1. On the centre of completion $\hat{U}$ of $\mathcal{U}(\mathfrak{g})$

A classical theorem by Chevalley states that for a semisimple Lie algebra the restriction of a non-zero $\mathfrak{g}$-invariant regular function on $\mathfrak{g}$ to $\mathfrak{h}$ is non-zero, that is for $f \in \mathcal{S}(\mathfrak{g}^*)^{\mathfrak{g}}$, $f \neq 0$ forces $f_{\mathfrak{h}} \neq 0$. This theorem was generalized by A. Sergeev (see [S3]) to all finite-dimensional Lie superalgebras. If $\mathfrak{g}$ admits an even non-degenerate invariant bilinear form, then $\mathfrak{g} \cong \mathfrak{g}^*$ as $\mathfrak{g}$-modules. Since $\operatorname{gr} z \in \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$ for $z \in \mathcal{Z}(\mathfrak{g})$ the Chevalley theorem can be reformulated as follows: $\deg \mathrm{HC}(z) = \deg(z)$ for any $z \in \mathcal{Z}(\mathfrak{g})$. In this section we will prove a similar statement for a certain completion $\hat{U}$ of $\mathcal{U}(\mathfrak{g})$ where $\mathfrak{g}$ is a Kac-Moody superalgebra with a symmetrizable Cartan matrix. For a finite-dimensional case this implies $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$.

The centre of $\hat{U}$ for $\mathfrak{g}$ being a Kac-Moody superalgebra with a symmetrizable Cartan matrix was described by Kac in [K3]. We elucidate his approach to Q-type algebras in Section 2; this will give us a description of $\mathcal{Z}(\mathfrak{g})$.

We denote by $\mathfrak{g}$ a a Kac-Moody superalgebra with a symmetrizable Cartan matrix.

1.1. **A construction $\hat{U}$.** Define on $\mathcal{U}(\mathfrak{g})$ a topology where a basis of neighborhoods of zero are left ideals $J(\nu) := \mathcal{U}(\mathfrak{g})\mathcal{U}(\mathfrak{n}^+)_{\geq\nu}$ for $\nu \in Q(\pi)$. Clearly, $J(\nu) = \mathcal{U}(\mathfrak{g})$ for $\nu \leq 0$ and $J(\nu) \subset J(\nu')$ if $\nu > \nu'$. Let $\hat{U}$ be the completion of $\mathcal{U}(\mathfrak{g})$ with respect to this topology. It is easy to see that the structure of associative algebra and the adjoint action of $\mathfrak{g}$ can be uniquely extended to $\hat{U}$. Clearly, $\mathcal{Z}(\hat{U}) = \hat{U}^{\mathfrak{g}}$.

Let N be a $\mathfrak{g}$-module with locally nilpotent action of $\mathfrak{n}^+$ [1]. Then the action of $\mathfrak{g}$ can be canonically extended to the action of $\hat{U}$.

1.1.1. Set

$$J'(\nu) := \mathcal{U}(\mathfrak{b}^-)\mathcal{U}(\mathfrak{n}^+)_\nu.$$

Then, by PBW theorem, $J(\nu) = \sum_{\mu\geq\nu} J'(\mu)$ and

$$\mathcal{U}(\mathfrak{g}) = \oplus_{\nu\in Q^+} J'(\nu), \quad \hat{U} = \prod_{\nu\in Q^+} J'(\nu). \tag{1}$$

We write the elements of $\hat{U}$ in the form $u = \sum_{\nu\in Q^+} u_\nu$ where $u_\nu \in J'(\nu)$.

1.1.2. Put $\mathfrak{b}^- := \mathfrak{n}^- + \mathfrak{h}$. Observe that $\mathcal{U}(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{b}^-)$ are closed with respect to our topology. One has $\hat{U} = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{b}^-) + \hat{U}\mathfrak{n}^+)$. Extend the Harish-Chandra projection to $\hat{U}$ along the above decomposition.

Take $u = \sum_{\nu\in Q^+} u_\nu$ where $u_\nu \in J'(\nu)$. Then $\mathrm{HC}(u) = \mathrm{HC}(u_0)$ and $\mathrm{HC}(u) = u_0$ if u has weight zero, because $J'(0) = \mathcal{U}(\mathfrak{b}^-)$.

[1] this means that $\dim \mathcal{U}(\mathfrak{n}^+)v < \infty$ for any $v \in N$.

1.1.3. The canonical filtration $\mathcal{F}$ on $\mathcal{U}(\mathfrak{g})$ can be naturally extended to $\hat{U}$, however, the resulted filtration is not exhausting: $\cup_{r=0}^{\infty}\mathcal{F}^r(\hat{U}) \neq \hat{U}$. Let $\deg u$ be the degree of $u \in \hat{U}$ with respect to this filtration ($\deg u = \infty$ if $u \notin \cup_{r=0}^{\infty}\mathcal{F}^r(\hat{U})$).

1.2. **Results.** The following theorem was suggested to the author by J. Bernstein.

