THE CENTRE OF A SIMPLE P-TYPE LIE SUPERALGEBRA.

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Abstract. We describe the centre of a simple Lie superalgebra of type $P(n)$. The description is based on the notion of anticentre.

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1. Introduction

1.1. The Lie superalgebra $\mathfrak{g} := P(n-1)$ is described in [K1]. It consists of the matrices of the block form

$$\begin{pmatrix} a & \mid & b \\ -- & - & -- \\ c & \mid & -a^T \end{pmatrix}$$

where a, b, c are $n \times n$-matrices over a base field k of characteristic zero such that a is traceless, b is symmetric and c is skew-symmetric. The even part $\mathfrak{g}_{\bar{0}}$ consists of the matrices with $b = c = 0$ and the odd part $\mathfrak{g}_{\bar{1}}$ consists of the matrices with $a = 0$. The Lie bracket on $\mathfrak{g}$ is given by the formula $[x, y] = xy - yx$ if x or y is even and $[x, y] = xy + yx$ if both x and y are odd. The Lie superalgebra $P(n-1)$ is simple for $n \geq 3$; its even part $\mathfrak{g}_{\bar{0}}$ is a simple Lie algebra $\mathfrak{sl}(n)$. The Lie superalgebra $\mathfrak{g}$ admits also a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ coincides with $\mathfrak{g}_{\bar{0}}$, $\mathfrak{g}_1$ consists of the matrices with $a = b = 0$ and $\mathfrak{g}_{-1}$ consists of the matrices with $a = c = 0$. The last grading induces a $\mathbb{Z}$-grading on the universal enveloping superalgebra $\mathcal{U}(\mathfrak{g})$.

The central elements of the P-type Lie superalgebras were investigated by Scheunert in [Sch]. It was shown that any central element without constant term is of degree $-n$ (with respect to the $\mathbb{Z}$-grading above) and its order is at least $\frac{1}{2}n(n+1)$. The first statement has the following important consequences. First, it shows that the centre $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is highly degenerate: the product of any two central elements without constant term vanishes. Second, this implies that such central elements annihilate all completely reducible representations. For $n = 3$ Scheunert constructed the lowest-order central element (of order 6).

1.2. Our goal is to determine the centre $\mathcal{Z}(\mathfrak{g})$. It turns out that for P-type Lie superalgebras the structure of the centre $\mathcal{Z}(\mathfrak{g})$ is similar to the structure of the *anticentre* $\mathcal{A}(\mathfrak{g})$. Recall that the even elements of the centre $\mathcal{Z}(\mathfrak{g})$ commute with all element of $\mathcal{U}(\mathfrak{g})$ and the odd elements of $\mathcal{Z}(\mathfrak{g})$ commute with the even elements of $\mathcal{U}(\mathfrak{g})$ and anticommute with the odd ones. By contrast, the odd elements of the anticentre $\mathcal{A}(\mathfrak{g})$ commute with all element of $\mathcal{U}(\mathfrak{g})$ and the even elements of $\mathcal{A}(\mathfrak{g})$ commute with the even elements of $\mathcal{U}(\mathfrak{g})$ and anticommute with the odd ones. In the situation when any even element of the Lie superalgebra $\mathfrak{p} = \mathfrak{p}_{\bar{0}} \oplus \mathfrak{p}_{\bar{1}}$ annihilates the one dimensional module $\Lambda^{\text{top}}\mathfrak{p}_{\bar{1}}$, the anticentre $\mathcal{A}(\mathfrak{p})$ can be easily determined — see [G1]. Namely, there is an explicit construction of a linear isomorphism from the centre $\mathcal{Z}(\mathfrak{p}_{\bar{0}})$ to the anticentre $\mathcal{A}(\mathfrak{p})$ and the image of $\mathcal{A}(\mathfrak{p})$ in the symmetric algebra $\mathcal{S}(\mathfrak{p})$ is equal to $\Lambda^{\text{top}}\mathfrak{p}_{\bar{1}}\mathcal{S}(\mathfrak{p}_{\bar{0}})^{\mathfrak{p}_{\bar{0}}}$.

