INVARIANTS AND SLICES FOR REDUCTIVE AND BIPARABOLIC COADJOINT ACTIONS.

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1. Outline and Historical Background

The base field will be assumed to be the complex numbers $\mathbb{C}$.

1.1 Let V be a finite dimensional vector space and G a group of linear automorphisms of V, Then G acts by transposition on the dual V^* of V and hence on the space of regular functions $R[V^*]$ on V^*, which may be identified with the symmetric algebra $S(V)$ of V, that is to say, the polynomial algebra generated by a basis of V. A basic problem of algebraic invariant theory is to determine $S(V)^G$ - the algebra of G-invariant functions on V^*.

1.2 One says that G is *reductive* on V if V is a direct sum of simple G modules. In this case $S(V)$ is also a direct sum of simple G modules. For example, $GL(V)$, the group of all linear automorphisms of V, is reductive since V is already a simple $GL(V)$ module. In addition because the ground field is assumed of characteristic zero, $GL(V)$ is also reductive on all the tensor products $V^{\otimes^n} : n \in \mathbb{N}$. One may further remark that up to twisting by one dimensional modules every simple finite dimensional $GL(V)$ module is obtained as a direct summand of $\oplus_{n\in\mathbb{N}} V^{\otimes^n}$, a result going back to Schur. (For a brief discussion of the proof see [32, 1.4.13].) The subgroup $SL(V) := \{a \in GL(V) | \det a = 1\}$ is simple. We write $GL(n), SL(n)$, when dim $V = n$. When $n = 2$, one obtains all simple finite dimensional $SL(2)$ modules by decomposing just the symmetric powers of V. For each integer $m \geq 0$. there is just one simple $SL(2)$ module V_m of dimension $m + 1$, up to isomorphism.

1.3 The study of $S(V_m)^{SL(2)}$ was pursued vigorously during the nineteenth century, full pages being devoted to the description of invariants. It is said, but perhaps only in retrospect, that the principal aim was to show that this algebra had finitely many generators. Then, in 1890, Hilbert showed that $S(V)^G$ was finitely generated for any reductive group G. This work was sarcastically described as being more like theology than mathematics. In any case, it had the unfortunate effect of closing the door on invariant theory.

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1.4 It is said that Algebra and Geometry progressed slowly and with failing momentum until they encountered one another. Invariant theory affords an excellent example of this fruitful marriage. As it turned out, invariant theory had to wait 75 years until notably Mumford pointed out that $S(V)^G$ being finitely generated allowed one to define the notion of the "categorical quotient" $V//G$, namely Spec $S(V)^G$. In this, Max $S(V)^G$ is deemed to be the set of points of $V//G$. Each element $\chi \in$ Max $S(V)^G$ assigns a scalar $\chi(f_i)$ to the i^{th} generator of $S(V)^G$, in a manner compatible with the relations they satisfy. Then we have the following geometric question. What is the nature of the subset V_χ of V on which f_i takes the value $\chi(f_i)$? It is immediate that V_χ is a union of G orbits and consequently we obtain a surjection φ of the geometric quotient V/G onto $V//G$. The fibres of this map are exactly the G orbits which cannot be separated by invariant functions.

1.5 The study of $V//G$ was the start of geometric invariant theory. However, it should immediately be said that this is not the aim of the present course. Rather, we want to investigate some cases in which G is not reductive. In general, $S(V)^G$ is no longer finitely generated, though it is no easy matter to find examples. There are, besides, many other technical difficulties and consequently, we must choose G and V carefully. For example we may take G to be a parabolic or biparabolic subgroup P of a simple algebraic group, mainly $SL(n)$, acting on the dual of its Lie algebra $\mathfrak{p}$. A second possibility is to take the centralizer of an element in a simple Lie algebra. In many such cases the algebra of polynomial invariants on the dual is a polynomial algebra [21], [35], [60], [38].

1.6 It may be useful to make some general and rather banal remarks concerning the nature of the categorical quotient $V//G$. From the definition of the Zariski topology, the inverse images of the points in $V//G$ are closed subsets (varieties) of V. Thus if $|\varphi^{-1}(\chi)| = 1$, the corresponding orbit is closed. Such orbits which are distinguished from other orbits by the invariant functions, are considered to be particularly good. At the other extreme, consider the augmentation ideal $S(V)^G_+$ generated by the homogeneous invariant polynomials of the positive degree. This is obviously a maximal ideal χ_0 of $S(V)^G$. The "worst" orbits are those in $\varphi^{-1}(\chi_0)$. One calls the corresponding preimage in V the nilfibre $\mathscr{N}$. It is presumed to be the worst behaved fibre. It is a cone and has zero as its unique closed orbit. It is presumed that the most difficult questions circulate around $\mathscr{N}$. One can ask if $\mathscr{N}$ is equidimensional, equivalently if all its components have the same codimension, and if this common value is equal to GK dim $S(\mathfrak{g})^G$. One can ask if the components are always complete intersections and even if $\mathscr{N}$ is irreducible. One can ask if $\mathscr{N}$ consists of finitely many orbits, or at least admits an orbit of minimal codimension. Such orbits are called regular. An important question is whether V admits a slice, that is to say, an affine subvariety which meets every regular orbit at exactly one point. Partial answers to these questions and their interrelationships for G reductive, are discussed in [36] -[39], [8], [60].

1.7 The case when G is a connected algebraic subgroup of $GL(V)$ acting on (Lie $G)^*$, is of particular importance. In this case an orbit (called a coadjoint orbit) admits (after Kirillov-Kostant-Souriau) the structure of a symplectic variety coming from the Lie bracket on $\mathfrak{g} := \text{Lie } G$. A symplectic variety is the natural habitat of classical (Hamiltonian) mechanics. The passage to Quantum Mechanics leads to the problem of quantizing a symplectic variety. This is interpreted (by mathematicians) as assigning an irreducible (unitary) representation to a coadjoint orbit.

A basic problem in Classical Mechanics is to solve Hamilton's equations of motion. This becomes particularly easy when we have sufficiently many variables which commute with the Hamiltonian - a circumstance which is known as a completely integrable system. Such systems are (in principle) naturally obtained from $S(\mathfrak{g})^G$ by "shift of argument". This procedure is particularly successful for G simple. However, one may show that it can work for certain parabolic subgroups P of a simple algebraic group. This leads to maximal Poisson commutative subalgebras of $S(\mathfrak{g})$ which are polynomial, providing thereby a completely integrable system. Of course, mathematicians cannot be expected to say that this corresponds to a physical system. This would involve identifying one of the generators as an acceptable Hamiltonian. Historically, the Toda lattice Hamiltonian was, in suitable co-ordinates, exactly one such generator in the case of $G = SL(3)$ acting on the dual of its Lie algebra.

1.8 These notes are organized as follows. First we describe the more classical theory when $\mathfrak{g}$ is a semisimple Lie algebra. The Chevalley theorem describing invariants is proved in Section 2.2. The Jacobson-Morosov theorem which leads to a description of all nilpotent orbits is proved in Section 2.3. Slice theory is outlined in Section 2.4 and shift of argument in Section 2.5. (For further details on slices we refer the reader to [39, Section 7].) Section 2.6 is devoted to Bolsinov's remarkable theorem concerning the construction of large Poisson commutative subalgebras. In Section 2.7 the Duflo-Vergne argument is used to show that the index of a stabilizer exceeds that of the algebra itself. In Section 2.8, this is recovered using Vinberg's inequality a proof of which is given following Panyushev [58]. The Rais theorem is proved in Section 2.9 and its application to computing the codimension of the variety of singular elements in the dual of a Lie algebra. Section 2.10 describes distinguished nilpotent orbits and the classification of all nilpotent orbits which results.

1.9 In the rewriting of these notes several new developments were taken into account. In particular the Ooms-van den Bergh sum rule [57], the description of maximal Poisson commutative subalgebras due to Panyushev and Yakimova [61] and the extension of Bolsinov's theorem to the "singular" case [42, Section 7]. For proofs the reader is referred to the articles cited.

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2. The Classical Theory - Coadjoint action for G semisimple.

2.1. **Structure theory for Semisimple Lie algebras.**

2.1.1 Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The Killing form $(x, y) \mapsto \operatorname{tr}(ad\ x)(ad\ y)$ on $\mathfrak{g}$ is bilinear, symmetric and invariant for the adjoint action, so in particular $((ad\ x)y, z)+ (y, (ad\ x)z) = 0,\ \forall\ x, y, z \in \mathfrak{g}$. (This becomes $(gy, gz) = (y, z)$, for all g in the adjoint group G of $\mathfrak{g}$). A basic fact is that the Killing form is non-degenerate if and only if $\mathfrak{g}$ is semisimple. This means that the adjoint and coadjoint actions coincide for $\mathfrak{g}$ semisimple and we may identify $\mathfrak{g}^*$ with $\mathfrak{g}$.

From now on $\mathfrak{g}$ denotes a semisimple Lie algebra. We recall some basic facts about the structure of $\mathfrak{g}$. More details can be found in [15, 1.5-1.10]

2.1.2 A Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ is a maximal commutative subalgebra whose adjoint action on $\mathfrak{g}$ is reductive. A basic fact is that the set of Cartan subalgebras is non-empty and forms a single orbit under the action of G. For each $\alpha \in \mathfrak{h}^*$, the root subspace $\mathfrak{g}_\alpha$ is defined by

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | (ad\ h)x = \alpha(h)x,\ \forall\ h \in \mathfrak{h}\}.$$

We call $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} | \mathfrak{g}_\alpha \neq 0\}$, the set of non-zero roots. It is immediate that the Killing form pairs $\mathfrak{g}_\alpha$ with $\mathfrak{g}_{-\alpha}$ and so restricts to a non-degenerate form on $\mathfrak{h}$. Thus, the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$ also identifies $\mathfrak{h}$ with $\mathfrak{h}^*$, and then the Killing form defines a scalar product $(\ ,\ )$ on $\mathfrak{h}^*$. Define $s_\alpha \in \operatorname{Aut} \mathfrak{h}^*$ through

$$s_\alpha \lambda = \lambda - 2((\alpha, \lambda)/(\alpha, \alpha))\alpha$$

for all $\lambda \in \mathfrak{h}^*$. Let W (the Weyl group of Δ) be the subgroup of Aut $\mathfrak{h}^*$ generated by the $s_\alpha : \alpha \in \Delta$. A basic fact is that Δ is W stable. Because Δ is a finite set, this condition is particularly restrictive and indeed, it is possible to classify all possible root systems which can be obtained from a semisimple Lie algebra. A root system decomposes into mutually orthogonal subsystems corresponding to the decomposition of $\mathfrak{g}$ into a product of simple Lie algebras. For a simple Lie algebra, the root system is one of four infinite families A, B, C, D indexed by dim $\mathfrak{h}$ or 5 exceptional cases E_6, E_7, E_8, F_4, G_2. The Lie algebra of $\mathfrak{sl}(n) : n \geq 2$ is of type A_{n-1}.

2.1.3 Retain the above hypotheses and notation. A basic fact is that $\dim \mathfrak{g}_\alpha = 1,\ \forall\ \alpha \in \Delta$. We can identify $\mathfrak{g}_0$ with $\mathfrak{h}$. If $\alpha, \beta \in \Delta$, then obviously $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and less obviously, one has equality exactly when $\alpha + \beta$ is a root. Finally, the $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ span $\mathfrak{h}$. These facts almost allow one to canonically determine the Lie algebra relations from the additive structure of Δ. Indeed call a root system Δ a finite subset of Euclidean n-space such that $2(\alpha, \beta)/(\alpha, \alpha) \in \mathbb{Z}, \forall \alpha, \beta \in \Delta$, such that $\alpha \in \Delta,\ n\alpha \in \Delta$ implies $n = \pm 1$ and such that Δ is stable under the resulting Weyl group defined as in 2.1.2. Then there is exactly one semisimple Lie algebra up to isomorphism with Δ as its set of non-zero roots. Up to signs its Lie algebra relations may be written down rather explicitly (see [24, Chap III, Thm. 5.5] for example). Allocating the signs and so proving existence and uniqueness needs a major effort. Perhaps a better approach to existence and uniqueness is through generators and relations. This extends to the general context of symmetrizable Kac-Moody Lie algebras [43].

We remark that a further possible development in the above is that the integrality condition on $2(\alpha,\beta)/(\alpha,\alpha)$ may be dropped and the *finite* groups so obtained be classified. These additional groups are the dihedral groups D_{2n} (not of course corresponding to the rank two semisimple Lie algebras) and two further groups customarily designated as H_3, H_4. All may be realized as subgroups of a larger rank Weyl group. See [26], [53] and [39, 2.10] for example. One may remark that this description of D_{10}, H_3, H_4 involves the Golden Section.

Naturally, the Weyl group plays a fundamental role in the structure theory of a semisimple Lie algebra, as well as in its representation theory.

2.1.4 A simple root system π for Δ is defined to be a basis for $\mathfrak{h}^*$ such that $\Delta = \Delta^+ \sqcup \Delta^-$ where $\Delta^- = -\Delta^+$ and

$$\Delta^+ = \mathbb{N}\pi \cap \Delta.$$

The existence of a simple root system is a remarkable fact. Moreover, the set of all simple root systems forms a single W orbit. Notice that if $\mathfrak{g}_\alpha = \mathbb{C}x_\alpha$, then

$$[x_\alpha, x_{-\beta}] = 0, \ \forall\ \alpha, \beta \in \pi \text{ distinct.} \tag{$*$}$$

2.1.5 Let π_1 be a subset of π. Define

$$\Delta_1^\pm = \pm\mathbb{N}\pi_1 \text{ and } \mathfrak{n}_{\pi_1}^\pm = \oplus_{\alpha\in\Delta_1^\pm}\ \mathfrak{g}_\alpha.$$

The latter is obviously a subalgebra of $\mathfrak{g}$. From the remarks in 2.1.3, it is generated by the $x_\alpha : \alpha \in \pm\pi_1$.
Then from 2.1.4($*$), one checks that

$$\mathfrak{q}_{\pi_1,\pi_2} := \mathfrak{n}_{\pi_2}^+ \oplus \mathfrak{h} \oplus \mathfrak{n}_{\pi_1}^-$$

is a subalgebra of $\mathfrak{g}$. It is called the standard biparabolic subalgebra determined by the pair $\pi_1, \pi_2 \subset \pi$. Such subalgebras and the abstraction of their properties were a main motivation for these lectures. Notice, we may identify $\mathfrak{q}_{\pi_1,\pi_2}^*$ with $\mathfrak{q}_{\pi_2,\pi_1}$ through the Killing form. Moreover, the coadjoint action of $\mathfrak{q} := \mathfrak{q}_{\pi_1,\pi_2}$ on $\mathfrak{q}^*$ just becomes commutation in $\mathfrak{g}$, modulo the orthogonal $\mathfrak{q}^\perp$ of $\mathfrak{q}$ taken with respect to the Killing form.

2.2. **Chevalley's Theorem.**

2.2.1 Since $S(\mathfrak{g})$ identifies with $R[\mathfrak{g}^*]$ and $S(\mathfrak{h})$ with $R[\mathfrak{h}^*]$, we obtain a canonical surjection of $S(\mathfrak{g})$ onto $S(\mathfrak{h})$ by restriction of functions. Let ψ denote the restriction of the above homomorphism to $Y(\mathfrak{g}) := S(\mathfrak{g})^G$.

Theorem (Chevalley [13])

$$\psi : S(\mathfrak{g})^G \xrightarrow{\sim} S(\mathfrak{h})^W.$$

Proof Set $G = \langle \exp ad\ x_\alpha : \alpha \in \Delta,\ x_\alpha \in \mathfrak{g}_\alpha \rangle$. Since $ad\ x_\alpha$ is nilpotent $\exp ad\ x_\alpha \in \text{Aut}\ \mathfrak{g}$ (over any base field of characteristic zero)). [More remarkably, if the x_α are restricted to belong to a Chevalley basis and $\mathfrak{g}_\mathbb{Z}$ is the $\mathbb{Z}$ module defined by that basis, then $G \subset \text{Aut}\ \mathfrak{g}_\mathbb{Z}$. The resulting image in $\text{Aut}(\mathfrak{g}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{Z}/ < p^n >)$ is a finite group called a Chevalley group-see [9, Chap. 4] for example.] If $x_\alpha, x_{-\alpha}$ together with $\alpha^\vee := [x_\alpha, x_{-\alpha}]$ form a standard basis for $\mathfrak{sl}(2)$, then under the image of $< \exp ad\ x_\alpha,\ \exp ad\ x_{-\alpha} >$ in $\mathbb{C}^2$, we obtain

$$\exp ad\ x_\alpha\ \exp ad\ x_{-\alpha}\ \exp ad - x_\alpha = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

With $\widetilde{s}_\alpha := \exp ad\ x_\alpha\ \exp ad\ x_{-\alpha}\ \exp ad - x_\alpha$, let N denote the subgroup of G generated by the $\widetilde{s}_\alpha : \alpha \in \Delta$. Using the fact that $\widetilde{s}_\alpha$ interchanges the eigenvectors of $\alpha^\vee$ in $\mathbb{C}^2$ one easily checks that $\widetilde{s}_\alpha \alpha^\vee = -\alpha^\vee$. Trivially, $\widetilde{s}_\alpha h = h$, if $h \in \mathfrak{h}$ satisfies $\alpha(h) = 0$. Hence $\widetilde{s}_\alpha h = s_\alpha h$, for all $h \in \mathfrak{h}$ and so $\widetilde{s}_\alpha$ sends $\mathfrak{g}_\beta$ to $\mathfrak{g}_{s_\alpha \beta}$, for all $\beta \in \Delta$. Consequently the action of N commutes with restriction of functions $S(\mathfrak{g}) \to S(\mathfrak{h})$ and so the image of $Y(\mathfrak{g})$ in $S(\mathfrak{h})$ is contained in $S(\mathfrak{h})^W$.

Injectivity of $\psi|_{Y(\mathfrak{g})}$.

Fix $h \in \mathfrak{h}$ and consider Gh. One checks that the tangent space $T_{Gh,h}$ to Gh at h coincides with $\oplus_{\alpha \in \Delta | h(\alpha) \neq 0}\ \mathfrak{g}_\alpha$. Set $\mathfrak{h}_{reg} = \{h \in \mathfrak{h} | h(\alpha) \neq 0,\ \forall\ \alpha \in \Delta\}$ which is Zariski open dense in $\mathfrak{h}$. From the above, it follows that $\dim G\mathfrak{h} = \dim \oplus_{\alpha \in \Delta}\ \mathfrak{g}_\alpha + \dim \mathfrak{h} = \dim \mathfrak{g}$. Since $\mathfrak{g}$ is irreducible (as a variety), we obtain $\overline{G\mathfrak{h}} = \mathfrak{g}$. Now, if $f \in Y(\mathfrak{g})$ vanishes on $\mathfrak{h}$, it vanishes on $G\mathfrak{h}$, hence on $\mathfrak{g}$. Thus $f = 0$, as required.

Surjectivity of $\psi|_{Y(\mathfrak{g})}$.

Take $a \in S(\mathfrak{h})$. Since $\mathfrak{g}$ is reductive, we can write

$$(ad\ U(\mathfrak{g}))a = I \oplus (ad\ U(\mathfrak{g})_+)a, \text{ with } I \subset Y(\mathfrak{g}), \dim I \leqslant 1. \tag{$*$}$$

By $(*)$, we may define a map $\theta_0 : S(\mathfrak{h}) \to Y(\mathfrak{g})$, by $a - \theta_0(a) \in (ad\ U(\mathfrak{g})_+)a$. This is plainly linear and furthermore, using N defined above, one checks that $\theta_0(wa) = \theta_0(a),\ \forall\ w \in W$. Thus θ_0 factors to a linear map θ of $S(\mathfrak{h})^W$ into $Y(\mathfrak{g})$. Similarly, we may define a linear map $\widetilde{\theta}$ of $U(\mathfrak{h})^W = S(\mathfrak{h})^W$ onto $Z(\mathfrak{g}) = \text{Cent}\ (U(g))$. Since symmetrization s commutes with adjoint action and with W, the diagram

$$\begin{array}{ccc} S(\mathfrak{h})^W & \xrightarrow{\theta} & Y(\mathfrak{g}) \\ s \downarrow & & \downarrow s \\ U(\mathfrak{h})^W & \xrightarrow{\widetilde{\theta}} & Z(\mathfrak{g}) \end{array}$$

is commutative. Recalling that the above restrictions of s (the vertical arrows,) are linear bijections, one obtains

$$\theta \text{ injective } \Leftrightarrow \widetilde{\theta} \text{ injective.}$$

The proof of surjectivity is then completed after the folowing step.

Injectivity of $\widetilde{\theta}$.

Let $\varepsilon : U(\mathfrak{g}) \to \mathbb{C}$ be the augmentation . One has $\ker \varepsilon = U(\mathfrak{g})_+$. For any finite dimensional $U(\mathfrak{g})$ module V, one has

$$\operatorname{tr}((ad\ a)b, V) = \varepsilon(a) \operatorname{tr}(b, V),\ \forall\ a, b \in U(\mathfrak{g}).$$

Hence

$$\operatorname{tr}(\widetilde{\theta}(a), V) = \operatorname{tr}(a, V) = \sum_{\nu \in \mathfrak{h}^*} a(\nu) \dim V_\nu,$$

where V_ν is the weight subspace of V of weight ν.

Let P (resp. P^+) denote the set of weights (resp. dominant weights). Given $\mu \in P^+$, there exists a unique up to isomorphism simple $U(\mathfrak{g})$ module $V(\mu)$ of highest weight μ, and which is furthermore finite dimensional. Define $\mu \leqslant \nu$ if $\nu - \mu \in \mathbb{N}\pi$ and, given $\mu \in P^+$, let

$$S(\mu) := \sum_{w \in W/\mathrm{Stab}_W \mu} e^{w\mu}$$

be the orbit sum it defines. The Weyl character formula asserts that, with respect to $\leqslant$, there exists a triangular matrix $m_{\mu,\nu}$, with ones on the diagonal, such that

$$ch\ V(\mu) = \sum_{\nu \leqslant \mu} m_{\mu,\nu} S(\nu).$$

Thus, if $tr(\widetilde{\theta}(a), V(\mu)) = 0,\ \forall\ \mu \in P^+$, we obtain

$$\sum_{w \in W} a(w\nu) = 0,\ \forall\ \nu \in P^+.$$

Since a is W invariant, we deduce that $a(\nu) = 0,\ \forall\ \nu \in P^+$ and since P^+ is Zariski dense in $\mathfrak{h}^*$, we obtain $a = 0$. Consequently $\widetilde{\theta}$ is injective.

Completing the proof of surjectivity.

From the above, we conclude that $\psi\theta$ is an injective linear map of $S(\mathfrak{h})^W$ to itself. Since it preserves degree and the space of invariant polynomials of degree $\leqslant n$ is finite dimensional, one concludes that $\psi\theta$ is bijective. Hence ψ is surjective. □

The advantage of the above proof is that it generalizes to a multivariable Chevalley type theorem. In particular, consider the corresponding map of $(S(\mathfrak{g}) \otimes S(\mathfrak{g}))^G$ onto $(S(\mathfrak{h}) \otimes S(\mathfrak{h}))^W$, where the groups G, W act diagonally. This map is no longer injective; but it is surjective, which is proved by showing that a similarly defined map to θ is injective [33]. For that we require more than the Weyl formula. Precisely, we require some detailed description of the tensor product of two simple $\mathfrak{g}$ modules, namely the truth of the refined PRV conjecture established by Kumar [49]. It even gives a deeper meaning to this conjecture.

The above description of invariants has been extended to the case of $S(\mathfrak{p})^{\mathfrak{k}}$, where $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of a semisimple Lie algebra $\mathfrak{g}$, by Tevelev [67].

2.2.2 A further theorem of Chevalley asserts that $S(\mathfrak{h})^W$ is a polynomial algebra. A key point in the proof is that if $p \in S(\mathfrak{h}^*)$ satisfies $s_\alpha p = -p$, then α divides p, as one easily checks. In particular, for all $p \in S(\mathfrak{h}^*)$ and all $\alpha \in \Delta$, the difference $s_\alpha p - p$ is divisible by

α. This fact enables one to apply a reasoning using induction on degree. The details are given below.

2.2.2.1 Let V be a $\mathbb{C}$-vector space and G a finite subgroup of $GL(V)$. Then G acts on V and by automorphisms on $S(V)$. The subalgebra $S(V)^G$ of invariant polynomials is easy to construct. For any $p \in S(V)$ note that

$$e(p) := \frac{1}{|G|} \sum_{g \in G} gp \in S(V)^G.$$

The advantage of the factor $|G|$ in the denominator (which can cause trouble in positive characteristic) is that it makes $e : S(V) \to S(V)^G$ so defined, an idempotent. In particular, e is surjective.

2.2.2.2 Now let W denote the Weyl group which we recall is a subgroup of $GL(V)$ with $V = \mathfrak{h}$. Identify $\mathfrak{h}$ with $\mathfrak{h}^*$ through the killing form. The following two results are valid for any finite group generated by reflections (or even pseudo-reflections). Let $S(\mathfrak{h})^W_+$ denote the ideal of $S(\mathfrak{h})^W$ generated by the homogeneous polynomials in $S(\mathfrak{h})^W$ of positive degree and set $I = S(\mathfrak{h})S(\mathfrak{h})^W_+$.

Lemma *Suppose $p_1, p_2, \ldots, p_n \in S(\mathfrak{h})^W$ satisfy $p_1 \notin \sum_{i=2}^n S(\mathfrak{h})^W p_i$ and $\sum_{i=1}^n q_i p_i = 0$ for some $q_i \in S(\mathfrak{h})$ homogeneous. Then $q_1 \in I$.*

Proof Induction on deg q_1. If deg $q_1 = 0$ then since

$$\sum_{i=1}^n e(q_i) p_i = 0$$

we must have $q_1 = e(q_1) = 0$ by the hypothesis on p_1. On the other hand, we can write $s_\alpha(q_i) - q_i = \alpha q_i'$, with q_i' homogeneous of degree $<$ deg q_i and moreover,

$$\sum_{i=1}^n q_i' p_i = 0.$$

Thus, by the induction hypothesis $q_1' \in I$ and so $s_\alpha q_1 - q_1 \in I$. Since this holds for $\alpha \in \pi$, we deduce that $q_1 = e(q_1) + I \subset I$, since deg $q_1 > 0$. □

2.2.2.3 Since $S(\mathfrak{h})$ is a polynomial algebra, the Hilbert basis theorem implies that any increasing sequence $I_1 \subset I_2 \subset \ldots$, of ideals of $S(\mathfrak{h})$ is stationary. Choose inductively $p_i : i = 1, 2, \ldots$, homogeneous and W invariant of positive degree, with $p_i \notin \sum_{j=1}^{i-1} S(\mathfrak{h}) p_j$ if the latter is strictly contained in I and so does not contain $S(\mathfrak{h})^W_+$. By the first remark, this gives $m \in \mathbb{N}^+$ and $p_1, p_2, \ldots, p_m \in S(\mathfrak{h})^W_+$ homogeneous such that

$$I = \sum_{j=1}^m S(\mathfrak{h}) p_i.$$

Furthermore, we can assume

$$p_i \notin I_i := \sum_{j(\neq i)=1}^{m} S(\mathfrak{h})p_j, \text{ for all } j = 1, 2, \ldots, m.$$

Lemma *The set $\{p_i\}_{i=1}^m$ generates $S(\mathfrak{h})^W$ as an algebra.*

Proof Let $S_n(\mathfrak{h})$ denote the subspace of $S(\mathfrak{h})$ of homogeneous polynomials of degree n. Since the action of W preserves degree, we have

$$S(\mathfrak{h})^W = \bigoplus_{n\in\mathbb{N}} S_n(\mathfrak{h})^W.$$

Let S_n denote the subspace of $S_n(\mathfrak{h})^W$ generated by the $\{p_i\}_{i=1}^m$. The assertion $S_n = S_n(\mathfrak{h})^W$ is trivial for $n = 0$. Assume it for some $n \geqslant 0$. Given $q \in S_{n+1}(\mathfrak{h})^W \subset S(\mathfrak{h})_+^W \subset I$, we can write

$$q = \sum_{i=1}^{m} q_i p_i, \text{ with deg } q_i = n + 1 - \deg p_i \leqslant n.$$

Then

$$q = e(q) = \sum_{i=1}^{m} e(q_i)p_i \subset \sum_{i=1}^{n} S_n p_i,$$

hence the assertion. □

Remark This follows closely Hilbert's proof that $S(V)^{SL(2)}$ is finitely generated, except that he used a different projection (called at the time, an Ω-process). A modern treatment replaces this with complete reducibility and is valid for any reductive subgroup of $GL(V)$.

