nilpotent and let $\mathcal{U}$ denote the set of unipotent elements of G. Clearly $\mathcal{U}$ is a disjoint union of G orbits (under conjugation) and if $char\ k = 0$, these are in bijection with the nilpotent orbits in $\mathfrak{g}$ via the exponential map. That there are also only finitely many G orbits in $\mathcal{U}$ also in positive characteristic, is due to G. Lusztig.

One calls

$$\mathcal{S} := \{(u, B_1, B_2) \in \mathcal{U}\times\mathcal{B}\times\mathcal{B} \,\big|\, u \in B_1 \cap B_2\}$$

the Steinberg variety. It is a closed subvariety of $\mathcal{U}\times\mathcal{B}\times\mathcal{B}$. Indeed it is the inverse image of the continuous map $(u, B_1, B_2) \mapsto (u, B_1, uB_1u^{-1}, B_2, uB_2u^{-1})$ of $\mathcal{U}\times\mathcal{B}\times\mathcal{B}$ into $\mathcal{U}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B}\times\mathcal{B}$ restricted to the diagonal $(u, B_1, B_1, B_2, B_2) : u \in \mathcal{U};\ B_1, B_2 \in \mathcal{B}$. A key role of $\mathcal{S}$ is to play off the above two decompositions of $\mathcal{U}$ and $\mathcal{B}\times\mathcal{B}$. However it also incorporates a number of other features, for example for each $u \in \mathcal{U}$ one may define the fixed point set $\mathcal{B}_u = \{B \in \mathcal{B} \,\big|\, uBu^{-1} = B\}$ which is a closed subvariety of $\mathcal{B}$. Since $N_G(B) = B$, this is just $\{B \in \mathcal{B} \,\big|\, u \in B\}$ and so coincides with projection onto the second (or third) factor of the fibre in $\mathcal{S}$ over u.

7.4.2. For each conjugacy class $\mathcal{C}$ (that is G orbit) in $\mathcal{U}$ let $\bar{\mathcal{C}}$ denote its closure in $\mathcal{U}$ and $\mathcal{S}(\mathcal{C})$ (resp. $\mathcal{S}(\bar{\mathcal{C}})$) denote the inverse image of $\mathcal{C}$ (resp. $\bar{\mathcal{C}}$) in $\mathcal{S}$ with respect to projection onto the first factor. Clearly $\mathcal{S}$ is a finite disjoint union of the $\mathcal{S}(\mathcal{C})$. As in 7.4.1 it follows that $\mathcal{S}(\bar{\mathcal{C}})$ is closed in $\mathcal{S}$. On the other hand $\mathcal{S}(\mathcal{C}) = G(u, \mathcal{B}_u, \mathcal{B}_u)$ and so is locally closed in $\mathcal{S}$.

Set $U = B \cap \mathcal{U}$ which is just the unipotent radical of B. For each $w \in W$, set $U_w = U \cap^w U$ which is isomorphic to $|\Delta^+| - \ell(w)$ copies of the affine line, so irreducible. Since G is connected GU_w is also irreducible, so by the finiteness of $\mathcal{U}/G$, it admits a unique dense orbit, denoted $St(w)$. This defines the Steinberg map of W into $\mathcal{U}/G$ (analogous to the Steinberg map of W onto $\mathcal{N}/G$ and equivalent to it in characteristic zero by the exponential map).

Set $\mathcal{S}_w(\mathcal{C}) = \{(u, B_1, B_2) \in \mathcal{S}(\mathcal{C}) \,\big|\, (B_1, B_2) \in Z_w\}$. Being an inverse image in $\mathcal{S}(\mathcal{C})$ of the locally closed set Z_w, it is locally closed.

Lemma. *$\mathcal{S}_w(\mathcal{C})$ has dimension $\leq$ (dim G – rk G). Equality holds exactly when $w \in St^{-1}(\mathcal{C})$ and in this case $\overline{\mathcal{S}_w(\mathcal{C})}$ is an irreducible component of $\mathcal{S}(\mathcal{C})$.*

Proof. The projection of $\mathcal{S}_w(\mathcal{C})$ onto Z_w has fibre $\mathcal{C} \cap B \cap^w B = \mathcal{C} \cap U_w$ at $(B,^w B)$. The remaining fibres are G translates of $\mathcal{C} \cap U_w$ and are so all isomorphic. Hence $dim\ \mathcal{S}_w(C) = dim\ Z_w + dim(\mathcal{C} \cap U_w)$. Since $N_G(B) = B$ (and of course $N_G(^wB) =^w B$), the element $(B,^w B) \in \mathcal{B} \times \mathcal{B}$ is fixed by $g \in G$ if and only if $g \in B \cap^w B$. Thus Z_w identifies with $G/(B \cap^w B)$. Yet $B \cap^w B = H \rtimes U_w$ and so we conclude that $dim\ \mathcal{S}_w(\mathcal{C}) \leq dim\ G - dim\ H$ with equality if and only if $\mathcal{C} \cap U_w$ is dense in U_w, that is if $w \in St^{-1}(\mathcal{C})$. Finally if this holds, $\mathcal{C} \cap U_w$ is irreducible and hence so is $St_w(\mathcal{C})$ being the G saturation of $(\mathcal{C} \cap U_w, B,^w B)$.

7.4.3. There is a second way to describe irreducible components of $\mathcal{S}(\mathcal{C})$. This relies on a remarkable result of N. Spaltenstein which asserts that the fixed point set $\mathcal{B}_u : u \in \mathcal{C}$ is equidimensional. To avoid a long digression we shall postpone the proof of this latter result till section 7.5. Let $\{X_\sigma\}_{\sigma \in \Sigma}$ denote the set of irreducible components of $\mathcal{B}_u$.

Let $Z_G(u)$ be the centralizer of u in G, and $Z^0_G(u)$ the connected component of the identity. The component group $A_u := Z_G(u)/Z^0_G(u)$ is finite. It plays a very significant role in the Springer classification theory of nilpotent orbits and this in turn has important implications for some fine details in the classification of $Prim\ U(\mathfrak{g})$. Here we just note that $Z_G(u)$ acts on $\mathcal{B}_u$. Being connected $Z^0_G(u)$ leaves each irreducible component $X_\sigma : \sigma \in \sum$, of $\mathcal{B}_u$ invariant. This leads to an induced action of A_u on $\sum$ which decomposes into finitely many orbits $\sum_i : i \in \mathcal{I}$.

Let $p : \mathcal{S}(\mathcal{C}) \to \mathcal{C}$ be the projection onto the first factor. Then $p^{-1}(u)$ identifies with $\mathcal{B}_u \times \mathcal{B}_u$. With respect to the diagonal action of G set

$$Y_{(\sigma,\tau)} = Z_G(u)(X_\sigma \times X_\tau), Z_{(\sigma,\tau)} = \overline{G(X_\sigma \times X_\tau)} : \sigma, \tau \in \sum .$$

Lemma. *The $Z_{(\sigma,\tau)} : (\sigma,\tau) \in (\sum \times \sum)/A_u$ are the distinct irreducible components of $\mathcal{S}(\mathcal{C})$. They have dimension $dim\ \mathcal{C} + 2dim\ \mathcal{B}_u$.*

Proof. Since $X_\sigma \times X_\tau$ is irreducible and G is connected, $Z_{(\sigma,\tau)}$ is irreducible and closed by construction. Since $G(\mathcal{B}_u \times \mathcal{B}_u) = \mathcal{S}(\mathcal{C})$, it follows that their union is $\mathcal{S}(\mathcal{C})$. Since G acts isomorphically on the fibres of p, it follows that $dim\ Z_{(\sigma,\tau)} = dim\ \mathcal{C} + dim(X_\sigma \times X_\tau) = dim\ \mathcal{C} + 2dim\ \mathcal{B}_u$, by 7.5.7 (ii). Thus the $\{Z_{(\sigma,\tau)} : \sigma, \tau \in \sum\}$ is the set of irreducible components of $\mathcal{S}(\mathcal{C})$. To determine which are distinct observe that $(u_1, B_1, B_2) \in G(X_\sigma \times X_\tau)$ if and only if $(B_1, B_2) \in Y_{(\sigma,\tau)}$. Since the $Y_{(\sigma,\tau)} : (\sigma,\tau) \in \sum \times \sum /A_u$ form an irredundent union of $\mathcal{B}_u \times \mathcal{B}_u$, it follows that $Z_{(\sigma,\tau)} = Z_{(\sigma',\tau')}$ if and only if (σ,τ) and (σ',τ') lie in the same A_u orbit.

Remark. Thus $\mathcal{S}(\mathcal{C})$ is equidimensional and by 7.4.2 its irreducible components are the $\overline{\mathcal{S}_w(\mathcal{C})} : w \in St^{-1}(\mathcal{C})$. □

7.4.4. The analysis in 7.4.1 - 7.4.3 is equally valid if $char\ k > 0$ (but k is still required to be algebraically closed, for one thing to define an algebraic group but also so that a Borel subgroup should be sufficiently large). Now we need that the Steinberg map is surjective which we have only proved (7.3.6) for $char\ k = 0$. This gives the following corollary.

Corollary. *Let $\mathcal{C} \subset \mathcal{U}$ be a conjugacy class. Every component of $\mathcal{S}(\mathcal{C})$ has dimension $dim\ G - rk\ G$ and furthermore $2dim\ \mathcal{B}_u + dim\ \mathcal{C} = dim\ G - rk\ G,\ \forall\ u \in \mathcal{U}$.*

Proof. Combine 7.4.2, 7.4.3 and the surjectivity of St. □

Remark. In particular $\mathcal{S}(\bar{\mathcal{C}}) = \coprod_{\mathcal{C}' \subset \bar{\mathcal{C}}} \mathcal{S}(\mathcal{C}')$, has dimension $|\Delta|$, so in general the inclusion $\mathcal{S}(\bar{\mathcal{C}}) \supset \overline{\mathcal{S}(\mathcal{C})}$ is strict.

7.5. The Fixed Point Set $\mathcal{B}_{\mathbf{u}}$.

7.5.1. Given $u \in \mathcal{U}$, set $\mathcal{B}_u = \{B \in \mathcal{B} \,\big|\, uBu^{-1} = B\}$. It is a closed subvariety of the projective variety $\mathcal{B}$, hence is projective. Our aim is to show that all the components of $\mathcal{B}_u$ have dimension $\frac{1}{2}(dim\ G - rk\ G - dim\ Gu)$. Our first result uses Bruhat decomposition to show that $\mathcal{B}_u$ is arc-wise connected.