1.2.1. **Theorem.** *For any $z \in \mathcal{Z}(\hat{U})$ one has*

$$\deg z = \deg \mathrm{HC}(z).$$

In other words, $z \in \mathcal{Z}(\hat{U})$ takes form $z = \sum_{\nu\in Q^+} z(\nu)$ where $z(\nu) \in J'(\nu)$ is such that $\deg z(\nu) \leq \deg z(0)$.

In particular, HC induces an algebra embedding $\mathcal{Z}(\hat{U}) \to \mathcal{U}(\mathfrak{h})$; the image of this embedding was described by V. Kac in [K3], Remark 3 and Section 8. Corollary 1.2.2 gives

$$\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$$

if $\mathfrak{g}$ is a finite-dimensional Kac-Moody (contragredient) superalgebra.

1.2.2. **Corollary.** *If $\mathfrak{g}$ is finite-dimensional $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$.*

Proof. From Theorem 1.2.1, $z(\nu) \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g}))$ where $r := \deg z(0)$. Then $z(\nu) = 0$ if ν is "sufficiently large" that is $\nu \notin \Delta^{(r)}$ where $\Delta^{(r)} := \{\sum_{i=1}^{s} \alpha_i |\ \alpha_i \in \Delta^+, s \leq r\}$. Since Δ is finite, $\Delta^{(r)}$ is also finite and thus z is a finite sum of elements of $\mathcal{U}(\mathfrak{g})$. Hence $z \in \mathcal{U}(\mathfrak{g})$. $\square$

1.2.3. View $\mathcal{U}(\mathfrak{h})$ as $\mathfrak{h}$-bimodule and extend the left action to the action of $\mathfrak{b}$ via the trivial action of $\mathfrak{n}^+$. View

$$\mathcal{M} := \mathrm{Ind}_{\mathfrak{b}}^{\mathfrak{g}} \mathcal{U}(\mathfrak{h})$$

as $\mathfrak{g}$–$\mathfrak{h}$ bimodule. Set

$$\mathcal{M}_\nu := \{v \in \mathcal{M} \mid hv - vh = \nu(h)v\ \forall h \in \mathfrak{h}\}.$$

Each Verma module $M(\lambda)$ can be obtained from $\mathcal{M}$ by the evaluation at λ and a weight space $M(\lambda)_{\lambda-\nu}$ is the image of $\mathcal{M}_\nu$.

Identify $\mathcal{M}$ with $\mathcal{U}(\mathfrak{b}^-)$ as $\mathfrak{b}^-$-modules and define the Shapovalov form $S : \mathcal{M} \otimes \mathcal{M} \to \mathcal{U}(\mathfrak{h})$ via this identification; recall that $\operatorname{Ker} S = 0$. Clearly, $S(\mathcal{M}_\mu, \mathcal{M}_\nu) = 0$ if $\mu \neq \nu$ and so $S = \sum S_\nu$ where S_ν is the restriction of S to $\mathcal{M}_\nu$. The Shapovalov form on $M(\lambda)$ becomes the evaluation of S at λ.

Since $\mathfrak{n}^+$ acts locally nilpotently on $\mathcal{M}$, the $\mathfrak{g}$-module structure on $\mathcal{M}$ can be canonically extended to $\hat{U}$-module structure.

Proposition.

(i) *$\mathcal{M}$ is a faithful $\hat{U}$-module.*

(ii) *For* $z \in \hat{U}$ *one has*

$$z \in \mathcal{Z}(\hat{U}) \iff zv = v\,\mathrm{HC}(z) \quad \textit{for any } v \in \mathcal{M}.$$

In other words, we have an embedding of $\mathfrak{g}$-modules $\hat{U} \to \mathrm{End}(\mathcal{M})$ and the image of a central element is an endomorphism induced by the right action.

1.3. **Proof of Theorem 1.2.1.** Take a non-zero $z \in \hat{U}^{\mathfrak{g}}$ and write $z = \sum_{\nu \in Q^+} z_\nu$ where $z_\nu \in J'(\nu)$. Set $r := \deg z_0$. Let us prove the inequality

$$\deg z_\nu \leq r \tag{2}$$

by induction on $\mathrm{ht}(\nu)$ where the function height on Q^+ is given by $\mathrm{ht}(\sum_{\alpha_i \in \pi} k_i \alpha_i) = \sum k_i$. The inequality trivially holds for $\mathrm{ht}(\nu) = 0$.

1.3.1. Suppose that the inequality (2) holds for all ν such that $\mathrm{ht}(\nu) \leq k$ and let us show that (2) holds for all weights of height $k+1$. Take μ of height $k+1$ and write $\mu = \nu + \alpha_0$ where α_0 is a simple root and $\nu \in Q^+$ is such that $\mathrm{ht}(\nu) = k$.

Denote by p_η the projection $\hat{U} \to J'(\eta)$ with respect to the decomposition (1). Since $z \in \hat{U}^{\mathfrak{g}}$ one has $[e_\beta, z] = 0$ for any simple root β and any $e_\beta \in \mathfrak{g}_\beta$. Therefore

$$0 = p_\mu\big([e_\beta, z]\big) = p_\mu\big([e_\beta, z_{\mu-\beta}]\big) + p_\mu\big([e_\beta, z_\mu]\big).$$

The canonical filtration is $\mathrm{ad}\,\mathfrak{g}$-stable and p_η-stable; thus, by the induction hypothesis, the first summand has degree at most r. Hence

$$p_\mu([e_\beta, z_\mu]) \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g})). \tag{3}$$

for all simple roots β. We need to deduce that $z_\mu \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g}))$.