2.2.2.4 Chevalley took the above construction one step further for reflection groups.

Lemma *The $\{p_i\}_{i=1}^m$ are algebraically independent.*

Proof Set $\deg p_i = m_i$. Define a grading on $\mathbb{C}[X_1, X_2, \ldots, X_m]$ by setting deg $X_i = m_i$ and let $\varphi : \mathbb{C}[X_1, X_2, \ldots, X_m] \to S(\mathfrak{h})^W$ be the graded algebra map defined by $\varphi(X_i) = p_i$. If the conclusion of the lemma is false, there exists $r \in \mathbb{C}[X_1, X_2, \ldots, X_m]$ homogeneous such that $\varphi(r) = 0$, and of minimal degree with this property. Set $r_i = \varphi(\frac{\partial r}{\partial X_i}) : i = 1, 2, \ldots, m$. These cannot all be zero. Let $R_i : i = 1, 2, \ldots, m$, denote the ideal of $S(\mathfrak{h})^W$ generated by the $r_j : j \neq i$. Relabel the r_i so that $r_i \notin R_i$, for $i = 1, 2, \ldots, s$ and $r_j \in \sum_{i=1}^s S(\mathfrak{h})^W r_i$, for all $j > s$. We may write

$$r_{s+j} = \sum_{i=1}^{s} u_{j,i} r_i : j = 1, 2, \ldots, m - s.$$

with $u_{j,i} \in S(\mathfrak{h})^W$ homogeneous of degree deg $r_{s+j} -$ deg r_i.

Let $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{h}$. Then

$$O = \frac{\partial r}{\partial x_k} = \sum_{i=1}^{m} \frac{\partial r}{\partial p_i} \frac{\partial p_i}{\partial x_k} = \sum_{i=1}^{s} r_i \left(\frac{\partial p_i}{\partial x_k} + \sum_{j=1}^{m-s} u_{j,i} \frac{\partial p_{s+j}}{\partial x_k} \right).$$

By construction and 2.2.2.2 we obtain

$$\frac{\partial p_i}{\partial x_k} + \sum_{j=1}^{m-s} u_{j,i} \frac{\partial p_{s+j}}{\partial x_k} \in I, \ i = 1, 2, \ldots, s, \ \forall \ k.$$

which moreover, is homogeneous of degree deg $p_i - 1$.

Multiplying by x_k and summing gives

$$m_i p_i + \sum_{j=1}^{m-s} u_{j,i} \ m_{s+j} \ p_{s+j} = \sum_{j=1}^{m} q_{i,j} p_j, \ i = 1, 2, \ldots, s,$$

for some $q_{i,j} \in S(\mathfrak{h})$ homogeneous of degree deg $p_i -$ deg $p_j \geqslant 1$. This forces $q_{i,i} = 0 : i \leqslant s$, and then the contradiction $p_i \in I_i$. □

2.2.2.5 Summarizing the above we have proved the following

Theorem *$S(\mathfrak{h})^W$ is a polynomial algebra*

2.2.2.6 The observation in 2.2.2 gives (see [34, 8.3] for example) a further result of Chevalley, namely that $S(\mathfrak{h})$ is free over $S(\mathfrak{h})^W$ (of rank $|W|$). Shephard and Todd [65] noted that the latter result gives a further proof of 2.2.2.5. Write $S = S(\mathfrak{h}), \ R = S(\mathfrak{h})^W$.

Lemma

$$S \otimes_R \operatorname{Tor}_n^R(\mathbb{C}, \mathbb{C}) = \operatorname{Tor}_n^S(S/I, S/I).$$

Proof Let $\to p_n \to \cdots \to p_0 \to \mathbb{C}$ be a free resolution of $\mathbb{C}$ as an R module. By definition, $\operatorname{Tor}_n^R(\mathbb{C}, \mathbb{C})$ is the n^{th} homology group [18, Chap. 6] of the complex $\to p_n \otimes_R \mathbb{C} \to \cdots \to p_0 \otimes_R \mathbb{C}$. Since S is free over R and hence $S \otimes_R -$ is exact, it follows that $\to S \otimes_R p_n \to \cdots \to S \otimes_R k$ is a free resolution of $S \otimes_R k \cong S/SAnn_R\mathbb{C} = S/I$. By definition $\operatorname{Tor}_n^S(S/I, S/I)$ is the n^{th} homology group of the complex $\to (S \otimes_R p_n) \otimes_S S/I \to \cdots \to S \otimes_R p_0 \otimes_S S/I$. Yet $(S \otimes_R p_n) \otimes_S (S \otimes_R k) = S \otimes_R (p_n \otimes_R k)$, so the assertion follows by again using exactness. □

Yet S is polynomial and so for any finitely generated S module, one has $\operatorname{Tor}_n^S(M, M) = 0$, for n sufficiently large [18, Chap. 15]. Consequently, by the exactness of $S \otimes_R -$, $\operatorname{Tor}_n^R(\mathbb{C}, \mathbb{C}) = 0$ for n sufficiently large. Since R is a finitely generated graded algebra, this forces R to be polynomial [18, Chap. 19].

2.2.2.7 Bernstein and Lunts [2] pointed out that $S(\mathfrak{h})$ being free over $S(\mathfrak{h})^W$ implied the Kostant theorem that $S(\mathfrak{g})$ is free over $S(\mathfrak{g})^G$, that is, we can write

$$S(\mathfrak{g}) = H \otimes S(\mathfrak{g})^G. \qquad (*)$$

Here reductivity allows one to take H to be $ad\,\mathfrak{g}$ stable and spanned by homogeneous polynomials. Further ramifications of this approach are pointed out in [34, 8.2]. We describe one below.

2.2.3 Combined with the Weyl character formula, one may deduce from $(*)$ the (known) degrees of the generators of $S(\mathfrak{g})^G$. Our treatment follows [34, 8.6,8.7]. It is inspired by a paper of Hesselink [25] which gave a less good result since he needed to know a formula of Kostant, that is 2.2.4 $(*)$ below.

First, for any graded vector space

$$V = \oplus_{n\in\mathbb{N}} V_n,$$

with V_n finite dimensional, we set

$$ch_q V = \sum_{n\in\mathbb{N}} q^n \dim V_n.$$

Similarly, if V is an $\mathfrak{h}$ module admitting a decomposition into finite dimensional weight spaces $V_\mu : \mu \in \mathfrak{h}^*$, then we write

$$ch\, V = \sum_{\mu\in\mathfrak{h}^*} (\dim V_\mu) e^\mu$$

where the e^μ are viewed as elements of the group ring of $\mathfrak{h}^*$. The latter is denoted by $\mathbb{Z}[e^{\mathfrak{h}^*}]$.

Now, since $S(\mathfrak{g})$ is both an $ad\,\mathfrak{h}$ module and a graded vector space through degree of homogeneous polynomials, it admits a q-character with values in $\mathbb{Z}[e^{\mathfrak{h}^*}]$ which, by the general structure theory described in 2.1.2 and 2.1.3, takes the form

$$ch_q S(\mathfrak{g}) = (1-q)^{-\dim\mathfrak{h}} \prod_{\alpha\in\Delta} (1-qe^\alpha)^{-1}. \qquad (*)$$

On the other hand, 2.2.2$(*)$ gives

$$ch_q S(\mathfrak{g}) = ch_q H \; ch_q S(\mathfrak{g})^G. \qquad (**)$$

Now decompose each graded subspace H_n into a direct sum of finite dimensional highest weight modules $V(\mu)$ and let $[H_n : V(\mu)]$ denote the multiplicity of $V(\mu)$ in H_n. Then, clearly,

$$ch_q H = \sum_{n\in\mathbb{N}} \sum_{\mu\in\mathfrak{h}^*} q^n [H_n : V(\mu)] ch\, V(\mu).$$

The $ch\, V(\mu)$ are given by the Weyl character formula. On the other hand, only trivial modules occur in $S(\mathfrak{g})^G$, whereas only H_0 contains the trivial module. Thus, substituting $(*)$ into $(**)$, we may compute both the multiplicities $[H_n : V(\mu)]$ and $ch_q S(\mathfrak{g})^G$. After a little combinatorics [34, 8.6], one obtains the

Theorem (Kostant)

$$ch_q S(\mathfrak{g})^G = \left[\left(\sum_{w\in W} q^{\ell(w)} \right) (1-q)^\ell \right]^{-1},$$

where $\ell(w)$ denotes the reduced length of $w \in W$.

2.2.4 Now let $\{d_i\}_{i=1}^{\ell} : \ell = \dim \mathfrak{h}$ denote the degrees of the homogeneous generators of $S(\mathfrak{g})^G$. By definition of ch_q, we must have

$$ch_q S(\mathfrak{g})^G = \prod_{i=1}^{\ell} (1 - q^{d_i})^{-1}$$

Substitution into Kostant's theorem determines the $\{d_i\}$ in terms of the length function $\ell(\cdot)$ on W. That is

$$\prod_{i=1}^{\ell} \left(\frac{1 - q^{d_i}}{1 - q} \right) = \sum_{w \in W} q^{\ell(w)}. \tag{$*$}$$

This is not the most convenient expression for the d_i. However, we note that W admits a unique longest element w_π and its length is $|\Delta^+|$. Now, for any finite dimensional Lie algebra $\mathfrak{a}$, let index $\mathfrak{a}$ denote the codimension of a regular orbit in $\mathfrak{a}^*$ and set

$$c(\mathfrak{a}) = \frac{1}{2}(\dim\ \mathfrak{a} + \text{index } \mathfrak{a})$$

which is an integer because any coadjoint orbit is even dimensional. For $\mathfrak{g}$ semisimple one has

$$\text{index } \mathfrak{g} = \text{rank } \mathfrak{g} = \dim \mathfrak{h}$$

and so

$$c(\mathfrak{g}) = |\Delta^+| + \dim \mathfrak{h}.$$

From the above we conclude that

$$\sum_{i=1}^{\ell} d_i = \ell + \ell(w_\pi) = c(\mathfrak{g})$$

For most biparabolics $\mathfrak{q}$, the semi-invariant algebra $Sy(\mathfrak{p})$ is polynomial [22], [35] and the degrees satisfy a similar sum formula. With some mild conditions on the Lie algebra $\mathfrak{a}$ this sum rule holds whenever the semi-invariant algebra $Sy(\mathfrak{a})$ is polynomial [57, Thm. 1.1, Prop. 1.4].

2.3. **The Nilpotent Cone and the Jacobson-Morosov Theorem.**

2.3.1 In the identification of $\mathfrak{g}$ with $\mathfrak{g}^*$, the nilfibre $\mathscr{N}$ identifies with the subset of ad-nilpotent elements of $\mathfrak{g}$ and is generally known as the nilpotent cone. To emphasize that it refers to $\mathfrak{g}$ and not to say a biparabolic subalgebra, we shall sometimes designate it as $\mathscr{N}_\pi$.

2.3.2 A central result in the study of $\mathscr{N}$ is the Jacobson-Morosov theorem, which allows one to conclude that $\mathscr{N}/G$ is finite, that $\mathscr{N}$ is irreducible and a complete intersection. Before turning to the general theory, let us first examine the case of $\mathfrak{sl}(n)$.

2.3.3 One easily checks that $x \in \mathfrak{sl}(n)$ is an ad-nilpotent if and only if it is nilpotent as an element of End $V : V = \mathbb{C}^n$. Choose $m \in \mathbb{N}^+$ such that $x^{m-1} \neq 0$, but $x^m = 0$. Choose $v \in V$ such that $x^{m-1}v \neq 0$. Then $v, xv, \ldots, x^{m-1}v$ is the basis of a subspace V_m of V of dimension m. With respect to this basis, the matrix representation of x in End V_m is a Jordan block, consisting of ones in the $(i, i+1)^{th}$ entries and zeros elsewhere. A non-trivial (though not too difficult) fact is that V_m admits a $\mathbb{C}[x]$ stable complement in V and so we may repeat the process. This eventually allows one to find a basis $\{v_i\}$ for V such that the matrix representation of x is a direct sum of Jordan blocks of decreasing size. Since $SL(n)$ permutes the bases of V, it follows that every element of $\mathscr{N}$ may be brought into this form by conjugation.

The sizes of the Jordan blocks of x form a decreasing sequence of integers $m_1, m_2, \ldots, m_k$ whose sum is n, and hence, a partition of n. Since any direct sum of Jordan blocks is nilpotent, every partition so obtains. The above construction gives a surjection of $\mathscr{N}$ onto the set $P(n)$ of partitions of n. Moreover, we can read off the dual partition directly from an element $x \in \mathscr{N}$ from the sequence of integers dim ker $x^i : i = 1, 2, \ldots$. Consequently, we obtain a bijection of $\mathscr{N}/G$ onto $P(n)$. Finally, we may give a recipe (though a rather redundant one - a better recipe has been given by Weyman [68]) for computing the ideal of definition of the closure of Gx. Namely, we note that dim ker $x^i \leqslant s$ if and only if all $(n-s) \times (n-s)$ minors of x^i vanish. These identities are called power rank identities. Since $|P(n)| \leqslant 2^n$, we conclude that $|\mathscr{N}/G| \leqslant 2^n$ and is in particular, finite.

2.3.4 Jordan block decomposition may be expressed by saying that up to conjugation, any ad-nilpotent element $x \in \mathfrak{sl}(n)$ may be written in the form

$$x = \sum_{\alpha \in \pi'} x_\alpha$$

for some subset $\pi' \subset \pi$. In this the components of π' describe the partition of n.

2.3.5 Now let x be a single Jordan block. It is easy to check that the centralizer of x in End V is just $\mathbb{C}[x]$ which has dimension n. Every element of $\mathbb{C}[x]$ can be presented as an upper triangular Toeplitz matrix. Those lying in $\mathfrak{g} = \mathfrak{sl}(n)$ must be strictly upper triangular. Thus $\mathfrak{g}^x$ has dimension $n-1$ and is commutative.

2.3.6 Jordan block decomposition implies that every $x \in \mathscr{N}$ can be conjugated into $\mathfrak{n}^+$. On the other hand, every element of $\mathscr{N}$ and hence of $G\mathfrak{n}^+$ is ad-nilpotent. Consequently, $\mathscr{N} = G\mathfrak{n}^+$. Since $\mathfrak{n}^+$ is a vector space and G is connected, we conclude that $\mathscr{N}$ is irreducible. Since $\mathscr{N}/G$ is finite, $\mathscr{N}$ must admit a unique dense orbit of dimension dim $G\mathfrak{n}^+$. Yet $\mathrm{Stab}_G\mathfrak{n}^+$ contains the group B of upper triangular matrices and G/B has dimension $|\Delta^-|$. Consequently, $\dim G\mathfrak{n}^+ \leqslant \dim G/B + \dim \mathfrak{n}^+ = |\Delta|$. On the other hand, we saw in 2.3.5 that the single Jordan block generates an orbit of codimension $\dim \mathfrak{h}$ and hence of dimension $|\Delta|$. Thus $\dim \mathscr{N} = |\Delta|$ and since $Y(\mathfrak{g})$ has $\dim \mathfrak{h}$ generators, $\mathscr{N}$ is a complete intersection.

2.3.7 All these pleasant conclusions may be deduced in the general case from the Jacobson-Morosov theorem. This asserts the following. Recall that we are identifying $\mathscr{N}$ with a subset of $\mathfrak{g}$.

Theorem *For each $x \in \mathscr{N}$ there exists $h, y \in \mathfrak{g}$ such that (x, h, y) form a standard basis of an $\mathfrak{sl}(2)$ subalgebra.*

We call (x, h, y) an s-triple containing x.

2.3.8 In the case of $\mathfrak{sl}(n)$, we may easily deduce the Jacobson-Morosov theorem from Jordan block decomposition. Indeed, it will be enough to do this for a single block. Then x takes the form

$$x = \sum_{\alpha \in \pi} x_\alpha$$

Now choose the unique $h \in \mathfrak{h}$ such that $h(\alpha) = 2, \ \forall \ \alpha \in \pi$. Then $[h, x] = 2x$, whilst $[h, y] = -2y$, for any $y \in \oplus_{\alpha \in \pi} \mathfrak{g}_{-\alpha}$. Finally, by the remark in 2.1.3 and 2.1.4$(*)$, we may choose y so that $[x, y] = h$.

2.3.9 The general case is somewhat more difficult. Here we shall indicate the main point in the proof due to Kostant, which is particularly astute. Observe that if an s-triple containing x exists, then $x \in (ad\ x)^2\mathfrak{g}$. Let us show this directly. It depends on a result in linear algebra which asserts that if a, b are endomorphisms of *a finite dimensional* vector space over a field of *characteristic zero* such that a is nilpotent and $[a, [a, b]] = 0$, then ab is nilpotent. (When these conditions are not met the pair $a = d/dx, b = x$ provides a counterexample.)

Take $a = (ad\, x)^2$ which is nilpotent, since $x \in \mathscr{N}$. Through the invariance of the Killing form $K(ay, z) = K(y, az), \ \forall \ y, z \in \mathfrak{g}$. Thus $Im\, a = (\ker\ a)^\perp$. Take $b = ad\, y$ with $y \in \ker\ a$. Then $ay = 0$ translates to $[x, [x, y]] = 0$ giving $[ad\ x, [ad\ x, ad\ y]] = 0$ and so $[ad\ x, [ad\ x, ad\ y]] = 0$. Consequently, $(ad\ x)(ad\ y)$ is nilpotent. Hence $K(x, y) = 0$, that is $x \in (\ker\ a)^\perp = Im\, a = Im(ad\, x)^2$, as required.

Now choose $y' \in \mathfrak{g}$ such that $[x, [x, y']] = -2x$, and set $h = [x, y']$. Then $[h, x] = 2x$. It is a relatively easy matter to adjust y' by an element of $\ker\ ad\, x$ to obtain an element $y \in y' + \ker\ ad\, x$ satisfying $[h, y] = -2y$. This completes Kostant's proof of the Jacobson-Morosov theorem.

A second proof of the Jacobson-Morosov theorem can be found in [9, 5.3]. It applies in positive characteristic p for p not too small. The basic idea is to construct a finite dimensional $\mathfrak{g}$ module V so that the image of x in $\operatorname{End} V$ is nilpotent. For example the adjoint representation will do. One constructs an $\mathfrak{sl}(2)$ subalgebra containing x in $\operatorname{End} V$ as in 2.3.8. Identify $\mathfrak{g}$ with its image in $\operatorname{End} V$. One must modify the $\mathfrak{sl}(2)$ subalgebra so it lies in $\mathfrak{g}$. For this one needs that the trace form on $\operatorname{End} V$ be non-degenerate, so requiring the characteristic to be not too small. Using orthogonal decomposition with respect to the trace form one effects this modification in a manner analogous to the last part of the proof of 2.3.9.

2.3.10 The power of the Jacobson-Morosov theorem derives from the representation theory of $\mathfrak{sl}(2)$. First of all, $\mathfrak{sl}(2)$ is reductive and so we may write $\mathfrak{g}$ itself as a direct sum of simple modules for the s-triple (x, h, y). Each simple $\mathfrak{sl}(2)$ module is a direct sum of its h eigenspaces and consequently, $ad\, h$ is a semisimple endomorphism of $\mathfrak{g}$. This further implies that $\mathfrak{g}^h$ contains a Cartan subalgebra for $\mathfrak{g}$ containing h itself. Then we may further use the Weyl group to conjugate h to a dominant element, satisfying $h(\alpha) \in \mathbb{N}$, for all $\alpha \in \pi$. Then a key (though simple) observation of Dynkin is

Lemma *If $h \in \mathfrak{h}$ is a dominant element of an s-triple (x, h, y) then $h(\alpha) \in \{0, 1, 2\}$ for all $\alpha \in \pi$.*

Proof Since $(ad\, h)y = -2y$, it follows that y is a sum of root vectors $x_{-\beta}$ with $h(\beta) = 2$. In particular $\beta \in \Delta^+$. We can assume $h(\alpha) > 0$. Then $[y, \mathfrak{g}_\alpha] \neq 0$ by $\mathfrak{sl}(2)$ theory. Thus there exists at least one root β occurring in y such that $-\beta+\alpha$ is a root. Since α is simple, it must be a negative root, or zero. Since h is dominant, we obtain $0 \geq h(-\beta+\alpha) = h(\alpha)-2$, as required. □

2.3.11 We may conclude from 2.3.10 that $|\mathscr{N}/G| \leqslant 3^\ell$ and so is finite. A further analysis shows that the values of the dominant element h on π, characterizes the nilpotent orbit Gx. Moreover, all the possible values have been listed. We show how to compute this data, called the Dynkin data, in Section 2.10. Note that $h(\alpha) = 2$, for all $\alpha \in \pi$, when $x = \sum_{\alpha\in\pi} x_\alpha$.

The conjugation of h into a dominant element of $\mathfrak{h}$ forces x into $\mathfrak{n}^+$ and so we obtain $\mathscr{N} = G\mathfrak{n}^+$ as before. Hence $\mathscr{N}$ is irreducible of dimension $\leqslant |\Delta|$. On the other hand, $\operatorname{codim} Gx = \dim \mathfrak{g}^x \geqslant \dim \mathfrak{g}^h$ by $\mathfrak{sl}(2)$ theory with equality if and only if each simple submodule of $\mathfrak{g}$ under the s-triple, is odd dimensional. This holds exactly when $ad\, h$ has just even integer eigenvalues. In particular, when $x = \sum_{\alpha\in\pi} x_\alpha$, one has $h(\alpha) \in 2\mathbb{Z}$, $\forall\ \alpha \in \Delta$. Then $\dim \mathfrak{g}^x = \dim \mathfrak{g}^h = \dim \mathfrak{h}$ and so, as before, we conclude that Gx has dimension $|\Delta|$ and is the unique dense orbit in $\mathscr{N}$, which is hence a complete intersection.

2.4. **Slices.**

2.4.1 Let G be an algebraic group acting by morphisms on a variety $\mathscr{V}$. The G orbits of maximal dimension in $\mathscr{V}$ form an open subset $\mathscr{V}_{reg}$ of $\mathscr{V}$. A slice is a closed subvariety W of $\mathscr{V}_{reg}$ meeting every G orbit at exactly one point. (For a more complete definition and discussion see [39, Section 7].) In principle the most obvious example of a slice arises when G is the group of rotations of a sphere S^2 around an axis, say the North-South axis. Then $\mathscr{V}_{reg}$ is the punctured sphere in which the North and South poles have been removed and the regular orbits are the longitudinal lines. As any sea captain can affirm, a latitude cuts every longitude at exactly one point. However this is not quite an example of a slice since a latitude is not an algebraic subvariety. Rather one should choose a great circle through the poles. However this has the disadvantage of meeting every regular orbit twice.

2.4.2 A bona fide example of a slice is provided by the set of companion matrices. This is a slice for $SL(n)$ acting by conjugation on its Lie algebra $\mathfrak{sl}(n)$. In detail set $\Omega = \{x \in End\ \mathbb{C}^n$ admitting a cyclic vector$\}$. Obviously Ω is a union of G orbits. Take $a \in \Omega$ and let $v \in \mathbb{C}^n$ be a cyclic vector for the action of $\mathbb{C}[a]$. One easily checks that $v_1 = v,\ v_i = a^{i-1}v : i = 1, 2, \ldots, n$, is a basis for $\mathbb{C}^n$. Then $x^n v = av_n = \sum_{i=1}^n c_i v_i$, for some $c_i \in \mathbb{C}$. Moreover, $c_n = 0$ if tr $a = 0$. We conclude that in this basis $a \in x + V$, where x is a Jordan block and V is the vector space lying in the bottom row of $\mathfrak{sl}(n)$. Since the adjoint action of $SL(n)$ just corresponds to permuting bases, it follows that $x + V$ meets every orbit in Ω at exactly one point. However, it is not quite obvious (though true - see 2.4.5) that $\Omega \cap \mathfrak{sl}(n)$ is the set of regular coadjoint in $\mathfrak{sl}(n)$. One calls $x + V$ the companion slice, since the elements of $x + V$ are known as companion matrices.

In the above example, V is by no means unique. We could replace it by the set of strictly lower triangular Toeplitz matrices, which is just $\mathfrak{g}^y$ where y is the element of the $\mathfrak{s}$ triple (x, h, y) containing x. This presentation generalizes for any semisimple Lie algebra and is due to Kostant. We outline the theory below.

2.4.3 An s-triple (x, h, y) in a semisimple Lie algebra $\mathfrak{g}$ is said to be principal if $x \in \mathfrak{g}_{reg}$. A principal s-triple is unique [44, Sect. 5], up to conjugation by G.

From the representation theory of $\mathfrak{sl}(2)$, one finds that
$Im\ ad\,y +$ ker $ad\,x = \mathfrak{g}$, for any s-triple (x, h, y). Set $V =$ ker $ad\,x = \mathfrak{g}^x$. It follows from the above sum rule that restriction of functions gives an injection ψ of $Y(\mathfrak{g})$ into $R[y + V]$. Since $GK \dim Y(\mathfrak{g}) = \dim \mathfrak{h} \leqslant \dim \mathfrak{g}^x$, one can only expect ψ to be surjective when y is regular. Indeed

Theorem (Kostant) *If (x, h, y) is a principal $\mathfrak{sl}(2)$ triple, then ψ is an isomorphism of $Y(\mathfrak{g})$ onto $R[y + V]$.*

Remark This resembles the Chevalley theorem though in fact is better. In effect the elements of $Y(\mathfrak{g})$ are linearized by this process. Indeed, we may find a set of generators $\{p_i\}_{i=1}^{\ell}$ of $Y(\mathfrak{g})$ such that $\{\psi(p_i)\}_{i=1}^{\ell}$ is a basis for V^*. The resulting subspace is called the Dynkin subspace. It is useful in the construction of $(\wedge^* \mathfrak{g})^G$ via transgression - see [47, below eq. 239].

2.4.4 The proof of 2.4.3 is based on a comparison of the set of degrees $\{d_i\}_{i=1}^{\ell}$ of the generators of $Y(\mathfrak{g})$ with the set of eigenvalues $\{e_i\}_{i=1}^{\ell}$ of h on $\mathfrak{g}^x$. It is convenient to rescale the elements of the principal s-triple (x, h, y) so that $[h, x] = x,\ [h, y] = -y$ and $[x, y] = h$. Then $h(\alpha) = 1,\ \forall\ \alpha \in \pi$. By $\mathfrak{sl}(2)$ theory one has $e_i \geqslant 0$, for all $i = 1, 2, \ldots, \ell$ and

$$\sum_{i=1}^{\ell} e_i = |\Delta^+|,$$

which, as before we saw, equals

$$\sum_{i=1}^{\ell} (d_i - 1).$$

Now suppose p_i is a homogenous generator of degree d_i. Let $\{x_i\}_{i=1}^{\ell}$ be a basis for $V = \mathfrak{g}^x$ and complete $\{x_i\}_{i=1}^{\ell} \cup \{y\}$ to a basis for $\mathfrak{g}$ by $ad\, h$ eigenvectors. Let $\{\xi_i\}_{i=1}^{\dim \mathfrak{g}}$ be a dual basis for $\mathfrak{g}^*$ with $\{\xi_i\}_{i=1}^{\ell}$ corresponding to $\{x_i\}_{i=1}^{\ell}$ and $\eta := \xi_{\ell+1}$ to y. If $(ad\, h)x_i = e_i x_i$, then $(ad\, h)\xi_i = -e_i\xi_i$. Moreover, $(ad\, h)\eta = \eta$.

Now consider the map $Y(\mathfrak{g}) \xrightarrow{\Psi} S[V \oplus \mathbb{C}y]$ defined by restriction. It is injective by the injectivity of ψ. Consider a monomial $(\Pi\xi_j^{s_{i,j}})\eta^{r_i}$ occurring in $\Psi(p_i)$. Then deg $p_i = d_i$ implies

$$(\sum_j s_{i,j}) + r_i = d_i$$

On the other hand, $(ad\, h)p_i = 0$ and Ψ commutes with the action of $ad\, h$ hence

$$\sum_j s_{i,j}e_j - r_i = 0,$$

which, combined with the previous relation implies that

$$\sum_j s_{i,j}(e_j + 1) = d_i.$$

Since every term on the left hand side is $\geqslant 0$, an easy induction argument shows that $\{s_{i,j}\}$ forms the entries of a triangular matrix with ones on the diagonal, up to ordering. This applies to each monomial occurring in $\Psi(p_i)$. If we choose the d_i (resp. e_i) to be increasing, one easily deduces that

$$d_i = e_i + 1, \ i = 1, 2, \ldots, \ell.$$

Moreover, one may find a new choice of generators q_i of the form $q_i = p_i \bmod \mathbb{C}[p_i, p_2, \ldots, p_{i-1}]$ such that $\psi(q_i) = \xi_i$ and hence form a basis for V^*. Surjectivity follows. $\square$

2.4.5 It is immediate from the above result that every G orbit in $G(y + V)$ meets $y + V$ at exactly one point. That $y + V \subset \mathfrak{g}_{reg}$ can be deduced from $y \in \mathfrak{g}_{reg}$ and that the $ad\, h$ eigenvalues in V are $\geqslant 0$ whilst $(ad\, h) = -y$, by a deformation argument. That $G(y + V)$ exhausts $\mathfrak{g}_{reg}$ is more delicate, but relies on the fact that $\mathscr{N}$ is irreducible (see [36, 8.7] for example). The same argument can be used to show that the companion slice exactly generates $\mathfrak{g}_{reg}$ via the action of G.