Since $\mathfrak{g}_{\bar{0}}$ is a simple Lie algebra $\mathfrak{sl}(n)$, $\Lambda^{\text{top}}\mathfrak{p}_{\bar{1}}$ meets the above condition and the isomorphism $\phi' : \mathcal{Z}(\mathfrak{g}_{\bar{0}}) \to \mathcal{A}(\mathfrak{g})$ can be easily written down. Note that $\mathcal{A}(\mathfrak{g})$ lies in the homogeneous component $\mathcal{U}(\mathfrak{g})_{-n}$ since the image of $\mathcal{A}(\mathfrak{g})$ in the symmetric algebra $\mathcal{S}(\mathfrak{g})$ is equal to $\Lambda^{\text{top}}\mathfrak{g}_{\bar{1}}\mathcal{S}(\mathfrak{g}_{\bar{0}})^{\mathfrak{g}_{\bar{0}}} = \Lambda^{\text{top}}\mathfrak{g}_1\Lambda^{\text{top}}\mathfrak{g}_{-1}\mathcal{S}(\mathfrak{g}_{\bar{0}})^{\mathfrak{g}_{\bar{0}}}$ and $\dim \mathfrak{g}_1 - \dim \mathfrak{g}_{-1} = -n$. Using the isomorphism ϕ', we construct a linear isomorphism $\phi : \mathcal{Z}(\mathfrak{g}_{\bar{0}}) \to \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{-n}$. This provides a full description of $\mathcal{Z}(\mathfrak{g})$ since, due to Scheunert, $\mathcal{Z}(\mathfrak{g}) = k \oplus \mathcal{Z}(\mathfrak{g})_{-n}$ where $\mathcal{Z}(\mathfrak{g})_{-n} := \mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{-n}$.

1.3. Remark that for other Lie superalgebras the structures of the centre and the anticentre are not so similar. However one might notice a certain connection. For instance, for a non-simple P-type Lie superalgebra $GP(n-1)$ ($n \geq 3$) (consisting of the block matrices of the same type as above but with an arbitrary matrix a) and for a Cartan type superalgebra $W(n)$ ($n \geq 3$) both centre and anticentre are trivial: the centre coincides with the base field— see [Sch], [Sh] and the anticentre is equal to zero ([G1]).

The sum of the centre and the anticentre is a subalgebra of $\mathcal{U}(\mathfrak{g})$ which we call *ghost centre*. Contrary to the case of basic classical Lie superalgebras where all central and anticentral elements are non-zero divisors, for $\mathfrak{g} = P(n-1)$ ($n \geq 3$) one has $\mathcal{Z}(\mathfrak{g})_{-n}\mathcal{Z}(\mathfrak{g})_{-n} = \mathcal{Z}(\mathfrak{g})_{-n}\mathcal{A}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})\mathcal{A}(\mathfrak{g}) = 0$. Thus the ghost centre $\widetilde{\mathcal{Z}}(\mathfrak{g}) := \mathcal{Z}(\mathfrak{g}) + \mathcal{A}(\mathfrak{g}) = k \oplus \mathcal{Z}(\mathfrak{g})_{-n} \oplus \mathcal{A}(\mathfrak{g})$ is an algebra with a trivial multiplication.

1.4. As in the cases of basic classical Lie superalgebras (that are the general linear, the special linear, and the orthosymplectic Lie superalgebras) we denote a lowest-order anticentral element by T (this is an element of $\mathcal{A}(\mathfrak{g})$ whose image in the symmetric algebra belongs to $\Lambda^{\mathrm{top}}\mathfrak{g}_{\bar{1}}$, the above condition determines T up to a scalar). For a basic classical Lie superalgebra $\mathfrak{p}$ the restrictions of Harish-Chandra projection P to the centre $\mathcal{Z}(\mathfrak{p})$ and to the anticentre $\mathcal{A}(\mathfrak{p})$ are injections. The image of $\mathcal{Z}(\mathfrak{p})$ is a subalgebra of the algebra of W-invariant polynomials $\mathcal{S}(\mathfrak{h})^{W.}$ described in [K2], [S], [BZV]. The image of $\mathcal{A}(\mathfrak{p})$ is simply $t\mathcal{S}(\mathfrak{h})^{W.}$ where $t := P(T)$ takes the form