1.3.2. Fix a basis $\{\mathbf{e}^{\mathbf{k}}\}$ in $\mathcal{U}(\mathfrak{n}^+)_\mu$, $\{\mathbf{h}^{\mathbf{s}}\}$ in $\mathcal{U}(\mathfrak{h})$ and write $z_\mu = \sum_{\mathbf{k},\mathbf{s}} a_{\mathbf{k},\mathbf{s}} \mathbf{h}^{\mathbf{s}} \mathbf{e}^{\mathbf{k}}$ where $a_{\mathbf{k},\mathbf{s}}$ are elements of $\mathcal{U}(\mathfrak{n}^-)_{-\mu}$. One has

$$p_\mu([e_\beta, z_\mu]) = \sum_{\mathbf{k},\mathbf{s}} [e_\beta, a_{\mathbf{k},\mathbf{s}}] \mathbf{h}^{\mathbf{s}} \mathbf{e}^{\mathbf{k}} \in \mathcal{F}^r(\mathcal{U}(\mathfrak{g})). \tag{4}$$

by (3). Recall that $\mathcal{S}(\mathfrak{g}) = \mathcal{S}(\mathfrak{b}^-) \otimes \mathcal{S}(\mathfrak{n}^+)$. As a consequence, for any linearly independent elements $\{s_i^+\} \subset \mathcal{S}(\mathfrak{n}^+)$ and any elements $\{s_i^-\} \subset \mathcal{S}(\mathfrak{b}^-)$ the degree of $\sum_i s_i^- s_i^+$ is the maximum of $\deg s_i^- + \deg s_i^+$. Denoting the degree of $\mathbf{e}^{\mathbf{k}}$ by $|\mathbf{k}|$ we obtain from (4)

$$\deg \sum_{\mathbf{s}} [e_\beta, a_{\mathbf{k},\mathbf{s}}] \mathbf{h}^{\mathbf{s}} \leq r - |\mathbf{k}| \tag{5}$$

for any $\mathbf{k}$. Identify $\mathfrak{b}^-$ with $\mathfrak{g}/\mathfrak{n}^+$. Then $\sum_{\mathbf{s}} [e_\beta, a_{\mathbf{k},\mathbf{s}}] \mathbf{h}^{\mathbf{s}}$ identifies with $(\mathrm{ad}\, e_\beta) \sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}} \mathbf{h}^{\mathbf{s}}$ where the adjoint action $\mathfrak{n}^+$ on $\mathfrak{g}/\mathfrak{n}^+$ is induced by the usual adjoint action.

Suppose that $\deg z_\mu > r$. Then

$$\deg \sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}} \mathbf{h}^{\mathbf{s}} > r - |\mathbf{k}| \tag{6}$$

for some $\mathbf{k}$. In the light of Lemma 1.4, the image of $\sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}}\mathbf{h}^{\mathbf{s}}$ in $\mathcal{S}(\mathfrak{g}/\mathfrak{n}^+)$ is not $\mathfrak{n}^+$-invariant and thus there exists a simple root β such that $\deg\sum_{\mathbf{s}}[e_\beta, a_{\mathbf{k},\mathbf{s}}]\mathbf{h}^{\mathbf{s}} = \deg\sum_{\mathbf{s}} a_{\mathbf{k},\mathbf{s}}\mathbf{h}^{\mathbf{s}}$. Comparing (5) with (6) we get a contradiction. The statement follows. $\square$

1.4. Put $\mathfrak{n} := \mathfrak{n}^+, \mathfrak{b} := \mathfrak{b}^+$.

Lemma. *View $\mathfrak{g}/\mathfrak{n}$ as $\mathfrak{n}$-module via the adjoint action. Then $\mathcal{S}(\mathfrak{g}/\mathfrak{n})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{h})$ where the embedding $\mathfrak{h} \to \mathfrak{g}/\mathfrak{n}$ is induces by the isomorphism $\mathfrak{b}/\mathfrak{n} \xrightarrow{\sim} \mathfrak{h}$.*

1.4.1. *Remark.* Assume that $\mathfrak{g}$ is finite-dimensional Lie algebra. Then $\mathfrak{g}/\mathfrak{n}$ identifies with $\mathfrak{b}^*$ via the invariant bilinear form and $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$ identifies with the set of regular functions on $\mathfrak{b}$, and $\mathcal{S}(\mathfrak{g}/\mathfrak{n})^{\mathfrak{n}}$ identifies with the invariant functions. The formula $\mathcal{S}(\mathfrak{g}/\mathfrak{n})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{b}/\mathfrak{n})$ means that for any invariant function ϕ one has $\phi(h+n) = \phi(h)$ for any $h \in \mathfrak{h}, n \in \mathfrak{n}$. This is a standard fact. Indeed, let ϕ be an invariant function and N be the Lie group corresponding to $\mathfrak{n}$. If h is a generic element of $\mathfrak{h}$ then the orbit $N.h$ is dense in $h+\mathfrak{n}$ and thus $\phi(h+n) = \phi(h)$ for generic h. Since the set of generic elements is dense in $\mathfrak{h}$, $\phi(h+n) = \phi(h)$ for any $h \in \mathfrak{h}, n \in \mathfrak{n}$.