7.5.2. A (standard) parabolic subgroup P of G is a subsgroup containing the (fixed) Borel B containing a maximal torus H. Given $\pi' \subset \pi$, let $W_{\pi'}$ be the subgroup of W generated by the reflections $s_\alpha : \alpha \in \pi'$. For each w let $n_w \in N_G(H)$ be a representation of $w \in W = N_G(H)/H$. Then Bruhat decomposition asserts that

$$(*) \qquad G = \coprod_{w \in W} Bn_wB.$$

For each $\pi' \subset \pi$, set

$$P_{\pi'} = \coprod_{w \in W_{\pi'}} Bn_wB.$$

From $(*)$ and the multiplication properties of the double cosets Bn_wB it is not difficult to show that the map $\pi' \mapsto P_{\pi'}$ is a bijection between the set of all subsets of π and the set of all subgroups of G containing B. From this it follows that $N_G(P) = P$, for any parabolic P of G. If $char\ k = 0$, then $P_{\pi'}$ is a unique connected subgroup of G with Lie algebra $\mathfrak{p}_{\pi'}$ (as defined in).

Let $L_{\pi'}$ denote the subgroup of G generated over H by the $U_\alpha : \alpha \in \Delta^+ \setminus \Delta^+_{\pi'}$. From the above description of $P_{\pi'}$ one shows that $P_{\pi'} = L_{\pi'}U_{\pi'}$. Moreover the latter is a semi-direct product with $U_{\pi'}$ the unipotent radical of $P_{\pi'}$ and $L_{\pi'}$ its so-called Levi facataor. In particular $P_{\pi'}/U_{\pi'} \cong L_{\pi'}$. In the particular case when $\pi' = \alpha$, we write $P_{\pi'} = P_\alpha$, $U_{\pi'} = U_\alpha$, $L_{\pi'} = L_\alpha$, Then $P_\alpha/U_\alpha \cong L_\alpha$ and the latter is a homomorphic image of $SL(2,k)$. It follows from the corresponding result for $SL(2,k)$ that P_α/B is isomorphic to the projective line $\mathbb{P}^1$.

7.5.3. Fix $\alpha \in \pi$. For all $g \in G$ one calls $g(P_\alpha/B) \subset G/B$, a line of type α. If $g(P_\alpha/B) \cap g'(P_\alpha/B) \neq \phi$, then $g^{-1}g' \in P_\alpha$ and so distinct lines of type α have null intersection. One says that $x, x' \in G/B$ are connected by a line of type α if there exists $g \in G$ such that $x, x' \in g(P_\alpha/B)$. Recall that each $u \in \mathcal{U}$ can be conjugated into U.

Proposition. *Take $u \in U$ and $x, x' \in \mathcal{B}_u$. Then there exists $t \in \mathbb{N}$ and a sequence $x = x_1, x_2, \cdots, x_{t+1} = x'$ such that for all $i = 1, 2, \cdots, t$, the points x_i, x_{i+1} are connected by a line of type $\alpha_i \in \pi$ contained in $\mathcal{B}_u$.*

Proof. We can assume $x = B$ and $x' = gBg^{-1}$. By Bruhat decomposition there exists $w \in W$ such that $g \in Bn_wB$ and we can take $g \in Bn_w$ without loss of generality.

Let $w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_t}$ be a reduced decomposition. Write u_{s_i} (resp. P_{s_i}) simply as u_i (resp. P_i). Then $g = bu_1u_2\cdots u_i$ and $x_i = g_iBg_i^{-1}$.

As $P_{i+1} = B \coprod Bu_{i+1}B$, the line $g_i(P_{i+1}/B)$ contains both x_i and x_{i+1}. It remains to show that it lies in $\mathcal{B}_u$. This is achieved by decreasing induction on i, which establishes in particular that $x_i \in \mathcal{B}_u$. Thus it suffices to consider the case $i = t$, all other cases being similar. Set $g' = g_\ell, \alpha' = \alpha_\ell$ and $P' = P_\ell$. Observe that $w\alpha' \in \Delta^-$.

Since $gBg^{-1} = x' \in \mathcal{B}_u$, one has $u \in gBg^{-1}$. Consequently

$$g^{-1}ug \in U \cap n_w^{-1}Bn_w = \prod_{\alpha \in \Delta^+ \mid w\alpha \in \Delta^+} U_\alpha \subset \prod_{\alpha \in \Delta^+ \setminus \{\alpha'\}} U_\alpha =: M_{\alpha'}.$$

Yet $M_{\alpha'}$ is the nilradical of $P_{\alpha'}$, so normal in $P_{\alpha'}$. Since $P'g'^{-1} = P'g^{-1}$ it follows that $pg'^{-1}ug'p^{-1} \in M_\alpha \subset B$ and so $u \in g'pBp^{-1}g'^{-1}$, for all $p \in P'$. Equivalently $g'(P'/B) \subset \mathcal{B}_u$. □

7.5.4. A key point in the proof of the equidimensionality of $\mathcal{B}_u$ is the application of Krull's theorem. Fix $\alpha \in \dot{\pi}$ and set $r = |\Delta^+|$. Then $U \cong \mathbb{A}^r$ in which M_α is a hyperplane. Thus if C is a closed irreducible subvariety of dimension d of U, then every irreducible component of $C \cap M$ has dimension $d - 1$. This will be used to prove that $\mathcal{C} \cap U$ is equidimensional for any conjugacy class $\mathcal{C} \subset \mathcal{U}$.

Set $P = P_\alpha$, $M = M_\alpha$ and let C be an irreducible component of $\mathcal{C} \cap U$. For any subset $S \subset G$ and any subgroup $G' \subset G$ set ${}^{G'}S = \{gsg^{-1} \mid g \in G', s \in S\}$. Since C and $C \cap M$ are B stable and P/B is projective, it follows that ${}^PC$ and ${}^P(C \cap M)$ are closed. Moreover ${}^P(C \cap M) = {}^P C \cap M$ since M is normal in P. Set $s_\alpha(C) = C \cup^P (C \cap M)$, which is a closed subvariety of $C \cap U$. (We shall eventually show how to interpret this as an action of W on the irreducible components of $\mathcal{C} \cap U$, or more precisely on its zero homology space). Set $d = dim(\mathcal{C} \cap U)$.

Lemma. *Let C be an irreducible component of $\mathcal{C} \cap U$ of maximal dimension d. Then $s_\alpha(C)$ is equidimensional of dimension d.*

Proof. Suppose $C \subset M$. Then ${}^P C \subset \mathcal{C} \cap M \subset \mathcal{C} \cap N$ and so $dim({}^P C) \leq d = dim\ C$. Since C is irreducible and P is connected, ${}^P C$ is irreducible and so ${}^P C = C$. (Thus C is P stable in this case.) Consequently $s_\alpha(C) = C$.

Suppose $C \not\subset M$. Then every irreducible component D of $C \cap M$ has dimension $d-1$. Moreover since P is connected the irreducible components of $s_\alpha(C)$ are C and the ${}^P D$ not contained in C.

Suppose ${}^P D = D$. Then ${}^P D \subsetneq C$ and so is not an irreducible component of $s_\alpha(C)$.

Suppose ${}^P D \supsetneq D$. Then $dim {}^P D > dim\ D = d-1$. Yet ${}^P D \subset {}^P (C \cap M) \subset \mathcal{C} \cap U$ and so $dim {}^P D = d$, as required. In this case only C is not P stable amongst the irreducible components of $s_\alpha(C)$. $\square$

7.5.5. Let $\mathcal{C} \subset \mathcal{U}$ be a conjugacy class and take $u \in \mathcal{C}$. We now compare components in $\mathcal{B}_u$ with components in $\mathcal{C} \cap U$. Then using 7.5.3 we show that every component of the latter can be reached by successively applying the operation $C \mapsto s_\alpha(C)$. Using 7.5.4 this will simultaneously prove equidimensionality in both set-ups.

7.5.6. Let $C_i : i \in \mathcal{I}$, be the irreducible components of $\mathcal{C} \cap U$ and $X_\sigma : \sigma \in \sum$, the irreducible components of $\mathcal{B}_u$. Let $\pi_1 : g \mapsto g^{-1}ug$ be the surjective morphism of G onto $\mathcal{C}$ and $\pi_2 : g \mapsto gBg^{-1}$ the surjective morphism of G onto $\mathcal{B}$. They are both closed maps since multiplication in G is a homeomorphism. Identifying $\mathcal{B}$ with G/B identifies $\pi_2^{-1}(X_\sigma)$ with $X_\sigma B/B$. Observe that $gBg^{-1} \in \mathcal{B}_u \iff u \in gBg^{-1} \iff g^{-1}ug \in U$. Then since $ker\ \pi_1 = Z_G(u)$ and $ker\ \pi_2 = N_G(B) = B$, the map $\varphi : X \mapsto \pi_1\pi_2^{-1}(X)$ induces a bijection between the $Z_G(u)$ orbits in $\mathcal{B}_u$ and the B orbits in $\mathcal{C} \cap U$. Recall that $A_u := Z_G(u)/Z_G^0(u)$ permutes the $X_\sigma : \sigma \in \sum$.

Lemma. *The map $\varphi : X_\alpha \mapsto \pi_1\pi_2^{-1}X_\sigma$ induces a bijection of $\sum/A_u$ onto $\mathcal{I}$. Moreover $dim\ X_\sigma + dim\ B = dim\ C_{\varphi(\sigma)} + dim\ Z_G(u)$.*

Proof. Take $\sigma \in \sum$. Since X_σ is irreducible and B is connected, $X_\sigma B$ is irreducible. Hence $X_\alpha B/B = \pi_2^{-1}(X_\sigma) =: Y_\sigma$, is irreducible. Moreover the $Y_\sigma : \sigma \in \sum$ form the irreducible components of $\pi_2^{-1}(\mathcal{B}_u) = \pi^{-1}(\mathcal{C} \cap U)$.