$$t = \prod_{\alpha \in \Delta_1^+} (\alpha^\vee + (\alpha, \rho)), \tag{1}$$

see [G1]. The element t is "in charge" of strong typicality. This means that for $\lambda \in \mathfrak{h}^*$ satisfying $t(\lambda) \neq 0$ the category of $\mathfrak{p}$ representations whose central character coincides with the one of a simple module of the highest weight λ is equivalent to the the category of $\mathfrak{p}_{\bar{0}}$ representations with a certain central character (see [PS],[P],[G2]).

A specific feature of $\mathfrak{g} = P(n-1)$ is the lack of symmetry: the fact that the anticentral elements as well central elements without constant terms are homogeneous of degree $-n$ reflects the fact that the dimensions of $\mathfrak{g}_1$ and $\mathfrak{g}_{-1}$ are not equal (the difference is exactly $-n$). Since the Harish-Chandra projection of an element of non-zero degree vanishes, we substitute the Harish-Chandra projection by another map $P_{-n} : \mathcal{U}(\mathfrak{g})_{-n} \to \mathcal{S}(\mathfrak{h})$. The restrictions of this map to the centre $\mathcal{Z}(\mathfrak{g}) \cap \mathcal{U}(\mathfrak{g})_{-n}$ and to the anticentre $\mathcal{A}(\mathfrak{g})$ are again injections. Moreover both images are equal to $t\mathcal{S}(\mathfrak{h})^{W.}$ where $t := P_{-n}(T)$ can be written in the form

$$t = \prod_{\alpha \in \Delta_0^+} (\alpha^\vee + (\alpha, \rho) - 1). \tag{2}$$

Observe that the linear factors of t correspond to the odd coroots in the formula (1) and to the even coroots in the formula (2). This difference is connected to the following fact: if x, y are odd elements of a basic classical Lie superalgebra of the opposite weights β and $-\beta$ respectively then $[x, y]$ is proportional to the odd coroot $\beta^\vee$; by contrast, if x, y are odd elements of $\mathfrak{g} = P(n-1)$ meeting the same condition then $[x, y]$ is proportional to a certain *even* coroot (see (3)).

1.5. For a basic classical Lie superalgebra $\mathfrak{p}$ a central (or anticentral) element z annihilates a Verma module of the highest weight λ iff $P(z)(\lambda) = 0$. Since $P(\mathcal{A}(\mathfrak{p})) = P(T)\mathcal{S}(\mathfrak{h})^{W.}$ it follows that a Verma module annihilated by T is annihilated by *any* anticentral element. This last property remains true for $\mathfrak{g} = P(n-1)$; moreover, if a Verma module is annihilated by T then it is annihilated not only by all anticentral elements but also by all central elements without constant term (this immediately follows from Theorem 4.1 (iii)). However, the equality $P_{-n}(z)(\lambda) = 0$ does not force that $z \in \mathcal{Z}(\mathfrak{g})_n \cup \mathcal{A}(\mathfrak{g})$ annihilates a Verma module of the highest weight λ (see 4.3).

2. Preliminaries

2.1. **Notation.** Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra $P(n-1)$ endowed with the $\mathbb{Z}$-grading described above. Extend this $\mathbb{Z}$-grading to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ and denote by $\mathcal{U}(\mathfrak{g})_r$ $(r \in \mathbb{Z})$ the corresponding graded component. For any subspace N of $\mathcal{U}(\mathfrak{g})$ set $N_r := N \cap \mathcal{U}(\mathfrak{g})_r$. Denote by ad the adjoint action of $\mathcal{U}(\mathfrak{g})$ on itself.