Proof of Lemma 1.4. Identify the image of $\mathfrak{g}/\mathfrak{n}$ in $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$ with $\mathfrak{b}^-$ and $\mathcal{S}(\mathfrak{g}/\mathfrak{n})$ with $\mathcal{S}(\mathfrak{b}^-)$. For an algebra S and its subspaces X, Y denote by XY the span of $xy, x \in X, y \in Y$.

1.4.2. First, let us check that for any $\alpha \in \Delta^+$ one has $\mathcal{S}(\mathfrak{g}_{-\alpha}+\mathfrak{h})^{\mathfrak{n}} = \mathcal{S}(\mathfrak{h})$. The algebra $\mathfrak{g}$ admits a non-degenerate invariant bilinear form $(-,-)$ and there exists $h \in \mathfrak{h}$ such that $[e,f] = (f,e)h$ for any $f \in \mathfrak{g}_{-\alpha}, e \in \mathfrak{g}_\alpha$. Let $\{e_i\} \subset \mathfrak{g}_\alpha$ and $\{f_i\} \subset \mathfrak{g}_{-\alpha}$ form dual bases with respect to $(-,-)$. Viewing $u \in \mathcal{S}(\mathfrak{g}_{-\alpha}+\mathfrak{h})$ as a polynomial in $\{f_i\}$ we obtain $(\operatorname{ad} e_i)(u) = h\frac{\partial u}{\partial f_i}$. Thus $(\operatorname{ad} e_i)(u) = 0$ for all i gives $u \in \mathcal{S}(\mathfrak{h})$ as required.

1.4.3. Now let us verify the statement of lemma. Extend the partial order on the root lattice $Q(\pi)$ to a total order compatible with the addition that is

$$\beta > \beta' \implies \beta+\alpha > \beta'+\alpha$$

(for instance, take an embedding $Q(\pi)$ into $\mathbb{R}$). Let Δ^+ be the set (not the multiset) of positive roots and set

$$\mathcal{P} := \{\sum_{\alpha\in\Delta^+} k_\alpha\alpha |\ k_\alpha \in \mathbb{Z}_{\geq 0},\ k_\alpha \in \{0,1\} \text{ if } \alpha \in \Delta_{\overline{1}}.\}$$

For $\mathbf{k} \in \mathcal{P}$ set $X_{\mathbf{k}} := \prod_{\alpha\in\Delta^+} \mathfrak{g}_{-\alpha}^{k_\alpha}\mathcal{S}(\mathfrak{h}) \subset \mathcal{S}(\mathfrak{b}^-)$. Then $\mathcal{S}(\mathfrak{b}^-) = \sum_{\mathbf{k}\in\mathcal{P}} X_{\mathbf{k}}$; let $p_{\mathbf{k}}$ be the projection with respect to this decomposition. For $u \in \mathcal{S}(\mathfrak{b}^-)$ set $\operatorname{supp} u := \{\mathbf{k}|\ p_{\mathbf{k}}(u) \neq 0\}$.

Define a lexicographic order on $\mathcal{P}$ by putting $\mathbf{k} > \mathbf{m}$ if for some $\beta \in \Delta^+$ one has $k_\beta > m_\beta$ and $k_\alpha = m_\alpha$ for all $\alpha > \beta$. Suppose that $u \in \mathcal{S}(\mathfrak{b}^-)^{\mathfrak{n}}$ is such that $u \notin \mathcal{S}(\mathfrak{h})$.

Let $\mathbf{k}$ be the maximal element in $\operatorname{supp} u$ and let $\alpha \in \Delta^+$ be the minimal root satisfying $k_\alpha \neq 0$. Put $\mathbf{k}' := \mathbf{k} - \alpha$ and let us compute $p_{\mathbf{k}'}(\operatorname{ad} e)(u)$ for $e \in \mathfrak{g}_\alpha$. One easily sees that

$$p_{\mathbf{k}'}(\operatorname{ad} e)(u) = p_{\mathbf{k}'}(\operatorname{ad} e)\big(p_{\mathbf{k}}(u)\big).$$

Fix a basis $\{f_i\}$ in $\mathfrak{g}_{-\alpha}^{k_\alpha}$ and write

$$p_{\mathbf{k}}(u) = \sum f_i a_i, \;\; a_i \in X_{\mathbf{k}-k_\alpha \alpha}.$$

Then for any $e \in \mathfrak{g}_\alpha$

$$0 = p_{\mathbf{k}'}(\operatorname{ad} e)\big(p_{\mathbf{k}}(u)\big) = \sum (\operatorname{ad} e)(f_i) a_i.$$