As π_1 is continuous, it follows that $\pi_1(Y_\sigma)$ is irreducible and so contained in some C_i. Set $\sum_i = \{\sigma \in \sum \Big| \pi_1(Y_\sigma) \subset C_i\}$. Obviously $\sum_i$ is A_u stable. By the first paragraph the $Y_\sigma : \sigma \in \sum_i$ are the irreducible components of $Y_i := \pi_1^{-1}(C_i)$. Since π_1 is a closed map, and the $\pi_1(Y_\sigma) : \sigma \in \sum$ are irreducible with union $\mathcal{C} \cap U$, it follows every C_j is some $\pi_1(Y_\sigma)$ and in particular $C_i = \pi_1(Y_\tau)$, for some $\tau \in \sum_i$. Then $Y_i = \pi_1^{-1}\pi_1(Y_\tau) = Z_G(u)Y_\tau$ and so $\sum_i$ is a single A_u orbit. This gives the first part.

Finally $dim\ X_\sigma + dim\ B = dim\ Y_\sigma = dim\ Y_i = dim\ C_i + dim\ Z_G(u)$, for any $\sigma \in \sum_i$.

□

7.5.7. The following result is due to N. Spaltenstein.

Theorem. *Let $\mathcal{C} \subset \mathcal{U}$ be a conjugacy class and $u \in \mathcal{C}$.*

(i) Every component of $\mathcal{C} \cap U$ has dimension $\frac{1}{2} dim\ \mathcal{C}$.

(ii) Every component of $\mathcal{B}_u$ has dimension $|\Delta^+| - dim \mathcal{C} \cap U$.

Proof. Let $X_\sigma : \sigma \in \sum$ be a component of the maximal dimension, that is $dim\ \mathcal{B}_u$. By 7.5.7, $C_{\varphi(\sigma)} := \pi_1\pi_2^{-1}(X_\sigma)$ is a component of maximal dimension of $\mathcal{C} \cap U$.

We claim that every $x' \in \mathcal{B}_u$ is contained in a component of maximal dimension. For this take $x \in X_\sigma$ and recover the notation of 7.5.3. In particular $x = x_1$ and there is a line $L \subset \mathcal{B}_u$ of type $\alpha = \alpha_1$ connecting x_1 and x_2. In this set-up we show below that L is contained in some X_τ, where $C_{\varphi(\tau)}$ is a component of $s_\alpha(C_{\varphi(\sigma)})$ and so has maximal dimension by 7.5.4. Then X_τ has maximal dimension by 7.5.6 and $x_2 \in X_\tau$. Proceeding inductively the claim follows.

In the above, set $P = P_\alpha,\ M = M_\alpha$. We can write $L = gP/B$, for some $g \in G$. Since $L \subset \mathcal{B}_u$ we have

$$g^{-1}ug \in \bigcap_{p \in B} pBp^{-1} \cap \mathcal{U} = M.$$

Then $\pi_1\pi_2^{-1}(L) = \{(gp)^{-1}u(gp) \Big| p \in P\}$ and so is the P orbit of $g^{-1}ug$ in M. Since $x_1 \in L \cap X_\sigma$, the latter is non-empty and so is $\pi_1\pi_2^{-1}(L \cap X_\sigma) = \pi_1\pi_2^{-1}(L) \cap C_{\varphi(\sigma)} \subset M \cap C_{\varphi(\sigma)}$. Now every component of ${}^P(M \cap C_{\varphi(\sigma)})$ is P stable and has the maximal dimension by 7.5.4. Thus the component C_i containing $\pi_1\pi_2^{-1}(x_1)$ contains $\pi_1\pi_2^{-1}(L)$. Thus $L \subset \bigcup_{\tau \in \Sigma_i} X_\sigma$. Yet

L being in $\mathcal{B}_u$ is pointwise invariant, whilst $\sum_i$ is a single A_u orbit.

Consequently there exists $\tau \in \sum_i$ such that $L \subset X_\tau$ as required.

The claim implies that the union of the irreducible components of maximal dimension of $\mathcal{B}_u$ is $\mathcal{B}_u$. Hence $\mathcal{B}_u$ is equidimensional. Then $U \cap \mathcal{C}$ is equidimensional by 7.4.6. Their asserted dimensions obtain by combining 7.5.6 and 7.4.4. □

7.5.8. For each $w \in W$, set

$$\mathcal{S}_w = \{(u, B_1, B_2) \in \mathcal{S} \Big| (B_1, B_2) \in Z_w\}.$$

The $\overline{\mathcal{S}_w} : w \in W$ *are the irreducible components of* $\mathcal{S}$ *and all have dimension* $dim\ G - rk\ G$.

Proof. Obviously $\mathcal{S}$ is a disjoint union of the $\mathcal{S}_w : w \in W$. Moreover $\mathcal{S}_w$ is locally closed being the inverse image in $\mathcal{S}$ of the locally closed set Z_w. It is irreducible being the G saturation of $(U_w, B, {}^w B)$.

Since $\mathcal{S}$ is a finite union of the $\mathcal{S}_w(\mathcal{C})$, we obtain $dim\ \mathcal{S} \leq dim\ G - rk\ G$ by 7.4.2. On the other hand for any $w \in W$ one has $dim\ \mathcal{S}_w \geq dim\ \mathcal{S}_w(St(w)) = dim\ G - rk\ G$, again by 7.4.2. Consequently $dim\ \overline{\mathcal{S}_w} = dim\ \mathcal{S}$, for all $w \in W$ and so the $\overline{\mathcal{S}_w} : w \in W$ must be the irreducible components of $\mathcal{S}$. □

7.6. The Springer Correspondence.

7.6.1. T. Springer showed how to obtain a Weyl group action on the etale homology spaces of the fixed point varieties. In particular the top homology space of $\mathcal{B}_u$ carries a representation of $W \times A_u$. Furthermore its A_u isotypical components are simple W modules which are pairwise isomorphic for different A_u components and for different conjugacy classes. Finally all simple W modules are so obtained.

A vast amount of work has been devoted to Springer's construction and related ways to obtain Weyl group representations. Here we shall describe just a few with only partial proofs; concentrating instead on some of the remaining open problems as well as its quite remarkable connection to Goldie rank representations.

7.6.2. The most immediate understanding of Springer's result comes from a study of the top homology of the Steinberg variety. This approach was suggested by Springer himself and carried out by Kazhdan and Lusztig, by Rossmann and by Ginsburg. All these authors used Borel-Moore homology and described the construction as "elementary". At least it does not involve derived categories nor perverse sheaves. Below we give a short introduction to Borel-Moore homology assuming some familiarity with singular homology and singular cohomology.

In what follows the coefficient ring is always $\mathbb{R}$. Topological spaces are assumed locally compact and Hausdorff. Recall that the singular homology spaces $H_*(X)$ of X are defined by continuous maps $\sigma : \Delta_i \to X$ of simplexes and the resulting boundary maps $C_i(X) \xrightarrow{d_i} C_{i-1}(X)$ on singular chains, setting $H_i(X) = Ker\ d_i/Im\ d_{i+1}$. If $f : X \to Y$ is continuous, then composition with σ gives a direct image $f_* : H_i(X) \to H_i(Y)$ which is linear and a covariant functor. If Y is a subspace of X, we obtain induced boundary maps $C_i(X)/C_i(Y) \xrightarrow{\bar{d}_i} C_{i-1}(X)/C_{i-1}(Y)$ defining the relative homology groups $H_i(X,Y) = ker\ \bar{d}_i/Im\ \bar{d}_{i+1}$. From this definition one obtains a long exact sequence for relative homology

$$\to H_i(Y) \to H_i(X) \to H_i(X,Y) \to H_{i-1}(Y) \to$$

which is functorial for the pair X, Y. Moreover if Z is a subspace of Y one similarly obtains

$$\to H_i(Y,Z) \to H_i(X,Z) \to H_i(X,Y) \to H_{i-1}(Y,Z) \to$$

which is functorial in the triple X, Y, Z.

If $\hat{X} = (X, \infty)$ is the one-point compactification of X, then the Borel-Moore homology spaces are defined by $H_i^{BM}(X) := H_i(\hat{X}, \infty)$. More generally if X is open in a compact set C then using excision one obtains $H_i^{BM}(X) = H_i(C, C \setminus X)$. In particular Borel-Moore and singular homology coincide on compact spaces. Recall that $f : X \to Y$ is said to be proper if $f^{-1}(K)$ is compact in X whenever K is compact in Y. Extend f to a map $\hat{f} : \hat{X} \to \hat{Y}$, by $\hat{f}(\infty) = \infty$. Then $\hat{f}$ is continuous if and only if f is continuous and proper. In this case the direct image $f_* : H_i^{BM}(X) \to H_i^{BM}(Y)$ is defined and gives a covariant functor.

If X is a smooth orientable connected real analytic manifold of dimension n then

$$(*) \qquad H_i^{BM}(X) = H^{n-i}(X), \quad for\ all\ \ i.$$

We shall just consider complex algebraic varieties viewed as real analytic (but not necessarily smooth) manifolds of real dimension twice their complex dimension. Such manifolds have a canonical orientation. Let X be a complex algebraic variety of complex dimension n. Replacing X by $\hat{X}$ and Y by $\{\infty\}$ in the first long exact sequence above we obtain $H_{2n}(\hat{X}) \xrightarrow{\sim} H_{2n}^{BM}(X)$. The fundamental class $[X]$ of X is then identified with the fundamental class $[\hat{X}] \in H_{2n}(\hat{X})$ of the compact orientable manifold $\hat{X}$.