For a superalgebra $\mathfrak{p}$ denote its universal enveloping algebra by $\mathcal{U}(\mathfrak{p})$. Since $\mathfrak{g}_{\pm 1}$ are supercommutative pure odd Lie superalgebras, $\mathcal{U}(\mathfrak{g}_{\pm 1})$ is canonically isomorphic to the exterior algebra $\Lambda \mathfrak{g}_{\pm 1}$.

2.1.1. Retain notation of 1.1. Denote by $\mathfrak{h}$ the set of diagonal matrices belonging to $\mathfrak{g}_0$, by $\mathfrak{n}_0^-$ (resp., $\mathfrak{n}_0^+$) the set of matrices whose upper-left block a is lower (resp., upper) triangular and both blocks b and c are equal to zero. Then $\mathfrak{g}_0 := \mathfrak{n}_0^- \oplus \mathfrak{h} \oplus \mathfrak{n}_0^+$ is a "standard" triangular decomposition of $\mathfrak{g}_0 \cong \mathfrak{sl}(n)$. As usual, it is convenient to present $\mathfrak{h}^*$ as the quotient of the n dimensional vector space with a basis $\{\varepsilon_i\}_1^n$ by the one-dimensional subspace spanned by $\sum_1^n \varepsilon_i$. Denote by W the Weyl group of $\mathfrak{g}_0$; it acts on the set $\{\varepsilon_i\}_1^n$ by the permutations. Denote by $(-,-)$ the canonical W-invariant bilinear form on $\mathfrak{h}^*$.

Set $\mathfrak{n}_1^+ := \mathfrak{g}_1$, $\mathfrak{n}_1^- := \mathfrak{g}_{-1}$ and $\mathfrak{n}^\pm := \mathfrak{n}_0^\pm + \mathfrak{n}_1^\pm$.

With this notation one has

$$\begin{array}{ll} \Delta_0^+ = \Omega(\mathfrak{n}_0^+) = & \{\varepsilon_i - \varepsilon_j\}_{1 \le i < j \le n}, \\ \Delta_1^+ = \Omega(\mathfrak{n}_1^+) = & \{-\varepsilon_i - \varepsilon_j\}_{1 \le i < j \le n}, \\ \Delta_1^- = \Omega(\mathfrak{n}_1^-) = & \{2\varepsilon_i; \varepsilon_i + \varepsilon_j\}_{1 \le i < j \le n} \end{array}$$

where $\Omega(N)$ stands for the multiset of $\mathfrak{h}$-weights of N. Set $\rho := \frac{1}{2}\sum_{\alpha \in \Delta_0^+} \alpha$.

For $r,s\in\{1,\dots,2n\}$ let $E_{r,s}$ be the $2n\times 2n$ matrix whose only non-zero entry is 1 at the place (r,s). For each pair (i,j) with $1\le i<j\le n$ set

$$\begin{aligned}
&e_{\varepsilon_i-\varepsilon_j}:=E_{i,j}-E_{n+j,n+i},\\
&f_{-\varepsilon_i+\varepsilon_j}:=E_{j,i}-E_{n+i,n+j},\\
&(\varepsilon_i-\varepsilon_j)^\vee=E_{i,i}-E_{j,j}-E_{n+i,n+i}+E_{n+j,n+j},\\
&x_{-\varepsilon_i-\varepsilon_j}:=E_{n+j,i}-E_{n+i,j},\\
&y_{\varepsilon_i+\varepsilon_j}:=E_{i,n+j}+E_{j,n+i},\\
&y_{2\varepsilon_i}:=E_{i,n+i}.
\end{aligned}$$

The set $\{e_\alpha\}_{\alpha\in\Delta_0^+}$ (resp., $\{f_{-\alpha}\}_{\alpha\in\Delta_0^+}$) forms a basis of $\mathfrak{n}_0^+$ (resp., $\mathfrak{n}_0^-$) and the set $\{x_\alpha\}_{\alpha\in\Delta_1^+}$ (resp., $\{y_\alpha\}_{\alpha\in\Delta_1^-}$) forms a basis of $\mathfrak{n}_1^+$ (resp., $\mathfrak{n}_1^-$). As always $(\varepsilon_i-\varepsilon_j)^\vee$ is the coroot corresponding to $(\varepsilon_i-\varepsilon_j)$ that is given by the formula $(\varepsilon_i-\varepsilon_j)^\vee(\mu)=(\varepsilon_i-\varepsilon_j,\mu)$ for any $\mu\in\mathfrak{h}^*$. One has