One has $\mathcal{S}(\mathfrak{b}^-) = \mathcal{S}(\mathfrak{g}_{-\alpha}) \otimes \mathcal{S}(\mathfrak{h}) \otimes \mathcal{S}'$ where $\mathcal{S}' := \otimes_{\beta \neq \alpha} \mathcal{S}(\mathfrak{g}_{-\beta})$. Notice that $(\operatorname{ad} e)(f_i) \in \mathcal{S}(\mathfrak{g}_{-\alpha})\mathcal{S}(\mathfrak{h})$ and $a_i \in \mathcal{S}(\mathfrak{h})\mathcal{S}'$. Thus $\sum (\operatorname{ad} e)(f_i) a_i = 0$ forces $\sum (\operatorname{ad} e)(f_i) b_i = 0$ for some non-zero $b_i \in \mathcal{S}(\mathfrak{h})$ which is impossible by 1.4.2. $\square$

1.5. **Proof of Proposition 1.2.3.** For (i) take a non-zero $u \in \mathcal{U}(\mathfrak{g})_\lambda$ and write $u = \sum u_\nu$ where $u_\nu \in J'(\nu)$. Let $\mu \in Q^+$ be a minimal element satisfying $u_\mu \neq 0$; note that $\mu \geq \lambda$. Fix a PBW basis $\mathbf{f}^{\mathbf{k}}, \mathbf{k} \in \mathcal{P}(\mu - \lambda)$ in $\mathcal{U}(\mathfrak{n}^-)_{\lambda-\mu}$. Write $u_\mu = \sum_{\mathbf{k} \in \mathcal{P}(\mu-\lambda)} \mathbf{f}^{\mathbf{k}} a_{\mathbf{k}}$ where $a_{\mathbf{k}}$ are elements of $\mathcal{U}(\mathfrak{b}^+)$. Identify $\mathcal{U}(\mathfrak{b}^-)$ and $\mathcal{M}$ as $\mathfrak{b}^-$-modules. Thanks to the minimality of μ, for $v \in \mathcal{M}_{-\mu}$ one has $uv = u_\mu v$. Then

$$u_\mu v = \sum_{\mathbf{k} \in \mathcal{P}(\mu-\lambda)} \mathbf{f}^{\mathbf{k}} a_{\mathbf{k}} v = \sum_{\mathbf{k}} \mathbf{f}^{\mathbf{k}} \operatorname{HC}(a_{\mathbf{k}} v).$$

Take $\mathbf{k}$ such that $a_{\mathbf{k}} \neq 0$. Since the restriction of Shapovalov form to $\mathcal{M}_\mu$ is non-degenerate, there exists $v \in \mathcal{M}_{-\mu}$ satisfying $\operatorname{HC}(a_{\mathbf{k}} v) \neq 0$. Hence $uv = u_\mu v \neq 0$ and this gives (i).

For (ii) recall that $\mathcal{M}$ is generated by the image of $1 \in \mathcal{U}(\mathfrak{h})$ which is annihilated by $\mathfrak{n}^+$. This gives the implication

$$z \in \mathcal{Z}(\hat{U}) \implies zv = v \operatorname{HC}(z) \;\; \text{for any } v \in \mathcal{M}.$$

The inverse implication follows from (i) and the fact that $\mathcal{M}$ is a $\hat{U}$-$\mathfrak{h}$ bimodule and so the map $v \mapsto v \operatorname{HC}(z)$ lies in $\operatorname{End}_{\hat{U}}(\mathcal{M})$. $\square$

2. The centre of a Q-type Lie superalgebra

3. The Kac theorem

3.1. **Assumptions.** Suppose that

(i) For any $\nu \in Q^+$, the zeroes of the Shapovalov determinant $\det S_\nu$ belong to the union of a countably many hyperplanes $\gamma_1, \gamma_2, \dots$ in $\mathfrak{h}^*$.

(ii) For each m, there is an open in γ_m set $\breve{\gamma}_m$ such that $\breve{\gamma}_m \cap \gamma_i = 0$ for all $i \neq m$ and for all $\lambda \in \breve{\gamma}_m$ we have
  (a) $\operatorname{corank} S_\nu(\lambda)$ is equal to the order of zero of the polynomial $\det S_\nu$ at λ;
  (b) $\overline{M}(\lambda)$ is a quotient of a Verma module $M(r(\lambda))$ for some weight $r(\lambda)$.

3.1.1. *Remark.* The order m of zero of a polynomial q at λ is defined to be $m = 0$ if $q(\lambda) \neq 0$; it is $m = 1$ if $q(\lambda) = 0$ and there exists a non-zero partial derivative $\frac{dq}{dx}(\lambda) \neq 0$, and so on.

It is clear that

$$\operatorname{corank} S_\nu(\lambda) \leq \text{ the order of zero of } \det S_\nu \text{ at } \lambda.$$

3.1.2. The above assumptions are valid for superalgebras $\mathfrak{g}(A)$; the first assumption easily follows from the existence of quadratic Casimir operator and the second assumption corresponds to the fact that the Jantzen filtration of $M(\lambda)$ has length two for $\lambda \in \breve{\gamma}_m$. The similar assumptions are valid in the quantum case (see [J], 4.1).