Let U be the set of smooth points of X. It is open in X and a disjoint union of irreducible subvarieties $U_\sigma : \sigma \in \sum$ whose closures X_σ are the irreducible components of X. Each U_σ is a smooth orientable connected real analytic manifold of real dimension $2dim_{\mathbb{C}}U_\sigma$. Furthermore $H^{BM}(U) = \Pi H^{BM}(U_\sigma)$. Consequently by $(*)$ we obtain $H_i^{BM}(U) = 0 : i > 2n$ and $H_{2n}^{BM}(U)$ has dimension equal to the cardinality of $\sum_n := \{\sigma \in \sum \,\big|\, dim_{\mathbb{C}}X_\sigma = n\}$. Set $V = X \setminus V$ which has dimension $< n$. In the second long exact sequence above replace X by $\hat{X}$, Y by $\hat{V} := V \cup \{\infty\}$, so then $\hat{X} \setminus \hat{V} = U$. By definition of $H_*^{BM}(\cdot)$, this gives the long exact sequence

$$\to H_i^{BM}(V) \to H_i^{BM}(X) \to H_i^{BM}(U) \to H_{i-1}^{BM}(V) \to$$

Then by the above and induction on dimension we obtain $H_i^{BM}(X) = 0$, for $i > 2n$. Thus $H_{2n}^{BM}(X) \xrightarrow{\sim} H_{2n}^{BM}$. We conclude that

$$H_{2n}^{BM}(X) = \bigoplus_{\sigma \in \sum} \mathbb{R}[X_\sigma].$$

Again if $j : U \hookrightarrow X$ is an open embedding the second long exact sequence gives an inverse image $j^* : H_i^{BM}(X) \to H_i^{BM}(U)$ which is a contravariant functor. Moreover setting $V = X \setminus U$, we obtain from the above a short exact sequence

$$(**) \qquad 0 \longrightarrow H_{2n}^{BM}(V) \longrightarrow H_{2n}^{BM}(X) \longrightarrow H_{2n}^{BM}(U) \longrightarrow 0.$$

Let $f, g : X \to Y$ be (proper) continuous maps. One says that f, g are (properly) homotopic (written $f \sim g$) if there exists a (proper) continuous map $F : X \times [0,1] \to Y$ such that $f(x) = F(x,0), g(x) = F(x,1)$. Given an embedding $X \overset{i}{\hookrightarrow} Y$ (which is automatically continuous) we say that X is a (proper) deformation retract in Y if there exists a neighbourhood U of X in Y and a (proper) continuous map $\tilde{i}i : U \to X$ such that

$\tilde{i} \sim id_X$ $i\tilde{i} \sim id_U$. Here we can assume $\tilde{i}i = 1d_X$, which means that $\tilde{i}\Big|_X = id_X$ that is to say $\tilde{i}$ is a retraction of U into X, and then $\tilde{i}$ is a retract (by definition) which is linked as above by a continuous family of maps to Id_U. The corresponding direct image maps $H_*(X) \underset{\tilde{i}_*}{\overset{i_*}{\rightleftarrows}} H_*(U)$ are mutual inverses. Observe that all maps above are automatically proper if U is compact and X is closed, so one obtains a corresponding isomorphism of Borel-Moore homologies.

Perhaps the most delicate point in the subsequent analysis is the existence of deformation retracts. These are provided by a general theorem of S. Lojasiewicz which asserts that if f is an analytic function on an open subset $S \subset \mathbb{R}^n$, then $f^{-1}(0)$ is a deformation retract. Thus one has a neighbourhood U of $f^{-1}(0)$ and a continuous function $F : U \times [0,1] \to U$ such that $F(\cdot,0) = 1d_U$ and $F(\cdot,1) : U \to f^{-1}(0)$ is a retraction. Here one may note that 0 is a minimum of f^2 and that $t \mapsto F(u,t)$ is the curve of steepest descent from u to $F(u,1) \in f^{-1}(0)$. Thus for example U must be sufficiently small to exclude other local minima. Again the $U_t := Im\ F(\cdot,t)$ form a continuous decreasing family of neighbourhoods of $f^{-1}(0)$.

Now let $X \overset{i}{\hookrightarrow} Y$ be a closed embedding of complex algebraic varieties. Then for some $m,n \in \mathbb{N}^+$ we may view X (and similarly Y) as a real algebraic subvariety of $\mathbb{R}^{2n}$ given by the common zeros of $2n$ real valued algebraic and hence analytic functions $f_i : i = 1,2,\cdots,2m$ and hence as $f^{-1}(0)$, where $f = \sum f_i^2$.

Now suppose Y is projective. Then X,Y are bounded real algebraic subsets of $\mathbb{R}^n$ and then one may choose U (and a similar neighbourhood V for Y) to be bounded and closed, hence compact. Let $r : V \to Y$ be the retraction so defined. Replacing U by $U \cap V$ we may assume that $U \subset V$. Then $F(U \cap Y, t) \subset V$ and so $(rF)(U \cap Y, t) \subset U \cap Y$, for all $t \in [0,1]$. Moreover $(rF)(u,1) = r(u) = u,\ \forall\ u \in U$ and $(rF)(x,0) = r(x) = x,\ \forall\ x \in X$. Yet $U \cap Y$ is a closed neighbourhood of X in Y, so X is a deformation retract in Y and all maps are proper.

Suppose we have a commuting diagram

$$\begin{array}{ccc} X' & \xrightarrow{i'} & Y' \\ h' \big\downarrow & & h \big\downarrow \\ X & \xrightarrow{i} & Y \end{array}$$

of closed embeddings of complex projective varieties. Choose $U, \tilde{i}$ (and analogously $U', \tilde{i}'$) as above replacing U' by $U \cap U'$ if necessary to ensure that $U' \subset U$. Then

$$(**) \qquad h'\tilde{i}' \sim \tilde{i}ih'\tilde{i}' = \tilde{i}hi'\tilde{i}' \sim \tilde{i}h.$$

This analysis further applies to giving a proper deformation retract of X in Y if $Y = P \times C$ then P is projective and C is a cone with X stable under scalar multiplication in C. This obtains by taking projectivizations.

In the sequel the Borel-Moore homology spaces will be simply denoted as $H_i(X)$.

7.6.3. In what follows G is assumed to be a complex connected simply-connected semisimple algebraic group. Set $n = |\Delta^+|$ and $n(u) = dim\ \mathcal{B}_u$.

The starting point is 7.5.8 which describes the irreducible components of the Steinberg variety $\mathcal{S}$. The corresponding classes $[\bar{\mathcal{S}}_w] : w \in W$, form a basis of the top Borel-Moore homology $H_{4n}(\mathcal{S})$ of $\mathcal{S}$. The first aim is to show that $H_{4n}(\mathcal{S})$ carries a $W \times W$ module structure making it isomorphic to the regular representation. (In fact even more is possible, namely that $H_{4n}(\mathcal{S})$ has an algebra structure making it isomorphic to the group algebra of W.) Moreover the $W \times W$ action can be defined just as well for $H_{4n}(\mathcal{S}(\mathcal{C})) : \mathcal{C} \in \mathcal{U}/G$ and then from the decomposition of $\mathcal{S}$ as a disjoint union of the $\mathcal{S}(\mathcal{C})$ one obtains (7.6.15) a decomposition of $H_{4n}(\mathcal{S})$ as a direct sum of $H_{4n}(\mathcal{S}(\mathcal{C}))$. Finally this construction gives an $A_u \times W$ action on $H_{2n(u)}(\mathcal{B}_u)$ and through 7.4.3 one obtains (7.6.16) an isomorphism

$$H_{4n}(\mathcal{S}(\mathcal{C})) \cong (H_{2n(u)}(\mathcal{B}_u) \otimes H_{2n(u)}(\mathcal{B}_u))^{A_u}$$

of $W \times W$ modules; for any $u \in \mathcal{C}$. It is clear that this implies that the A_u isotypical components of the $H_{2n(u)}(\mathcal{B}_u) : u \in \mathcal{U}/G$ run over the distinct W modules.

7.6.4. To carry out the above program one considers $\hat{E} := E \times \mathcal{B}$, where E is successively $\mathcal{U}, \mathcal{C}, u$ and where in the first two cases $\mathcal{B}$ may be replaced by $\mathcal{B} \times \mathcal{B}$. If $E = \{u\}$, then $\hat{E}$

is already projective. Otherwise we may identify $\mathcal{U}$ (resp. $\mathcal{C}$) with $\mathcal{N}$ (resp. a nilpotent orbit $\mathcal{O}$) by the exponential map. The latter are locally closed algebraic subvarieties of $\mathfrak{g}$ stable under scalar multiplication which induces the action $e \mapsto e^s$ of $\mathbb{C}$ on E.

Let $\varphi : \mathbb{C} \times \hat{E} \to \hat{E}$ be the algebraic action of $\mathbb{C}$ on E given by conjugation by u^t. Typically $\varphi(t, u, B_1) = (u, u^t B_1 u^{-t})$, for all $t \in \mathbb{C}, u \in \mathcal{U}, B \in \mathcal{B}_1$. The fixed point set $\hat{E}_\varphi := \{\hat{e} \in \hat{E} \mid \varphi(t, \hat{e}) = \hat{e},\ \forall e \in \mathbb{C}\}$ is a closed subvariety of $\hat{E}$. When $E = \mathcal{U}$ or $\mathcal{C}$, then $\hat{E}_\varphi$ is stable under the action $e \mapsto e^s$ of the previous paragraph and so may be viewed as a closed subvariety of the projectivization of $\hat{E}$. When $E = \{u\}$, then $\hat{E}_\varphi = \mathcal{B}_u$ and when $E = \mathcal{U}$ (identified with $\mathcal{N}$) then by 7.2.14(i) it follows that $\hat{E}_\varphi$ identifies with $G \times_B \mathfrak{n}^+$, which is the cotangent bundle $T^*(\mathcal{B})$ of $\mathcal{B}$. If $\hat{E} = \mathcal{U} \times \mathcal{B} \times \mathcal{B}$ and $\varphi(t, u, B_1, B_2) = (u, u^t B_1 u^{-t}, u^t B_2 u^{-t})$, then $\hat{E}_\varphi = \mathcal{S}$.

7.6.5. Given $\hat{E}, \varphi, \hat{E}_\varphi$ as in 7.6.4, one may construct for each simple root $\alpha \in \pi$ an involution s_*^α on the Borel-Moore homology spaces $H_{2i}(\hat{E}_\varphi)$ as follows.

Set $\mathcal{P}^\alpha = \{gP_\alpha g^{-1} \mid g \in G\}$ which identifies with G/P_α whilst $\mathcal{B}$ identifies with G/B. Set $\hat{E}^\alpha = E \otimes \mathcal{P}^\alpha$. The action of $\mathbb{C}$ on $\hat{E}$ defined by φ descends to $\hat{E}^\alpha$ and we let $\hat{E}^\alpha_\varphi$ denote its fixed point set. The inverse image of $\hat{E}^\alpha_\varphi$ under the surjection $\hat{E}^\alpha \to \hat{E}$ is just $\hat{F}^\alpha := \{(e, gBg^{-1}) : e \in E, g \in G \mid \varphi(t, e, gBg^{-1}) \in (e, gP_\alpha g^{-1}), \forall t \in \mathbb{C}\}$. It is φ stable with fixed point set $\hat{F}^\alpha_\varphi$ equal to $\hat{E}_\varphi$.