$$[e_{\varepsilon_i-\varepsilon_j},f_{-\varepsilon_i+\varepsilon_j}]=[x_{\varepsilon_i+\varepsilon_j},y_{-\varepsilon_i-\varepsilon_j}]=(\varepsilon_i-\varepsilon_j)^\vee. \tag{3}$$

Denote by J the set of odd positive roots (that is Δ_1^+) with a fixed total order. For a subset $J'\subseteq J$ denote by $x_{J'}$ and $y_{J'}$ respectively the products $\prod_{\beta\in J'}x_\beta,\prod_{\beta\in J'}y_{-\beta}$ taken with respect to the total order. Set also

$$\begin{aligned}
y_{I\setminus J}&:=\prod_{i=1}^n y_{-2\varepsilon_i},\\
y_I&:=y_{I\setminus J}y_J
\end{aligned}$$

Since $\mathfrak{n}_1^\pm$ are supercommutative one has $y_I=\pm y_Jy_{I\setminus J}$, $x_{J'}x_{J''}=\pm x_{J'\cup J''}$, $y_{J'}y_{J''}=\pm y_{J'\cup J''}$ if $J'\cap J''=\emptyset$ and $x_{J'}x_{J''}=y_{J'}y_{J''}=0$ if $J'\cap J''\neq\emptyset$. Note that the elements $y_I,y_J,y_{I\setminus J},x_J$ lie in $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$. Moreover $x_J\in\Lambda^{\mathrm{top}}\mathfrak{n}_1^+,y_I\in\Lambda^{\mathrm{top}}\mathfrak{n}_1^-$ and, in particular, x_J,y_I are $\mathfrak{g}_0$-invariant.

2.1.2. For $\mu\in\mathfrak{h}^*$ and a vector subspace $N\subset\mathcal{U}(\mathfrak{g})$ denote by $N|_\mu$ the corresponding $\mathfrak{h}$-weight subspace of N.

We identify $\mathcal{U}(\mathfrak{h})$ with $\mathcal{S}(\mathfrak{h})$. Define a twisted action of the Weyl group W on $\mathcal{S}(\mathfrak{h})$ by setting

$$w.p(\lambda)=p(w^{-1}(\lambda+\rho)-\rho)$$

for any $w\in W,p\in\mathcal{S}(\mathfrak{h}),\lambda\in\mathfrak{h}^*$.

Recall that the elements y_β ($\beta\in\Delta_1^-$) have degree -1 and the elements x_β ($\beta\in\Delta_1^+$) have degree 1; thus one has $\mathcal{U}(\mathfrak{g})_r=0$ if $r<-\#\Delta_1^-=-\frac{n(n+1)}{2}$ or $r>\#\Delta_1^+=\frac{n(n-1)}{2}$ (where # stands for the cardinality).

The universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ admits the canonical filtration given by $\mathcal{F}^k(\mathcal{U}(\mathfrak{g}))=\mathfrak{g}^k$; the associated graded algebra is the symmetric algebra $\mathcal{S}(\mathfrak{g})=\mathcal{S}(\mathfrak{g}_0)\Lambda\mathfrak{g}_{\bar 1}$. For any $u\in\mathcal{U}(\mathfrak{g})$ denote by $\operatorname{gr}u$ the image of u in the symmetric algebra $\mathcal{S}(\mathfrak{g})$.

2.1.3. *Verma modules.* For $\lambda \in \mathfrak{h}^*$ denote by k_λ a one-dimensional $(\mathfrak{h}+\mathfrak{n}^+)$-module such that $\mathfrak{n}^+ v = 0$ and $hv = \lambda(h)v$ for any $h \in \mathfrak{h}, v \in k_\lambda$. Define a Verma module $\widetilde{M}(\lambda)$ by setting $\widetilde{M}(\lambda) := \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b})} k_\lambda$. Call the image of a fixed non-zero element of k_λ in $\widetilde{M}(\lambda)$ *a canonical generator* of $\widetilde{M}(\lambda)$. Similarly, denote by $M(\lambda)$ a Verma $\mathfrak{g}_0$-module of the highest weight λ.