3.1.3. The property (i) is equivalent to the fact that each Shapovalov determinant $\det S_\nu$ admits a linear factorization: up to a constant factor, we have

$$\det S_\nu = \prod_m (h_m - c_m)^{d_m(\nu)},$$

where $d_m(\nu) \in \mathbb{Z}_{\geq 0}$ and $h_m \in \mathfrak{h}$, $c_m \in \mathbb{C}$. The pairs (h_m, c_m) define the hyperplanes γ_m from (i):

$$\gamma_m = \{\mu \in \mathfrak{h}^* \mid \mu(h_m) - c_m = 0\}.$$

The order of zero of $\det S_\nu$ at λ is equal to the sum

$$\sum_{m \mid \lambda \in \gamma_m} d_m(\nu).$$

Thus (ii)(a) states that $d_m(\nu) = \operatorname{corank} S_\nu(\lambda)$ for all $\lambda \in \breve{\gamma}_m$.

3.2. **Theorem (Kac, '84).** *Set*

$$X := \bigcup_m \breve{\gamma}_m.$$

The restriction of HC *to* $\mathcal{Z}(\mathfrak{g})$ *is an algebra isomorphism*

$$\mathrm{HC} : \mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} \{\varphi \in \mathcal{U}(\mathfrak{h}) \mid \varphi(\lambda) = \varphi(r(\lambda)) \text{ for any } \lambda \in X\} \qquad (*)$$

where $r(\lambda)$ *is introduced in the assumption (ii) (b).*

We prove this theorem using a modification of Kac method presented in [K3]. Actually we prove that $\mathrm{HC}(\mathcal{Z}(\hat{U})) = Z$ (see Section 1 for the definition of $\hat{U}$) and then use the equality $\mathcal{Z}(\hat{U}) = \mathcal{Z}(\mathfrak{g})$ obtained in Corollary 1.2.2.

3.3. **Proof of Theorem 3.2.** By Theorem 1.2.1 the restriction of HC to $\mathcal{Z}(\hat{U})$ is injective. Since a central element z acts on $M(\lambda)$ by $\mathrm{HC}(z)(\lambda)$, the assumption (ii) (b) of 3.1 implies $\mathrm{HC}(\mathcal{Z}(\hat{U})) \subseteq Z$. To prove the opposite inclusion $\mathrm{HC}(\mathcal{Z}(\hat{U})) \supseteq Z$ we assign to each $\phi \in Z$ an element $z \in \hat{U}$ with the property: $\mathrm{HC}(z) = \phi$ and $zv = v\phi$ for all $v \in \mathcal{M}$. By Proposition 1.2.3, z is central.

We construct the element $z = \sum z_\nu$ by a recursive procedure; the summands $z_\nu \in J'(\nu)$ (see 1.1.1 for the notation) are chosen to fulfill the condition $\sum_{\mu \leq \nu} z_\mu v = v\phi$ for all $v \in \mathcal{M}_{-\nu}$. For any $\nu \in Q^+$ and any $v \in \mathcal{M}_{-\nu}$ one has $zv = \sum_{\mu \leq \nu} z_\mu v$. Hence $zv = v\phi$ for all $v \in \mathcal{M}$.

Putting $z_{<\nu} := \sum_{\mu<\nu} z_\mu$ we can rewrite the above condition as

$$z_\nu v = v\phi - z_{<\nu} v, \ \forall v \in \mathcal{M}_{-\nu}\,. \tag{7}$$

In the rest of the proof we show the existence of z_ν satisfying (7).

3.4. We proceed by induction on $\nu \in Q^+(\pi)$. Recall that $\varphi_{0,0} = \varphi$. Assume that for all $0 \leq \mu < \nu$ we have constructed $\varphi_{\mathbf{k},\mathbf{m}}$ for any partitions $\mathbf{k}, \mathbf{m} \in \mathcal{P}(\mu)$. Therefore the sum

$$z_{<\nu} := \sum_{\mu<\nu} z_\mu$$

is known. Observe that, for $\mu < \nu$ and $v \in \mathcal{M}_{-\mu}$, we have

$$z_{\leq\nu} v = z_{\leq\mu} v = z_{<\nu} v.$$

Therefore, for such v, the equality $z_{\leq\nu} = \varphi(\lambda)v$ follows from the induction hypothesis. Hence the property (7) is equivalent to

$$z_{\leq\nu} v = v\varphi \ \text{ for any } v \in \mathcal{M}_{-\nu}. \tag{8}$$

Denote by v_0 the canonical generator of $\mathcal{M}$. The set $\{\mathbf{f}^{\mathbf{s}} v_0 \mid \mathbf{s} \in \mathcal{P}(\nu)\}$ forms an $\mathcal{U}(\mathfrak{h})$-basis of $\mathcal{M}_{-\nu}$ (which is a free $\mathcal{U}(\mathfrak{h})$-module). Notice that $z_{\leq\nu} v = v\varphi$ implies $z_{\leq\nu}(va) = (va)\varphi$ for any $a \in \mathcal{U}(\mathfrak{h})$ since $\mathcal{U}(\mathfrak{h})$ is commutative. Therefore the equation (8) is equivalent to the system whose equations are labeled by the partitions of ν: for each partition $\mathbf{s} \in \mathcal{P}(\nu)$ the corresponding equation is

$$z_{\leq\nu} \mathbf{f}^{\mathbf{s}} v_0 = \mathbf{f}^{\mathbf{s}} v_0 \varphi. \tag{9}$$