Suppose $(u, gBg^{-1}) \in \hat{F}^\alpha$, for some $u \in \mathcal{U}, g \in G$. Then $g^{-1}ugB(g^{-1}ug)^{-1} \subset P_\alpha$ and so by [, 8.3.1] we obtain $g^{-1}ug \in P_\alpha$. This forces $(u, gB'g^{-1}) \in \hat{F}^\alpha$, for every Borel B' lying in P_α. Since $P_\alpha/B \cong \mathbb{P}^1$, it means that $\hat{F}^\alpha \to \hat{E}^\alpha_\varphi$ is a $\mathbb{P}^1$ bundle with structure group $SL(2, \mathbb{C})$.

Identify $\mathbb{P}^1$ with the Riemann sphere and let j_α be a continuous map of the $\mathbb{P}^1$ bundle $E \times G/B \to E \times G/P_\alpha$ which is the antipodal involution on each fibre. (We shall use the specific choice as described in 7.6.11. It is right multiplication by a representative of s_α in $N_K(T)$ when we realize G/B as K/T.) Since j_α is involutive and continuous it is proper. Of course $\hat{F}^\alpha_\varphi = \hat{E}_\varphi$ is not j_α stable. This difficulty is overcome as follows. First by the remarks in 7.6.4 one may view $\hat{F}^\alpha_\varphi$ and $\hat{F}^\alpha$ as complex subvarieties of the projectivization of $E \times \mathcal{B}$. Consequently $\hat{F}^\alpha_\varphi$ admits a closed retractable neighbourhood $U \subset \hat{F}^\alpha$. Thus that the inclusion $i : \hat{F}^\alpha_\varphi \hookrightarrow U$ is a proper homotopy equivalence and we let $\tilde{i} : U \to \hat{F}^\alpha_\varphi$ be a proper homotopy inverse to i. It now remains to "squeeze" $j_\alpha(F^\alpha_\varphi)$ back into U. This is done as follows.

Consider the action of P_α on the non-zero vectors of its standard two dimensional representation. This commutes with the $\mathbb{C}^*$ action defined by scalar multiplication. Moreover the set of equivalence classes under this latter action identifies with $\mathbb{P}^1$. In this $\binom{1}{0} \in \mathbb{P}^1$ is the B fixed point (and so $B/B \in P_\alpha/B$ identifies with $\left(\binom{1}{0}\right)$). Let n_{s_α} be a representative of $s_\alpha \in W$ lying in $N_{P_\alpha}(H)$. Then $n_{s_\alpha}B/B \in P_\alpha/B$ identifies with $\binom{0}{1}$ and is moreover the image of B/B under j_α.

Set $B_1 = gBg^{-1}$. Then $(u, B_1) \in \hat{F}^\alpha$ (resp. $\hat{F}^\alpha_\varphi$) exactly when $g^{-1}ug \in P_\alpha$ (resp. B). Assume $(u, B_1) \in \hat{F}^\alpha_\varphi$. Then the projection of $g^{-1}ug$ onto $P_\alpha/M_\alpha \cong SL(2,\mathbb{C})$ is unipotent and lies in B/M_α. Thus it defines a continuous function $\mu : \mathcal{U} \times G \to \mathbb{C}$ such that the action $B_1 \mapsto u^t B_1 u^{-t}$ on gP_α/B becomes multiplication by $\left(\begin{smallmatrix} 1 & t\mu(u,g) \\ 0 & 1 \end{smallmatrix}\right)$. The case $\mu(u,g) = 0$ is just when the complete projection line $(u, gP_\alpha/B)$ lies in $\hat{F}^\alpha_\varphi$ and nothing needs to be done. Otherwise since U is a neighbourhood of $\hat{F}^\alpha_\varphi$ there exists a continuous function $\nu : \mathcal{U} \times G \to \mathbb{R}^+$ so that $\binom{1}{\varepsilon}$ belongs to U in the fibre over $(u, gP_\alpha g^{-1})$ given $|\varepsilon| \leq \nu(u,g)/|\mu(u,g)|$.

Now
$$\begin{pmatrix} 1 & t\mu(u,g) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} t\mu(u,g) \\ 1 \end{pmatrix} \sim \begin{pmatrix} 1 \\ t^{-1}\mu(u,g)^{-1} \end{pmatrix}$$

and so $\varphi(t, u, g({}^{s_\alpha}B)g^{-1}) \in U$, given $|t| \geq \nu(u,g)$. Thus the function $\psi : \hat{F}^\alpha \to \hat{F}^\alpha$ defined by $\psi(u, gBg^{-1}) = \varphi(\nu(u,g), u, gBg^{-1})$ restricts to a proper continuous map of $j_\alpha(\hat{E}_\varphi)$ into U. Finally let s^α be the composition $s^\alpha = \tilde{i}\psi j_\alpha i$, where we note that

$$\hat{E}_\varphi = \hat{F}^\alpha_\varphi \overset{j_\alpha}{\to} j_\alpha(\hat{F}^\alpha_\varphi) \xrightarrow{\psi} U \xrightarrow{\tilde{i}} \hat{F}^\alpha_\varphi \ .$$

It is a proper continuous map. Since ψ, j_α commute $(s^\alpha)^2$ is homotopy equivalent to the identity. Thus its direct image s^α_* defines an involution on any $H_{2i}(\hat{E}_\varphi)$ and in particular an involution on $H_{2m}(\hat{E}_\varphi) : m = dim\ \hat{E}_\varphi$. It is independent of the above choices.

One may roughly describe how s^α acts as follows. Either the complete fibre belongs to $\hat{F}^\alpha_\varphi$ and ψj_α is the antipodal involution, or only one point belongs to $\hat{F}^\alpha_\varphi$ and ψj_α becomes the identity map in the limit $t \to \infty$. Of course such a map would not be continuous and this is where the deformation retract is needed. It cannot simultaneously preserve singleton fibres approaching $\mathbb{P}^1$ fibres. If all fibres are reduced to a point, s_*^α is the identity map, whilst if all fibres belong to $\hat{F}_\varphi^\alpha$ then s_*^α is the identity map up to a sign.

Suppose $E' \overset{h}{\hookrightarrow} E$ is a closed embedding (for example $E' = \{u\}, E = \mathcal{U}$) and extend h by the identity to $\hat{E}$. Then $\hat{E}'_\varphi \overset{h'}{\hookrightarrow} \hat{E}_\varphi$ with $h' = h \Big|_{\hat{E}'_\varphi}$ is a closed embedding. Take $X = \hat{E}_\varphi, Y = \hat{E}$ in the last part of 7.6.2 and choose U' as described there. Define i', j'_α, ψ' by restriction and $\tilde{i}' : U' \to \hat{E}'_\varphi$ to be a proper homotopy inverse to i, setting $s'^\alpha = \tilde{i}'\psi' j'_\alpha i'$. Then from 7.6.2($*$) we obtain $s^\alpha h' = \tilde{i}\psi j_\alpha i h' = \tilde{i}\psi j_\alpha h i = \tilde{i} h \psi' j'_\alpha i' \sim \tilde{h}\tilde{i}'\psi' j'_\alpha i' = \tilde{h} s'^\alpha$. Functoriality gives a commuting diagram

($*$)
$$\begin{array}{ccccc}
H_i(\hat{E}'_\varphi) & \xrightarrow{k'_*} & H_i(\hat{E}_\varphi) & \longrightarrow & H_i(\hat{E}''_\varphi) \\
\big\downarrow s_*'^\alpha & & \big\downarrow s_*^\alpha & & \big\downarrow s_*''^\alpha \\
H_i(\hat{E}'_\varphi) & \xrightarrow{k'_*} & H_i(\hat{E}_\varphi) & \longrightarrow & H_i(\hat{E}''_\varphi)
\end{array}$$

where s''^α is defined by similarly applying the above construction to $E'' := E \backslash E'$. Moreover by 7.6.2($**$) the rows become short exact if $i = 2dim_{\mathbb{C}} E_\varphi$.

7.6.6. A difficulty with the above construction is that it is not immediate that the s_*^α satisfy the relations in W. Elementary ways to overcome this all involve the real algebraic action of W on $\mathcal{B}$. The latter obtains as follows.

Let $\{x_\alpha : \alpha \in \Delta, h_\alpha : \alpha \in \pi\}$ be a Chevalley basis for $\mathfrak{g}$ and let $z \mapsto \bar{z}$ be complex conjugation with respect to that basis. Define the (compact) Cartan involution θ for $\mathfrak{g}$ by composing complex conjugation with the product of the Chevalley and principal antiautomorphisms. Then $\mathfrak{k} = \{x + \mathfrak{g} \,\big|\, \theta(x) = x\}$, $\mathfrak{p} = \{x \in \mathfrak{g} \,\big|\, \theta(x) = -x\}$, $\mathfrak{t} = \mathfrak{k} \cap \mathfrak{h}$, $\mathfrak{a} = \mathfrak{p} \cap \mathfrak{h}$ are real subspaces of $\mathfrak{g}$. All but $\mathfrak{p}$ are Lie subalgebras. Let K, T, A be the connected real Lie subgroups of G (viewed as a real Lie group of twice the complex dimension of G). Again let H (resp. N) be the connected real Lie group corresponding to $\mathfrak{h}$ (resp. $\mathfrak{n}^+$). The Iwasawa decomposition asserts that the multiplication map is a bijection of $K \times A \times N$ onto G. Moreover the image $T \times A \times N$ is just $HN = B$. Thus $\mathcal{B} = G/B$ identifies as a real homogeneous space with K/T. Yet W identifies with $N_K(T)/T$ and so right multiplication gives a real algebraic action of W on $\mathcal{B}$. One may check that the Killing form is negative definite on $\mathfrak{k}$ and so K is a compact real Lie group. The Killing form is non-degenerate on $\mathfrak{t}$ and we let $\mathfrak{t}^\perp$ denote the orthogonal of $\mathfrak{t}$ with respect to that form.