2.5. **Shift of Argument.**

2.5.1 Shift of argument is a natural prolongation of Cayley-Hamilton theory for the construction of $A := (S((End\ V)^*))^{GL(V)}$. In this case we note that the determinant function det : $a \mapsto \det a$ is a polynomial function on End V of degree equal to $\dim V$ and invariant under conjugation by $GL(V)$. Thus det $\in A$. Similarly $a \mapsto \det(a - \lambda Id)$ is invariant for all $\lambda \in \mathbb{C}$. Develop $\det(a - \lambda 1)$ in powers of λ and let $p_i(a)$ denote the coefficient of λ^{n-1}, that is

$$\det(a - \lambda Id) = \sum_{i=0}^{n} \lambda^{n-i} p_i(a).$$

Then $a \mapsto p_i(a)$ is an invariant polynomial homogenous of degree i. The Cayley-Hamilton theorem asserts that

$$A = \mathbb{C}[p_1, p_2, \dots, p_n].$$

Obviously $p_n(a) = \det a$. We can also usefully describe the p_i in another fashion. Observe that the action of an element $a \in \operatorname{End} V$ extends to a diagonal action of a on $\wedge^i V$. Then

$$a \mapsto \operatorname{tr}(a, \wedge^i V)$$

is a polynomial function of degree i. It coincides with p_i. Observe further that the matrix elements of a as an endomorphism of $\wedge^i V$, are the $i \times i$ minors of a. Thus p_i is an appropriate sum of certain $i \times i$ minors of a. This presentation allows to more fully appreciate the companion slice. Indeed, let $\widetilde{V}$ denote the bottom row of End V, that is the linear span of the $\{x_{n,i}\}_{i=1}^n$, where $x_{i,j}$ are the usual matrix units. Let $\mathfrak{p}$ be the maximal (parabolic) subalgebra of $\mathfrak{g} = \mathfrak{gl}(V)$ consisting of all matrices whose bottom rows have zero entries. Obviously $\mathfrak{p} \oplus \widetilde{V} = \text{End } V$, so $S(\mathfrak{p}) \otimes S(\widetilde{V}) = S(\text{End } V)$. Dualizing gives $S(\mathfrak{p}^*) \otimes S(\widetilde{V}^*) = S((\text{End } V)^*)$

Now the presentation of the $p_i \in (S((\text{End } V)^*))^{GL(V)}$ as minors, implies that they lie in the subspace

$$S(\mathfrak{p}^*) \oplus S(\mathfrak{p}^*) \otimes \widetilde{V}^*.$$

Moreover, they generate this subspace over $S(\mathfrak{p}^*)$. This is a refinement of the linearization process described by the companion slice $y + \widetilde{V}$, which shows that when we evaluate the matrix coefficients in $S(\mathfrak{p}^*)$ on y, the resulting matrix remains of rank n.

One can conclude from the above discussion that

$$(\text{Fract } S(\mathfrak{p}))\ S(\widetilde{V}) = (\text{Fract } S(\mathfrak{p}))\ Y(\mathfrak{g}). \tag{$*$}$$

Actually, one may do better. Instead of the rather drastic inversion of all non-zero elements of $S(\mathfrak{p})$, Dixmier showed that it suffices to localize at just one very particular element which we denote by d. (For a proof see A.2.) It turns out that the semi-invariant subalgebra $Sy(\mathfrak{p})$, as defined in 2.5.6, is a polynomial algebra on one homogeneous generator, and this is d. The construction of d (obtained independently by Dixmier [14] and by Joseph [27]) is itself an interesting exercise. One shows, in particular, that d has degree $\frac{1}{2}n(n-1)$ which is just the number of positive roots in $\mathfrak{gl}(n)$.

At the time, this was the first known example when $Sy(\mathfrak{p})$ could be determined and shown to be polynomial. Shortly afterward [28, 4.12, 4.14] it was shown that $Sy(\mathfrak{b})$,where $\mathfrak{b} = \mathfrak{p}_\emptyset$ is a Borel subalgebra is polynomial, on rank $\mathfrak{g}$ generators (for any semisimple Lie algebra). Now we have a much more complete theory establishing polynomiality of $Sy(\mathfrak{q})$ for most cases when $\mathfrak{q}$ is a biparabolic or centalizer [21], [35], [60].

2.5.2 Let $\mathfrak{a}$ be any finite dimensional Lie algebra and recall $c(\mathfrak{a})$ defined in 2.2.4. Its significance is the following. The commutative algebra $S(\mathfrak{a})$ admits a Poisson algebra structure coming from the Lie bracket on $\mathfrak{a}$. Specifically, choose a basis $\{x_i\}$ of $\mathfrak{a}$ and define

$$\{f, g\} = \sum \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j} [x_i, x_j],$$

for all $f, g \in S(\mathfrak{a})$. One checks that $\{\ ,\ \}$ is a Lie bracket on $S(\mathfrak{g})$. It gives $\mathfrak{g}^*$ the structure of a Poisson variety. Let A be the adjoint group of $\mathfrak{a}$ acting on $\mathfrak{a}^*$ by transposition. Then the A orbits in $\mathfrak{a}^*$ are the Poisson leaves of $\mathfrak{a}^*$, which are (by definition) symplectic subvarieties. In this, note that

$$Y(\mathfrak{a}) := S(\mathfrak{a})^A = \{f \in S(\mathfrak{a}) |\ \{f, g\} = 0, \forall g \in S(\mathfrak{a})\}.$$

Moreover, the elements of $Y(\mathfrak{a})$ become constants on a given A orbit in $\mathfrak{a}^*$.

For physicists a symplectic variety $\mathscr{S}$ is phase space, that is to say the natural habitat of Hamiltonian or Lagrangian mechanics. On such a variety $\mathscr{S}$ one may choose Darboux co-ordinates $q_i, p_i\ :\ i = 1, 2, \ldots, \frac{1}{2}\dim \mathscr{S}$ defined locally in some open neighbourhood N_s of a point $s \in \mathscr{S}$. In terms of these co-ordinates, the Poisson bracket takes a particularly simple form

$$\{q_i, q_j\} = \{p_i, p_j\} = 0,\ \{q_i, p_j\} = \delta_{i,j}. \qquad (*)$$

Finally, any analytic function in a neighbourhood of s can be expressed as analytic functions of these co-ordinates in some possibly smaller neighbour.

Of course, we would prefer to stick to a purely algebraic context and require that $q_i, p_j \in$ Fract $(S(\mathfrak{a})/I)$, where I is the ideal of definition of a coadjoint orbit closure, satisfying $(*)$ above with $\mathbb{C}(q_i, p_j\ :\ i, j = 1, 2, \ldots, \frac{1}{2}\dim \mathscr{S}) =$ Fract $(S(\mathfrak{a})/I)$. For this it is necessary to assume that $\mathfrak{a}$ is an algebraic Lie algebra. Then this result is known for $\mathfrak{a}$ solvable [51], [29] and for Richardson orbits in $\mathfrak{a}$ semisimple [30]. (The arguments given in these references refer mainly to the enveloping algebra; but go over without too much change (cf [41, Section 6]) to the symmetric algebra with its Poisson structure. One may further ask if the Goldie skew of a primitive quotient is a Weyl skew field. This question is discussed in [31], where it is completely resolved for $\mathfrak{a} = \mathfrak{sl}(n)$. Some further results of this nature have been reported by Premet [62].)

The point of the above is that Hamilton's equations of motion become particularly easy to solve in Darboux co-ordinates if the Hamiltonian itself can be arranged to become one of these co-ordinates. The resulting system is said to be completely integrable.

The above circumstance raises the following important question. Can one find a maximal Poisson commutative subalgebra C of $S(\mathfrak{a})$ which specializes on the regular orbits to a subalgebra whose fraction field contains a maximal commuting family of Darboux co-ordinates, for example $\mathbb{C}(q_1, q_2, \ldots, q_s) : 2s = \dim \mathfrak{a} - \text{index } \mathfrak{a}$. If, in addition, $\mathfrak{a}$ is unimodular, we must have $GK \dim Y(\mathfrak{a}) = \text{index } \mathfrak{a}$ and so we require $GK \dim C = s + \text{index } \mathfrak{a} = c(\mathfrak{a})$. Preferentially, C should itself be polynomial. We remark that any Poisson commutative subalgebra of $S(\mathfrak{a})$ must satisfy $GK \dim C \leqslant c(\mathfrak{a})$ and moreover Sadetov [63] has shown that the bound can be saturated; but it is not obvious that the algebra he obtains will have any particularly good properties, for example provide the required co-ordinates.

2.5.3 An elegant approach to the above problem obtains via shift of argument. Consider $f \in Y(\mathfrak{a})$ and fix $\eta \in \mathfrak{a}^*$. Define new functions $f_i \in S(\mathfrak{a})$ by

$$f(\xi + \lambda\eta) = \sum f_i(\xi)\lambda^i,\ \forall\ \xi \in \mathfrak{a}^*. \qquad (*)$$

Let $T_\eta(\mathfrak{a})$ denote the subalgebra of $S(\mathfrak{a})$ generated by all such f_i as f runs over $Y(\mathfrak{a})$. One easily checks that it suffices to restrict to generators of $Y(\mathfrak{a})$ in computing $T_\eta(\mathfrak{a})$.

Lemma *For any $\eta \in \mathfrak{a}^*$, the algebra $T_\eta(\mathfrak{a})$ is Poisson commutative.*

The proof is an easy induction argument (see [22, Appendice no.1] for example), though not quite as straightforward as it might first seem. An inductive-free proof may be found in 2.6.5.

2.5.4 In general, $T_\eta(\mathfrak{a})$ has too small a GK dimension. For example, suppose that $Y(\mathfrak{a})$ is a polynomial algebra on generators p_i of degree d_i. Now shift of argument is essentially differentiation (with respect to the "dual" variable to η). A given polynomial p_i can therefore provide at most d_i derivatives, and so we require

$$\sum d_i \geqslant c(\mathfrak{a}). \qquad (*)$$

However, it is not at all obvious that the resulting derivatives are algebraically independent. Indeed, this necessarily fails if the inequality in $(*)$ is strict. Moreover, we should further expect to require that η be in "general position", specifically that $\eta \in \mathfrak{a}^*_{reg}$. Indeed this is shown in 2.6.13.

2.5.5 An optimal situation would seem to arise if equality holds in $(*)$. This situation does in fact arise when $\mathfrak{a}$ is semisimple. Moreover, it also holds for most (truncated) biparabolics. Below we sketch a formulism for showing that $T_\eta(\mathfrak{a})$ has the required properties if η is appropriately chosen.

2.5.6 Let $\mathfrak{a}$ be an algebraic Lie algebra, A its algebraic adjoint group. One calls $a \in S(\mathfrak{a})$ a semi-invariant if it generates a one-dimensional ad A module. Let $Sy(\mathfrak{a})$ denote the span of all semi-invariant elements of $S(\mathfrak{a})$. One calls $\mathfrak{a}$ semi-invariant free if $Sy(\mathfrak{a}) = Y(\mathfrak{a})$. Assume $\mathfrak{a}$ to be semi-invariant free. Under these conditions, the Chevalley-Dixmier lemma (sometimes known as the Rosenlicht theorem) asserts that

$$GK \dim Y(\mathfrak{a}) = \text{index } \mathfrak{a}.$$

2.5.7 One cannot expect to get too far unless $Y(\mathfrak{a})$ is polynomial and furthermore, that $\mathfrak{a}^*$ admits a slice. In the semisimple case, a slice was constructed from an s-triple. However, this is too much structure to expect in general. Instead we will make do with an adapted pair (h, y) which consists of $h \in \mathfrak{a}$, for which $ad\, h$ on $\mathfrak{a}^*$ is semisimple with rational eigenvalues, and $y \in \mathfrak{a}^*_{reg}$ such that

(1) $(ad\, h)y = -y$.

Since $ad\, h$ is semisimple, we can find an $ad\, h$ stable complement V to $(ad\, \mathfrak{a})y$ in $\mathfrak{a}^*$. Note that $\dim V = \text{index } \mathfrak{a}$. We further assume

(2) The eigenvalues e_i of $(ad\, h)$ on V are $\geqslant 0$.

Now recall that we have assumed $Y(\mathfrak{a})$ to be polynomial and let $\{d_i\}$ be the degrees of a set of homogeneous generators. Then we shall also require that

(3) $\sum_{i=1}^{\text{index } \mathfrak{a}} (d_i - 1) = \sum_{i=1}^{\text{index } \mathfrak{a}} e_i$.

It is possible that 1), 2) imply 3) and even that 1) implies 2). Indeed this has now been shown in [42].

For any $\xi \in \mathfrak{a}^*$ set
$$\mathfrak{a}^\xi = \{a \in \mathfrak{a} | (ad\, a)\xi = 0\}.$$

Lemma *The natural ad h invariant pairing* $\mathfrak{a}^* \times \mathfrak{a} \to \mathbb{C}$ *(which is non-degenerate) restricts to a non-degenerate pairing*
$$V \times \mathfrak{a}^y \to \mathbb{C}.$$

Proof Of course V identifies with $\mathfrak{a}^*/(ad\, \mathfrak{a})y$. Nevertheless, the argument is not quite standard, since y itself is not an endomorphism of $\mathfrak{a}$. We must show that the orthogonal $((ad\, \mathfrak{a})y)^\perp$ of $(ad\, \mathfrak{a})y$ in $\mathfrak{a}$, is just $\mathfrak{a}^y$.

Consider $a \in \mathfrak{a}^y$. Then $((ad\, a)y)(b) = 0,\ \forall\ b \in \mathfrak{a}$. Yet
$$((ad\, a)y)(b) = -y([a, b]) = -((ad\, b)y)(a).$$
Thus $a \in \mathfrak{a}^y \Leftrightarrow ((ad\, b)y)(a) = 0, \forall\ b \in \mathfrak{a} \Leftrightarrow ((ad\, a)y)(b) = 0 \Leftrightarrow$
$a \in ((ad\, a)y)^\perp$, as required. $\square$

2.5.8 For any $ad\, h$ invariant subspace $\mathfrak{b}$ of $\mathfrak{a}$ (or $\mathfrak{a}^*$), let $\mathfrak{b}_>(\mathfrak{b}_\geqslant,\ \mathfrak{b}_<)$ denote the sum of $ad\, h$ eigenspaces of $\mathfrak{b}$ with positive (resp. non-negative, negative) eigenvalues. Clearly, hypothesis 2) of 2.5.7 is equivalent to $V_< = 0$. Through 2.5.7 it is also equivalent to $\mathfrak{a}^y_> = 0$. This is a property which can be deduced from $\mathfrak{sl}(2)$ theory if the pair (h, y) forms part of an s-triple (x, h, y).

For $\mathfrak{a}$ semisimple, pairs satisfying (1), (2); but *without* y needing to be regular have been classified by A. Elashvili and V. Kac [19], who called such pairs "good". They are a first step in the construction of vertex operator algebras. Of course given y nilpotent the Jacobson-Morosov theorem at least one such pairs (h, y) containing y. On the other hand for an arbitrary finite dimensional Lie algebra $\mathfrak{a}$ it is not at all obvious if an element $y \in \mathscr{N}$, admits an element $h \in \mathfrak{a}$ such that $(\mathrm{ad}\, h)y = -y$.

2.5.9 Suppose (h, y) is an adapted pair. Then by exactly the argument of Kostant described in 2.4.4, we obtain the

Theorem

(1) *Restriction of functions induces an isomorphism*
$$Y(\mathfrak{a}) \tilde{\to} R[y + V]$$

(2) $d_i = e_i + 1$, *for a suitable ordering.*

2.5.10 The hypothesis for an adapted pair further imply the

Lemma *Suppose* (h, y) *is an adapted pair for* $\mathfrak{a}$ *and define* $\mathfrak{a}^*_{\geqslant 0}$ *as in 2.5.8. One has*

$$\dim \mathfrak{a}^*_{\geqslant 0} = c(\mathfrak{a}).$$

Proof As noted in 2.5.8, one has $\mathfrak{a}^y_{>} = 0$. Since $\dim V = \text{index } \mathfrak{a}$, V has eigenvalues $\geqslant 0$ and the sum $(ad\, \mathfrak{a})y + V = \mathfrak{a}^*$ is direct, we obtain

$$\dim \mathfrak{a}^*_{\geqslant} = \dim \mathfrak{a}_{>} + \text{index } \mathfrak{a}.$$

$$\begin{aligned} Yet \quad c(\mathfrak{a}) &= \tfrac{1}{2}(\dim \mathfrak{a} + \text{index } \mathfrak{a}) \\ &= \tfrac{1}{2}(\dim \mathfrak{a}^* + \text{index } \mathfrak{a}) \\ &= \tfrac{1}{2}(\dim \mathfrak{a}^*_{\geqslant} + \dim a^*_{<} + \text{index } \mathfrak{a}) \\ &= \dim \mathfrak{a}^*_{\geqslant}, \text{ by the above.} \end{aligned}$$

□

2.5.11 The above conclusion suggests that for suitable $\eta \in \mathfrak{a}^*$, the map

$$T_\eta(\mathfrak{a}) \to R[y + \mathfrak{a}^*_{\geqslant}]$$

defined by restriction, is an isomorphism. However as noted in the proof of 2.6.17 we certainly need some extra hypotheses. First we have

$$GK \dim T_\eta(\mathfrak{a}) \leq \sum d_i,$$

so we need the sum on the right hand side to be greater than or equal to $c(\mathfrak{a})$. Since we would like linear independence of the translated functions, we impose equality, that is

(H1) $\sum_{i=1}^{\text{index } \mathfrak{a}} d_i = c(\mathfrak{a})$.

Next, in order that eigenvalues match, we must require η to have zero weight for $ad\, h$, that is

(H2) $\eta \in \mathfrak{a}^*_0$.

To optimize the algebraic independence of the generators of $T_\eta(\mathfrak{a})$, we further require

(H3) $\eta \in \mathfrak{a}^*_{reg}$.

Finally, we impose that

(H4) $\mathfrak{a}^\eta = \mathfrak{a}_0$.

It is possible that (H2),(H3) imply (H4). One may show that (H2), (H4) imply (H3) [42, Section 3]. Again (H1) is a consequence of [57, Prop. 1.4], since by construction a truncated biparabolic is semi-invariant free. On the other hand, (H2) and (H3) are generally speaking mutually exclusive - generic elements do not like to be eigenvectors! When $\mathfrak{g}$ is semisimple, we can identify $\mathfrak{g}$ with $\mathfrak{g}^*$. Further, let (x, h, y) be a principal s-triple. Then (h, y) is an adapted pair, (H1) holds and we can satisfy (H2)-(H3) with $\eta = h$. One may remark that $\mathfrak{g}_{\geqslant}$ is a Borel subalgebra, for $\mathfrak{g}$ semisimple.

Theorem *Under the additional hypotheses, (H1)-(H4), the restriction map induces an isomorphism of $T_\eta(\mathfrak{a})$ onto $R[y + \mathfrak{a}^*_{\geqslant}]$.*

2.5.12 The key point in the proof is the adaption of a construction due to Mishchenko and Fomenko, described below.

Choose a basis $\{x_i\}_{i=1}^n$ for $\mathfrak{a}$ and $\xi \in \mathfrak{a}^*$. For all $f \in S(\mathfrak{a})$ define (grad $f)(\xi) \in \mathfrak{a}$ by

$$(\mathrm{grad} f)(\xi) = \sum_{i=1}^n x_i \left(\frac{\partial f}{\partial x_i}(\xi)\right).$$

A key, and almost obvious, fact is that if $f \in Y(\mathfrak{a})$, then (grad $f)(\xi) \in \mathfrak{a}^\xi$. In particular, this holds if ξ is replaced by $\xi + \lambda\eta$. With respect to the expansion given in 2.5.3($*$), a development of powers of λ gives the following crucial recurrence relation.

$$(grad f_j)(\xi) \cdot \xi + (grad f_{j-1})(\xi) \cdot \eta = 0. \qquad (*)$$

where the dot denotes co-adjoint action.

2.5.13 Let (h, y) be an adapted pair. In general, the eigenvalues of $ad\, h$ on $\mathfrak{a}^*$ have a pattern very different to the case when (h, y) can be completed to an s-triple. However, this nice pattern is partly recovered when (H4) holds.

Lemma *Suppose (H4) holds. Then for all $k \geqslant 0$ one has*
$\dim \mathfrak{a}_{-k} - \dim \mathfrak{a}^y_{-k} = \dim \mathfrak{a}_{-(k+1)}$.

Proof By 2.5.8 one has $V_{-(k+1)} = \{0\}$, if $k \geqslant 0$. Hence the direct sum $(ad\, \mathfrak{a})y \oplus V = \mathfrak{a}^*$ gives

$$(ad\, \mathfrak{a}_{-k})y = \mathfrak{a}^*_{-(k+1)}, \ \forall\ k \geqslant 0.$$

Hence

$$\dim \mathfrak{a}_{-k} - \dim \mathfrak{a}^y_{-k} = \dim \mathfrak{a}^*_{-(k+1)}, \ \forall\ k \geqslant 0. \qquad (**)$$

Under the hypothesis of (H4)

$$(ad\, \mathfrak{a})\eta \oplus \mathfrak{a}^*_0 = \mathfrak{a}^*,$$

and so

$$(ad\, a_{-k})\eta = \mathfrak{a}^*_{-k}, \text{ for all } k > 0.$$

and hence, by (H4) again,

$$\dim \mathfrak{a}_{-k} = \dim \mathfrak{a}^*_{-k}, \text{ for all } k > 0.$$

Substitution into ($**$) gives the required conclusion. □

2.5.14 Let (h, y) be an adapted pair for $\mathfrak{a}$. (For the moment we only need 1),2) of 2.5.7. Then the assertions of Theorems 2.5.9(i) and 2.5.10 are respectively equivalent to

1) *The inclusion* $\mathbb{C}\{(grad f)(y) | f \in Y(\mathfrak{a})\} \subset \mathfrak{a}^y$, *is an equality.*
2) *The inclusion* $\mathbb{C}\{(grad f)(y) | f \in T_\eta(\mathfrak{a})\} \subset \mathfrak{a}_\leqslant$, *is an equality.*

2.5.15 Recall that $Y(\mathfrak{a})$ is assumed polynomial and that $\mathfrak{a}$ is algebraic and semi-invariant free. Recall 2.5.6 and set $\ell = \text{index}\, \mathfrak{a}$. Let $\{f_i : i = 1, 2, \dots, \ell\}$ be a set of homogeneous generators of $Y(\mathfrak{a})$ with f_i of degree $e_i + 1$. Define $f_{i,j} : j = 0, 1, 2, \dots, e_i$ as in 2.5.3, with respect to each f_i above and $\eta \in \mathfrak{a}^*$.

Theorem *Let (h, y) be an adapted pair and assume (H1)-(H4) to hold. Then the following three conditions are equivalent*
(i) $\{(grad f_i)(y)\}_{i=1}^{\ell}$ *is a basis for* $\mathfrak{a}^y$.
(ii) $\{(grad f_{i,e_i})(y)\}_{i=1}^{\ell}$ *is a basis for* $\mathfrak{a}^\eta = \mathfrak{a}_0$.
(iii) $\{\{(grad f_{i,j})(y)\}_{j=0}^{m_i}\}_{i=1}^{\ell}$ *is a basis for* $\mathfrak{a}_\leqslant$.

Remarks on the Proof That (ii)$\Rightarrow$ (iii) follows from 2.5.12($*$) by downward induction (note change in notation). Here one also uses (H4), 2.5.10 and 2.5.11. That (iii) implies (i) (or (ii)) is essentially trivial. That (i)$\Rightarrow$ (iii) follows from 2.5.12($*$), 2.5.10, the regularity of η and (H1).

2.5.16 Combining Theorem 2.5.9(1) with 2.5.14 and 2.5.15, gives Theorem 2.5.11.

2.5.17 Let us observe that the conclusion of Theorem 2.5.11 implies that $T_\eta(\mathfrak{a})$ is maximal Poisson commutative.

Corollary $T_\eta(\mathfrak{a})$ *equals its Poisson commutant in* $S(\mathfrak{a})$.

Proof Take $f \in S(\mathfrak{a})$ such that $\{f, T_\eta(\mathfrak{a})\} = 0$. Since GK dim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$ and that this is the maximal GK dim of a Poisson commutative subalgebra of $S(\mathfrak{a})$, we conclude that f is algebraic over $T_\eta(\mathfrak{a})$. Let $\psi : S(\mathfrak{a}) \rightarrow R[y + \mathfrak{a}^*_\geqslant]$ be defined by restriction. Recall by Theorem 2.5.11 that $\psi|_{T_\eta(\mathfrak{a})}$ is surjective. Thus, there exists $a \in T_\eta(\mathfrak{a})$ such that $\psi(f - a) = 0$. Replacing f by $f - a$ we can assume $\psi(f) = 0$. Suppose $f \neq 0$. Let s be minimal such that

$$\sum_{i=0}^{s} a_i f^i = 0 : a_i \in T_\eta(\mathfrak{a}) \text{ not all zero.}$$

Then $a_0 \neq 0$, for otherwise we could cancel f. Yet $\psi(f) = 0$, so we also have $\psi(a_0) = 0$. Yet $\psi|_{T_\eta(\mathfrak{a})}$ is injective by Theorem 2.5.11, so $a_0 = 0$ and a contradiction results. Hence $f = 0$, as required. $\square$

Remark. Apart from [31] one may also obtain some results on completing these co-ordinates to Darboux co-ordinates from [48].

2.6. **A Theorem of Bolsinov.**

2.6.1 Let $\mathfrak{a}$ be an algebraic semi-invariant free Lie algebra and recall 2.5.6. In this situation A. V. Bolsinov [3] has given a remarkably simple and seemingly weak criterion for when shift of argument provides a Poisson commutative subalgebra $T_\eta(\mathfrak{a})$ of $S(\mathfrak{a})$ having the maximal GK dimension. The proof is also remarkably simple though there are some subtle points to which careful attention should be given. The conclusion is weaker than Theorem 2.5.9 and indeed there are situations in which the shifted algebra $T_\eta(\mathfrak{a})$ has the maximal GK dimension; but is not itself maximal Poisson commutative - see 2.6.17. When the former is true and $Y(\mathfrak{a})$ is polynomial, then the degrees d_i of the generators must satisfy the sum rule in $(*)$ of 2.5.4. This is quite remarkable in that Bolsinov's criterion has nothing a priori to do with degrees of generators. Below we examine the proof in detail.

2.6.2 It is convenient to replace x in the right hand side of 2.5.12 by dx. Then the left hand side becomes $df(\xi+\lambda\eta)$. This is of course a trivial change of notation. Less trivially (though this was used implicitly in the derivation of 2.5.12$(*)$) differentiation commutes with translation and so with respect to the development in 2.5.3$(*)$ we obtain

$$df(\xi+\lambda\eta)=\sum df_i(\xi)\lambda^i. \qquad (*)$$

This leads to an important change of emphasis. Instead of trying to establish enough linear independence of the df_i at a fixed point ξ, we try to establish enough linear independence of the df on the affine line $(\xi+\lambda\eta)$. Correspondingly, instead of just considering the standard Poisson structure on $S(\mathfrak{a})$ given by a Lie bracket, we consider a family of Poisson structures obtained by translation.

2.6.3 Poisson Structures.

Let V be a finite dimensional vector space with basis $\{x_i\}_{i=1}^n$. A Poisson structure A on $S(V)$ is a matrix with entries $A_{i,j}\in S(V)$ such that

$$\{f,g\}:=\sum A_{i,j}\partial f/\partial x_i\,\partial g/\partial x_j$$

is a Lie bracket on $S(V)$. Obviously this generalizes 2.5.2, which one may view as the linear case. Historically Sophus Lie had tried to classify all Poisson structures (still an unresolved problem) and later just considered the linear case which led to Lie algebras.

If A,B are two Poisson structures, we say that A,B are compatible if $\alpha A+\beta B$ is a Poisson structure for all $\alpha,\beta\in\mathbb{C}$. Notice that then

$$\{f,g\}_{\alpha A+\beta B}=\alpha\{f,g\}_A+\beta\{f,g\}_B.$$

The Poisson centre $Y_A(V)$ with respect to A is defined to be

$$Y_A(V)=\{f\in S(V)|\{f,g\}_A=0,\quad\forall\, g\in S(V)\}.$$

Obviously $Y_A(\mathfrak{a})=Y(\mathfrak{a})$, when $A_{i,j}=[x_i,x_j]$.

Notice that if $f \in Y_A(V),\ g \in Y_B(V)$ with A, B compatible, then

$$\{f,g\}_{\alpha A+\beta B} = \alpha\{f,g\}_A + \beta\{f,g\}_B = 0 \text{ for all } \alpha,\beta \in \mathbb{C}. \tag{$*$}$$

This construction will be used to obtain Poisson commutative subalgebras of $S(V)$. In general it may not be enough since, for example, if $f, g \in Y_A(V)$, it is not immediate that $\{f,g\}_B = 0$.

2.6.4 Take $V = \mathfrak{a}$ in the above and

$$A_{i,j} = [x_i, x_j] = \sum c_{i,j}^k x_k$$

which is a Poisson structure A on $S(\mathfrak{a})$. One may obtain a second Poisson structure B by evaluation at $\eta \in \mathfrak{a}^*$, that is

$$B_{i,j} = \sum c_{i,j}^k\ \eta(x_k).$$

It is compatible with A.