For any $u \in \mathcal{U}(\mathfrak{g})$, we have

$$u(\mathbf{f}^{\mathbf{s}} v_0) = P_+(u\mathbf{f}^{\mathbf{s}}) v_0,$$

where P_+ is the projection $\mathcal{U}(\mathfrak{g}) \to U(\mathfrak{n}^- \oplus \mathfrak{h})$ with the kernel $\mathcal{U}(\mathfrak{g})U(\mathfrak{n}^+)$. Formula (9) can be rewritten as

$$P_+(z_{\leq\nu} \mathbf{f}^{\mathbf{s}}) = \mathbf{f}^{\mathbf{s}} \varphi.$$

Using the formula $z_{\leq\nu} = z_{<\nu} + z_\nu$ we obtain for each partition $\mathbf{s} \in \mathcal{P}(\nu)$ the equation

$$P_+\Big(\sum_{\mathbf{k},\mathbf{m} \in \mathcal{P}(\nu)} \mathbf{f}^{\mathbf{k}} \varphi_{\mathbf{k},\mathbf{m}} \sigma(\mathbf{f}^{\mathbf{m}}) \mathbf{f}^{\mathbf{s}} \Big) = \mathbf{f}^{\mathbf{s}} \varphi - P_+(z_{<\nu} \mathbf{f}^{\mathbf{s}})$$

which can be rewritten as

$$\sum_{\mathbf{k},\mathbf{m}\in\mathcal{P}(\nu)} \mathbf{f}^{\mathbf{k}}\varphi_{\mathbf{k},\mathbf{m}}\,\mathrm{HC}\big(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}\big) = \mathbf{f}^{\mathbf{s}}\varphi - P_+(z_{<\nu}\mathbf{f}^{\mathbf{s}}) \tag{10}$$

since $P_+(u_-u_0) = u_-\,\mathrm{HC}(u)$ for any $u_- \in U(\mathfrak{n}^-)$ and $u \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$. Recall that

$$\mathrm{HC}\big(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}\big) = (S_\nu)_{(\mathbf{m},\mathbf{s})}.$$

Both sides of the equation (10) are elements of $L := U(\mathfrak{n}^-)_{-\nu}\mathcal{U}(\mathfrak{h})$. Note that L is a right free $\mathcal{S}(\mathfrak{h})$-module and the elements $\mathbf{f}^{\mathbf{k}}$, where $\mathbf{k} \in \mathcal{P}(\nu)$, form a system of free generators. By the induction hypothesis, the right-hand side of (10) is known. Writing

$$\mathbf{f}^{\mathbf{s}}\varphi - P_+(z_{<\nu}\mathbf{f}^{\mathbf{s}}) = \sum_{\mathbf{k}\in\mathcal{P}(\nu)} \mathbf{f}^{\mathbf{k}} c_{\mathbf{k},\mathbf{s}},$$

where $c_{\mathbf{k},\mathbf{s}} \in \mathcal{S}(\mathfrak{h})$, we obtain, for each pair $\mathbf{k},\mathbf{s} \in \mathcal{P}(\nu)$, the equation

$$\sum_{\mathbf{m}\in\mathcal{P}(\nu)} \varphi_{\mathbf{k},\mathbf{m}}(S_\nu)_{\mathbf{m},\mathbf{s}} = c_{\mathbf{k},\mathbf{s}}.$$

Hence we get a system of linear equations which in the matrix form is

$$\Phi S_\nu = C, \tag{11}$$

where $\Phi := (\varphi_{\mathbf{k},\mathbf{m}})$ is the matrix of indeterminates and $C := (c_{\mathbf{k},\mathbf{s}})$ is a known matrix. Since $\det S_\nu \neq 0$ the system (11) has a unique solution

$$\Phi = CS_\nu^{-1},$$

where Φ is a matrix with entries in $\mathrm{Fract}\,\mathcal{S}(\mathfrak{h})$, the field of fraction of $\mathcal{S}(\mathfrak{h})$. Our next goal is to show that the entries of Φ actually lie in $\mathcal{S}(\mathfrak{h})$.

3.5. We want to prove that the $\varphi_{\mathbf{k},\mathbf{m}}$ are polynomials. We have $\Phi = CS_\nu^{-1}$ and $\det S_\nu(\xi) \neq 0$ for $\xi \in \mathfrak{h}^* \setminus (\cup\gamma_m)$. Thus the $\varphi_{\mathbf{k},\mathbf{m}}$ are rational functions holomorphic on $\mathfrak{h}^* \setminus (\cup_m\gamma_m)$. Since $\cup\breve{\gamma}_m$ is open in $\cup_m\gamma_m$, it suffices to show that the functions $\varphi_{\mathbf{k},\mathbf{m}}$ are holomorphic in a neighbourhood of any $\lambda \in \cup\breve{\gamma}_m$. Kac proved the following lemma.