Suppose that $\lambda \in \mathfrak{h}^*$ is such that $(\lambda+\rho, \alpha)$ is a positive integer for some $\alpha \in \Delta_0^+$. Let v be a canonical generator of $\widetilde{M}(\lambda)$; then $\mathcal{U}(\mathfrak{g}_0)v \cong M(\lambda)$ contains an $\mathfrak{n}_0$-invariant vector uv (with $u \in \mathcal{U}(\mathfrak{g}_0)$) of the weight $s_\alpha.\lambda$ (here $s_\alpha \in W$ is the reflection corresponding to the root α). The vector uv is $\mathfrak{n}$-invariant because $\mathfrak{n}_1^+$ is $\operatorname{ad}\mathfrak{g}_0$-invariant. Since $u \in \mathcal{U}(\mathfrak{g}_0)$ is a non-zero divisor in $\mathcal{U}(\mathfrak{g})$, the vector uv generates a submodule isomorphic to $\widetilde{M}(s_\alpha.\lambda)$. Hence $\widetilde{M}(s_\alpha.\lambda) \subset \widetilde{M}(\lambda)$.

Caution: The module $\widetilde{M}(\lambda)$ is never simple because $[y_{2\varepsilon_1}, \mathfrak{n}^+] \subset \mathfrak{n}^+$ and so for a canonical generator $v \in \widetilde{M}(\lambda)$ the subspace $\mathcal{U}(\mathfrak{g})(y_{2\varepsilon_1}v)$ is a proper submodule.

2.1.4. *Projections P_+, P.* Denote by P_+ the projection $\mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{n}^- + \mathfrak{h})$ with respect to the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{n}^- + \mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$ and by P the Harish-Chandra projection $P : \mathcal{U}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{h})$ with respect to the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathfrak{n}^-\mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})\mathfrak{n}^+)$. The restrictions of P_+ and P to the subalgebra $\mathcal{U}(\mathfrak{g})_0^{\mathfrak{h}}$ coincide and give an algebra homomorphism. Note that the restrictions of P_+ and P to the subalgebra $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ do not coincide and are not algebra homomorphisms — for instance, both y_J, x_J lie in $\mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$ and $P(y_J) = 0$, but $P_+(y_J) = y_J$ and $P(x_J y_J) \neq 0$ by Lemma 3.1 below.

The inclusion $\mathcal{U}(\mathfrak{g})\mathfrak{n}^+\mathcal{U}(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g})\mathfrak{n}^+$ implies that

$$P_+(ab) = P_+(a)P(b), \quad \forall a \in \mathcal{U}(\mathfrak{g}), b \in \mathcal{U}(\mathfrak{g})_0^{\mathfrak{h}}. \tag{4}$$

2.2. **Anticentre $\mathcal{A}(\mathfrak{g})$.** The anticentre $\mathcal{A}(\mathfrak{g})$ can be defined as the set of invariants of $\mathcal{U}(\mathfrak{g})$ with respect to a twisted adjoint action: $\mathcal{A}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^{\operatorname{ad}' \mathfrak{g}}$ where ad' is given by the formula

$$(\operatorname{ad}' g)u = gu - (-1)^{d(g)(d(u)+1)}ug$$

for all homogeneous $g \in \mathfrak{g}, u \in \mathcal{U}(\mathfrak{g})$ (here $d(.)$ stands for the $\mathbb{Z}_2$-degree of the element that is $d(u) = 0$ for $u \in \mathcal{U}(\mathfrak{g})_{\bar{0}}$ and $d(u) = 1$ for $u \in \mathcal{U}(\mathfrak{g})_{\bar{1}}$). Note that the odd elements of the anticentre $\mathcal{A}(\mathfrak{g})$ commute with all element of $\mathcal{U}(\mathfrak{g})$ and the even elements of $\mathcal{A}(\mathfrak{g})$ commute with the even elements of $\mathcal{U}(\mathfrak{g})$ and anticommute with the odd ones.