For all $\lambda \in \mathbb{C},\ f \in S(\mathfrak{a})$, set

$$(T_{\lambda\eta} f)(\xi) = f(\xi + \lambda\eta). \tag{$*$}$$

Lemma *For all $f, g \in S(\mathfrak{a})$ one has*

$$\{T_{\lambda\eta} f,\ T_{\lambda\eta} g\}_{A+\lambda B} = T_{\lambda\eta}\{f,g\}_A.$$

In particular $T_{\lambda\eta} f \in Y_{A+\lambda B}(\mathfrak{a}) \iff f \in Y_A(\mathfrak{a})$.

Proof Indeed

$$\begin{aligned}
\{T_{\lambda\eta} f,\ T_{\lambda\eta} g\}_{A+\lambda B} &= \sum c_{i,j}^k (x_k + \lambda\eta(x_k)) \partial T_{\lambda\eta} f/\partial x_i\ \partial T_{\lambda\eta} g/\partial x_j, \\
&= \sum c_{i,j}^k T_{\lambda\eta}(x_k) T_{\lambda\eta}(\partial f/\partial x_i)\ T_{\lambda\eta}(\partial g/\partial x_j), \\
&= T_{\lambda\eta}\{f,g\}_A.
\end{aligned}$$

Hence the assertion. □

2.6.5 Retain the above conventions.

Lemma *For all $\lambda, \mu \in \mathbb{C}$ distinct, one has*

$$\{T_{\lambda\eta} f, T_{\mu\eta} g\}_A = 0.$$

Proof Suppose $f, g \in Y_A(\mathfrak{a})$. Since $\lambda, \mu \in \mathbb{C}$ are distinct, we can choose $\alpha, \beta \in \mathbb{C}$, such that

$$\alpha(A + \lambda B) + \beta(A + \mu B) = A.$$

By 2.6.4 one obtains $T_{\lambda\eta} f \in Y_{A+\lambda B}(\mathfrak{a})$ and $T_{\mu\eta} g \in Y_{A+\mu B}$. Then by 2.6.3($*$) the required result obtains. □

Remark Treating λ and μ as independent parameters and expanding in their powers one obtains a (rather elegant) proof of 2.5.3.

2.6.6 Families of Bilinear Forms.

The proof of the Bolsinov theorem results from two crucial lemmas on bilinear forms. We remark that antisymmetry (or symmetry) is needed for 2.6.7(i). Eventually antisymmetry is needed to calculate $\dim I_0$ through 2.6.8.

Let V be a finite dimensional vector space and $\mathscr{A}$ a two dimensional space of antisymmetric bilinear forms on V. Given $A \in \mathscr{A}$, we obtain an element $\varphi(A) \in \operatorname{Hom}(V, V^*)$ by $\varphi(A)(v)(v') = A(v, v')$. One has

$$\ker A := \{v \in V | A(v, v') = 0,\ \forall\ v' \in V\} = \ker\ \varphi(A).$$

Again if W is a subspace of V, then

$$W_A^\perp := \{v \in V | A(w, v) = 0,\ \forall\ w \in W\} = \varphi(A)^\perp.$$

Let $\mathscr{A}_0$ denote the subset $\mathscr{A}$ of all forms of maximal rank, equivalently all forms for which $\dim \ker A$ is minimal. It is an open dense Zariski subset of $\mathscr{A}$. Hence $\dim(\mathscr{A} \smallsetminus \mathscr{A}_0) \leqslant 1$. Consequently either $\mathscr{A} \smallsetminus \mathscr{A}_0 = \{0\}$, or the projectivisation of $(\mathscr{A} \smallsetminus \mathscr{A}_0)$ is zero dimensional, equivalently there exists a finite subset $\mathscr{B} \subset \mathscr{A} \smallsetminus \mathscr{A}_0$, such that

$$(\mathscr{A} \smallsetminus \mathscr{A}_0) \smallsetminus \{0\} = \coprod_{B \in \mathscr{B}} \mathbb{C}^* B.$$

Set

$$I_0 := \sum_{A \in \mathscr{A}_0} \ker A \subset I := \sum_{A \in \mathscr{A} \smallsetminus \{0\}} \ker A.$$

Lemma *$\varphi(A)(I_0)$ is independent of the choice of $A \in \mathscr{A} \smallsetminus \{0\}$.*

Proof Obviously I_0 is a subspace of V. Since the latter is finite dimensional, we can choose a finite set $A_1, A_2, \dots, A_t \in \mathscr{A}_0$, such that

$$I_0 = \sum_{i=1}^{t} \ker A_i.$$

In particular

$$\mathscr{A}' := \mathscr{A} \smallsetminus \bigcup_{i=1}^{t} \{\mathbb{C} A_i\},$$

is a two dimensional subvariety of $\mathscr{A}$. Choose $C, D \in \mathscr{A}'$, linearly independent. Then we may write

$$A_i = \alpha_i C + \beta_i D, \text{ with } \alpha_i, \beta_i \in \mathbb{C}^*.$$

Then $\varphi(C)(\ker A_i) = \varphi(D)(\ker A_i)$, for all i, and so $\varphi(C)(I_0) = \varphi(D)(I_0) =: W$.

Consider $E \in \mathscr{A} \setminus \{0\}$. Since $\{C, D\}$ is a basis for $\mathscr{A}$, we can write $E = \alpha C + \beta D$. Then

$$\varphi(E)(I_0) = \alpha\varphi(C)(I_0) + \beta\varphi(D)(I_0) \subset W. \qquad (*)$$

If $E \in \mathscr{A}'$ and not a multiple of D, then $E \notin (\bigcup_i \mathbb{C}^* A_i) \cup \mathbb{C}^* D$. Thus we may replace C by E in the first argument to give equality in $(*)$. Thus we are reduced to the case $E \in \bigcup_{i=1}^t \mathbb{C}^* A_i$. Then $I_0 \supset \ker E$, so $\dim \varphi(E)(I_0) = \dim I_0 - \dim \ker \varphi(E)$. The latter is independent of $E \in \mathscr{A}_0$ so by comparison with the case $E' \in \mathscr{A}' \cap \mathscr{A}_0$, we conclude from $(*)$ that equality also holds for E. $\square$

Remark The assertion is false for I_0 replaced by I. Here the last part of the argument fails.

2.6.7 Set $I_0^\perp = \{v \in V | B(v, I_0) = 0\} = \{v \in V | B(I_0, v) = 0\} = \varphi(B)(I_0)^\perp$, which by 2.6.6 is independent of the choice of $B \in \mathscr{A} \setminus \{0\}$.

Corollary

(i) $I_0^\perp \supset \ker B, \ \forall\ B \in \mathscr{A} \setminus \{0\}$.

(ii) $I_0^\perp \supset I \supset I_0$.

(iii) $\varphi(B)(I_0^\perp) \subset \varphi(A)(I_0^\perp), \ \forall\ B \in \mathscr{A}, \ A \in \mathscr{A}_0$.

Proof (i) and (ii) are clear. For (iii) observe that $(I_0^\perp)_B^\perp = I_0 + \ker B$, whilst $(I_0^\perp)_A^\perp = I_0 + \ker A = I_0$, since $A \in \mathscr{A}_0$. Thus $(I_0^\perp)_B^\perp \supset (I_0^\perp)_A^\perp$. Then $\{v \in V | v(\varphi(B)(I_0^\perp)) = 0\} = \{v \in V | B(v, I_0^\perp) = 0\} = (I_0^\perp)_B^\perp \supset (I_0^\perp)_A^\perp = \{v \in V | v(\varphi(A)(I_0^\perp)) = 0\}$. Hence (iii). $\square$

2.6.8 Corollary 2.6.7 (ii) asserts that I_0 is isotropic. The next result shows under what conditions it is maximal isotropic and hence of dimension $(\dim V + \dim \ker A)$, for any $A \in \mathscr{A}_0$.

Lemma $I_0^\perp = I_0 \Leftrightarrow \mathscr{A}_0 = \mathscr{A} \setminus \{0\}$.

Proof Take $A \in \mathscr{A}_0, \ B \in \mathscr{A} \setminus \{0\}$ distinct from A. Since $\varphi(A)(I_0) = \varphi(B)(I_0)$, by 2.6.6, it follows that $\ker B \subset I_0$ if and only if $B \in \mathscr{A}_0$. Hence $\Rightarrow$.

For $\Leftarrow$ assume that A, B span $\mathscr{A}$. Moreover up to a non-zero multiple every element of $(\mathscr{A} \setminus \{0\}) \setminus \mathscr{A}_0$ takes the form $B - \lambda A : \lambda \in \mathbb{C}$.

We construct an element $\Phi \in$ End $I_0^\perp / I_0$. Let $\pi : I_0^\perp \to I_0^\perp / I_0$ be the natural projection. Take $v \in I_0^\perp$. By 2.6.7(iii) there exists $v' \in I_0^\perp$ such that $\varphi(A)v' = \varphi(B)v$. Moreover v' is unique up to $\ker \varphi(A) \subset I_0$. If $v \in I_0$, we can choose $v' \in I_0$ since $A(I_0) = B(I_0)$ by 2.6.6. Thus we may define Φ through

$$\Phi(v + I_0) = v' + I_0, \text{ given } \varphi(A)v' = \varphi(B)v.$$

If $I_0^\perp / I_0 \neq 0$, there exists $v + I_0 \in I_0^\perp / I_0$ non-zero such that $\Phi(v + I_0) = \lambda(v + I_0)$, for some $\lambda \in \mathbb{C}$. Set $W = \varphi(A)(I_0)$. Then

$$\varphi(A)\Phi(v + I_0) = \lambda(v + I_0) = \lambda\varphi(A)v + W,$$

whilst

$$\varphi(A)\Phi(v+I_0) = \varphi(A)(v'+I_0) = \varphi(A)v' + W = \varphi(B)v + W.$$

Hence $\varphi(B-\lambda A)v \in W$. Yet $(B-\lambda A) \in \mathscr{A} \setminus \{0\}$ and so by 2.6.6, there exists $v'' \in I_0$ such that $\varphi(B-\lambda A)(v-v'') = 0$. Since $v \notin I_0$, we conclude that $\ker(B-\lambda A) \not\subset I_0$. Hence $(B-\lambda A) \notin \mathscr{A}_0$, as required. □

2.6.9 Return to the case when V is an algebraic semi-invariant free Lie algebra $\mathfrak{a}$ with basis $\{x_i\}_{i=1}^n$. The space of differentials $dS(V)$ of $S(V)$ is by definition the free $S(V)$ module with generators $dx_i : i = 1, 2, \ldots, n$. For a given Poisson structure A on $S(\mathfrak{a})$ one may define an antisymmetric bilinear form, which we denote by $\widehat{A}$, on $dS(V)$ through

$$\widehat{A}(dx_i, dx_j) := A_{i,j}.$$

Now let A denote the Poisson structure on $S(\mathfrak{a})$ defined by the Lie bracket, that is $A_{i,j} = [x_i, x_j]$. For all $\xi \in \mathfrak{a}^*$, let A_ξ denote its evaluation at ξ, namely

$$(A_\xi)_{i,j} = \xi([x_i, x_j]).$$

The map $\xi \mapsto A_\xi$ is linear. Hence a two dimensional subspace of $\mathfrak{a}^*$ gives rise to a two-dimensional subspace of Poisson structures and hence to a two dimensional subspace of antisymmetric bilinear forms on $\mathbb{C}\{dx_i : i = 1, 2, \ldots, n\}$.

Return for the moment to our general formalism. The easy though crucial identity

$$\{f, g\}_A = \widehat{A}(df, dg),$$

implies that the linear map $f \mapsto df$ sends the Poisson centre $Y_A(V)$ into $\ker \widehat{A}$. Its kernel is the space of the constant functions. One cannot expect this map to be surjective. However some form of surjectivity is required for the Bolsinov theorem.

Recall that a set $f_1, f_2, \ldots, f_t$ of elements of $S(V)$ are algebraically independent if and only if their differentials $df_1, df_2, \ldots, df_t$, are linearly independent over $F(V) := \text{Fract}\, S(V)$. This is again equivalent to their differentials $df_1(\xi), df_2(\xi), \ldots, df_t(\xi)$, being linearly independent at some point $\xi \in V^*$ and hence on some Zariski open dense subset of V^*.

Now take $\xi \in \mathfrak{a}^*$. It is clear that $rkA \geq rkA_\xi$ with equality on some Zariski open dense subset Ω_0 of $\mathfrak{a}^*$. On the other hand rkA is just the dimension of the co-adjoint orbit through ξ. (This easy though crucial fact is due to A. A. Kirillov. It implies that every coadjoint orbit is even dimensional because an antisymmetric bilinear form can only be non-degenerate on an even dimensional vector space.) We conclude that $\Omega_0 = \mathfrak{a}^*_{reg}$ and that

$$rkA = \dim \mathfrak{a} - \text{index}\, \mathfrak{a}.$$

Now (if a is algebraic and semi-invariant free) then GKdim $Y(\mathfrak{a}) = \text{index}\, \mathfrak{a}$. Dimensionality then implies that

$$F(V)\{df | f \in Y(\mathfrak{a})\} = F(V) \ker \widehat{A}. \tag{1}$$

This is a weakened version of the required surjectivity in a special case.

Notice that the above dimensionality estimate on $Y(\mathfrak{a})$ also implies that the set

$$\Omega := \{\xi \in \mathfrak{a}^* | \dim\{df(\xi) | f \in Y(\mathfrak{a})\} = \text{index}\, \mathfrak{a}\} \tag{2}$$

is Zariski open dense in $\mathfrak{a}^*$. However is not immediate that $\Omega \supset \mathfrak{a}^*_{reg}$. Indeed already for $\mathfrak{a}$ semisimple it is a deep result of Kostant that a regular nilpotent element belongs to Ω. Thus we do not immediately have an analogue of (1) above, namely

$$\{df(\xi) | f \in Y(\mathfrak{a})\} = \ker A_\xi, \text{for all } \xi \in \mathfrak{a}^*_{reg}, \tag{3}$$

may *fail* to hold. An example of this failure is given in [42, 8.3, Remark].

2.6.10 Set $\mathfrak{a}^*_{sing} := \mathfrak{a}^* \setminus \mathfrak{a}^*_{reg}$. We remark that as a variety $\mathfrak{a}^*_{sing}$ is given by the vanishing of the minors of A of the maximal rank $\dim \mathfrak{a} -$ index $\mathfrak{a}$. In this sense $\mathfrak{a}^*_{sing}$ is calculable. The Bolsinov criterion for the translated algebra $T_\eta(\mathfrak{a})$ to have the maximal GK dimension $c(\mathfrak{a})$ at some point $\eta \in \mathfrak{a}^*$ is that $\operatorname{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} \geqslant 2$. As noted below this implies that we can find a two dimensional subspace W of $\mathfrak{a}^*$ such that $W \setminus \{0\} \subset \mathfrak{a}^*_{reg}$.

View W as a two dimensional subspace $\mathscr{A}$ of forms. Then by 2.6.6 and 2.6.8 it follows that the sum I of the kernels of the forms $A \in \mathscr{A} \setminus \{0\}$ is maximal isotropic for any one of these forms and so has dimension equal to $c(\mathfrak{a})$. However because of a possible failure of 2.6.9 (3) we cannot immediately deduce that GKdim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$ for all $\eta \in W \setminus \{0\}$.

2.6.11 The above difficulty is overcome through intersection theory in projective space. Recall that the projectivisation $\mathbb{P}(\mathfrak{a}^*)$ of $\mathfrak{a}^*$ is obtained by removing $\{0\}$ and identifying points which are non-zero scalar multiples of each other, that is

$$\mathbb{P}(\mathfrak{a}^*) = \{\mathfrak{a}^* \setminus \{0\}\}/\mathbb{C}^*.$$

Now Ω and $\mathfrak{a}^*_{sing}$ defined in 2.6.9, 2.6.10 are stable by this action of $\mathbb{C}^*$ and so admit projectivisations $\mathbb{P}(\Omega)$ and $\mathbb{P}(\mathfrak{a}^*_{sing})$, which are respectively open and closed in $\mathbb{P}(\mathfrak{a}^*)$.

Lemma *Suppose* $\operatorname{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} \geqslant 2$ *and* $\eta \in \mathfrak{a}^*_{reg} \cap \Omega$. *Then there exists* $\xi \in \mathfrak{a}^*_{reg}$ *and a finite subset* $F \subset \mathbb{C}$ *such that* $W := \mathbb{C}\xi + \mathbb{C}\eta$ *satisfies*

(i) $W \cap \mathfrak{a}^*_{sing} = \{0\}$,

(ii) $\Omega \cap W \supset \{\xi + \lambda\eta | \lambda \in \mathbb{C} \setminus F\}$.

Proof Set $n = \dim \mathbb{P}(\mathfrak{a}^*), m = \dim \mathbb{P}(\mathfrak{a}^*_{sing})$. The hypothesis translates to $m \leqslant n-2$. Since $\eta \notin \mathfrak{a}^*_{sing}$, there exists a linear function f_1 vanishing at η but not vanishing identically on $\mathfrak{a}^*_{sing}$. Let $\mathbb{H}_1$ be the hypersurface in $\mathbb{P}(\mathfrak{a}^*)$ defined by the zeros of f_1. Since $\mathbb{P}(\mathfrak{a}^*_{sing})$ is closed in $\mathbb{P}(\mathfrak{a}^*)$, dimension theory [64, Sect. 6.2,Thm. 4] gives $\dim(\mathbb{P}(\mathfrak{a}^*_{sing}) \cap \mathbb{H}_1) = \dim \mathbb{P}(\mathfrak{a}^*_{sing}) - 1$. Repeating this argument m times and then recalling that a variety of dimension zero is finite, it follows that there exist linear functions $f_1, f_2, \dots, f_{m+1}$ not vanishing at η so their set of common zeros $\mathbb{L}$ has null intersection with $\mathbb{P}(\mathfrak{a}^*_{sing})$. Since $m+1 \leqslant n-1$, we may add a further $n-(m+2)$ linear functions, so that $\mathbb{L}$ becomes a projective line. Let W be the two-dimension subspace of $\mathfrak{a}^*$ whose projectivisation is $\mathbb{L}$. Then $\eta \in W$, whilst $\mathbb{L} \cap \mathbb{P}(\mathfrak{a}^*_{sing}) = \emptyset$. This gives (i).

Since $\mathbb{P}(\Omega)$ is open in $\mathbb{P}(\mathfrak{a}^*)$, it follows that $\mathbb{P}(\Omega) \cap \mathbb{L}$ is open in $\mathbb{L}$, while it is non-empty by construction. Hence $\mathbb{P}(\Omega) \cap \mathbb{L}$ is open dense in $\mathbb{L}$. This translates to (ii). $\square$

2.6.12 Let $\mathscr{A}$ be an m dimensional space of linear transformations on a $\mathbb{C}$-vector space V of dimension n. Let $\{x_i\}_{i=1}^n$ be a basis of V. Let $\mathscr{A}_0$ be the subset of $\mathscr{A}$ of matrices of maximal rank, say r. Let $A_i : i = 1, 2, \dots, m$, be a basis of $\mathscr{A}$. We identify $\mathscr{A}$ with $\mathbb{C}^m$ through the linear map $(c_1, c_2, \dots, c_m) \mapsto \Sigma c_i A_i =: A(c)$. Then $\mathscr{A}_0$ identifies with an irreducible dense open subset Ω_0 of $\mathbb{C}^m$.

Set $I_0 = \sum_{A \in \mathscr{A}_0} \ker A$. For each $s \in \mathbb{N}^+$, let $\mathscr{A}_1^s$ be the subset of the s-fold Cartesian product defined by

$$\mathscr{A}_1^s := \{(A_1, A_2, \dots, A_s) \in \mathscr{A}_0^s \mid \sum_{i=1}^s \ker A_i = I_0\}. \tag{$*$}$$

Lemma

(i) *$\mathscr{A}_1^s$ is Zariski open in $\mathscr{A}_0^s$.*

(ii) *$\mathscr{A}_1^s$ is Zariski dense in $\mathscr{A}_0^s$ (and hence in $\mathscr{A}^s$), if $s \geqslant n$.*

Proof Let $\{M_t(c)\}_{t=1}^u$ be the set of $r \times r$ minors of $A(c)$ which are not identically zero. Let $D_t \subset \mathbb{C}^m$ be the set of zeros of $M_t(c)$. It is Zariski closed. Clearly

$$\bigcup_{t=1}^u D_t = \mathbb{C}^m \setminus \Omega_0.$$

Equivalently setting $O_t = \mathbb{C}^m \setminus D_t$, one has

$$\bigcup_{t=1}^u O_t = \Omega_0,$$

giving a finite open cover of Ω_0. For each $s \in \mathbb{N}^+$, let $O_{\mathfrak{t}}^s : \mathfrak{t} \in \{1, \dots, u\}^s$ be the resulting finite open cover of Ω_0^s.

Let $R[O_t]$ denote the algebra of regular functions on O_t. Solving the linear equations for $\ker A(c)$ in O_t, we obtain $f_{i,j}^t \in R[O_t]$ such that

$$\ker A(c) = \sum_{j=1}^n f_{i,j}^t(c) x_j, \ \forall\, c \in O_t.$$

Thus the condition that

$$\sum_{i=1}^s \ker A(c_i)$$

has maximal possible dimension defines a Zariski open set in each O_t and hence a Zariski open set in Ω_0^s. On the other hand, the above sum is contained in I_0 with equality if and only if dimensions coincide. Combined with our previous observation this proves (i).

Since $\dim I_0 \leqslant \dim V = n$, it follows that $\mathscr{A}_1^s$ is non-empty for all $s \geqslant n$. Since $\mathscr{A}_1^s$ is irreducible, (i) follows from (ii). $\square$

2.6.13 Recall the definition of Ω in 2.6.9 and the construction of W in 2.6.11. Let $\mathscr{A}$ be the subspace of antisymmetric forms corresponding to W. Identify V with its dual and with $m=2$, identify $\mathscr{A}$ of 2.6.12 with $\mathscr{A}$ above. Obviously $\mathscr{A}_1^s$ defined in 2.6.12 $(*)$ identifies with a subset Ω_1^s of W^s stable by the action of $\mathbb{C}^*$ in each factor and stable under permutations. Then by the conclusion of 2.6.12, the set

$$\Lambda_n := \{\boldsymbol{\lambda} := (\lambda_1, \lambda_2, \dots, \lambda_n) \in (\mathbb{C} \setminus F)^n | (\xi + \lambda_i \eta)_{i=1}^n \in \Omega_1^n\},$$

is Zariski open dense in $\mathbb{C}^n$.

Fix $\boldsymbol{\lambda} \in \Lambda_n$. In view of 2.6.8, 2.6.11 and the definitions of Ω, F and Λ_n we conclude that

$$M := \sum_{i=1}^n \sum_{f \in Y(\mathfrak{a})} \mathbb{C} df(\xi + \lambda_i \eta),$$

is maximal isotropic with respect to A_ξ, for any $\xi \in W \setminus \{0\}$. In particular it has dimension $c(\mathfrak{a})$. Moreover it remains maximal isotropic and hence unchanged if $\boldsymbol{\lambda}$ is augmented to any m-tuple $(\lambda_1, \lambda_2, \dots, \lambda_m)$ of distinct elements. On the other hand by 2.6.2$(*)$ and 2.6.4$(*)$, one has

$$M := \sum_{i=1}^m \sum_{f \in Y(\mathfrak{a})} \mathbb{C} d(T_{\lambda_i \eta} f)(\xi).$$

This space having dimension $c(\mathfrak{a})$ means that the subalgebra of $S(\mathfrak{a})$ generated by the set $\{T_{\lambda_i \eta} f | i = 1, 2, \dots, n, f \in Y(\mathfrak{a})\}$ has GKdimension $c(\mathfrak{a})$. Since the invariants are polynomial and m can be made arbitrarily large this algebra is just $T_\eta(\mathfrak{a})$. Summarizing we obtain the following result of Bolsinov [3, Thm. 1.3], [4, Thm. 3.1].

Theorem *Suppose $\mathfrak{a}$ is an algebraic, semi-invariant free Lie algebra. Then there exists $\eta \in \mathfrak{a}^*$ such that GKdim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$, if and only if $\mathrm{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} \geqslant 2$.*

Proof It remains to verify "only if". Suppose GKdim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$, for some $\eta \in \mathfrak{a}^*$. Then the space of differentials of $T_\eta(\mathfrak{a})$ has dimension $c(\mathfrak{a})$ at any point ξ of a Zariski open dense subset Ω. We may therefore choose $\xi \in \mathfrak{a}^*_{reg} \cap \Omega$, so that $W = \mathbb{C}\xi + \mathbb{C}\eta$, is two dimensional. Reversing the argument in the proof of "if" , it follows that

$$N := \sum_{\xi \in W \cap \mathfrak{a}^*_{reg}} \ker \widehat{A}_\xi,$$

has dimension $c(\mathfrak{a})$ and hence is maximal isotropic. By 2.6.8 this forces $W \cap \mathfrak{a}^*_{reg} = W \setminus \{0\}$. Thus the projectivisation $\mathbb{P}(W)$ of W must have null intersection with $\mathbb{P}(\mathfrak{a}^*_{sing})$. By intersection theory [64, Sect. 6.2, Cor. 5], this forces $\mathrm{codim}_{\mathbb{P}(\mathfrak{a}^*)} \mathbb{P}(\mathfrak{a}^*_{sing}) \geqslant 2$ and hence $\mathrm{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} \geqslant 2$, as required.

Remark 1. Notice that we have also shown that GKdim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$ forces $\eta \in \mathfrak{a}^*_{reg}$. Conversely, if $\mathrm{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} \geqslant 2$, then GK dim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$, for all $\eta \in \mathfrak{a}^*_{reg}$, via 2.6.11.

Remark 2. We only need $\mathfrak{a}$ to be algebraic to ensure that GKdim $Y(\mathfrak{a})$ = index $\mathfrak{a}$. However under the hypothesis that $\mathfrak{a}$ is semi-invariant free this is automatic [57, Prop. 4.1].

Remark 3. For a formulation and proof of this theorem when $\operatorname{codim}_{\mathfrak{a}^*}\mathfrak{a}^*_{sing} = 1$, see [42, Theorem 7.2].

2.6.14 Consider the Heisenberg algebra $\mathfrak{a}$ of dimension $2n+1$ and centre $\mathbb{C}z$. Being nilpotent it is algebraic and semi-invariant free. Moreover $\mathfrak{a}^*_{sing}$ is just the zero set Z of z and so is a hyperplane. Thus $\mathfrak{a}$ does not satisfy the Bolsinov criterion. Correspondingly $c(\mathfrak{a}) = n+1$, whilst $Y(\mathfrak{a})$ has the single generator z and this is of degree 1, so $T_\eta(\mathfrak{a}) = Y(\mathfrak{a})$ for all $\eta \in \mathfrak{a}^*$.

By contrast if $\mathfrak{g}$ is semisimple, then $\operatorname{codim}_{\mathfrak{g}^*}\mathfrak{g}^*_{sing} = 3$. More generally suppose there exists $h \in \mathfrak{a}$ and regular elements x, h^*, y of $\mathfrak{a}^*$ being $ad\,h$ eigenvectors of eigenvalues 1,0,-1 respectively. We call (h, x, h^*, y) an adapted quadruple. In this case a deformation argument (as in [36, 8.2], or better [39, 7.8]) shows that $(\mathbb{C}x + \mathbb{C}h^* + \mathbb{C}y) \setminus \{0\} \subset \mathfrak{a}^*_{reg}$. As in 2.6.13, this forces $\operatorname{codim}_{\mathfrak{a}^*}\mathfrak{a}^*_{sing} \geqslant 3$. For $\mathfrak{g}$ semisimple we can identify $\mathfrak{g}^*$ with $\mathfrak{g}$ through the Killing form and take $h^* = h$ with (x, h, y) a principal s-triple.

Let $\mathfrak{a}^*_{subreg}$ denote the union of subregular orbits in $\mathfrak{a}^*$, that those of codimension index $\mathfrak{a}+2$. For $\mathfrak{g}$ semisimple $\mathfrak{g}^*_{subreg}$ is irreducible. It is called the subregular sheet [7]. The orbit space $\mathfrak{g}^*_{subreg}/G$ is naturally isomorphic to the quotient of a vector space by a finite group. Thus $\mathfrak{g}^*_{subreg}$ has codimension 3. For $\mathfrak{g}$ semisimple, the remaining orbits can similarly be shown to make a much smaller contribution to $\mathfrak{g}^*_{sing}$. Consequently $\operatorname{codim}_{\mathfrak{g}^*}\mathfrak{g}^*_{sing} = 3$. Recently Ooms and van den Bergh [57, Prop 5.11] have shown that

$$\operatorname{codim}^*_{\mathfrak{a}}\mathfrak{a}^*_{sing} \leq 3, \qquad (*)$$

for any finite dimensional Lie algebra $\mathfrak{a}$ satisfying the hypotheses of Theorem 2.6.13 and for which $Y(\mathfrak{a})$ is polynomial. However unlike the semisimple case it is not necessarily orbits of codimension (index $\mathfrak{a}$) + 2, which make the largest contribution to $\mathfrak{a}^*_{sing}$.