3.5.1. **Lemma.** *Let $B = (b_{ij})$ and $C = (c_{ij})$ be two $N \times N$ matrices, where b_{ij} and c_{ij} are functions in $z_1, \dots, z_n$ holomorphic on a neighbourhood U of the origin. Set $V := U \cap \{z_1 = 0\}$. Let B be invertible on $U \setminus V$ and for all $\lambda \in V$ one has (a) the order of zero of $\det B$ at λ is equal to $\dim \operatorname{Ker} B(\lambda)$, (b) $\operatorname{Ker} B(\lambda) \subset \operatorname{Ker} C(\lambda)$. Then CB^{-1} is holomorphic on U.*

Proof. Since B is invertible on $U \setminus V$, it suffices to show that CB^{-1} has no pole on the hyperplane $z_1 = 0$. Denote by R the local ring obtained from the ring of holomorphic functions in $z_1, \dots, z_n$ by localization at the principal prime ideal (z_1). Consider B and C as matrices over R. There exists an invertible matrix D_1 over R such that the matrix $B' := D_1B$ is upper-triangular and the diagonal entries are of the form z_1^i where $i \geq 0$.

We can choose an open set $U' \subset U$ such that the entries of D_1 and D_1^{-1} are holomorphic in U', and $V' := U' \cap V$ is open in V. Then, for all $\lambda \in V'$, the order of zero of $\det B'$ at λ is equal to $\dim \operatorname{Ker} B'(\lambda)$. As a consequence, any diagonal entry of B' is either 1 or z_1. Moreover, all entries of the column with diagonal entry z_1 lie in the maximal ideal Rz_1. So there exists an invertible matrix D_2 over R such that the matrix $B'' := D_2 B'$ is diagonal and any its diagonal entry is either 1 or z_1.

We can choose an open set $U'' \subset U'$ such that the entries of D_2 and D_2^{-1} are holomorphic in U'', and $V'' := U'' \cap V$ is open in V. For any $\lambda \in V''$, we have $\operatorname{Ker} B''(\lambda) = \operatorname{Ker} B(\lambda) \subset \operatorname{Ker} C(\lambda)$. This means that if $b'_{ii} = z_1$ then $c_{ji}(\lambda) = 0$ for all $\lambda \in V''$ and $j = 1, \ldots, N$, and hence $c_{ji} \in R(z_1)$, and so $C(B'')^{-1}$ is a matrix over R. Thus $CB^{-1} = C(B'')^{-1} D_2 D_1$ is a matrix over R and so CB^{-1} has no pole on the hyperplane $z_1 = 0$. □

3.5.2. The lemma is proven, so it remains to verify the fulfillment of (a) and (b). Let us view $\mathfrak{h}$ as an affine space. Set $N := |\mathcal{P}(\nu)|$ and $B := S_\nu$. Take a hyperplane γ_m and fix $\lambda' \in \breve{\gamma}_m$. Let the hyperplane $z_1 = 0$ coincides with γ_m. Choose U to be a neighbourhood of λ' such that $U \cap \gamma_i = \emptyset$ for $i \neq m$ and $U \cap \gamma_m \subset \breve{\gamma}_m$.

Set $V := U \cap \gamma_m$. Since $B = S_\nu$, the first assumption of the lemma is exactly the assumption (ii) (a). To verify the second assumption of the lemma, fix $\lambda \in V$. Let us identify $M(\lambda)$ and $U(\mathfrak{n}^-)$, as $U(\mathfrak{n}^-)$-modules. Consider the matrices $B(\lambda)$ and $C(\lambda)$ as endomorphisms of the N-dimensional vector space $M(\lambda)_{\lambda-\nu}$ with the basis $\mathbf{f}^{\mathbf{s}}$. Recall that

$$\operatorname{Ker} B(\lambda) = \operatorname{Ker} S_\nu(\lambda) = \overline{M}(\lambda)_{\lambda-\nu}.$$

The equality

$$\sum_{\mathbf{k}} c_{\mathbf{k},\mathbf{s}} \mathbf{f}^{\mathbf{k}} = \mathbf{f}^{\mathbf{s}} \varphi - P_+(z_{<\nu} \mathbf{f}^{\mathbf{s}})$$

shows that $C(\lambda)(v) = (\varphi(\lambda) - z_{<\nu})v$ for all $v \in M(\lambda)_{\lambda-\nu}$.

Recall that $\lambda \in V \subset \lambda \in \breve{\gamma}_m$. The assumption (ii) (b) of 3.1, ensures that $\overline{M}(\lambda)$ is a quotient of $M(r(\lambda))$, where $r(\lambda) < \lambda$. Thus, for any $v \in \operatorname{Ker} B(\lambda) = \overline{M}(\lambda)_{\lambda-\nu}$, we have

$$z_{<\nu} v = z_{\leq\mu} v,$$

where $\lambda - \nu = r(\lambda) - \mu$. Then $\mu = \nu + r(\lambda) - \lambda < \nu$ and, by the induction hypothesis, $z_\mu v = \varphi(r(\lambda))v = \varphi(\lambda)v$. Hence $(\varphi(\lambda) - z_{<\nu})v = 0$, and so $C(\lambda)v = 0$. This establishes the second assumption of the lemma. Finally, $\Phi = CS_\nu^{-1}$ is holomorphic on U.

3.5.3. **Corollary.** *The functions $\varphi_{\mathbf{k},\mathbf{m}}$ are polynomial.*

□

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