By Theorem 3.3 of [G1], $\operatorname{gr}\mathcal{A}(\mathfrak{g}) = \Lambda^{\mathrm{top}}\mathfrak{g}_{\bar{1}} \operatorname{gr}(\mathcal{Z}(\mathfrak{g}_0))$. This can be rewritten as

$$\operatorname{gr}\mathcal{A}(\mathfrak{g}) = \operatorname{gr}(x_J y_I \mathcal{Z}(\mathfrak{g}_0))$$

because $\operatorname{gr} x_J y_I$ spans $\Lambda^{\mathrm{top}}\mathfrak{g}_{\bar{1}}$. Since $\mathcal{A}(\mathfrak{g})$ is a graded subspace of $\mathcal{U}(\mathfrak{g})$, the elements of $\mathcal{A}(\mathfrak{g})$ have degree equal to $\#J - \#I = -n$. Therefore,

$$\mathcal{A}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})_{-n}.$$

3. Useful assertions.

The element

$$t := P(x_J y_J)$$

plays an important role in the description of the centre and seems to be instrumental in the study of representations of $\mathfrak{g}$.

3.1. **Proposition.** *One has*

$$t = \pm \prod_{\alpha\in\Delta_0^+} (\alpha^\vee + (\alpha,\rho) - 1).$$

Proof. The proof has two steps. As a first step, let us prove by induction that for all $r = 2, \dots, n$ one has

$$P\Big(\prod_{1<j\leq r} x_{-\varepsilon_1-\varepsilon_j} \prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j}\Big) = \pm \prod_{1<j\leq r} \Big((\varepsilon_1-\varepsilon_j)^\vee + j - 2\Big). \tag{5}$$

For $r = 2$ the assertion immediately follows from the equality (3). For the induction step observe that

$$\begin{aligned} &P\Big(\textstyle\prod_{1<j\leq r+1} x_{-\varepsilon_1-\varepsilon_j} \prod_{1<j\leq r+1} y_{\varepsilon_1+\varepsilon_j}\Big) = \pm P\Big(\prod_{1<j\leq r} x_{-\varepsilon_1-\varepsilon_j}\cdot x_{-\varepsilon_1-\varepsilon_{r+1}}\cdot y_{\varepsilon_1+\varepsilon_{r+1}}\prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j}\Big) \\ &= \pm P\Big(\textstyle\prod_{1<j\leq r} x_{-\varepsilon_1-\varepsilon_j}((\varepsilon_1-\varepsilon_{r+1})^\vee - y_{\varepsilon_1+\varepsilon_{r+1}} x_{-\varepsilon_1-\varepsilon_{r+1}})\prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j}\Big) \end{aligned} \tag{6}$$

It is easy to see that $\mathcal{U}(\mathfrak{n}^-)|_\mu \neq 0$ for $\mu = \sum_i c_i\varepsilon_i$ only if $c_s + c_{s+1} + \dots + c_n \geq 0$ for all $s = 1, \dots, n$. In particular, $\mathcal{U}(\mathfrak{n}^- + \mathfrak{h})$ does not contain a non-zero element of weight $(r-2)\varepsilon_1 + \varepsilon_2 + \dots \varepsilon_r - \varepsilon_{r+1}$. Therefore the element $x_{-\varepsilon_1-\varepsilon_{r+1}}\prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j}$ belongs to $\mathcal{U}(\mathfrak{g})\mathfrak{n}^+$ and thus (6) implies

$$P\Big(\prod_{1<j\leq r+1} x_{-\varepsilon_1-\varepsilon_j} \prod_{1<j\leq r+1} y_{\varepsilon_1+\varepsilon_j}\Big) = \pm P\Big(\prod_{1<j\leq r} x_{-\varepsilon_1-\varepsilon_j}(\varepsilon_1-\varepsilon_{r+1})^\vee \prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j}\Big).$$

Taking into account that

$$(\varepsilon_1-\varepsilon_{r+1})^\vee \prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j} = \prod_{1<j\leq r} y_{\varepsilon_1+\varepsilon_j}((\varepsilon_1-\varepsilon_{r+1})^\vee + r - 1),$$

one obtains the required equality (5) by induction.