2.6.15 A more comprehensive formulation of the last part of 2.6.14 is the following. Fix a finite dimensional Lie algebra $\mathfrak{a}$ of dimension n and a non-negative integer k. Consider the union $C(k)$ of all co-adjoint orbits in $\mathfrak{a}^*$ of dimension $< k$. Clearly $C(k)$ is the Zariski closed set defined by the vanishing all $k \times k$ minors of the $n \times n$ matrix with entries $[x_i, x_j] : i, j = 1, 2, \dots, n$. Thus the set $S(k)$ of all co-adjoint orbits of dimension exactly k is a Zariski locally closed subset of $\mathfrak{a}^*$. A sheet is an irreducible component of some $S(k)$. This concept was first introduced by Dixmier who classified sheets for $\mathfrak{g}$ simple of type A. More generally for $\mathfrak{g}$ semisimple, sheets were classified by Borho [5] and later with Kraft [6], they showed that every sheet is the orbit space of some affine space $\mathbb{A}^t$ by a finite group (coming from the Weyl group). We may then regard $\mathbb{A}^t$ as a parameter set for the orbits in the corresponding sheet with t parameters. Key points in their analysis is the possibility to identify $\mathfrak{g}$ with $\mathfrak{g}^*$, to use Jordan decomposition and to use the classification of the nilpotent orbits.

A goal would be to classify sheets for biparabolic subalgebras. This will need a different approach and is liable to be very difficult.

2.6.16 Let $\mathfrak{a}(n) : n \geqslant 2$, be the Lie algebra spanned by $a, x_1, x_2, \ldots, x_n$ with non-zero relations given by $[a, x_i] = x_{i+1} : i = 1, 2, \ldots, n$, with x_{n+1} set equal to zero. If $n = 2$, it is just the Heisenberg Lie algebra of dimension 3. If $n = 3$, it is the truncated Borel for $\mathfrak{g}$ simple of type C_2. Being nilpotent it is algebraic and semi-invariant free. One checks that $c(\mathfrak{a}(n)) = n$, whilst the $\mathfrak{a}(n)^*_{sing}$ is just the zero set of $< x_2, x_3, \ldots, x_n >$ and so has codimension $(n-1)$. In particular it satisfies the Bolsinov criterion for $n \geqslant 3$. On the other hand, Dixmier verified by direct computation that $Y(\mathfrak{a}(4))$ fails to be polynomial. Recently Ooms and van den Bergh obtain this as a consequence of a general criterion for polynomiality [57, Prop. 1.6 (1)]. For $n > 4$ polynomiality is excluded by a further result of theirs, namely $(*)$ of 2.6.14.

When $n = 3$ the sum of the degrees of the generators equals $c(\mathfrak{a}(n))$. That their sum is at least $c(\mathfrak{a}(n))$ is forced by Theorem 2.6.13. That it is exactly three follows from a general sum rule of Ooms and van den Bergh [57, Prop. 1.4] for a Lie algebra satisfying the hypotheses of 2.6.15 with $Y(\mathfrak{a})$ polynomial on index $\mathfrak{a}$ generators. This sum rule has been further extended to the case when $\mathfrak{a}$ being semi-invariant free is replaced by the weaker condition of being unimodular with its fundamental semi-invariant (see Remark 2.8.4) being an invariant [42, Thm. 5.7].

On may further remark that whereas $T_\eta(\mathfrak{a}(3))$ must lie in the Poisson commutative subalgebra generated by x_1, x_2, x_3, it can at best have generators of degrees 1,1,2 and so cannot be maximal Poisson commutative. This has been recently in a more general framework by Panyushev and Yakimova [61] who show that a Lie algebra $\mathfrak{a}$ satisfying the hypotheses of 2.6.13, the shifted algebra $T_\eta(\mathfrak{a}) : \eta \in \mathfrak{a}^*_{reg}$ is maximal Poisson commutative if $\operatorname{codim}^*_{\mathfrak{a}} \mathfrak{a}^*_{sing} \geq 3$.

One of the simplest examples of a Lie algebra $\mathfrak{a}$ admitting an adapted quadruple is given by the truncated Borel of $\mathfrak{sl}(3)$. It has basis $\{x, y, z, h\}$ with non-zero Lie brackets being given by $[x, y] = z, [h, x] = x, [h, y] = -y$. Let $\{x^*, y^*, z^*, h^*\}$ be the dual basis. Then $\{h, y^*, z^*, x^*\}$ is an adapted quadruple. In this case the conclusion of Theorem 2.5.11 holds.

2.6.17 Now suppose that $\mathfrak{q}$ is a truncated biparabolic. In most cases $Y(\mathfrak{q})$ is polynomial and the sum of the degrees of the generators equals $c(\mathfrak{q})$. After Ooms van den Bergh [57, Prop 1.4] the first property implies that the sum of the degrees of the generators is $\leq c(\mathfrak{q})$ with equality if and only if $\operatorname{codim}_{\mathfrak{q}^*} \mathfrak{q}^*_{sing} \geq 2$. Unfortunately it is not always true that $Y(\mathfrak{q})$ is polynomial as a counterexample was found by Yakimova [70] in type E_8 by direct computation (for the truncated parabolic which is the centralizer of a highest weight vector). So far this has not been understood in a more general context.

An even more interesting question is how to construct these generators in the polynomial case. One finds that a subalgebra (and a surprisingly large one) $Y_0(\mathfrak{q})$ of $Y(\mathfrak{q})$ obtains from corresponding invariants in the Hopf dual of $U(\mathfrak{q})$. Moreover in most cases (even more surprisingly) all of $Y(\mathfrak{q})$ obtains in this fashion. A crucial question is how to obtain the missing part of $Y(\mathfrak{q})$. When Bolsinov's criterion holds it is natural to suggest that $Y(\mathfrak{q})$ lies in Fract $Y_0(\mathfrak{q})$. Otherwise it would seem we need to take square roots of generators, which in general can be expected to imply that the sum of the degrees is strictly less than $c(\mathfrak{q})$.

As an example suppose $\mathfrak{q}$ is the truncated Borel of a simple Lie algebra $\mathfrak{g}$. Such an algebra takes the form $\mathfrak{q}=\mathfrak{n}+\mathfrak{t}$, where $\mathfrak{n}$ is the nilradical of the Borel and $\mathfrak{t}$ is the subspace of $\mathfrak{h}$ of elements vanishing on the weights of $Y(\mathfrak{n})$. Set $r = \text{rk }\mathfrak{g}$ and $t=\dim\mathfrak{t}$. Let $\{\varpi_i : i = 1,2,\dots,r\}$, be the set of fundamental weights and w_0 the unique longest element of the Weyl group. When $\varpi_i=-w_0\varpi_i$, there is just one (up to scalars) element of $Y(\mathfrak{q})$ of this weight of degree. Using the notation of [21, Lemme 4.9] we denote its degree by s_i. Otherwise there are two elements of degrees s_i and $t_i = s_i + 1$, moreover this occurs exactly t times. Let s be the sum of the degrees of these elements. By [22, Lemme 4.9] it follows that

$$2s-t=\frac{1}{2}(\dim\mathfrak{g}+\text{rk }\mathfrak{g}).$$

On the other hand the right hand side of the above is just $\dim\mathfrak{q}+\text{index }\mathfrak{q}-t$ and so

$$s=\frac{1}{2}(\dim\mathfrak{q}+\text{index }\mathfrak{q}).$$

When $\varpi_i=-w_0\varpi_i$ there can sometimes be an element of $Y(\mathfrak{q})$ of this weight (and of degree $s_k/2$). Let p denote the sum of the degrees of such elements. It turns out [28, 4.12, 4.14] that these elements together with those described above which are not squares of the former form a system of polynomial generators of $Y(\mathfrak{q})$. In particular the sum $s_{\mathfrak{q}}$ of the degrees of the generators of $Y(\mathfrak{q})$ satisfies

$$s_{\mathfrak{q}}=\frac{1}{2}(\dim\mathfrak{q}+\text{index }\mathfrak{q})-p.$$

In the language of [57], let $p_{\mathfrak{q}}$ be the fundamental semi-invariant (see 2.8.4, Remark) of $\mathfrak{q}$. Since $\mathfrak{q}$ is semi-invariant free (by construction) and $Y(\mathfrak{q})$ is polynomial, the Ooms-van den Bergh sum rule [57, Prop. 1.4] gives

$$s_{\mathfrak{q}}=\frac{1}{2}(\dim\mathfrak{q}+\text{index }\mathfrak{q})-\deg p_{\mathfrak{q}}.$$

We conclude that $p=\deg p_{\mathfrak{q}}$. Moreover this strongly suggests that $p_{\mathfrak{q}}$ is just the product of the generators of $Y(\mathfrak{q})$ which are square roots of the elements of weight $2\varpi_i$ coming from invariants in the Hopf dual of $U(\mathfrak{q})$. Through the tables in [28] (as corrected in [21] !) one may note that $p=0$ exactly when $\mathfrak{g}$ is of type A or C.

One may remark that the fundamental semi-invariant of a Lie algebra $\mathfrak{a}$ is scalar if and only if $\text{codim}_{\mathfrak{a}^*}\,\mathfrak{a}^*_{sing}\geq 2$.

From the above we obtain the following improvement of a result appearing in the original version of these notes. Set $c:=\text{codim}_{\mathfrak{q}^*}\,\mathfrak{q}^*_{sing}$, where we recall that $\mathfrak{q}$ is the truncated Borel of a simple Lie algebra $\mathfrak{g}$.

Lemma. *If $\mathfrak{g}$ is of type A (resp. $C_n : n\geq 2$) then $c=3$(resp. $c=2$). Otherwise $c=1$.*

Proof. By the above it remains to consider types A and C. By the Ooms-van den Bergh result noted in $(*)$ of 2.6.14 and the existence [40, Section 4] of an adapted quadruple in type A, the assertion for type A results. Then for type C we just have to show that the inequality $\text{codim}_{\mathfrak{q}^*} \mathfrak{q}^*_{sing} \leq 3$ is strict. If not a result of Panyushev and Yakimova [61] implies that $T_\eta(\mathfrak{q})$ is maximal Poisson commutative for all $\eta \in \mathfrak{q}^*_{reg}$.

In type C_n the truncated Borel is just its nilradical $\mathfrak{n}_n$. One may write $n_n = n_{n-1} + h_n$, with h_n an ideal isomorphic to a Heisenberg subalgebra of dimension $2n-1$ having a unique up to scalars central element z_n.

One may choose $\eta \in (n_n)^*_{reg}$ to be one on each $z_m : m \leq n$ and zero on the remaining root vectors. (These roots form the so-called Kostant cascade. It is a general fact that if $\eta \in \mathfrak{q}^*$ is non-zero on the root vectors of the Kostant cascade and zero on the remaining root vectors, then η is regular.)

Adopt the Bourbaki [7, Planche III] notation. Let $\mathfrak{n}_n^+$ be the subalgebra of $\mathfrak{n}_n$ with basis the root vectors with roots in the set $\{\varepsilon_i + \varepsilon_j : i, j = 1, 2, \ldots, n\}$. It is commutative of dimension $c(\mathfrak{n}_n)$. One may remark that $S(\mathfrak{n}_n^+)$ is maximal Poisson commutative in $S(\mathfrak{n}_n)$.

On the other hand from the tables in [28], [21], $Y(\mathfrak{n}_n)$ has a generator of degree $m : m = 1, 2, \ldots, n$ of weight $2(\varepsilon_1 + \varepsilon_2, \ldots + \varepsilon_m)$. It follows that $Y(\mathfrak{n}_n)$ is contained in $S(\mathfrak{n}_n^+)$ and a fortiori so in $T_\eta(\mathfrak{n}_n)$. Yet since $n \geq 2$ the generators of $T_\eta(\mathfrak{n}_n)$ cannot all be linear, so this inclusion is strict and so $T_\eta(\mathfrak{n}_n)$ is not maximal Poisson commutative. □

Remark. Let $\mathfrak{q}$ be a truncated Borel of a simple Lie algebra. As in type C the nilradical $\mathfrak{n}$ of $\mathfrak{b}$ may be written as a direct sum of Heisenberg subalgebra whose centres run over the root vectors whose roots form the Kostant cascade. Let z be the (unique up to scalars) highest root vector and Z the zero set of z. It is a union of co-adjoint orbits. The Heisenberg subalgebra with centre z has dimension $d \geqslant 2\text{rk}\ \mathfrak{g} - 1$, with equality if and only if $\mathfrak{g}$ is simple of type A or C. Now $\text{rk}\ \mathfrak{g} = \text{index}\ \mathfrak{q}$ by [28, 4.14]. On the other hand the number of Heisenberg subalgebras occurring in the decomposition of n equals $\text{rk}\ \mathfrak{g} - \dim \mathfrak{t}$. In particular, the dimension of the space spanned by $\mathfrak{t}$ and the centres of the remaining Heisenberg subalgebras equals $\text{rk}\ \mathfrak{g} - 1$. Let $\{x_i\}$ be a basis for $\mathfrak{q}$. One checks that if $\xi \in Z$, then

$$\dim \mathfrak{q} - \text{rk}\ \xi[x_i, x_j] \geqslant d - (\text{rk}\ \mathfrak{g} - 1).$$

From this and the remark above it follows that Z is an irreducible component of $\mathfrak{q}^*_{sing}$, outside types A and C.

2.6.18 Assume that $\mathfrak{a}$ is algebraic and semi-invariant free. Assume further that $Y(\mathfrak{a})$ is polynomial.

Recall 2.5.7 and let $(h, \eta) \in \mathfrak{a} \times \mathfrak{a}^*_{reg}$ be an adapted pair. Then the conclusion of 2.5.9 combined with 2.5.14(1) implies that $\eta \in \mathfrak{a}^*_{reg} \cap \Omega$ (with $\Omega \subset \mathfrak{a}^*$ defined as in 2.6.9).

2.6.19 Since we always have $\text{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} \geqslant 1$, Lie algebras not satisfying the Bolsinov criterion are very special and it would be interesting to classify them. Call an algebraic Lie algebra $\mathfrak{a}$ singular if $\text{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_{sing} = 1$. Set $\dim \mathfrak{a} = n$ and let $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{a}$. Set $m = n -$ index $\mathfrak{a}$.

Lemma *Suppose $\mathfrak{a}$ is a singular Lie algebra. Let $\mathscr{V}$ be an irreducible component of $\mathfrak{a}^*_{sing}$ of dimension $n-1$. Then $\mathscr{V}$ is the zero variety of some homogeneous semi-invariant element f of $S(\mathfrak{a})$ of positive degree. Moreover, f is a divisor of every non-zero $m \times m$ minor of $\{[x_i, x_j]\}_{i,j=1}^n$.*

Proof Let A be the algebraic adjoint group of $\mathfrak{a}$. It acts on $\mathfrak{a}^*$ by transposition. Clearly $\mathfrak{a}^*_{sing}$ is A stable and since A is irreducible, so is any irreducible component, for example $\mathscr{V}$. Similarly, $\mathfrak{a}^*_{sing}$, being stable for the natural action of $\mathbb{C}^*$, forces the same of $\mathscr{V}$. By Krull's theorem, there exists a unique up to scalars polynomial f on $\mathfrak{a}^*$ such that $\mathscr{V}$ is the variety of zeros of f. Then $I(\mathscr{V}) = S(\mathfrak{a})f$. Take $a \in A$. Since $a^{-1}\mathscr{V} \subset \mathscr{V}$ it follows that $a.f$ vanishes on $\mathscr{V}$, and so $a.f \in I(\mathscr{V})$. Hence $a.f$ divides f. Yet, deg $a.f$ = deg f and so $a.f \in \mathbb{C}^* f$, which shows that f must be a semi-invariant. Again, $\mathscr{V}$ being $\mathbb{C}^*$ stable, forces f to be homogeneous. Finally, $\mathfrak{a}^*_{sing}$ is just the set of common zeros of the $m \times m$ minors of $\{[x_i, x_j]\}_{i,j=1}^n$, each of which must be divisible by f. □

2.6.20 Of course, if $\mathfrak{a}$ is semi-invariant free, then f in the conclusion of 2.6.19 must lie in $Y(\mathfrak{a})$ and indeed, in its augmentation $Y(\mathfrak{a})_+$. Let $\mathscr{N}_\mathfrak{a}$ denote the set of zeros of $S(\mathfrak{a})Y(\mathfrak{a})_+$ which we called (1.6) the nilfibre (of the categorical quotient map). If $\mathfrak{g}$ is semisimple, then identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ through the Killing form, $\mathscr{N}_\mathfrak{g}$ identifies with the nilpotent cone (2.3). As we have seen, in this case $\mathscr{N}_\mathfrak{g} \cap \mathfrak{g}^*_{reg} \neq \emptyset$. More generally, recall (2.5.7) that if $(h, \eta) \in \mathfrak{a} \times \mathfrak{a}^*_{reg}$ is an adapted pair then $(ad\, h)\eta = -\eta$.

Notice this latter condition implies that $\eta \in \mathscr{N}_\mathfrak{a}$. The Jacobson-Morosov theorem gives an adapted pair for $\mathfrak{g}$ semisimple, starting from a regular nilpotent element. More generally, one may construct an adapted pair for any (truncated) biparabolic subalgebra of $\mathfrak{sl}(n)$ and even for certain maximal parabolics in general. Thus it is worthwhile to point out the following

Lemma *Suppose $\mathfrak{a}$ is semi-invariant free. If $\mathfrak{a}$ is singular, then $\mathscr{N}_\mathfrak{a} \cap \mathfrak{a}^*_{reg} = \emptyset$.*

Proof Let $\mathscr{V}, f$ be as in 2.6.19. Then $\mathfrak{a}^*_{sing} \supset \mathscr{V} = \mathscr{V}(S(\mathfrak{a})f) \supset \mathscr{N}_\mathfrak{a}$, since f is a homogeneous invariant of positive degree. Hence the assertion. □

Remark Suppose $\mathfrak{a}$ is semi-invariant free. If $\mathfrak{a}$ admits an adapted pair (h, η) one has GKdim $T_\eta(\mathfrak{a}) = c(\mathfrak{a})$, via 2.6.13 and Remark 3 of the latter.

2.7. **A Comparison Inequality.**

2.7.1 Let $\mathfrak{a}$ be a finite dimensional Lie algebra. Then $\mathfrak{a}$ acts on $\mathfrak{a}^*$ by transposition of the adjoint action. For all $\xi \in \mathfrak{a}^*$ set $\mathfrak{a}^\xi = \{x \in \mathfrak{a} | x\,.\,\xi = 0\}$. Duflo and Vergne [17] showed that $\mathfrak{a}^\xi$ is commutative if $\xi \in \mathfrak{a}^*_{reg}$.

Bolsinov [3, see after Prop. 3.1] notes the following, namely index $\mathfrak{a}^\xi \geqslant$ index $\mathfrak{a}$, for all $\xi \in \mathfrak{a}^*$ for which he refers the reader to a book of Arnold and Givental. Here we remark that if $\xi \in \mathfrak{a}^*_{reg}$, then index $\mathfrak{a} = \dim \mathfrak{a}^\xi$. yet index $\mathfrak{b} \geqslant \dim \mathfrak{b}$ for a Lie algebra $\mathfrak{b}$ implies that $\mathfrak{b}$ is commutative. Hence the above extends the Duflo-Vergne result.

The proof of the above generalization may be obtained by adapting the Duflo-Vergne argument and is given below. (We later learnt that it is also a consequence of an inequality attributed to Vinberg. This is discussed in the next section.)

2.7.2 Fix $\xi \in \mathfrak{a}^*$ and let $\iota : \mathfrak{a}^* \to (\mathfrak{a}^\xi)^*$ be the surjective linear map defined by restriction. Since $(\mathfrak{a}^\xi)^*_{reg}$ is open dense in $(\mathfrak{a}^\xi)^*$ it follows that $\iota^{-1}((\mathfrak{a}^\xi)^*_{reg})$ is non-empty and open in $\mathfrak{a}^*$. Hence it meets the open dense set $\mathfrak{a}^*_{reg}$. Thus we can choose $\eta \in \mathfrak{a}^*_{reg}$ such that $\iota(\eta) \in (\mathfrak{a}^\xi)^*_{reg}$, which we fix from now on. Recall that $\dim \mathfrak{a}^\eta = \operatorname{index} \mathfrak{a}$.

2.7.3 Fix a complementary subspace $\mathfrak{b}$ to $\mathfrak{a}^\eta$ in $\mathfrak{a}$. Let $\{x_i\}_{i=1}^n$ be a basis for $\mathfrak{a}$ such that $\{x_i\}_{i=1}^m$ is a basis for $\mathfrak{b}$ and $\{x_i\}_{i=m+1}^n$ as basis for $\mathfrak{a}^\eta$. The relation $\eta([x_i, x_j]) = -(x_i \,.\, \eta)(x_j)$ implies that $\mathfrak{a}^\eta$ is just the kernel of the matrix with entries $\eta([x_i, x_j]) : i, j = 1, 2, \dots, n$. In particular, $m = \operatorname{codim} \mathfrak{a}^\eta = \dim \mathfrak{a} - \operatorname{index} \mathfrak{a}$. Again there exists a subset J of cardinality m such that $\det\{\eta([x_i, x_j])\}_{i\in I, j\in J} \neq 0$, where $I = \{1, 2, \dots, m\}$.

Since η is regular, one has for all $\lambda \in \mathbb{C}$ that

$$\mathfrak{a} = \mathfrak{b} \oplus \mathfrak{a}^{\eta+\lambda\xi} \iff \mathfrak{b} \cap \mathfrak{a}^{\eta+\lambda\xi} = 0.$$

Moreover, the set Ω of all $\lambda \in \mathbb{C}$ satisfying this latter condition is open in $\mathbb{C}$ and contains 0 by construction.

2.7.4 Given $x \in \mathfrak{a}^\eta$, set $x = \sum_{i=m+1}^n c_i x_i$. For all $\lambda \in \Omega$, we may write $x = x_1(\lambda) + x_2(\lambda)$ with $x_2(\lambda) \in \mathfrak{a}^{\eta+\lambda\xi}$ and $x_1(\lambda) = \sum_{i=1}^m d_i(\lambda) x_i \in \mathfrak{b}$. Then the condition $x_2(\lambda) \,.\, (\eta + \lambda\xi) = 0$, becomes

$$\sum_{i=1}^m d_i(\lambda)(\eta + \lambda\xi)([x_i, x_j]) = \sum_{i=m+1}^n c_i(\eta + \lambda\xi)([x_i, x_j]), \tag{$*$}$$

which we solve for the $d_i(\lambda) : i = 1, 2, \dots, m$. Since $d(\lambda) := \det\{\eta + \lambda\xi)([x_i, x_j]\}_{i\in I, j\in J} \neq 0$ at $\lambda = 0$, it is also non-zero in some $\Omega' \subset \Omega$ open in $\mathbb{C}$. It follows that we may solve $(*)$ for the $d_i(\lambda) : i = 1, 2, \dots, m$, at any $\lambda \in \Omega'$ and that these are rational functions in λ. We can write $d_i(\lambda)$ uniquely in the form $d_i(\lambda) = p_i(\lambda)/q_i(\lambda)$, with p_i, q_i coprime polynomials and $q_i(\lambda)$ monic. Then each $q_i(\lambda)$ is a factor of $d(\lambda)$ and so is non-zero at $\lambda = 0$. Let $q(\lambda)$ be their lowest common multiple, set $m_i(\lambda) = q(\lambda)p_i(\lambda)/q_i(\lambda)q(0) : i = 1, 2, \dots, m$, and

$$x(\lambda) := \frac{q(\lambda)}{q(0)} x - \sum_{i=1}^m m_i(\lambda) x_i. \tag{$*$}$$

Then $x(\lambda)$ is just the component of $\frac{q(\lambda)}{q(0)} x$ in the direct summand $\mathfrak{b}$ of $\mathfrak{b} \oplus \mathfrak{a}^{\eta+\lambda\xi}$. Obviously $x(0) = x$.

2.7.5 Now consider λ as an independent parameter t and let $\mathfrak{a}(t)$ be the $\mathbb{C}[t]$ module generated by the polynomials $x(t)$ in 2.7.4$(*)$ above, as x runs through a basis of $\mathfrak{a}^\eta$. Now $\mathfrak{a}(t)$ is finitely generated and torsion free over the principal ideal domain $\mathbb{C}[t]$, so is free of rank equal to $\dim \mathfrak{a}^\eta$. Let $\{x_i(t)\}_{i=1}^{n-m}$ be a free basis over $\mathbb{C}[t]$ such that $\sum_{i=1}^{n-m} \deg x_i(t)$ is minimal over all possible choices.

Lemma *The leading coefficients x_{i,d_i} of the $x_i(t) : i = 1, 2, \ldots, n-m$, are linearly independent.*

Proof Suppose

$$\sum_{k \in K} c_k \; x_{k,d_k} = 0 \tag{$*$}$$

for some non-empty subset $K \subset \{1, 2, \ldots, n-m\}$, with $c_k \neq 0 \;\forall\; k \in K$. Choose $\ell \in K$ such that d_ℓ is maximal and replace just $x_\ell(t)$ in the given basis by

$$x_\ell(t) + \sum_{k \in K \setminus \{\ell\}} (c_k/c_\ell) x_k(t) t^{d_\ell - d_k}$$

which through $(*)$ has degree $< d_\ell$. This gives a new $\mathbb{C}[t]$ basis contradicting the hypothesis on the given one. □

2.7.6 Fix a basis element $y(t)$ of $\mathfrak{a}(t)$ and write

$$y(t) = \sum_{i=0}^{d} y_i t^i : y_d \neq 0.$$

Lemma $y_d \in (\mathfrak{a}^\xi)^{\iota(\eta)}$.

Proof Equating powers of t in the equation $y(t) \,.\, (\eta + t\xi) = 0$ gives

$$y_d \,.\, \xi = 0, \; y_d \,.\, \eta + y_{d-1} \,.\, \xi = 0.$$

The first relation means that $y_d \in \mathfrak{a}^\xi$. Evaluated on $\mathfrak{a}^\xi$, the second relation gives

$$\begin{aligned} 0 = (y_d \,.\, \eta)(\mathfrak{a}^\xi) \; &= \eta([y_d \;,\; \mathfrak{a}^\xi]) \\ &= \iota(\eta)([y_d \;,\; \mathfrak{a}^\xi]), \text{ since } y_d \in \mathfrak{a}^\xi, \\ &= (y_d \,.\, \iota(\eta))(\mathfrak{a}^\xi), \end{aligned}$$

as required. □

2.7.7 The desired result now obtains.

Proposition *Let $\mathfrak{a}$ be a finite dimensional Lie algebra. Then for all $\xi \in \mathfrak{a}^*$ one has*

$$\text{index } \mathfrak{a}^\xi \geqslant \text{index } \mathfrak{a}.$$

Proof By the choice of η and 2.7.5, 2.7.6 we obtain

$$\text{index } \mathfrak{a} = \dim \mathfrak{a}^\eta \leqslant \dim (\mathfrak{a}^\xi)^{\iota(\eta)} = \text{index } \mathfrak{a}^\xi.$$

□

2.7.8 By contrast, suppose we take $y \in \mathfrak{a}$ and set $\mathfrak{a}^y = \{x \in \mathfrak{a} | [x,y] = 0\}$. Unless $\mathfrak{a}$ is semisimple, it can be false that index $\mathfrak{a} \leqslant$ index $\mathfrak{a}^y$, even if y is ad-semisimple. For example, let $\mathfrak{n}$ denote the nilradical of the Borel subalgebra of a semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$. By [28, 4.14] one may easily deduce that $\frac{1}{2}$ rk $\mathfrak{g} \leqslant$ index $\mathfrak{n} \leqslant$ rk $\mathfrak{g}$. Now choose $h \in \mathfrak{h}$ such that $[h,x] = x$, for every simple root vector $x \in \mathfrak{n}$. Then $(\mathfrak{n} \oplus \mathbb{C}h)^h = \mathbb{C}h$ and so has index 1. On the other hand, index $(\mathfrak{n} \oplus \mathbb{C}h) \geqslant$ index $\mathfrak{n} - 1$.

2.7.9 One may appreciate the difficulty raised in 2.7.8 by the following. Let $\mathfrak{g}$ be semisimple with Cartan subalgebra $\mathfrak{h}$. Let $\mathfrak{a}$ be a subalgebra of $\mathfrak{g}$ and $h \in \mathfrak{h}$ satisfying $(ad\, h)\mathfrak{a} \subset \mathfrak{a}$. Let $\mathfrak{a}^\perp$ denote the orthogonal with respect to the Killing form K. Let κ be the $ad\,\mathfrak{g}$ isomorphism of $\mathfrak{g}$ onto $\mathfrak{g}^*$ defined by K, namely $\kappa(x)(y) = K(x,y)$, $\forall\ x, y \in \mathfrak{g}$. Let ι be the surjection of $\mathfrak{g}^*$ onto $\mathfrak{a}^*$ defined by restriction.

Lemma $\mathfrak{a}^{\iota\kappa(h)} = \mathfrak{a}^h + \mathfrak{a} \cap \mathfrak{a}^\perp$.