7.6.7. Let us examine briefly the action of W on $H_*(\mathcal{B})$ which results from the real algebraic action of W on $\mathcal{B}$ described in 7.6.6. The main results are due to Borel who developed a general theory for the (singular) cohomology theory of homogeneous spaces of a compact Lie group.

The Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ is just the real vector space with basis $\{ih_\alpha : \alpha \in \pi\}$. Thus if $\lambda \in \mathfrak{h}^*$ is such that $i\lambda \in \mathfrak{t}^*$, then λ takes real values on the $h_\alpha : \alpha \in \pi$. Extend $i\lambda$ to an element of $\mathfrak{k}^*$ via $\lambda(\mathfrak{t}^\perp) = 0$. View $x, y \in \mathfrak{k}$ as C^∞ vector fields on K/T and define the 2-form σ_λ on K/T by

$$\sigma_\lambda(x,y)(kT) =< [x,y], k\lambda > .$$

It is closed, alternating and if λ is regular (i.e. $\lambda(h_\alpha) \neq 0,\ \forall \alpha \in \Delta$) also non-degenerate. In particular K/T is a symmetric manifold. In the latter case it is just the Kirillov-Kostant form on the regular semisimple coadjoint orbit $K\lambda = K/T$; but this special case was discovered by Borel [ ] ten years previously.

Let Z be a closed irreducible subvariety of $\mathcal{B}$ of complex dimension r. Then Z can be viewed as a compact real subvariety of K/T of dimension $2r$ and one may integrate Z with measure defined by $\frac{1}{r!}\sigma_\lambda^r$. It is convenient to write this integral as

$$c_Z(\lambda) := \int_Z exp\ \sigma_\lambda$$

where by convention only the r^{th} power of σ_λ is retained. Let $[Z]$ denote the fundamental class of Z in $H_{2r}(\mathcal{B})$.

Let $S(\mathfrak{h})^W_+$ denote the augmentation ideal of $S(\mathfrak{h})^W$ (which is a polynomial algebra). The space $\mathbb{H}$ of W harmonic elements of $S(\mathfrak{h})$ can be viewed as the set of polynomials on $\mathfrak{h}^*$ annihilated by the homogeneous W invariant differential operators of degree > 0. It is a graded complement to $S(\mathfrak{h})S(\mathfrak{h})^W_+$ and carries the regular representation of W. A useful characterization of $\mathbb{H}$ is that it may be generated by applying constant coefficient differential operators on $\mathfrak{h}$ to the polynomial

$$p = \prod_{\alpha \in \Delta^+} \alpha$$

Indeed p must divide any polynomial transforming by the sign representation and so is of the lowest degree amongst the polynomials having the property. Thus for example p and all its derivatives lie in $\mathbb{H}$.

One has the following remarkable result of Borel. Note that $deg\ p = |\Delta^+|$ which is just the complex dimension of $\mathcal{B}$.

Theorem. *The map $[Z] \mapsto (\lambda \mapsto \int_Z exp\ \sigma_\lambda)$ extends linearly to a W module isomorphism of $H_*(\mathcal{B})$ onto $\mathbb{H}$ sending $H_{2i}(\mathcal{B})$ onto $\mathbb{H}_i$ for all $i = 0, 1, 2, \cdots, deg\ p$.*

7.6.8. The result is that 7.6.7 is elegant but not immediately useful especially to algebraists who have not yet learnt to integrate. However by some relatively standard sheaf technology one can cast it into the more useful form described below.

Let Z be as in 7.6.7. Given $\lambda \in P^+(\pi)$, let $V(\lambda)$ be the corresponding simple finite dimensional $\mathfrak{g}$ module with highest weight vector v_λ. Since for λ regular, $\mathcal{B}$ identifies with the image of Gv_λ in the projectivization of $V(\lambda)$, it is natural to consider the subset $Zv_\lambda = \{zv_\lambda \,\big|\, z \in Z\}$ and the dimension $q_Z(\lambda)$ of its $\mathbb{C}$ linear span (better the codimension of the orthogonal of Zv_λ in $V(\lambda)^*$). By taking resolutions of the ideal sheaf of Z one shows that $\lambda \mapsto q_Z(\lambda)$ is polynomial for very dominant λ (that is when the $\lambda(h_\alpha) : \alpha \in \pi$ are all sufficiently large). Thus the leading degree term $gr\ q_Z$ is defined and one further obtains the

Proposition. *Let Z be a closed irreducible subvariety of $\mathcal{B}$, then $c_Z = gr\ q_Z$.*

Remark. The embedding of a fixed point set $\mathcal{B}_u \overset{\dot{\varphi}}{\hookrightarrow} \mathcal{B}$ gives by functoriality linear maps $H_{2i}(\mathcal{B}_u) \overset{\varphi_*}{\hookrightarrow} H_{2i}(\mathcal{B})$. In particular the irreducible components $X_\sigma : \sigma \in \sum$ which all have dimension $n(u) = dim\ \mathcal{B}_u$ give rise to harmonic polynomials $c_{X_\sigma} : \sigma \in \sum$ of degree $n(u)$. One can ask if they define a W submodule of $H_{n(u)}$. In fact this is true but there is no known easy proof. Moreover the module is simple and just corresponds to the isotypical component of $H_{2n(u)}(\mathcal{B}_u)$ with respect to the trivial representation of A_u, the remaining part lying in $ker\ \varphi_*$. In bried there is no Royal Route to the Springer correspondence.

7.6.9. We may apply the preceding theory to $\mathcal{B} \times \mathcal{B}$ which is the flag variety of $G \times G$. Let $\mathbf{\Delta} := \{(B, B) \,\big|\, B \in \mathcal{B}\}$ be the diagonal. It is a complex irreducible subvariety isomorphic to $\mathcal{B}$ and so of dimension $n := |\Delta^+|$. Let ρ be the half sum of the positive roots.

Proposition. *The fundamental class $[\mathbf{\Delta}]$ generates the two-sided regular representation of W in $H_{2n}(\mathcal{B} \times \mathcal{B})$, up to a twist by the sign representation.*

Proof. Fix $\lambda, \mu \in P^+(\pi)$. Through the Weyl dimension formula one has

$$q_{\mathbf{\Delta}}(\lambda, \mu) = \prod_{\alpha \in \Delta^+} \frac{(\alpha, \lambda + \mu + \rho)}{(\alpha, \rho)}.$$

Thus up to a scalar

$$c_{\mathbf{\Delta}}(\lambda, \mu) = \prod_{\alpha \in \Delta^+} (\alpha, \lambda + \mu).$$

Introducing two sets $\alpha^{(i)} : i = 1, 2$ of positive roots, one for the left hand and one for the right hand factor we may write

$$c_{\mathbf{\Delta}} = \prod_{\alpha \in \Delta^+} (\alpha^{(1)} + \alpha^{(2)}).$$

Now the coefficients of the powers of the simple roots $\alpha^{(2)} : \alpha \in \pi$ in $c_{\mathbf{\Delta}}$ are up to scalars just the corresponding derivatives of $p = \prod_{\alpha \in \Delta^+} \alpha^{(1)}$. From the characterization of $\mathbb{H}$ given in 7.6.7 we conclude that their linear span admits a cyclic vector e for the regular representation. On the other hand the left action of W on $c_{\mathbf{\Delta}}$ is given by

$$w.c_{\mathbf{\Delta}} = \prod_{\alpha \in \Delta^+} (w\alpha^{(1)} + \alpha^{(2)})$$

and so a relation of the form

$$\sum a_w w.c_{\mathbf{\Delta}} = 0 : a_w \in \mathbb{C}$$

would imply by taking coefficients with respect to powers of the $\alpha^{(2)} : \alpha \in \pi$ that $\sum a_w we = 0$. Hence $a_w = 0$, for all $w \in W$ as required. Finally the right action of W on $c_{\mathbf{\Delta}}$ is given by

$$c_{\mathbf{\Delta}}.w = \prod_{\alpha \in \Delta^+} (\alpha^{(1)} + w^{-1}\alpha^{(2)}) = (-1)^{\ell(w)} \prod_{\alpha \in \Delta^+} (w\alpha^{(1)} + \alpha^{(2)}) = (-1)^{\ell(w)} w.c_{\mathbf{\Delta}}.$$

These two facts prove the proposition. □

7.6.10. Consider the special case $G = SL(2, \mathbb{C})$. Its Lie algebra $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ admits a Chevalley basis

$$x_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; h_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \; y_\alpha = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then the real subalgebra $\mathfrak{k}$ is spanned by the Pauli matrices

$$\sigma_1 = i(x_\alpha + y_\alpha) = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \ \sigma_2 = x_\alpha - y_\alpha = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \sigma_3 = ih = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Identify the projective line $\mathbb{P}^1$ with equivalence classes of non-zero column vectors as in 7.6.5. Then $T = \{exp\ \alpha\sigma_3 : \alpha \in \mathbb{R})$ leaves $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{P}^1$ invariant and in this manner K/T identifies with $\mathbb{P}^1$. We may write $\mathbb{P}^1 \setminus \left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\} = \{\left(\begin{smallmatrix} z \\ 1 \end{smallmatrix}\right)\} : z \in \mathbb{C}\}$, and from this one may compute the action of the $exp\ \alpha_j\sigma_j : j = 1, 2, 3,\ \alpha_j \in \mathbb{R}$ on $\mathbb{P}^1$. If we write $z = x + iy$ and make the inverse stereographic projection onto the (Riemann) sphere given by

$$a_1 = \frac{2x}{x^2 + y^2 + 1}\ , \ a_2 = \frac{2y}{x^2 + y^2 + 1}\ , \ a_3 = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}$$

then one may check that $exp\ \alpha_j\sigma_j$ leaves a_j fixed and induces a rotation by $2\alpha_j$ about the j^{th} axis. This doubling corresponds to K being the double cover of $SO(3, \mathbb{R})$.

From the above one may describe the W action on K/T. Take the representative $n_s = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in $N_K(T)$ of the non-trivial element s. This sends the T fixed point $\begin{pmatrix} 1 \\ 0 \end{pmatrix} \in \mathbb{P}^1$ to be T fixed point $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \in \mathbb{P}^1$. On the Riemann sphere these points are respectively the "north pole" $(0, 0, 1)$ and the "south pole" $(0, 0, -1)$. Since this action induced by right multiplication, commutes with the action of K on the sphere induced by left multiplication. It follows that W acts by antipodal involution.