Proof One has

$$\begin{aligned}
\mathfrak{a}^{\iota\kappa(h)} &= \{y \in \mathfrak{a} | y \,.\, \iota\kappa(h) = 0\} \\
&= \{y \in \mathfrak{a} | \kappa(h)([y, \mathfrak{a}] = 0)\} \\
&= \{y \in \mathfrak{a} | K((ad\, h)y, \mathfrak{a}) = 0\} \\
&= \{y \in \mathfrak{a} | (ad\, h)y \in \mathfrak{a}^\perp\} \supset \mathfrak{a}^h.
\end{aligned}$$

On the other hand, $\mathfrak{a}$ and $\mathfrak{a}^\perp$ are $ad\, h$ stable and hence so is $\mathfrak{a}^{\iota\kappa(h)}$. Consequently, $\mathfrak{b} := \mathfrak{a} \cap \mathfrak{a}^\perp \subset \mathfrak{a}^{\iota\kappa(h)}$. Conversely an $ad\, h$ stable complement $\mathfrak{c}$ to $\mathfrak{b}$ in $\mathfrak{a}^{\iota\kappa(h)}$ must satisfy $(ad\, h)\mathfrak{c} \subset \mathfrak{b} \cap \mathfrak{c} = 0$ and so lies in $\mathfrak{a}^h$. $\square$

2.8. **Vinberg's Inequality.**

2.8.1 It turns out that there is a more efficient way to obtain 2.7.7. This derives from an inequality attributed to E.B. Vinberg by A.G. Elashvili. Our exposition follows that of D.I. Panyushev [58, Prop. 1.6].

2.8.2 Let $\mathfrak{a}$ be a finite dimensional Lie algebra and V an $\mathfrak{a}$ module. Fix $w \in V$. One checks that $\mathfrak{a}w$ is a $\mathfrak{a}^w$ submodule of V. Let $\pi : V \to V/\mathfrak{a}w$ be the canonical projection.

Proposition (Vinberg) *For all $w \in V$ one has*

$$\max_{x \in V} \dim \mathfrak{a}x \geqslant \max_{y \in \pi(V)} \dim \mathfrak{a}^w y + \dim \mathfrak{a}w.$$

Proof Let $\mathfrak{b}$ be a complement to $\mathfrak{a}^w$ in $\mathfrak{a}$. Choose $y = v + \mathfrak{a}w \in \pi(V)$ so that $\dim \mathfrak{a}^w y$ is maximal. For all $t \in \mathbb{C} \setminus \{0\}$ one has $\mathfrak{a}(tv + w) = \mathfrak{a}^w v + \mathfrak{b}(tv + w)$. The condition that the right hand side does not have its maximal dimension for $t \in \mathbb{C}$ is given by a polynomial in t and so defines a finite subset F of $\mathbb{C}$. When $t = 0$ (which may or may not belong to F)

the right hand side equals $\mathfrak{a}^w v + \mathfrak{b}w \supset \mathfrak{b}w = \mathfrak{a}w$. Hence at $t = 0$ the dimension of the right hand side equals $\dim \mathfrak{a}w + \dim \mathfrak{a}^w \pi(v) = \max_{y \in \pi(V)} \dim \mathfrak{a}^w y + \dim \mathfrak{a}w$, by the choice of v. Consequently, for all $t \notin F \setminus \{0\}$ one has

$$\max_{x \in V} \dim \mathfrak{a}x \geqslant \dim \mathfrak{a}(tv + w) \geqslant \max_{y \in \pi(v)} \dim \mathfrak{a}^w y + \dim \mathfrak{a}w.$$

$\square$

2.8.3 In the above proposition we may take $V = \mathfrak{a}^*$, given its coadjoint action. The kernel of the restriction map $\mathfrak{a}^* \to (\mathfrak{a}^w)^*$, which is surjective equals $\{\xi \in \mathfrak{a}^* | \xi(\mathfrak{a}^w) = 0\}$ and has dimension $\dim \mathfrak{a} - \dim \mathfrak{a}^w = \dim \mathfrak{a}w$. On the other hand, $(\mathfrak{a}.w)(\mathfrak{a}^w) = w[\mathfrak{a}, \mathfrak{a}^w] = (\mathfrak{a}^w w)(\mathfrak{a}) = 0$ and so the above kernel contains $\mathfrak{a}.w$. Hence $(\mathfrak{a}^w)^*$ identifies with $\mathfrak{a}^*/\mathfrak{a}w = V/\mathfrak{a}w$. Then, by the Vinberg proposition

$$\begin{aligned}
\text{index } \mathfrak{a} &= \dim \mathfrak{a} - \max_{x \in V} \dim \mathfrak{a}x \\
&\leqslant (\dim \mathfrak{a} - \dim \mathfrak{a}.w) - \max_{y \in (\mathfrak{a}^w)^*} \dim \mathfrak{a}^w y \\
&= \text{index } \mathfrak{a}^w.
\end{aligned}$$

Hence 2.7.7.

2.8.4 A slightly more effective way of calculating index than just computing $\mathrm{rk}\{[x_i, x_j]\}_{i,j=1}^n$ derives from the following.

Let V be an even dimensional vector space and $\{x_i\}_{i=1}^{2n}$ a basis for V. Let $\xi_{i,j} : 1 \leqslant i < j \leqslant 2n$ be indeterminates and set $\xi_{j,i} = -\xi_{i,j}$ for $i < j$ and $\xi_{i,i} = 0$.

Let M_ξ be the matrix with entries $\{\xi_{i,j}\}_{i,j=1}^{2n}$ and set $q_\xi = \det M_\xi$. Define $\omega_\xi \in \mathbb{C}[\xi_{i,j} : i, j = 1, 2, \ldots, 2n] \otimes \wedge^* \vee$ by

$$\omega_\xi = \sum_{1 \leqslant i < j \leqslant 2n} \xi_{i,j}(x_i \wedge x_j).$$

We may write

$$\omega_\xi^n = n! p_\xi (x_1 \wedge x_2 \wedge \ldots \wedge x_{2n})$$

Lemma $q_\xi = p_\xi^2$.

Proof View the $\xi_{i,j} : 1 \leqslant i < j \leqslant 2n$ as the co-ordinate functions on the affine space of dimension $n(2n-1)$ which we denote simply by $\mathbb{A}$. Then $q_\xi, p_\xi \in R[\mathbb{A}]$. Let us show that q_ξ, p_ξ have the same set of zeros on $\mathbb{A}$. Observe that $M_{\xi(a)}$ defines an antisymmetric bilinear form on V^* through

$$M_{\xi(a)}(\xi_i, \xi_j) = \xi_{i,j}(a), \text{for all } i, j = 1, 2, \ldots, 2n,$$

where $\{\xi_i\}_{i=1}^{2n}$ is a dual basis. Moreover, $M_{\xi(a)}$ is non-degenerate if and only if $q_\xi(a) \neq 0$.

We may write

$$\omega_{\xi(a)} = \sum_{1\leqslant i<j\leqslant 2n} \xi_{i,j}(a)(x_i \wedge x_j) = \frac{1}{2}\sum_{i,j=1}^{2n} M_{\xi(a)}(\xi_i, \xi_j)(x_i \wedge x_j),$$

which is independent of choice of basis.

Suppose $q_\xi(a) = 0$ and set $W = \ker M_{\xi(a)}$ which is even-dimensional. Choose a basis $\{y_i\}_{i=1}^{2n}$ for V such that with respect to the dual basis $\{\eta_i\}_{i=1}^{2n}$, the subset $\{\eta_i\}_{i=1}^{2m}$ is a basis for W. Then

$$\omega_{\xi(a)} = \sum_{i=1}^{2n} M_{\xi(a)}(\eta_i, \eta_j)(y_i \wedge y_j)$$

and so $\omega_{\xi(a)}^t = 0$, for $t > n - m$. In particular, $\omega_{\xi(a)}^n = 0$ and so $p_\xi(a) = 0$.

Conversely, suppose that $q_\xi(a) \neq 0$. Scaling we can assume $q_\xi(a) = 1$. By base change we may reduce $M_{\xi(a)}$ to canonical form, namely

$$1 = M_{\xi(a)}(\eta_{2i-1}, \eta_{2i}) = -M_{\xi(a)}(\eta_{2i}, \eta_{2i-1}) : i = 1, 2, \ldots, n$$

with all other entries zero. However, in this case, it is immediate that $\omega^n = n!$ and so $p_{\xi(a)} = 1$.

From the above it follows that p, q have the same irreducible factors. On the other hand, p is multilinear in the $\xi_{i,j} : 1 \leqslant i < j \leqslant 2n$, that is to say in each monomial of p every exponent is at most one. Consequently, p is a product of distinct irreducible factors, say $p_\xi = p_1 p_2 \ldots p_r$. Again, since q_ξ is obtained by evaluating a determinant with entries $\xi_{i,j}$, where $\xi_{j,i} = -\xi_{i,j}$ every monomial in q_ξ has exponent at most two. We conclude that up to a non-zero scalar $q_\xi = p_1^{s_1} p_2^{s_2} \ldots p_t^{s_r} : 1 \leqslant s_i \leqslant 2$, for all $i = 1, 2, \ldots, t$.

It is clear that p is homogeneous of degree n whilst q is homogeneous of degree $2n$. Combined with the previous observation, we conclude that $q_\xi = p_\xi^2$, up to a non-zero scalar. Yet as we have seen, there exists $a \in \mathbb{A}$ such that $q_\xi(a) = p_\xi(a) = 1$. Hence $q_\xi = p_\xi^2$, as required. $\square$

Remark. Recall 2.6.19 and let $q_\mathfrak{a}$ denote the greatest common divisor of the minors of $\{[x_i, x_j]\}_{i,j=1}^n$. Then by the above there exists a polynomial $p_\mathfrak{a}$ whose square is $q_\mathfrak{a}$. Moreover $p_\mathfrak{a}$ is semi-invariant. It is what Ooms-van den Bergh [57] call the fundamental semi-invariant of $\mathfrak{a}$.

2.9. **The Rais theorem, index and the singular set.**

Let $\mathfrak{a}$ be a finite dimensional Lie algebra. In some good situations, particularly for semi-direct products, the Rais theorem can be used to describe $\mathfrak{a}^*_{sing}$.

2.9.1 Let $\mathfrak{p}$ be a finite dimensional Lie algebra. Take $p \in \mathfrak{p}^*$, $x \in \mathfrak{p}$ and let $(x, p) \mapsto x.p$ designate the coadjoint action of $\mathfrak{p}$ on $\mathfrak{p}^*$. Set $\mathfrak{p}^p = \{x \in \mathfrak{p} | x.p = 0\}$. Recall the alternating two form $B_p : (x, y) \mapsto p([x, y])$ on $\mathfrak{p}$. Given a subspace $\mathfrak{a}$ of $\mathfrak{p}$, let $\mathfrak{a}^\perp$ denote its orthogonal in $\mathfrak{p}$ with respect to this form. One has $\mathfrak{a}^p = \mathfrak{a} \cap \mathfrak{p}^\perp$. If $\mathfrak{a}$ is a subalgebra, let $a \in \mathfrak{a}^*$ be the restriction of p to $\mathfrak{a}$. Then $\mathfrak{a}^a = \mathfrak{a} \cap \mathfrak{a}^\perp$.

2.9.2 From now on we suppose that $\mathfrak{a}$ is an ideal of $\mathfrak{p}$. Then $\mathfrak{h} := \mathfrak{a}^\perp$ is a subalgebra of $\mathfrak{p}$.

Lemma $\mathfrak{h}^h = \mathfrak{p}^p + \mathfrak{a}^a$.

Proof By 2.9.1, one has

$$\begin{aligned} \mathfrak{h}^h = \mathfrak{h} \cap \mathfrak{h}^\perp = \mathfrak{a}^\perp \cap \mathfrak{a}^{\perp\perp} &= \mathfrak{a}^\perp \cap (\mathfrak{p}^\perp + \mathfrak{a}) \\ &= \mathfrak{p}^\perp + \mathfrak{a} \cap \mathfrak{a}^\perp, \text{ since } \mathfrak{p}^\perp \subset \mathfrak{a}^\perp \\ &= \mathfrak{p}^p + \mathfrak{a}^a, \text{ as required.} \end{aligned}$$

2.9.3 From now on we suppose that $\mathfrak{a}$ is an abelian ideal of $\mathfrak{p}$. Then in 2.9.2 one has $\mathfrak{a}^a = \mathfrak{a}$. Moreover, we assume that $\mathfrak{a}$ is complemented in $\mathfrak{p}$ by a subalgebra $\mathfrak{g}$. We may write $p = g + a$, where g (resp. a) is extended to $\mathfrak{p}$ by setting $g(\mathfrak{a}) = 0$ (resp. $a(\mathfrak{g}) = 0$). Then $\mathfrak{g}^g$ is independent of whether we view g as an element of $\mathfrak{g}^*$ or of $\mathfrak{p}^*$ (the same does *not* apply to $\mathfrak{a}^a$). Set $\mathfrak{g}(a) = \{x \in \mathfrak{g} | x.a = 0\}$ which is also independent of whether we view a as element of $\mathfrak{a}^*$ or of $\mathfrak{p}^*$. It is a subalgebra of $\mathfrak{g}$. Let g_0 denote the restriction g, equivalently of $\mathfrak{p}$, to $\mathfrak{g}(a)$.

Lemma *Set* $\mathfrak{h} = \mathfrak{a}^\perp,\ h = p|_{\mathfrak{h}}$. *Then*

(i) $\mathfrak{h} = \mathfrak{g}(a) \oplus \mathfrak{a}$,

(ii) $\mathfrak{h}^h = \mathfrak{g}(a)^{g_0} \oplus \mathfrak{a}$.

Proof Clearly $\mathfrak{h} \supset \mathfrak{a}$, whilst $\mathfrak{h} \cap \mathfrak{g} = \{x \in \mathfrak{g} | p[x, \mathfrak{a}] = 0\} = \{x \in \mathfrak{g} | (x.a)(\mathfrak{a}) = 0\} = \mathfrak{g}(a)$, since $(x.g)(\mathfrak{a}) = 0$. Hence (i). By 2.9.2 or directly $\mathfrak{h}^h \supset \mathfrak{a}$, whilst

$\mathfrak{h}^h \cap \mathfrak{g}(a) = \{x \in \mathfrak{g}(a) | p([x, y]) = 0,\ \forall\ y \in \mathfrak{h}\}$
$= \{x \in \mathfrak{g}(a) | g_0([x, y]) = 0,\ \forall\ y \in \mathfrak{g}(a)\} = \mathfrak{g}(a)^{g_0}$,

since $a([x, y]) = 0$ for $x \in \mathfrak{g}(a)$ and $g([x, y]) = 0$ for $y \in \mathfrak{a}$. Hence (ii). □

2.9.4 Combining 2.9.2 and 2.9.3, we obtain the

Proposition *For all* $p \in \mathfrak{p}^*$ *with* $a = p|_{\mathfrak{a}},\ g_0 = p|_{\mathfrak{g}(a)}$ *one has*

$$\dim \mathfrak{p}^p = \dim \mathfrak{g}(a)^{g_0} + \operatorname{codim}_{\mathfrak{a}^*} \mathfrak{g}.a.$$

Proof Observe that $\mathfrak{a} \cap \mathfrak{p}^p = \{x \in \mathfrak{a} | a([x, \mathfrak{g}]) = 0\}$ which is just the orthogonal of $\mathfrak{g}.a$ in $\mathfrak{a}$. Hence $\dim(\mathfrak{a} \cap \mathfrak{p}^p) = \dim \mathfrak{a} - \dim \mathfrak{g}.a$. Then, by 2.9.2, we obtain

$$\begin{aligned} \dim \mathfrak{h}^h &= \dim \mathfrak{p}^p + \dim \mathfrak{a} - \dim \mathfrak{a} \cap \mathfrak{p}^p \\ &= \dim \mathfrak{p}^p + \dim(\mathfrak{g}.a) \end{aligned}$$

Substitution from 2.9.3 gives the required assertion. □

2.9.5 Significantly g_0 only occurs in the first factor on the right hand side of 2.9.4. This gives the

Corollary

(i) $p \in \mathfrak{p}^*_{reg} \Rightarrow \mathfrak{g}_0 \in \mathfrak{g}(a)^*_{reg}$,

(ii) $g_0 \in \mathfrak{g}(a)^*_{sing} \Rightarrow p \in \mathfrak{p}^*_{sing}$.

2.9.6 Let ρ denote the representation of $\mathfrak{g}$ defined by its action on $\mathfrak{a}^*$. Let $\mathfrak{a}^*_r$ denote the subset of $a \in \mathfrak{a}^*$ such that $\text{codim}_{\mathfrak{a}^*}(\mathfrak{g}.a)$ is minimal and call this minimal value index ρ. Obviously $\mathfrak{a}^*_r$ is open dense in $\mathfrak{a}^*$ and hence so is its inverse image in $\mathfrak{p}^*$ under the restriction map ι. Set $\Omega = \iota^{-1}(\mathfrak{a}^*_r) \cap \mathfrak{p}^*_{reg}$, which is open dense in $\mathfrak{p}^*$.

Theorem(Rais) *Suppose $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}$, with $\mathfrak{g}$ a subalgebra and $\mathfrak{a}$ an abelian ideal. Then*

(i) *index $\mathfrak{p}$ = index $\mathfrak{g}(a)$ + codim$_{\mathfrak{a}^*}\mathfrak{g}.a$, if $\iota^{-1}(a) \cap \mathfrak{p}^*_{reg} \neq \emptyset$.*

(ii) *index $\mathfrak{p}$ = index $\mathfrak{g}(a)$ + index ρ, if $\iota^{-1}(a) \cap \Omega \neq \emptyset$.*

(iii) *index $\mathfrak{p}$ − index $\rho = \inf_{\iota^{-1}(a) \cap \mathfrak{p}^*_{reg} \neq \emptyset}$ index $\mathfrak{g}(a)$.*

Proof For (i) take $p \in \mathfrak{p}^*_{reg}$ and apply 2.9.4 and 2.9.5(i). Then (ii) follows from (i) by definition of Ω. By (i) if $\iota^{-1}(a) \cap \mathfrak{p}^*_{reg} \neq \emptyset$, one has

$$\begin{aligned} \text{index } \mathfrak{g}(a) &= \text{index } \mathfrak{p} - \text{codim}_{\mathfrak{a}}\mathfrak{g}.a, \\ &\geqslant \text{index } \mathfrak{p} - \text{index } \rho. \end{aligned}$$

By (ii) equality holds if $\iota^{-1}(a) \cap \Omega \neq \emptyset$. Hence (iii). □

2.9.7 Recall the definition of $\mathfrak{a}^*_r$ given in 2.9.6. Set $\mathfrak{a}^*_s = \mathfrak{a}^* \setminus \mathfrak{a}^*_r$. Since $\mathfrak{a}^*_r$ is open dense in $\mathfrak{a}^*$ one has $\text{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_s \geqslant 1$. Equality forces a similar conclusion to 2.6.19, namely an irreducible subvariety of $\mathfrak{a}^*_s$ of codimension 1 in $\mathfrak{a}^*$ is the zero set of a $\mathfrak{g}$ semi-invariant homogeneous element of $S(\mathfrak{a})$ dividing every maximal rank minor of the matrix describing the action of $\mathfrak{g}$ on $\mathfrak{a}$. In particular suppose that there are no $\mathfrak{g}$ semi-invariants in $S(\mathfrak{a})$. Then if there exists a pair $(h, y) \in \mathfrak{g} \times \mathfrak{a}^*_r$ with $[h, y] \in \mathbb{C}^* y$, then $\text{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_s \geqslant 2$. Moreover, $\text{codim}_{\mathfrak{p}^*} \iota^{-1}\mathfrak{a}^*_s = \text{codim}_{\mathfrak{a}^*} \mathfrak{a}^*_s$, so in the latter case it suffices to compute from the following to determine if $\mathfrak{p}$ is singular.

Corollary *For all $a \in \mathfrak{a}^*_r$ one has*

$$\dim(\mathfrak{p}^*_{sing} \cap \iota^{-1}(a)) = \dim \mathfrak{g}(a)^*_{sing}.$$

Proof It is enough to show that the reverse implications, to those given in 2.9.5, hold, under the hypothesis that $a \in \mathfrak{a}^*_r$. By the hypothesis $\text{codim}_{\mathfrak{a}^*} \mathfrak{g}.a = \text{index } \rho$. Suppose $g_0 \in \mathfrak{g}(a)^*_{reg}$. Then $\dim \mathfrak{g}(a)^{g_0} = \text{index } \mathfrak{g}(a)$. Substitution into 2.9.4 and 2.9.6(ii) gives index $\mathfrak{p} = \dim \mathfrak{p}^p$ and so $p \in \mathfrak{p}^*_{reg}$, as required. □

2.9.8 The condition $a \in \mathfrak{a}_r^* \Rightarrow \mathfrak{g}(a) = 0$, is independent of the choice of $a \in \mathfrak{a}_r^*$. Suppose it holds (which is often the case if $\dim \mathfrak{a}$ is large enough). Then by the corollary $\iota^{-1}(\mathfrak{a}_r^*) \subset \mathfrak{p}_{reg}^*$. Thus if in addition $\operatorname{codim}_{\mathfrak{a}^*} \mathfrak{a}_s^* \geq 2$, then $\mathfrak{p}$ is non-singular.

This result may also be used to prove that $\mathfrak{p}$ is non-singular even if the above conditions do not hold. For example the first condition can be replaced by $\mathfrak{g}(a)$ being commutative for all $a \in \mathfrak{a}_r^*$, for then $\mathfrak{g}(a)_{sing}^* = \emptyset$. An intermediate case is provided by the following example. For all $n \geqslant 1$, let $\mathfrak{p}_n$ denote the derived algebra of a (standard) parabolic subalgebra of $\mathfrak{sl}(n+1)$ of codimension n. Then $\mathfrak{p}_n$ is the semi-direct product of $\mathfrak{sl}(n)$ and an abelian ideal $\mathfrak{a}_n$ of dimension n on which $\mathfrak{sl}(n)$ acts by its standard representation. Observe that $\dim \mathfrak{p}_1 = 1$ and so index $\mathfrak{p}_1 = 1$, whilst $(\mathfrak{p}_1)_2^* = \emptyset$.

Lemma *For all $n \geqslant 2$ one has*

(i) index $\mathfrak{p}_n = 1$,
(ii) codim $(\mathfrak{p}_n)_{sing}^* = 2$.

Proof Both statements are proved by induction on n. Set $\mathfrak{g} = \mathfrak{sl}(n), \mathfrak{a} = a_n, \mathfrak{p} = \mathfrak{g} + \mathfrak{a}$. Notice that $\mathfrak{a}_r^* = \mathfrak{a}^* \setminus \{0\}$. Moreover, if $a \in \mathfrak{a}_r^*$, then $\operatorname{codim}_{\mathfrak{a}^*}(\mathfrak{g}.a) = 0$, whilst $\mathfrak{g}(a) \cong \mathfrak{p}_{n-1}$. Thus, by 2.9.6, we obtain
index $\mathfrak{p}_n =$ index $\mathfrak{p}_{n-1}$. Since index $\mathfrak{p}_1 = 1$, this gives (i).

For (ii) write $\mathfrak{p}_{sing}^*$ as the disjoint union of $\mathfrak{p}_{s,0}^* := \iota^{-1}(0) \cap \mathfrak{p}_{sing}^*$ and its complement

$$\mathfrak{p}_{s,1}^* := \bigsqcup_{a \in \mathfrak{a}^* \setminus \{0\}} \iota^{-1}(a) \cap \mathfrak{p}_{sing}^*.$$

Take $\xi \in \iota^{-1}(0) = \mathfrak{g}^*$. Then $\operatorname{codim}_{\mathfrak{p}^*}(\mathfrak{p}.\xi) \geqslant \operatorname{codim}_{\mathfrak{g}^*}(\mathfrak{g}.\xi) \geqslant \operatorname{rk} \mathfrak{g}$. Yet index $\mathfrak{p} = 1$. So if $n \geqslant 2$ we obtain $\iota^{-1}(0) \subset \mathfrak{p}_{sing}^*$, that is $\mathfrak{p}_{s,0}^* = \iota^{-1}(0)$ and hence

$$\operatorname{codim}_{\mathfrak{p}^*} \mathfrak{p}_{s,0}^* = n. \tag{$*$}$$

(Notice that here the reverse implication in 2.9.5 fails.)

Now suppose $\mathfrak{p}_{s,1}^*$ is non-empty and take $p \in \mathfrak{p}_{s,1}^*$. Denote $a := p|\mathfrak{a}$ and $g_0 = p|\mathfrak{g}(a)$. Then $\operatorname{codim}_{\mathfrak{a}^*}(\mathfrak{g}.a) = 0$ and so, by 2.9.4, we have $\dim \mathfrak{g}(a)^{g_0} = \dim \mathfrak{p}^p >$ index $\mathfrak{p} = 1$. Yet $\mathfrak{g}(a) = \mathfrak{p}_{n-1} = 0$, so our hypothesis forces $n > 2$. Thus for $n = 2$, $\mathfrak{p}_{s,1}^* = \emptyset$ and so the assertion follows from $(*)$.

Now suppose $n > 2$ and take $a \in \mathfrak{a}_r^*$. Then $\mathfrak{g}(a) = \mathfrak{p}_{n-1}$ and so $\operatorname{codim}_{\mathfrak{g}(a)^*} \mathfrak{g}(a)_{sing}^* = 2$, by the induction hypothesis. Hence $\dim(\iota^{-1}(a) \cap \mathfrak{p}_{sing}^*) = \dim p_{n-1} - 2$, by 2.9.7. Thus $\operatorname{codim}_{\mathfrak{p}^*} \mathfrak{p}_{s,1}^* = 2$. Combined with $(*)$, this gives (ii). $\square$

2.10. **Distinguished Orbits - Bala-Carter Theory.**

Let $\mathfrak{g}$ be a reductive Lie algebra and $x \in \mathfrak{g}$. Elashvili conjectured that index $\mathfrak{g}^x = \operatorname{rank} \mathfrak{g}$. Through Jordan decomposition we may assume x nilpotent and of course that $\mathfrak{g}$ is simple. Through the Killing form and Proposition 2.7.7 (say via the Vinberg inequality 2.8.2) we immediately obtain index $\mathfrak{g}^x \geq \operatorname{rank} \mathfrak{g}$. The opposite inequality is apparently much more difficult, although it may just be that we are not as smart as Vinberg. In any case

Yakimova [69] checked this last inequality in classical type. This is effected by taking a Jordan decomposition compatible with an invariant symmetric (for $\mathfrak{so}(n)$) or antisymmetric (for $\mathfrak{sp}(n)$) form on the underlying vector space. Here type A is straightforward, type C doable and types B, D, simply "épouvantable". In addition van den Graaf [23] has verified the conjecture for the exceptional Lie algebras by computer. More recently Charbonnel and Moreau [12] have reported a proof using much less computer aided computations. Indeed they reduce the question to the case of so-called rigid orbits. On the other Elashvili suggested it is possible to also reduce the question to the case of distinguished orbits. Since a distinguished orbit is always a Richardson orbit in a smaller Lie algebra, together this would give a computer free proof. Unfortunately we could not see how to achieve Elashvili's suggestion. However we describe below the theory of distinguished orbits. It allows one describe all the Dynkin data. Our treatment follows closely that given in [9].

2.10.1 Recall 2.3.7. Let $\mathfrak{g}$ be a semisimple Lie algebra and (x,h,y) an s-triple. Set $\mathfrak{s} = \mathbb{C}x \oplus \mathbb{C}h \oplus \mathbb{C}y$, which is isomorphic to $\mathfrak{sl}(2)$. By $\mathfrak{sl}(2)$ theory, the eigenvalues of $ad\, h$ on $\mathfrak{g}$ take integer values. Thus, if we set

$$\mathfrak{g}_i = \{z \in \mathfrak{g} | (ad\, h)z = iz\},\ \forall\ i \in \mathbb{Z}.$$

Then

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i,$$

is a grading of $\mathfrak{g}$ as a Lie algebra. The subalgebra

$$\mathfrak{p} := \bigoplus_{i \geqslant 0} \mathfrak{g}_i$$

is a parabolic subalgebra, called the Dynkin parabolic. Its Levi factor is $\mathfrak{g}_0$ and its nilradical is

$$\mathfrak{m} := \bigoplus_{i > 0} \mathfrak{g}_i.$$

By $\mathfrak{sl}(2)$ theory, $\mathfrak{p}^x = \mathfrak{g}^x$. (One may further prove that $P^x = G^x$ for the corresponding connected groups P, G. This is stronger if G^x is not connected.)

One calls (x,h,y) distinguished if $\mathfrak{p}^x = \mathfrak{m}^x$. Now $\mathfrak{g}_0^x = \mathfrak{g}^{\mathfrak{s}}$, by $\mathfrak{sl}(2)$ theory, and the latter is reductive. Thus (x,h,y) is distinguished if and only if $\mathfrak{g}^x$ is a nilpotent Lie algebra.

2.10.2 Suppose (x,h,y) is not distinguished. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}^{\mathfrak{s}}$ and consider $\mathfrak{r} := \mathfrak{g}^{\mathfrak{t}}$, which is again reductive. Thus $\mathfrak{r}$ is a direct sum of its centre $\mathfrak{z}$ and its derived algebra $\mathfrak{r}'$ which is semisimple. Since $\mathfrak{t} \neq 0$, it follows that $\text{rk}\ \mathfrak{r}' < \text{rk}\ \mathfrak{g}$. Since x, h, y commute with elements of $\mathfrak{g}^{\mathfrak{s}}$ and hence with $\mathfrak{t}$, we obtain $x, h, y \in \mathfrak{r}$. Moreover, $adx = adz + adx'$, with $z \in \mathfrak{z},\ x' \in \mathfrak{r}'$ is a Jordan decomposition of $adx \in \text{End}\ \mathfrak{g}$. Hence $z = 0$. Consequently x and similarly, y lie in $\mathfrak{r}'$. Hence so does h.