Return to the general case for each $\alpha \in \pi$ let K_α be the real connected subgroup of G with Lie algebra $\mathfrak{t} \oplus \mathbb{R}(x_\alpha - y_\alpha) \oplus i\mathbb{R}(x_\alpha + y_\alpha)$. Then G/P_α identifies with K/K_α and the projective line P_α/B with the Riemann sphere K_α/T. Thus the $\mathbb{P}^1$ bundle $G/B \to G/P_\alpha$ admits an involution j_α continuous along the fibres which on each fibre is the antipodal involution.

7.6.11. Recall the definition of $\mathfrak{t}^\perp$ given in 7.6.6. One checks that $\{i(x_\alpha + y_\alpha), (x_\alpha - y_\alpha) : \alpha \in \Delta^+\}$ is an $\mathbb{R}$ basis for $\mathfrak{t}^\perp$. The linear map $1 + \theta$ on $\mathfrak{g}$ commutes with T and restricts to a bijection of $\mathfrak{n}^+$ onto $\mathfrak{t}^\perp$. Indeed if we write

$$z = \sum_{\alpha \in \Delta^+} z_\alpha x_\alpha : z_\alpha \in \mathbb{C}$$

then

$$(1 + \theta)(z) = \sum_{\alpha \in \Delta^+} (Re\ z_\alpha)(x_\alpha - y_\alpha) + i(Im\ z_\alpha)(x_\alpha + y_\alpha).$$

Thus one may define a map $\Theta : K \times_T \mathfrak{t}^\perp \to G \times_B \mathfrak{n}^+$ by

$$\Theta(k, x) = (k, (1+\theta)^{-1}(x)).$$

It is bijective, commutes with left multiplication by K and is linear in the second factor Consequently it is an isomorphism of real vector bundles.

7.6.12. Since $W = N_K(T)/T$ and the adjoint action of $N_K(T)$ on $\mathfrak{k}$ leaves $\mathfrak{t}$ and hence $\mathfrak{t}^\perp$ invariant, there is both an algebraic action of W on K/T and on $K \times_T \mathfrak{t}^\perp$. If we choose a representative $n_w \in N_K(T)$ of $w \in W$, then these are given respectively by $k \overset{r_w}{\longmapsto} kn_{w^{-1}}$, $(k, x) \overset{r_w}{\longmapsto} (kn_{w^{-1}}, n_w(x))$, for all $k \in K$, $x \in \mathfrak{t}^+$. Let $(k, x) \overset{p}{\longmapsto} kT$ be the projection onto the first factor and $kT \overset{s}{\longmapsto} (k, 0)$ be the zero section. Then obviously the diagram

$$\begin{array}{ccc}
K \times_T \mathfrak{t}^\perp & \xrightarrow{r_w} & K \times_T \mathfrak{t}^\perp \\
{\scriptstyle s}\uparrow\downarrow{\scriptstyle p} & & {\scriptstyle s}\uparrow\downarrow{\scriptstyle p} \\
K/T & \xrightarrow{r_w} & K/T
\end{array}$$

commutes. Clearly $ps = 1d_{K/T}$, whilst sp is homotopy equivalent to the identity on $K \times_T \mathfrak{t}^\perp$. Inverse image in singular cohomology gives by functionality commuting diagrams

$$\begin{array}{ccc}
H^i(K \times_T \mathfrak{t}^\perp) & \xleftarrow{r_w^*} & H^i(K \times_T \mathfrak{t}^\perp) \\
{\scriptstyle s^*}\downarrow\uparrow{\scriptstyle p^*} & & {\scriptstyle s^*}\downarrow\uparrow{\scriptstyle p^*} \\
H^i(K/T) & \xleftarrow{r_w^*} & H^i(K \times_T \mathfrak{t}^\perp)
\end{array}$$

where all maps are now isomorphisms. Yet $K \times_T \mathfrak{t}^\perp$ and K/T are smooth (with the latter also compact). Thus by 7.6.3(∗), noting $dim_{\mathbb{R}}(K \times_T \mathfrak{t}^\perp) = 4n$, $dim_{\mathbb{R}}(K/T) = 2n$ and $(r_w^*)^{-1} = (r_w)_*$, we obtain a commuting diagram of Borel-Moore homologies

$$\begin{array}{ccc}
H_i(K \times_T \mathfrak{t}^\perp) & \xrightarrow{(r_w)_*} & H_i(K \times_T \mathfrak{t}^\perp) \\
\int \downarrow & & \int \downarrow \\
H_{i-2n}(K/T) & \xrightarrow{(r_w)_*} & H_{i-2n}(K/T).
\end{array}$$

The vertical map is known as the Thom isomorphism.

Remarks. The Thom isomorphism holds true for arbitrary algebraic vector bundles but its proof requires the dualizing sheaf.

Let V, F be a complex algebraic varieties and N a neighbourhood of $v \in V$ which we can assume smooth and contractible. Then the zero section $s : F \to v \times F$ induces an isomorphism $s^* : H_{top}(N \times F) \xrightarrow{\sim} H_{top}(F)$. This obtains as above since by 7.6.2 one may replace F by its set F_0 of smooth points.

7.6.13. Given $w = s_\alpha : \alpha \in \pi$, write r_w simply as r_α. Consider the case when $E = \mathcal{U}$ and the fixed point set $\hat{E}_\varphi = (\mathcal{U} \times \mathcal{B})_\varphi$ is just $T^*(\mathcal{B}) = G \times_B \mathfrak{n}^+ = K \times_T t^\perp$.

Lemma. *The maps $ps_\alpha, r_\alpha p : K \times_T \mathfrak{t}^\perp \to K/T$ are homotopy equivalent.*

Proof. Through the Iwasawa decomposition we can write an element of $G \times_B \mathfrak{n}^+$ in the form $(k, x) : k \in K, x \in \mathfrak{n}^+$. Then the image of $k, x)$ in $K \times_T \mathfrak{t}^\perp$ is just $(k, (1 + \theta)(x))$. Write n_{s_α} simply as n. Then the $r_\alpha p$ is the map $(k, x) \mapsto kn^{-1}$. Take $u = exp\ x$ in 7.6.5 and set $t(x, k) = \nu(u, k)$. View (k, x) as an element of $K/T \times \mathcal{N}$. We can take j_α to be the map $(k, x) \mapsto (kn^{-1}x)$ on the latter space. View p as the projection onto the first factor. Then ps_α is the map $(k, x) \mapsto pi'((exp\ t(x, k)x)kn^{-1}, x)$. Since i' is a deformation retract the above map is homotopy equivalent to the map $(k, x) \mapsto (exp\ t(x, k)x)kn^{-1}$ which is in turn homotopy equivalent to $r_\alpha p$. □

7.6.14. Define a new $\mathbb{C}^*$ action on $E := \mathcal{U} \times \mathcal{B} \times \mathcal{B}$ through $\tilde{\varphi}(t, u, B_1, B_2) = (u, u^t B_1 u^{-t}, B_2)$. Then E_φ is a closed subvariety of $E_{\tilde{\varphi}}$ whilst the latter is just $(\mathcal{U} \times \mathcal{B})_\varphi \times \mathcal{B}$. Either directly or using 7.6.12, 7.6.13 and the Kunneth decomposition we obtain a commuting diagram

$$\begin{array}{ccc} H_i((\mathcal{U} \times \mathcal{B} \times \mathcal{B})_{\tilde{\varphi}}) & \xrightarrow{s_*^\alpha} & H_i((\mathcal{U} \times \mathcal{B} \times \mathcal{B})_{\tilde{\varphi}}) \\ \int \Big\downarrow & & \int \Big\downarrow \\ H_{i-2n}(\mathcal{B} \times \mathcal{B}) & \xrightarrow{(r_\alpha)_*} & H_{i-2n}(\mathcal{B} \times \mathcal{B}) \end{array}$$

where s^α is constructed as in 7.6.5 and r_α is the real algebraic action of the simple reflection $s_\alpha \in W$ on the left hand factor in $\mathcal{B} \times \mathcal{B}$.

As in 7.6.5 (taking $X = Y'$ in the last part of 7.6.2) functoriality gives a commuting diagram

$$\begin{array}{ccc} H_i((\mathcal{U} \times \mathcal{B} \times \mathcal{B})_\varphi) & \xrightarrow{s_*^\alpha} & H_i((\mathcal{U} \times \mathcal{B} \times \mathcal{B})_\varphi) \\ \downarrow & & \downarrow \\ H_i((\mathcal{U} \times \mathcal{B} \times \mathcal{B})_{\tilde{\varphi}}) & \xrightarrow{s_*^\alpha} & H_i((\mathcal{U} \times \mathcal{B} \times \mathcal{B})_{\tilde{\varphi}}). \end{array}$$

Combined this means that we obtain a left action of W on $H_*(\mathcal{S})$. A right action is similarly obtained.

Recall 7.5.8 and let e be the identity in W. Then Z_e is just the diagonal copy $\boldsymbol{\Delta}$ of $\mathcal{B}$ in $\mathcal{B} \times \mathcal{B}$. Moreover it is closed and so is its inverse image $\mathcal{S}_e$ in $\mathcal{S}$. Through the above diagrams and 7.6.9 the fundamental class $[\mathcal{S}_e]$ of $\mathcal{S}_e$ in $H_{4n}(\mathcal{S})$ generates the regular representation of W under either left or right action. On the other hand by 7.5.8 one has $dim\ H_{4n}(\mathcal{S}) = |W|$. Thus $H_{4n}(\mathcal{S})$ carries the regular representation of W with respect to either left or right action.