Lemma *(x,h,y) is distinguished in $\mathfrak{r}'$.*

Proof Set $\mathfrak{r}'_i = \mathfrak{r}' \cap \mathfrak{g}_i$, which is just the Dynkin grading of $\mathfrak{r}'$. It remains to show that $(\mathfrak{r}'_0)^x = 0$. Now $(\mathfrak{g}^\mathfrak{s})^\mathfrak{t} = \mathfrak{t}$, because $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}^\mathfrak{s}$. Then $(\mathfrak{r}'_0)^x = \mathfrak{r}' \cap \mathfrak{g}_0^x = \mathfrak{r}' \cap \mathfrak{g}^\mathfrak{s} = \mathfrak{r}' \cap (\mathfrak{g}^\mathfrak{s})^\mathfrak{t} = \mathfrak{r}' \cap \mathfrak{t} = \mathfrak{r}' \cap (\mathfrak{t} \cap \mathfrak{g}^\mathfrak{t}) = \mathfrak{r}' \cap \mathfrak{z} = 0$. □

Remark Similarly $\mathfrak{r}_0^x = \mathfrak{r} \cap \mathfrak{g}^\mathfrak{s} = \mathfrak{z}$.

2.10.3 The above result allows one to reduce the classification of nilpotent orbits to the distinguished ones. The Bala-Carter theory described below reduces the latter to an easily solved purely combinatorial problem.

2.10.4 Recall (2.3.11) that $\dim \mathfrak{g}^h \leqslant \dim \mathfrak{g}^x$ with equality if and only if all eigenvalues of $ad\, h$ on $\mathfrak{g}$ are even. In the latter case one calls (x, h, y) an even s-triple and Gx an even orbit. In terms of Dynkin data (2.3.11), this means that $h(\alpha) \in \{0, 2\}$, for all $\alpha \in \pi$ (instead of $h(\alpha) \in \{0, 1, 2\}$). A key tool in analyzing nilpotent orbits is provided by a theorem of Richardson. It asserts that any parabolic subgroup P has a dense orbit on the nilradical $\mathfrak{m}$ of its Lie algebra $\mathfrak{p}$, and that $\dim Gx = 2 \dim Px$. We shall not give the proof. However, we remark that it results from Bruhat decomposition, particularly that $B \backslash G / B$ is finite, where B is a Borel subgroup, the finiteness of $\mathscr{N}/G$, and finally using the Steinberg triple variety. In Lie algebraic terms, the first part of Richardson's theorem is equivalent to the following assertion. There exists $m \in \mathfrak{m}$ such that $[\mathfrak{p}, m] = \mathfrak{m}$. As observed by Jantzen, this already gives an easy proof of the following key result.

Lemma *A distinguished orbit is even.*

Proof Let $\mathfrak{p}$ be the Dyknin parabolic and $\mathfrak{m}$ its nilradical. Write m in the conclusion of Richardson's theorem as $m = m_1 + m_2 + \ldots$, with $m_i \in \mathfrak{g}_i$. Then $[\mathfrak{p}, m] = \mathfrak{m}$ implies

$$[\mathfrak{g}_0, m_1] = \mathfrak{g}_1,\ [\mathfrak{g}_1, m_1] + [\mathfrak{g}_0,\ m_1 + m_2] = \mathfrak{g}_1 + \mathfrak{g}_2.$$

If (x, h, y) is not even, then $\mathfrak{g}_1 \neq 0$, by $\mathfrak{sl}(2)$ theory. This forces $m_1 \neq 0$, which in turn implies $\dim[\mathfrak{g}_1, m_1] < \dim \mathfrak{g}_1$. Then the second relation above implies that

$$\dim \mathfrak{g}_1 + \dim \mathfrak{g}_0 > \dim \mathfrak{g}_1 + \dim \mathfrak{g}_2,$$

forcing $\dim \mathfrak{g}_0 > \dim \mathfrak{g}_2$. By $\mathfrak{sl}(2)$ theory, one has $[\mathfrak{g}_0, x] = \mathfrak{g}_2$ and so $\mathfrak{g}_0^x \neq 0$ and hence (x, h, y) is not distinguished. □

2.10.5 For the moment let h denote an element of $\mathfrak{h}$ satisfying $h(\alpha) = \{0, 2\},\ \forall\ \alpha \in \pi$. Any such function obtains from a standard parabolic subalgebra $\mathfrak{p}_{\pi'}$ by setting $h(\alpha) = 0,\ \forall\ \alpha \in \pi',\ h(\alpha) = 2,\ \forall\ \alpha \in \pi \setminus \pi'$.

Conversely, suppose h is as above and set $\Delta_i = \{\alpha \in \Delta | h(\alpha) = i\}$ with

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Delta_0} \mathfrak{g}_\alpha,\ \mathfrak{g}_i = \bigoplus_{\alpha \in \Delta_i} \mathfrak{g}_\alpha : i \neq 0.$$

Then

$$\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{2i},$$

and is a grading of $\mathfrak{g}$. As in 2.10.1, a grading defines a parabolic subalgebra $\mathfrak{p} = \oplus_{i\geqslant 0}\, \mathfrak{g}_{2i}$, with nilradical $\mathfrak{m} = \oplus_{i>0}\, \mathfrak{g}_{2i}$ and Levi factor $\mathfrak{g}_0$. Moreover, $\mathfrak{p} = \mathfrak{p}_{\pi'}$, where $\pi' = h^{-1}(0)$. Obviously each $\mathfrak{g}_{2i}$ is a $\mathfrak{g}_0$ module. Recall that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$. if $\alpha, \beta, \alpha+\beta$ are non-zero roots. Since π generates Δ^+ additively and lies in $\Delta_0 \cup \Delta_2$, it follows that $\mathfrak{g}_2$ (which is $\mathfrak{g}_0$ stable) generates $\mathfrak{m}$. In particular, the injection $\mathfrak{g}_2 \hookrightarrow \mathfrak{m}$ factors to an isomorphism $\mathfrak{g}_2 \tilde{\rightarrow} \mathfrak{m}/\mathfrak{m}'$. On the other hand, by the first part of the previous lemma (replacing 1 by 2) we see that Richardson's theorem implies that $\dim \mathfrak{g}_0 \geqslant \dim \mathfrak{g}_2$.

One says that a parabolic subalgebra $\mathfrak{p}$ with Levi decomposition $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ is distinguished if $\dim \mathfrak{r} = \dim \mathfrak{m}/\mathfrak{m}'$.

The previous analysis shows that the construction of all (standard) distinguished parabolics is the purely combinatorial problem of finding all h as above, satisfying

$$|\Delta_2| = |\Delta_0| + \dim \mathfrak{h}.$$

Moreover, we shall see in (2.10.9), that this problem is rather easy.

2.10.6 The Bala-Carter theory asserts that there is a natural one to one correspondence between distinguished orbits and distinguished parabolics, namely every distinguished parabolic is the Dynkin parabolic of a distinguished orbit. By 2.10.4, a distinguished orbit gives rise to a distinguished parabolic. Below, the converse is established. Since Dynkin data uniquely determines a nilpotent orbit, this will establish the correspondence.

2.10.7 If (x, h, y) in an even s-triple, then by definition of the Dynkin grading, one has $x \in \mathfrak{g}_2$. Now let $\mathfrak{p} = \mathfrak{r} \oplus \mathfrak{m}$ be the corresponding Dynkin parabolic. Then, recalling 2.10.4, one has $\dim \mathfrak{p}^x = \dim \mathfrak{g}^x \leq \dim \mathfrak{g}^h$ and so the inclusion $[\mathfrak{p}, x] \subset \mathfrak{m}$ is an equality. This translates to imply that the unique dense P orbit in $\mathfrak{m}$ must be Px and so in particular, meets $\mathfrak{g}_2$. A key fact is that this property holds automatically for a distinguished parabolic.

Proposition *Let $\mathfrak{p} = \mathfrak{p}_{\pi'}$ be a distinguished (standard) parabolic and Px its unique dense orbit in $\mathfrak{m}$. Then with respect to the grading defined by π', one has $Px \cap \mathfrak{g}_2 \neq \emptyset$.*

Proof Clearly $[\mathfrak{m}, x] \subset \mathfrak{m}'$. Yet

$$\begin{aligned}
\dim[\mathfrak{m}, x] &= \dim \mathfrak{m} - \dim \mathfrak{m}^x, \\
&\geqslant \dim \mathfrak{m} - \dim \mathfrak{p}^x, \\
&= 2 \dim \mathfrak{m} - \dim \mathfrak{p}, \text{ since } [\mathfrak{p}, x] = \mathfrak{m}, \\
&= \dim \mathfrak{m}', \text{ by 2.10.5 and since } \dim \mathfrak{g}_0 = \dim \mathfrak{g}_2.
\end{aligned}$$

Hence $[\mathfrak{m}, x] = \mathfrak{m}'$.

This translates to imply that Mx, which lies in the affine space $x + \mathfrak{m}'$, has the same dimension as the latter. Yet M is unipotent so Mx is closed by a theorem of Rosenlicht. Since $x + \mathfrak{m}'$ is irreducible, we conclude that $Mx = x + \mathfrak{m}'$. Yet $(x + \mathfrak{m}') \cap \mathfrak{g}_2 \neq 0$, hence the assertion. □

Remark We can write $x = x_2 + x_4 + \ldots$, with $x_2 \in \mathfrak{g}_{2:}$. Since $x_4 \in \mathfrak{m}'$, there exists $x_2' \in \mathfrak{m}$ such that $[x_2', x] = x_4 + x_6' + \ldots$. Yet $ad\, x_2'$ is a nilpotent endomorphism of $\mathfrak{g}$ and so exp $ad\, x_2'$ is a well-defined automorphism of $\mathfrak{g}$ (over characteristic zero) lying in M whilst (exp - $ad\, x_2'$)$x = x_2 + x_6'' + \ldots$. Eventually one obtains $m \in M$ such that $mx = x_2$. This avoids Rosenlicht's theorem.

2.10.8 Retain the hypotheses and notations of 2.10.7 and choose $x' \in Px \cap \mathfrak{g}_2$. Obviously $Px' = Px$. By the first part of Richardson's theorem, the inequality $\dim Gx' \leqslant \dim G/P + \dim Px'$, is an equality and of course, equals $2\dim \mathfrak{m}$. Thus $\dim \mathfrak{g}^{x'} = \dim \mathfrak{g}_0 = \dim \mathfrak{g}_2 = \dim \mathfrak{m}^{x'}$. Thus the inclusion $\mathfrak{m}^{x'} \subset \mathfrak{g}^{x'}$ is an equality, so x' is distinguished. Replace x' by x.

Set $n_{\mathfrak{g}}(x) = \{z \in \mathfrak{g} | [z, x] \in \mathbb{C}x\}$. Obviously $\mathfrak{g}^x \subset n_{\mathfrak{g}}(x)$ and $\dim n_{\mathfrak{g}}(x)/\mathfrak{g}^x = 1$. Since $x \in \mathfrak{g}_2$ we have $[h, x] = 2x$ and so we conclude that $n_{\mathfrak{g}}(x) = \mathbb{C}h \oplus \mathfrak{m}^x$. On the other hand, by the Jacobson-Morosov theorem (2.3.7), we can embed x in an s-triple (x, h', y') with $[h', x] = 2x$. Thus $h' - h \in \mathfrak{m}^x$. Yet M^x is unipotent and since $\mathfrak{m}^x \subset \oplus_{i>0}\mathfrak{g}_{2i}$ it follows that $(\mathfrak{m}^x)^h = 0$. Thus $\dim M^x h = \dim \mathfrak{m}^x$, whilst $M^x h \subset h + \mathfrak{m}^x$. Then by Rosenlicht's theorem (or as in the Remark following 2.10.7) we can find $m \in M^x$ such that $h' = mh$. Thus we can replace our s-triple by (x, h, y) with $y' = my$. This completes the proof of the Bala-Carter correspondence.

2.10.9 Let us describe the distinguished parabolic subalgebras for $\mathfrak{g}$ of classical type. In type A every nilpotent orbit is generated by an element of the form

$$x_{\pi'} := \sum_{\alpha \in \pi'} x_\alpha.$$

Such an element is defined in general and said to be of Bala-Carter type. A Bala-Carter orbit is regular and distinguished if $\pi' = \pi$. Otherwise $\mathfrak{h} \cap \mathfrak{g}^{x_{\pi'}}$ is generated by the fundamental weights $\varpi_\alpha : \alpha \in \pi \setminus \pi'$ and hence non-zero. Thus if $\pi' \subsetneq \pi$ such an orbit is not distinguished.

Now suppose $\pi = \{\alpha_i\}_{i=1}^{\ell}$ is of type B_ℓ with α_ℓ the short root. Choose positive integers $n_1, n_2, \ldots, n_{k-1}$ and n_k non-negative, whose sum is ℓ. For all $1 \leqslant r < k$, set $m_r := \sum_{i \leqslant r} n_i$. Define $\pi' \subset \pi$ by setting $\pi \setminus \pi' := \{\alpha_{m_r}\}_{r=1}^{k-1}$. Define $h \in \mathfrak{h}$ by

$$h(\alpha) = \begin{cases} 2 & : \quad \alpha \in \pi \setminus \pi' \\ 0 & : \quad \alpha \in \pi' \end{cases}$$

One easily checks that

$$\dim \mathfrak{g}_2 = \sum_{i=1}^{k-2} n_i n_{i+1} + n_{k-1}(2n_k + 1),$$

whilst

$$\dim \mathfrak{g}_0 = \sum_{i=1}^{k-1} n_i(n_i - 1) + 2n_k^2 + \ell.$$

Thus

$$\dim \mathfrak{g}_0 - \dim \mathfrak{g}_2 = \frac{1}{2}\sum_{i=1}^{k-2}(n_i - n_{i+1})^2 + \frac{1}{2}(n_{k-1} - 2n_k)^2 + n_k - n_{k-1} + \frac{1}{2}n_1^2$$

Now the first term is greater than $\frac{1}{2}(n_{k-1} - n_1)$ with equality if and only if the $n_i : i = 1, 2, \dots, k-1$ are increasing and by at most one at each step. The second term is greater than $\frac{1}{2}(n_{k-1} - 2n_k)$ with equality if and only if $n_{k-1} - 2n_k$ equals 0 or 1, the choice being dictated by whether n_{k-1} is even or odd. Thus the overall sum is greater than $\frac{1}{2}n_1(n_1 - 1)$ with equality exactly when the above two conditions hold. Thus the distinguished parabolics corresponds exactly to the choices $n_1 = 1,\ n_{i+1} = n_i$ or $n_i + 1$, for $i = 1, 2, \dots, k-1$ and $n_k = [\frac{n_{k-1}}{2}]$. The regular orbit corresponds to the choice $n_i = 1 : i = 1, 2, \dots, k-1,\ n_k = 0$.

The result for D_n is similar. The result for C_n is slightly different; but the proof is very similar.

The complete set of distinguished parabolics in all types was computed by P.Bala and R.W. Carter. The list can be found in [9, pp. 174-177].

2.10.10 Suppose $\pi = \{\alpha_i\}_{i=1}^{\ell}$ is the set of roots of a simple Lie algebra $\mathfrak{g}$ with unique highest root β. Set $\alpha_0 = -\beta$. Then $\pi_0 := \{\alpha_i\}_{i=0}^{\ell}$ is the set of simple roots of the affinisation of $\mathfrak{g}$. Of course, as described above, these roots are not linearly independent, though any proper subset π' of π_0 does have this property. In particular, if $|\pi'| = |\pi|$ then the element

$$\sum_{\alpha \in \pi'} x_\alpha \qquad (*)$$

has a chance of being distinguished. For example, the unique non-regular distinguished orbit in type C_2 and type G_2 occurs in this fashion. However, in type D_4 such an orbit is either regular or not distinguished. Moreover in type D_4 apart from the regular orbit, just the subregular orbit is distinguished. The latter possesses a representative of the form

$$x = \sum_{\alpha \in S} x_\alpha.$$

For example, we can take $S = \{\alpha_1, \alpha_3, \alpha_4, \alpha_1 + \alpha_2\}$ in the Bourbaki convention. In general one can always write a distinguished element as the sum of rank $\mathfrak{g}$ root vectors with roots spanning $\mathfrak{h}^*$ [11, Section 1].

2.10.11 Let us briefly describe how to obtain all Dynkin data from those of distinguished orbits. Recall the notation of 2.10.2 and let (x, h, y) be a distinguished s-triple in $\mathfrak{r}'$. Since $\mathfrak{r} = \mathfrak{g}^t$, by choosing t dominant (and correspondingly conjugating $\mathfrak{r}$) we can assume that $\mathfrak{r}$ is a Levi factor $\mathfrak{g}_{\pi'}$ of some standard parabolic. Then $h \in \mathfrak{g}'_{\pi'}$ is a sum of coroot vectors $\alpha^\vee : \alpha \in \pi'$ and so uniquely determined by its values on π'. Through W we may conjugate h into a unique dominant element h_d. Then the required Dynkin data of Gx is simply $\{h_d(\alpha)\}_{\alpha\in\pi}$. For example, consider $\mathfrak{g}$ of type A_3 with $\pi = \{\alpha_1, \alpha_3, \alpha_3\}$. Then regular orbit in $\mathfrak{g}'_{\{\alpha_1\}}$ is given by $h = \alpha_1^\vee$ and so $\{h_d(\alpha_i)\}_{i=1}^3 = \{1, 0, 1\}$. For the regular orbit in $\mathfrak{g}'_{\{\alpha_1,\alpha_2\}}$

one has $h = 2(\alpha_1^\vee + \alpha_2^\vee)$ and so $\{h_d(\alpha_i)\}_{i=1}^3 = \{2, 0, 2\}$. Finally, for the regular orbit in $\mathfrak{g}'_{\{\alpha_1,\alpha_3\}}$ one obtains $\{h_d(\alpha_i)\}_{i=1}^3 = \{0, 2, 0\}$. For the regular (resp. zero) orbit, the Dynkin data is $\{2, 2, 2\}$ (resp. $\{0, 0, 0\}$). This describes all possible Dynkin data in type A_3. All possible Dynkin data for all $\mathfrak{g}$ simple is listed by Carter [9, pp. 418-433].

2.10.12 The signficance of Dynkin data is rather mysterious. However, we do have the following observation of G. Lusztig and N. Spaltenstein which is easily checked. Let $Gx \subset \mathfrak{g}_\pi$ be a nilpotent orbit. Retain only the subset π' of π being the inverse image of $\{0, 1\}$. *Assume* that the resulting data on π' is the Dynkin data of a nilpotent element x' of the corresponding Levi factor $\mathfrak{g}_{\pi'}$. Then x is "induced" from x', that is to say it is the generator of the unique dense orbit in $G(x' + \mathfrak{m})$, where $\mathfrak{m}$ is the nilradical of the parabolic with Levi factor $\mathfrak{g}_{\pi'}$. A nilpotent orbit of a (Levi factor of $\mathfrak{g}$) is said to be *rigid* if it cannot be induced from a smaller Levi factor. The subset of the Dynkin data corresponding to the set of rigid orbits has been described, though there does not seem to be an easy or elegant way to do this as for distinguished orbits. Only the zero orbit is rigid in type A and more generally, there are no distinguished orbits which are rigid, as a consequence (of an easy case) of the Lusztig-Spaltenstein [50] result above. If $x' \in \mathfrak{g}_{\pi'}$ generates a rigid orbit, then it defines a sheet $\mathscr{S}$ in $\mathfrak{g}_\pi$ in the sense of 2.6.15. Let $\mathfrak{z}$ be the centre of $\mathfrak{p}_{\pi'}$. Then $\mathscr{S}$ is the union of all orbits of maximal dimension in $G(x' + \mathfrak{m} + \mathfrak{z})$. It contains the induced orbit as its unique nilpotent orbit. All sheets are so obtained.

Outside type A, sheets may intersect non-trivially. If $W_\mathfrak{z}$ is the normalizer of $\mathfrak{z}$ in the Weyl group, then the natural map $\mathfrak{z} \to \mathscr{S}/G$ factors to a homeomorphism of $z/W_\mathfrak{z} \tilde{\to} \mathscr{S}/G$. (For more details see [5] and [6]). Sheets are important to the Borho-Dixmier programme of studying the prime spectrum Spec $U(\mathfrak{g})$ of the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$. For example, there is a decomposition of Spec $U(\mathfrak{g})$ into sheets [1] which are further parameterized by a positive integer (the Goldie rank). There are finitely many sheets up to a given Goldie rank [1, Prop. 6.13]. In particular there are finitely many Goldie rank one sheets. It would be of some interest to classify the latter. Their exact relation with the finitely many (though generally less) sheets in $\mathfrak{g}$ (more properly in $\mathfrak{g}^*$) is an important open question [1, Conj. 7.11].

Notice that for the sheet $\mathscr{S}$ consisting of all regular co-adjoint orbits on has $\mathfrak{h}/W \tilde{\to} \mathscr{S}/G$. However, in this case, the Kostant slice better describes $\mathscr{S}/G$ as $x + \mathfrak{g}^y$. A similar description is not known for the remaining sheets and must fail outside type A since $\mathfrak{z}/W_\mathfrak{g}$ need not be an affine space [7].

A.1 Let $\mathfrak{a}$ be a finite dimensional Lie algebra. Given an $\mathfrak{a}$ module M, set

$$F_a(M) := \{m \in M \mid \dim U(\mathfrak{a})m < \infty\}.$$

For all $\lambda \in (\mathfrak{a}/[\mathfrak{a}, \mathfrak{a}])^*$, set

$$M_\lambda = \{m \in M | am = \lambda(a)m, \ \forall\ a \in \mathfrak{a}\}.$$

Let $D(\mathfrak{a})$ be the multiplicative set generated by the non-zero elements of

$$\bigcup_{\lambda\in(\mathfrak{a}/[\mathfrak{a},\mathfrak{a}])^*} S(\mathfrak{a})_\lambda.$$

Lemma $F_{\mathfrak{a}}(\mathrm{F}\mathit{ract}\ S(\mathfrak{a})) = D(\mathfrak{a})^{-1}S(\mathfrak{a})$.

Proof The inclusion $\supset$ is immediate. For the reverse inclusion, let $\widehat{\mathfrak{a}}$ denote the almost ad-algebraic hull of $\mathfrak{a}$. Any $ad\,\mathfrak{a}$ module is automatically an $ad\,\widehat{\mathfrak{a}}$ module and we can write $\widehat{\mathfrak{a}} = \mathfrak{r}\oplus\mathfrak{n}$, where $ad\,\mathfrak{r}$ acts reductively and $ad\,\mathfrak{n}$ acts nilpotently. Moreover $\mathfrak{r} = \mathfrak{s}\oplus\mathfrak{z}$ with $\mathfrak{z}$ the centre of $\mathfrak{r}$ and $\mathfrak{s}$ semisimple. Let Δ be the set non-zero roots of $\mathfrak{s}$ with respect to a Cartan subalgebra. Given $x\in\widehat{\mathfrak{a}}$, write $X = ad\,x$. Let $M\subset F_{\mathfrak{a}}$ (Fract $S(\mathfrak{a})$) be a finite dimensional submodule.

Set $I = \{a\in S(\mathfrak{a})|aM\subset S(\mathfrak{a})\}$. An element of I is a common denominator of the elements of M, which we simply call a denominator of M. We must show that $I\cap D(\mathfrak{a})\neq\emptyset$. Since M is a finite sum of its cyclic submodules we can assume M cyclic.

Step 1 Since M is finite dimensional, I is non-zero. Again I is an ideal of $S(\mathfrak{a})$ and is $ad\,\widehat{\mathfrak{a}}$ stable, since M is an $ad\,\widehat{\mathfrak{a}}$ module. Since $\mathfrak{n}$ is a finite dimensional nilpotent ideal of $\widehat{\mathfrak{a}}$ acting by locally nilpotent derivations, one obtains $I^{\mathfrak{n}}\neq 0$. Thus we can choose a denominator of M to lie in $J := I^{\mathfrak{n}}$. Let K denote the set of $\mathfrak{z}$ semi-invariant elements of J.

Step 2 Now suppose that c is an X eigenvector, so then c divides Xc. Decompose c as a product of its irreducible factors c_i with appropriate multiplicites. Since $S(\mathfrak{a})$ is factorial it follows that c_i divides Xc_i. Yet deg $Xc_i\leqslant$ deg c_i, so c_i is again an X eigenvector. Moreover since the base field is of characteristic zero if X annihilates c, then it annihilates c_i. Notice that this also holds if X is replaced by a collection of elements.

Step 3 Let ξ be a cyclic generator for M. We can write $\xi = a^{-1}b$. Then $X\xi = -a^{-2}(Xa)b + a^{-1}(Xb)$. Repeating this argument and using the finite dimensionality of M, shows that there exists $n\in\mathbb{N}$ such that $a^n\in I$.

Step 4 Take $X = ad\,x : x\in\widehat{\mathfrak{a}}$. Suppose $\xi\in$ Fract $S(\mathfrak{a})$ satisfies $X\xi = \mu\xi$, for some μ scalar. Write $\xi = a^{-1}b$, with a,b coprime. Then $a(Xb) = (Xa)b + \mu ab$ and since $S(\mathfrak{a})$ is factorial, a divides Xa. Then as in step 2, we obtain $Xa = \lambda a$. In particular, $\lambda = 0$ if X is nilpotent.

Step 5 Let S be a simple $\mathfrak{r}$ submodule of M. Since $\mathfrak{z}$ acts by scalars on S, it follows from steps 1 and 4 that we can choose a denominator of S to be a $\mathfrak{z}$ semi-invariant element of J.

Fix $\alpha\in\Delta$ and set $X = ad\,x_\alpha$. Fix a cyclic generator $\xi\in S$, satisfying $X\xi = 0$. By step 5 we can write $\xi = a^{-1}b$ with $a\in K$. Eliminating common factors we can assume that a,b are coprime with a being $\mathfrak{z}+\mathfrak{n}$ semi-invariant. Then by step 4, one has $Xa = 0$. Then by step 3, S admits a denominator which is annihilated by X and is $\mathfrak{z}+\mathfrak{n}$ semi-invariant.

Since $\alpha \in \Delta$ is arbitrary, when we write $\xi \in S$ as $a^{-1}b$, with a, b coprime, it follows from step 2, that a is $\mathfrak{a}$ semi-invariant, that is lies in $D(\mathfrak{a})$.

Since $\mathfrak{r}$ is reductive, M is a direct sum of its simple submodules, so this completes the proof. $\square$

A.2 We may now obtain Dixmier's result mentioned on 2.5.1.

Take $\mathfrak{g}, \mathfrak{p}, \widetilde{V}$ as in 2.5.1, and recall that $\mathfrak{p} \oplus \widetilde{V} = \mathfrak{g}$. Consequently, $S(\mathfrak{g}) \subset S(\mathfrak{p}) \otimes S(\widetilde{V}) \subset (\text{Fract } S(\mathfrak{p}))Y(\mathfrak{g})$ by 2.5.1 $(*)$, and so $D(\mathfrak{p})^{-1}S(\mathfrak{g}) \subset (\text{Fract } S(\mathfrak{p}))Y(\mathfrak{g})$. Now $ad\,\mathfrak{p}$ acts locally finitely on $S(\mathfrak{g})$ and hence locally finitely on the left hand side. On the other hand, the multiplication map Fract $S(\mathfrak{p}) \otimes Y(\mathfrak{g}) \to (\text{Fract } S(\mathfrak{p}))Y(\mathfrak{g})$ is injective (via the discussion in 2.5.1) and so $F_{\mathfrak{p}}(\text{Fract } S(\mathfrak{p})Y(\mathfrak{g})) = F_{\mathfrak{p}}(\text{Fract } S(\mathfrak{p}))Y(\mathfrak{g}) = (D(\mathfrak{p})^{-1}S(\mathfrak{p}))Y(\mathfrak{g}) \subset D(\mathfrak{p})^{-1}S(\mathfrak{g})$. Recalling that $D(\mathfrak{p})^{-1} = \bigcup_{n\in\mathbb{N}} d^n$, we can write $D(\mathfrak{p})^{-1}S(\mathfrak{g})$ simply as $S(\mathfrak{g})_d$. We have proved the

Corollary (Dixmier)

$$S(\mathfrak{g})_d = S(\mathfrak{p})_d Y(\mathfrak{g}).$$

A.3 A more rapid though less elementary proof of A.1 obtains from the following result of Dixmier, Duflo and Vergne [16].

Theorem *Let $\mathfrak{a}$ be a finite dimensional Lie algebra. Any non-zero ad $\mathfrak{a}$ invariant ideal I of $S(\mathfrak{a})$ contains a non-zero semi-invariant element.*

Remark. Let M be a finite dimensional ad $\mathfrak{a}$ invariant subspace of $F_{\mathfrak{a}}(\mathrm{F}ract\ S(\mathfrak{a}))$. Observe that the set $\{a \in S(\mathfrak{a})\ aM \subset S(\mathfrak{a})\}$ is a non-zero ad $\mathfrak{a}$ invariant ideal I of $S(\mathfrak{a})$¿ Apply the conclusion of the theorem to I.

A.4 Actually, Dixmier also proved an enveloping algebra version of A.2 basically by explicit computation. One can obtain this latter result of Dixmier as above using the following result of C. Moeglin [54].

Theorem *Let $\mathfrak{a}$ be a finite dimensional Lie algebra. Any non-zero two-sided ideal I of $U(\mathfrak{a})$ meets its semi-centre $Sz(\mathfrak{a})$.*

Moeglin's proof of this theorem uses Duflo's classification of minimal primitive ideals (see [15, Chap. 10] and as a consequence, is rather long and complicated. One might hope to shorten it using A.3 but this is not immediate. The difficulty is the following.

Let I be as in the hypothesis of the Moeglin's theorem. Define $\widehat{\mathfrak{a}}$ as in A.1. Just as in Step 1 of A.1, it follows that $I^{\mathfrak{n}} \neq 0$. Since $\mathfrak{r}$ is reductive, $I^{\mathfrak{n}}$ and $gr_{\mathscr{F}} I^{\mathfrak{n}}$ are isomorphic as $\mathfrak{r}$ modules. Thus, to prove Moeglin's theorem, it suffices to show that $gr_{\mathscr{F}} I^{\mathfrak{n}}$ admits a one-dimensional $\mathfrak{r}$ module. By A.3 it follows that $(gr_{\mathscr{F}} I)^{\mathfrak{n}}$ admits a one dimensional $\mathfrak{r}$ module. With respect to the canonical filtration $\mathscr{F}$ of $U(\mathfrak{a})$, one has $gr_{\mathscr{F}} I^{\mathfrak{n}} \subset (gr_{\mathscr{F}} I)^{\mathfrak{n}}$. However this inclusion is generally not an equality.

Moeglin used the above theorem to prove a further important result, namely that every completely prime primitive ideal of $U(\mathfrak{sl}(n))$ is induced [55]. This fails for $\mathfrak{g}$ semisimple in general, yet can be replaced by the conjecture [1, 7.13] that if a completely prime ideal is rigid (that is not induced) then it is completely rigid (not minimal over an induced ideal). The point is that completely rigid primitive ideals are relatively easy to classify. They come in "coherent families" [1, Lemma 6.4] and such families are identifiable by the simple W module M spanned by their associated Goldie rank polynomials. Indeed let $W_{\pi'}$ be the Weyl subgroup generated by a subset π' of the set π of simple roots. Then the family consists of completely rigid ideals if and only if M cannot be generated by a simple $W_{\pi'}$ module with $\pi' \subsetneq \pi$.

A.5 Fauquant-Millet [20] obtained a quantized enveloping algebra version of A.2 again by explicit computation. To avoid this as above one would need the following genealization of Moeglin's theorem which we state as a conjecture.

Let H be a Hopf algebra. In particular H admits a coproduct Δ and an antipode σ. We recall that Δ is an algebra homomorphism of H to $H \otimes H$ and we write $\Delta(h) = \sum h_{(1)} \otimes h_{(2)}$ using the Sweedler sum convention. Again σ is an algebra endomorphism of H. Then one may define an adjoint action of H on itself by

$$(\text{ad } h)h' := \sum h_{(1)} h' \sigma(h_{(2)}), \forall h, h' \in H.$$

One may check for example the ad-invariant elements of H form its centre $Z(H)$ and that a left or right ideal of I is two sided if and only if it is ad-invariant.

Conjecture *Let H be a Hopf algebra. Any non-zero two-sided ideal I of H admits a non-zero ad H eigenvector.*

Remark. M. Gorelik has pointed out (see [39, 7.12]) that this holds for the Drinfeld-Jimbo quantized enveloping algebra $U_q(\mathfrak{g})$. For the result of Fauquant-Millet mentioned above we would like to know that this also holds for the quantized enveloping algebra $U_q(\mathfrak{p})$ of a parabolic subalgebra $\mathfrak{p}$, specifically the parabolic described in A.2.

A.6 Take $\mathfrak{p} = \mathfrak{g} \oplus \mathfrak{a}$ as in 2.9.6. In principle one may use the Rais theorem to compute both index $\mathfrak{p}$ and codim $\mathfrak{p}_{sing}$. However this may not be too easy in practice. In [3, Corollary to Theorem 1.3] Bolsinov claims that codim $\mathfrak{p}_{sing} \geq 2$ if $\mathfrak{g}$ is a classical simple Lie algebra and $\mathfrak{a}$ is a simple $\mathfrak{g}$ module. This seems to have been verified case by case along the lines indicated in 2.9.8.

Non-singularity can fail if $\mathfrak{a}$ is not a simple $\mathfrak{g}$ module. The example below was obtained following a suggestion of V. Kac.

Take $\mathfrak{g}$ to be $\mathfrak{sl}(n)$ and $\mathfrak{a}$ to be n copies of its fundamental module $\mathbb{C}^n$. The action of $\mathfrak{g}$ on $S(\mathfrak{a})$ admits an invariant d of degree n which can be viewed as the determinant of the matrix formed by the n copies of the fundamental module $\mathbb{C}^n$.

Adopt the notation of 2.9.6.

Lemma.
(i) index $\mathfrak{p} = 1$.
(ii) $S(\mathfrak{p}) = \mathbb{C}[d]$.
(iii) $\operatorname{codim} \mathfrak{p}_{sing} = 1$.
(iv) the fundamental semi-invariant of $\mathfrak{p}$ *is* d^{n-1}.

Proof. Take $a \in \mathfrak{a}^*$ in general position. One checks that $\mathrm{Stab}_{\mathfrak{g}} a = \{0\}$. Indeed in the matrix presentation of $\mathfrak{a}$ indicated above, it is enough that a to be non-zero on all diagonal entries for it to admit a trivial stabilizer. Thus the $SL(n)$ orbits in $\mathfrak{a}_r^*$ have codimension one. Consequently index $\mathfrak{p}$ = index $\rho = 1$, by 2.9.6 (iii).

One obtains GKdim $Y(\mathfrak{p}) \leq 1$ from (i). Thus the elements of $Y(\mathfrak{p})$ are algebraic over $\mathbb{C}[d]$. Taking account of the $\mathbb{C}^*$ action coming from the action of $GL(n)$ on $\mathfrak{a}$ gives (ii).

Let $p_{\mathfrak{p}}$ denote the fundamental semi-invariant of $\mathfrak{p}$. Since $\mathfrak{p}$ is semi-invariant free the Ooms-van den Bergh sum rule [57, Prop. 1.4] gives deg $p_{\mathfrak{p}} = c(\mathfrak{p}) - n = n(n-1)$, by (i) and (ii). This gives (iii) and since $p_{\mathfrak{p}}$ is homogeneous of degree n we also obtain (iv). $\square$

Remark. At least for $n = 2$ it is easy enough to check (iv) by explicit computation.

A.7 Let $\mathfrak{a}$ be a finite dimensional Lie algebra. Fix a basis $\{x_i\}_{i=1}^n$ for $\mathfrak{a}$ and let $\{\xi_i\}_{i=1}^n$ be a dual basis for $\mathfrak{a}^*$. Define the structure constants $\{c_{i,j}^k : i, j, k = 1, 2, \dots, n\}$ for $\mathfrak{a}$ by

$$[x_i, x_j] = \sum_{k=1}^n c_{i,j}^k x_k.$$

Let $\mathfrak{a}$ act on $\wedge^* \mathfrak{a}^*$ through transposition and the Leibnitz rule. Set

$$\Psi(\xi_k) = \sum_{1 \leq i < j \leq n} c_{i,j}^k (\xi_i \wedge \xi_j).$$

Observe that

$$\sum_{k=1}^n x_k \otimes \Psi(\xi_k) = \sum_{i,j=1}^n x_i x_j \otimes (\xi_i \wedge \xi_j), \qquad (*)$$

where the left hand factors are taken in the enveloping algebra.

From the Jacobi identities it follows that Ψ is an $\mathfrak{a}$ module map and hence extends to a homomorphism of $S(\mathfrak{a}^*) \to \wedge^* \mathfrak{a}^*$ of $\mathfrak{a}$ algebras.

A.8 Let $\mathfrak{g}$ be a semisimple Lie algebra. In the notation of 2.2.1, we define $I_{m,\lambda}(x) = \mathrm{tr}_V(\lambda) x^m, \forall m \in \mathbb{N}, \lambda \in P^+$. The Weyl character formula implies that the $\psi(I_{m,\lambda}) : m \in \mathbb{N}, \lambda \in P^+$ span $S(\mathfrak{h}^*)^W$. Then by the Chevalley theorem (2.2.1) it follows that the $I_{m,\lambda} : m \in \mathbb{N}, \lambda \in P^+$ span $S(\mathfrak{g})^G$. Let $S(\mathfrak{g})_+$ denote the augmentation ideal of $S(\mathfrak{g})$. Apply the notation of A.7 to $\mathfrak{g}$.

Lemma. $\Psi(S(\mathfrak{g})_+^G) = 0$.

Proof. Written out in terms of bases one has

$$I_{m,\lambda} = \sum_{i_1,i_2,\dots,i_m=1}^{n} (\mathrm{tr}_{V(\lambda)} x_{i_1} \dots x_{i_m}) \xi_{i_1} \dots \xi_{i_m}.$$

From the definition of Ψ, the fact that it is an algebra map and A.7 $(*)$ we obtain

$$\Psi(I_{m,\lambda}) = \sum_{i_1,i_2,\dots,i_{2m}=1}^{n} (\mathrm{tr}_{V(\lambda)} x_{i_1} \dots x_{i_{2m}}) \xi_{i_1} \wedge \dots \wedge \xi_{i_{2m}}.$$

Assume $m \geq 1$. Then the right hand side above vanishes because trace is cyclically symmetric whereas the wedge product of an even number of factors is cyclically antisymmetric. Hence the assertion of the lemma.

□

A.9 In the above identify ξ_i with $\partial/\partial x_i$ and hence $S(\mathfrak{g}^*)$ with the algebra of constant coefficient differential operators acting on $S(\mathfrak{g})$. Given $f \in S(\mathfrak{g})$ let $f(0)$ denote its value at $0 \in \mathfrak{g}^*$. Then given $\partial \in S(\mathfrak{g}^*)$ set

$$\langle \partial, f \rangle = (\partial f)(0).$$

The subspace $H \subset S(\mathfrak{g})$ of harmonic functions is defined by

$$H := \{a \in S(\mathfrak{g}) | \partial a = 0, \forall \partial \in S(\mathfrak{g}^*)^G_+\}.$$

Lemma. *H is the orthogonal of $S(\mathfrak{g}^*)S(\mathfrak{g}^*)^G_+$ in $S(\mathfrak{g}^*)$ with respect to $\langle , \rangle$.*

Proof. It is immediate that

$$\langle S(\mathfrak{g}^*)S(\mathfrak{g}^*)^G_+, H \rangle = 0.$$

On the other hand if $h \notin H$, then there exists $\partial \in S(\mathfrak{g}^*)^G_+$ such that $\partial h \neq 0$ and so there exists $\partial' \in S(\mathfrak{g}^*)$ such that $\partial' \partial h$ is a non-zero scalar necessarily equal to $\langle \partial' \partial, h \rangle$, which is therefore non-zero, as required.

□

A.10 Let S_m denote the permutation group on m letters. Identify $\wedge^m \mathfrak{g}^*$ with $(\wedge^m \mathfrak{g})^*$ through the pairing

$$\langle (\xi_{j_1} \wedge \dots \wedge \xi_{j_m}), (x_{i_1} \wedge \dots \wedge x_{i_m}) \rangle = \sum_{\sigma \in S_m} \frac{sg(\sigma)}{m!} \prod_{r=1}^{m} \langle \xi_{j_r}, x_{\sigma(i_r)} \rangle.$$

Identify $S(\mathfrak{g})$ with the graded dual of $S(\mathfrak{g}^*)$. Let $\Psi^* : \wedge(\mathfrak{g}) \to S(\mathfrak{g})$ be defined by transport of structure.

Lemma. *Im $\Psi^* \subset H$.*

Proof. Since Ψ is an algebra map this follows from Lemmas A.8 and A.9. □

Remark. This result is due to Kostant [46, Thm. 3.4], though with a different proof. Kostant has asked if one can determine the precise image.

A.11 Define $\omega \in \mathfrak{g} \otimes \wedge^2\mathfrak{g}$ by

$$\omega = \sum_{1 \leq i < j \leq n} [x_i, x_j] \otimes (\xi_i \wedge \xi_j) = \sum_{k=1}^{n} x_k \Psi(\xi_k).$$

Lemma. *For all $m \in \mathbb{N}^+$, one has*

$$\omega^m = \sum_{1 \leq i_1 < i_2 < \ldots < i_{2m} \leq n} \Psi^*(x_{i_1} \wedge \ldots \wedge x_{i_{2m}}) \otimes \xi_{i_1} \wedge \ldots \wedge \xi_{i_{2m}}.$$

In particular

$$\omega^m \in H \otimes \wedge^{2m}\mathfrak{g}^*.$$

Proof. One has

$$\begin{aligned} \langle \omega^m, x_{i_1} \wedge \ldots \wedge x_{i_{2m}} \rangle &= \textstyle\sum_{j_1 \ldots j_m = 1}^{n} \langle \Psi(\xi_{j_1} \ldots \xi_{j_m}), x_{i_1} \wedge \ldots \wedge x_{i_{2m}} \rangle x_{j_1} \ldots x_{j_m}, \\ &= \textstyle\sum_{j_1 \ldots j_m = 1}^{n} \langle \xi_{j_1} \ldots \xi_{j_m}, \Psi^*(x_{i_1} \wedge \ldots \wedge x_{i_{2m}} \rangle x_{j_1} \ldots x_{j_m}, \\ &= \Psi^*(x_{i_1} \wedge \ldots \wedge x_{i_{2m}}). \end{aligned}$$

Hence the assertion. The last part follows from A.10. $\square$

A.12 It is clear from 2.2.2.7 $(*)$, that a non-scalar element of cannot divide an element of H. In view of A.11, it follows that the fundamental semi-invariant of a semisimple Lie algebra is scalar. Of course we knew this already but the above proof has in principle more mileage to be extracted out of it. For example take $\mathfrak{p}$ as in A.6 with $\mathfrak{g}$ semisimple and $\mathfrak{a}$ a finite dimensional $\mathfrak{g}$ module. One possibility to show $\operatorname{codim}_{\mathfrak{a}^*} \mathfrak{a}_s^* \neq 1$ is via an adapted pair as described in 2.9.7. An alternative is to show that the largest common factor of the largest non-vanishing minors of the matrix with entries describing the action of $\mathfrak{g}$ on $\mathfrak{a}$, is scalar. Now exactly as in the case of the fundamental semi-invariant, such a factor must be a semi-invariant for the action of G on $S(\mathfrak{a})$.

Now suppose $\mathfrak{a}$ is the adjoint representation. Then our previous result implies that the above semi-invariant is indeed scalar and so $\operatorname{codim}_{\mathfrak{a}^*} \mathfrak{a}_s^* \geq 2$. (By contrast in A.6 we exhibited an example for which this semi-invariant was invariant and non-scalar.)

To show that $\mathfrak{p}$ is non-singular it remains to show (cf 2.9.7) that $\iota^{-1}(\mathfrak{a}_r^*) \subset \mathfrak{p}_{reg}^*$.

By our hypothesis there exists a $\mathfrak{g}$ module isomorphism $\iota : \mathfrak{g}^* \to \mathfrak{a}^*$.

Adopt the notation of 2.9.6. Take $x \in \mathfrak{g}_{reg}^*$ and set $a := \iota(x)$. Then $\operatorname{Stab}_{\mathfrak{g}} x = \operatorname{Stab}_{\mathfrak{g}} a =: \mathfrak{g}(a)$ and has dimension $\operatorname{rank} \mathfrak{g}$. Then index $\mathfrak{g}(a) = \operatorname{rank} \mathfrak{g}$, by 2.7.7. We conclude that $\mathfrak{g}(a)$ is commutative and so $\mathfrak{p}$ is non-singular by the discussion in the first part of 2.9.8.

We remark that index $\rho = \operatorname{rank} \mathfrak{g}$. In particular it follows from 2.9.6 (ii) that index $\mathfrak{p} = 2 \operatorname{rank} \mathfrak{g}$.

References

[1] W. Borho and A. Joseph, Sheets and topology of primitive spectra for semisimple Lie algebras. J. Algebra 244 (2001), no. 1, 76–167. Corrigendum, J. Algebra 259 (2003), no. 1, 310–311.

[2] J. Bernstein and V. Lunts, A simple proof of Kostant's theorem that $U(\mathfrak{g})$ is free over its center. Amer. J. Math. 118 (1996), no. 5, 979-987.

[3] A. V. Bolsinov, Compatible Poisson brackets on Lie algebras and the completeness of families of functions in involution. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 55 (1991), no. 1, 68-92; translation in Math. USSR-Izv. 38 (1992), no. 1, 69-90.

[4] A. V. Bolsinov, Commutative families of functions related to consistent Poisson brackets. Acta Appl. Math. 24 (1991), no. 3, 253-274.

[5] W. Borho, Über Schichten halbeinfacher Lie-Algebren. (German) [On sheets of semisimple Lie algebras] Invent. Math. 65 (1981/82), no. 2, 283–317.

[6] W. Borho and H. Kraft, Hanspeter Über Bahnen und deren Deformationen bei linearen Aktionen reduktiver Gruppen.(German) Comment. Math. Helv. 54 (1979), no. 1, 61-104.

[7] N. Bourbaki, Éléments de mathématique. (French) [Elements of mathematics] Groupes et algèbres de Lie. Chapitres 4, 5 et 6. [Lie groups and Lie algebras. Chapters 4, 5 and 6] Masson, Paris, 1981.

[8] J. Brown and J. Brundan, Elementary invariants for centralizers of nilpotent matrices. J. Aust. Math. Soc. 86 (2009), no. 1, 1-15.

[9] R. W. Carter, Simple groups of Lie type. Pure and Applied Mathematics, Vol. 28. John Wiley and Sons, London-New York-Sydney, 1972.

[10] R. W. Carter, Finite groups of Lie type. Conjugacy classes and complex characters. Pure and Applied Mathematics (New York). A Wiley-Interscience Publication. John Wiley and Sons, Inc., New York, 1985.

[11] R. W. Carter and G. B. Elkington, A note on the parametrization of conjugacy classes. J. Algebra 20 (1972), 350-354.

[12] J.-Y. Charbonnel and A. Moreau, The index of centralizers of elements of reductive Lie algebras, arXiv:0904.1778

[13] C. Chevalley, Invariants of finite groups generated by reflections. Amer. J. Math. 77 (1955), 778–782.

[14] J. Dixmier, Jacques Sur les algebres enveloppantes de $\mathfrak{sl}(n,\mathbb{C})$ et $\mathfrak{af}(n,\mathbb{C})$. Bull. Sci. Math. (2) 100 (1976), no. 1, 57-95.

[15] J. Dixmier, Algèbres enveloppantes. (French) [Enveloping algebras] Reprint of the 1974 original. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics] Éditions Jacques Gabay, Paris, 1996.

[16] J. Dixmier, M. Duflo, M. Vergne, Sur la représentation coadjointe d'une algèbre de Lie. (French) Compositio Math. 29 (1974), 309-323.

[17] M. Duflo and M. Vergne, Une proprietéde la represéntation coadjointe d'une algebre de Lie. (French) C. R. Acad. Sci. Paris Seŕ. A-B 268 (1969), A583–A585.

[18] D. Eisenbud, Commutative algebra. With a view toward algebraic geometry. Graduate Texts in Mathematics, 150. Springer-Verlag, New York, 1995.

[19] A. G. Elashvili and V. G. Kac, Classification of good gradings of simple Lie algebras. Lie groups and invariant theory, 85–104, Amer. Math. Soc. Transl. Ser. 2, 213, Amer. Math. Soc., Providence, RI, 2005.

[20] F. Fauquant-Millet, Quantification de la localisation de Dixmier de $U(\mathrm{sl}_{n+1}(\mathbb{C}))$. (French) [Quantization of the Dixmier localization of $U(\mathrm{sl}_{n+1}(\mathbb{C}))$] J. Algebra 218 (1999), no. 1, 93-116.

[21] F. Fauquant-Millet and A. Joseph, Semi-centre de l'algèbre enveloppante d'une sous-algèbre parabolique d'une algèbre de Lie semi-simple, Annales Scientifiques de l' École Normale Suprieure, Volume 38 (2) (2005), pp. 155-191.

[22] F. Fauquant-Millet and A. Joseph, La somme dans faux degrés—un mystère en théorie des invariants. (French) [The sum of the false degrees - a mystery in the theory of invariants] Adv. Math. 217 (2008), no. 4, 1476–1520.
[23] W. A. de Graaf, Computing with nilpotent orbits in simple Lie algebras of exceptional type. LMS J. Comput. Math. 11 (2008), 280-297.
[24] S. Helgason, Differential geometry and symmetric spaces. Pure and Applied Mathematics, Vol. XII. Academic Press, New York-London 1962.
[25] W. H. Hesselink, Characters of the nullcone. Math. Ann. 252 (1980), no. 3, 179-182.
[26] C. Hoyt, Private communication.
[27] A. Joseph, Second commutant theorems in enveloping algebras. Amer. J. Math. 99 (1977), no. 6, 1167–1192.
[28] A. Joseph, A preparation theorem for the prime spectrum of a semisimple Lie algebra, Journal of Algebra, 48(2) (1977), pp. 241-289.
[29] A. Joseph, A generalization of the Gelfand-Kirillov conjecture. Amer. J. Math. 99 (1977), no. 6, 1151-1165.
[30] A. Joseph, On the Gelfand-Kirillov conjecture for induced ideals in the semisimple case. Bull. Soc. Math. France 107 (1979), no. 2, 139–159.
[31] A. Joseph, A. Goldie rank in the enveloping algebra of a semisimple Lie algebra. I, J. Algebra 65 (1980), no. 2, 269–283.
[32] A. Joseph, Quantum Groups and their Primitive Ideals, Springer-Verlag, 1995.
[33] A. Joseph, On a Harish-Chandra homomorphism. C. R. Acad. Sci. Paris Se'r. I Math. 324 (1997), no. 7, 759-764.
[34] A. Joseph, Sur l'annulateur d'un module de Verma. (French) [On the annihilator of a Verma module] With an outline of the annihilation theorem by M. Gorelik and E. Lanzmann. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 514, Representation theories and algebraic geometry (Montreal, PQ, 1997), 237-300, Kluwer Acad. Publ., Dordrecht, 1998.
[35] A. Joseph, On semi-invariants and index for biparabolic (seaweed) algebras. II. J. Algebra 312 (2007), no. 1, 158-193.
[36] A. Joseph, Parabolic actions in type A and their eigenslices. Transform. Groups 12 (2007), no. 3, 515-547.
[37] A. Joseph, Slices for biparabolic coadjoint actions in type A, Journal of Algebra, Volume 319 (12) (2008),pp. 5060-5100.
[38] A. Joseph, Compatible adapted pairs and a common slice theorem for some centralizers. Transform. Groups 13 (2008), no. 3-4, 637-669.
[39] A. Joseph, An Algebraic Slice in the Coadjoint Space of the Borel and the Coxeter Element, preprint, Weizmann 2009.
[40] A. Joseph and P. Lamprou, Maximal Poisson commutative subalgebras for truncated parabolic subalgebras of maximal index in $\mathfrak{sl}_n$. Transform. Groups 12 (2007), no. 3, 549–571.
[41] A. Joseph and G. Letzter, On the Kostant-Parthasarathy-Ranga Rao-Varadarajan determinants.II. Construction of the KPRV determinants. J. Algebra 241 (2001), no. 1, 46–66.
[42] A. Joseph and D. Shafrir, Polynomiality of Invariants, Unimodularlity and Adapted pairs. (In preparation).
[43] V. G. Kac, Infinite-dimensional Lie algebras. Third edition. Cambridge University Press, Cambridge, 1990.
[44] B. Kostant, The principal three-dimensional subgroup and the Betti numbers of a complex simple Lie group. Amer. J. Math. 81 (1959), 973-1032.
[45] B. Kostant, Lie group representations on polynomial rings. Amer. J. Math. 85 (1963), 327-404.
[46] B. Kostant, A Lie algebra generalization of the Amitsur-Levitski theorem. Adv. in Math. 40 (1981), no. 2, 155–175.

[47] B. Kostant, Clifford algebra analogue of the Hopf-Koszul-Samelson theorem, the ρ-decomposition $C(\mathfrak{g}) = \operatorname{End} V_\rho \otimes C(P)$, and the $\mathfrak{g}$-module structure of $\wedge\mathfrak{g}$, Adv. Math. 125 (1997), no. 2, 275-350.
[48] B. Kostant and N. Wallach, Gelfand-Zeitlin theory from the perspective of classical mechanics. I. Studies in Lie theory, 319–364, Progr. Math., 243, Birkha"user Boston, Boston, MA, 2006.
[49] S. Kumar, A refinement of the PRV conjecture. Invent. Math. 97 (1989), no. 2, 305-311.
[50] G. Lusztig and N. Spaltenstein, Induced unipotent classes. J. London Math. Soc. (2) 19 (1979), no. 1, 41–52.
[51] J. C. McConnell, Representations of solvable Lie algebras. IV. An elementary proof of the $(U/P)_E$-structure theorem. Proc. Amer. Math. Soc. 64 (1977), no. 1, 8–12.
[52] A. S. Mishchenko, A. T. Fomenko, Euler equation on finite-dimensional Lie groups. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 42 (1978), no. 2, 396–415, 471.
[53] R.V. Moody and J. Patera, J. Quasicrystals and icosians. J. Phys. A 26 (1993), no. 12, 2829-2853.
[54] C. Moeglin, Eĺeḿents centraux dans les ideáux d'une algeb̀re enveloppante. (French) C. R. Acad. Sci. Paris Seŕ. A-B 286. (1978), no. 12, A539–A541.
[55] C. Moeglin, Idéaux complètement premiers de l'algèbre enveloppante de $\mathfrak{gl}_n(\mathbb{C})$. (French) [Completely prime ideals of the enveloping algebra of $\mathfrak{gl}_n(\mathbb{C})$] J. Algebra 106 (1987), no. 2, 287–366.
[56] A. I. Ooms, Computing invariants and semi-invariants by means of Frobenius Lie algebras. J. Algebra 321 (2009), no. 4, 1293–1312.
[57] A. Ooms and M. van den Bergh, A degree inequality for Lie algebras with a regular Poisson semi-center, arXiv :0805.1342.
[58] D. I. Panyushev, The index of a Lie algebra, the centralizer of a nilpotent element, and the normalizer of the centralizer. Math. Proc. Cambridge Philos. Soc. 134 (2003), no. 1, 41–59.
[59] D. Panyushev, On the coadjoint representation of $\mathbb{Z}_2$-contractions of reductive Lie algebras. Adv. Math. 213 (2007), no. 1, 380-404.
[60] D. Panyushev, A. Premet and O. Yakimova, On symmetric invariants of centralisers in reductive Lie algebras. J. Algebra 313 (2007), no. 1, 343-391.
[61] D. Panyushev, O. Yakimova, The argument shift method and maximal commutative subalgebras of Poisson algebras. Math. Res. Lett. 15 (2008), no. 2, 239-249.
[62] A. Premet, Modular Lie algebras and the Gelfand-Kirillov conjecture, arXiv:0907.2500.
[63] Sadètov, S. T. A proof of the Mishchenko-Fomenko conjecture (1981). (Russian) Dokl. Akad. Nauk 397 (2004), no. 6, 751-754.
[64] I. R. Shafarevich, Basic algebraic geometry. Translated from the Russian by K. A. Hirsch. Revised printing of Grundlehren der mathematischen Wissenschaften, Vol. 213, 1974. Springer Study Edition. Springer-Verlag, Berlin-New York, 1977.
[65] G. C. Shephard and J. A. Todd, Finite unitary reflection groups. Canadian J. Math. 6, (1954). 274-304.
[66] A. A. Tarasov, The maximality of some commutative subalgebras in Poisson algebras of semisimple Lie algebras. (Russian) Uspekhi Mat. Nauk 57 (2002), no. 5(347), 165-166; translation in Russian Math. Surveys 57 (2002), no. 5, 1013-1014.
[67] E. A. Tevelev, On the Chevalley restriction theorem. J. Lie Theory 10 (2000), no. 2, 323-330.
[68] J. Weyman, The equations of conjugacy classes of nilpotent matrices. Invent. Math. 98 (1989), no. 2, 229–245.
[69] O. S. Yakimova, The index of centralizers of elements in classical Lie algebras. (Russian) Funktsional. Anal. i Prilozhen. 40 (2006), no. 1, 52-64, 96; translation in Funct. Anal. Appl. 40 (2006), no. 1, 42-51.
[70] O. S. Yakimova, A counterexample to Premet's and Joseph's conjectures. Bull. Lond. Math. Soc. 39 (2007), no. 5, 749-754.