In particular there exists an invertible matrix $A(w,y) : w, y \in W$ with entries in the coefficient ring such that for say the left action of W one has

$$[\bar{\mathcal{S}}_w] = \sum_{y \in W} A(w,y) y.[\bar{\mathcal{S}}_1].$$

One may show that $A(w,y)$ are integers (and even of sign $(-1)^{\ell(w)-\ell(y)}$) satisfying $A(w,y) = A(w^{-1}, y^{-1})$. Since interchanging the $\mathcal{B}$ factors in $\mathcal{S}$ sends $\mathcal{S}_y$ to $\mathcal{S}_{y^{-1}}$ it follows that

$$w.[\bar{\mathcal{S}}_1] = [\bar{\mathcal{S}}_1] \cdot w^{-1}.$$

Summarizing we obtain the

Theorem. *$H_{4n}(\mathcal{S})$ is isomorphic to $\mathbb{R}W$ as a two-sided W module with basis the fundamental classes $[\bar{\mathcal{S}}_w] : w \in W$.*

7.6.15. Let $\mathcal{C} \subset \mathcal{U}$ be a conjugacy class. It is open in its closure $\bar{\mathcal{C}}$. Again $(\mathcal{C} \times \mathcal{B} \times \mathcal{B})_\varphi = \mathcal{S}(\mathcal{C}), (\bar{\mathcal{C}} \times \mathcal{B} \times \mathcal{B})_\varphi = \mathcal{S}(\bar{\mathcal{C}})$ and these spaces have all complex dimension $2n$. Then by 7.6.2(∗∗) and 7.6.5(∗) taking $E = \bar{\mathcal{C}} \times \mathcal{B} \times \mathcal{B}$, $E' = (\bar{\mathcal{C}} \setminus \mathcal{C}) \times \mathcal{B} \times \mathcal{B}$ and by induction on $dim\ \mathcal{C}$ we obtain a $W \times W$ module structure on $H_{4n}(\mathcal{S}(\mathcal{C}))$ induced from that on $H_{2n}(\mathcal{S})$. Moreover since $W \times W$ is completely reducible it follows that $H_{4n}(\mathcal{S})$ is isomorphic (as a $W \times W$ module) to a direct sum of the $H_{4n}(\mathcal{S}(\mathcal{C})) : \mathcal{C} \subset \mathcal{U}$. Again by 7.4.2 and 7.5.8 one has $\overline{\mathcal{S}_w(\mathcal{C})} = \bar{\mathcal{S}}_w$ if and only if $\mathcal{C} \in St(w)$. Since the Steinberg map is surjective we conclude that the $[\overline{\mathcal{S}_w(\mathcal{C})}] : w \in W,\ \mathcal{C} \subset \mathcal{U}$ are just the $[\bar{\mathcal{S}}_w]$ and so form a basis of $H_{4n}(\mathcal{S})$ by 7.6.14. Of course the fibres of the Steinberg map form a disjoint union of W so this basis also gives a direct sum decomposition

$$H_{4n}(\mathcal{S}) = \bigoplus_{\mathcal{C} \subset \mathcal{U}} H_{4n}(\mathcal{S}(\mathcal{C}))$$

However this is not a $W \times W$ module isomorphism because the $H_{4n}(\mathcal{S}(\mathcal{C}))$ have only a quotient $W \times W$ module structure.

On the other hand taking $E = \mathcal{U},\ E' = \bar{\mathcal{C}}$ in 7.6.5($*$) and using 7.6.2($**$) we obtain an exact sequence

$$0 \to H_{4n}(\mathcal{S}(\hat{\mathcal{C}})) \to H_{4n}(\mathcal{S}) \to H_{4n}(\mathcal{S}(\mathcal{U} \setminus \bar{\mathcal{C}})) \to 0$$

of $W \times W$ modules. Thus $H_{4n}(\mathcal{S}(\bar{\mathcal{C}}))$ is a $W \times W$ submodule of $H_{4n}(\mathcal{S})$ and admits as a basis $\{[\bar{\mathcal{S}}] : w \in \bigcup_{\mathcal{C} \subset \bar{\mathcal{C}}} St^{-1}(\mathcal{C}')\}$. One may conclude that the matrix $A(w,y)$ gives some information on conjugacy class closures just as the Kazhdan-Lusztig polynomials determine the inclusions of primitive ideals, though the result falls rather short of the more precise conjecture announced in . We remark that the latter has now been proved using in particular that $H_{4n}(\mathcal{S})$ has an algebra structure making it isomorphic to $\mathbb{C}W$.

7.6.16. Since $Z_G(u)$ acts algebraically on $\mathcal{B}_u$, it acts on each homology space $H_{2i}(\mathcal{B}_u)$ by functoriality of the direct image. Since this action (which corresponds to left multiplication in K/T) commutes with j_α (which corresponds to right multiplication by n_{s_α}) and since s^α does not depend on the choices of ψ and U (defined in 7.6.4) it follows that the action of $Z_G(n)$ commutes with that of s_*^α. Since $Z_G^0(u)$ fixes the irreducible components of $\mathcal{B}_u$ (whose fundamental classes are a basis of $H_{2n(n)}(\mathcal{B}_u)$) the action of $Z_G(u)$ on $H_{2n(u)}(\mathcal{B}_u)$ factors to an action of A_u which commutes with that of s_*^α.

As in 7.4.3 let $p : \mathcal{S}(\mathcal{C}) \to \mathcal{C}$ be the projection onto the first factor. The fibres of p are all isomorphic to a given one say $p^{-1}(u) = \mathcal{B}_u \times \mathcal{B}_u =: F$.

Let $\mathbf{v}$ be a complementary subspace to the Lie algebra of $Z_G(u)$ in $\mathfrak{g}$. For $\varepsilon > 0$ set $N_\varepsilon = \{exp\ tv \mid v \in \mathbf{v}, \mathbf{t} \in \mathbb{C}, |\mathbf{t}| < \varepsilon\}$. For ε sufficiently small the multiplication map

$N_\varepsilon \times Z_G(u) \to G$ is injective. Since $\mathcal{C} = Gu = G/Z_G(u)$, we may identify N_ε with an open neighbourhood of u in $\mathcal{C}$. Through the diagonal action of G on $\mathcal{S}(\mathcal{C})$ we may identify $p^{-1}(V_\varepsilon)$ with $N_\varepsilon(u, \mathcal{B}_u \times \mathcal{B}_u) \cong N_\varepsilon \times F$. In other words p is a locally trivial fibration of $\mathcal{S}(\mathcal{C})$ with fibre F.

Let Y be an irreducible component of $\mathcal{S}(\mathcal{C})$ associatd by 7.4.3 to an A orbit in $\sum \times \sum$. Since G is connected, Y is G stable. Take $y \in Y$. Since $Gp(y) = \mathcal{C}$, there exists $g \in G$ such that $u = gp(y) = p(gy)$. Thus $gy \in p^{-1}(u) \cap Y = F \cap Y$ which by 7.4.3 is just the union $\cup X_i$ of the irreducible components of F corresponding to the given A orbit. Let $j : N_\varepsilon \times F \hookrightarrow \mathcal{S}(\mathcal{C})$ be the embedding and $s : F \to N_\varepsilon \times F$ be the zero section. Then the image of $[Y] \in H_{4n}(\mathcal{S}(\mathcal{C}))$ under the composed map $H_{4n}(\mathcal{S}(\mathcal{C})) \xrightarrow{j^*} H_{4n}(N_\varepsilon \times F) \xrightarrow[\sim]{s^*} H_{4n(u)}(F)$ is just $\sum_i [X_i]$. Consequently $s^* j^* = (js)^*$ induces a linear isomorphism of $(H_{2n(u)}(\mathcal{B}) \otimes H_{2n(u)}(\mathcal{B}_u))^{A_u} \cong (H_{4n(u)}(\mathcal{B}_u \times \mathcal{B}_u))^{A_u}$ onto $H_{4n}(\mathcal{S}(\mathcal{C}))$. Through 7.6.5($*$) this transports the action of s_*^α on the left (resp. right) hand factor in $H_{2n(u)}(\mathcal{B}_u) \otimes H_{2n(u)}(\mathcal{B}_u)$ to the left (resp. right) action of s_*^α on $H_{4n}(\mathcal{S}(\mathcal{C}))$.

One knows that A_u is always a group of the form $\mathbb{Z}_2^n, S_3, S_4, S_5$, where S_n denotes the permutation group on n elements. Consequently every simply A_u module is defined over $\mathbb{R}$ and so is self-dual. Thus if $M_{\mathcal{C},\tau}$ denotes the $\tau \in \hat{A}_u$ isotypical component of $M_{\mathcal{C}} := H_{2n(u)}(\mathcal{B}_u)$, we obtain

$$(H_{2n(u)}(\mathcal{B}_u) \otimes H_{2n(u)}(\mathcal{B}_u))^{A_u} = \bigoplus_{\tau \in \hat{A}_u} (M_{\mathcal{C},\tau} \otimes M_{\mathcal{C},\tau}).$$

Then from the previous paragraph we conclude that the action of the $s_*^\alpha : \alpha \in \pi$ on $H_{2n(u)}(\mathcal{B}_u)$ is a W action and that

$$(H_{4n}(\mathcal{S}(\mathcal{C})) \xrightarrow{\sim} (H_{2n(u)}(\mathcal{B}_u) \otimes H_{2n(u)}(\mathcal{B}_u))^{A_u}$$

is an isomorphism of $W \times W$ bimodules.

7.6.17. We now obtain Springer's theorem which can be expressed in the form.

Theorem. *The $M_{\mathcal{C},\tau} : \mathcal{C} \subset \mathcal{U}, \tau \in \hat{A}_u$ are pairwise non-isomorphic and exhaust the simple W modules which are furthermore defined over $\mathbb{R}$.*

Proof. Combining 7.6.14 - 7.6.16 we obtain a decomposition of the two-sided regular representation of W over $\mathbb{R}$, namely

$$\mathbb{R}W \cong \bigoplus_{\mathcal{C}\subset\mathcal{U},\tau\in\hat{A}_u} (M_{\mathcal{C},\tau} \otimes M_{\mathcal{C},\tau}).$$

Let a given $\sum M_{\mathcal{C},\tau,i}$ denote the decomposition over $\mathbb{C}$ of $M_{\mathcal{C},\tau}$ into simple modules. By the above a given $M_{\mathcal{C},\tau,i}$ occurs at least $dim\ M_{\mathcal{C},\tau}$ times in the regular representation of W. This forces $M_{\mathcal{C},\tau,i} = M_{\mathcal{C},\tau}$ and so the latter is a simple W module defined over $\mathbb{R}$. For the same reason it cannot ocur as a different $M_{\mathcal{C}',\tau'}$ and all simples must be so obtained. $\square$

Remark. It does not follow that $M_{\mathcal{C},\tau} \neq 0$, for every $\tau \in A_u$ and indeed this is false.