The second step of the proof is to show that for $r = 1, \dots, n-1$ the term

$$t_r := P\Big(\prod_{n-r\leq i<j\leq n} x_{-\varepsilon_i-\varepsilon_j} \prod_{n-r\leq i<j\leq n} y_{\varepsilon_i+\varepsilon_j}\Big)$$

is given by the formula

$$t_r = \pm \prod_{n-r\leq i<j\leq n} \Big((\varepsilon_i-\varepsilon_j)^\vee + (\varepsilon_i-\varepsilon_j,\rho) - 1\Big). \tag{7}$$

We again proceed by induction. For $r=1$ the equality (3) gives

$$t_1 = P(x_{-\varepsilon_{n-1}-\varepsilon_n} y_{\varepsilon_{n-1}+\varepsilon_n}) = (\varepsilon_{n-1}-\varepsilon_n)^\vee = \Big((\varepsilon_{n-1}-\varepsilon_n)^\vee + (\varepsilon_{n-1}-\varepsilon_n,\rho) - 1\Big)$$

as required.

Assume that (7) holds for some $r<n$. For $s=1,\dots,n-1$ let $\mathfrak{n}^-_{(s)}$ (resp., $\mathfrak{n}^+_{(s)}$) be the Lie subalgebra of $\mathfrak{n}^-$ (resp., $\mathfrak{n}^+$) spanned by the elements $f_{-\varepsilon_i+\varepsilon_j}, y_{2\varepsilon_i}, y_{\varepsilon_i+\varepsilon_j}$ (resp., $e_{\varepsilon_i-\varepsilon_j}, x_{-\varepsilon_i-\varepsilon_j}$) with $n-s\leq i<j\leq n$. Set $X_{(r)} := \prod_{n-r\leq j\leq n} x_{-\varepsilon_{n-r-1}-\varepsilon_j}$. Let us show that $(\operatorname{ad}\mathfrak{n}^-_{(r)})X_{(r)} \in \mathfrak{n}^-\mathcal{U}(\mathfrak{g})$. Indeed, fix a pair (i,j) with $n-r\leq i<j\leq n$. The equality $(\operatorname{ad} f_{-\varepsilon_i+\varepsilon_j})X_{(r)} = 0$ immediately follows from the supercommutativity of $\mathfrak{n}^+$. Furthermore it is easy to see that a homogeneous element of degree m belonging to the algebra $\mathcal{U}(\mathfrak{h}+\mathfrak{n}^+_{(r+1)})$ has a weight of the form $\sum_{j=n-r-1}^n c_j\varepsilon_j$ with $c_{n-r-1}\geq -m$. Combining the facts that the terms $(\operatorname{ad} y_{\varepsilon_i+\varepsilon_j})X_{(r)}$, $(\operatorname{ad} y_{2\varepsilon_i})X_{(r)}$ lie in $\mathcal{U}' := \mathcal{U}(\mathfrak{n}^-_{(r+1)}+\mathfrak{h}+\mathfrak{n}^+_{(r+1)})$, have degree r and weights of the form $(-(r+1)\varepsilon_{n-r-1}+\dots)$, one concludes that these terms lie in $\mathfrak{n}^-\mathcal{U}(\mathfrak{g})$ because $\mathcal{U}' = \mathfrak{n}^-_{(r+1)}\mathcal{U}' \oplus \mathcal{U}(\mathfrak{n}^+_{(r+1)}+\mathfrak{h})$ and the rth graded component of the algebra $\mathcal{U}(\mathfrak{n}^+_{(r+1)}+\mathfrak{h})$ does not contain non-zero elements of the weights of the above form.

One has

$$\begin{aligned} t_{r+1} &= P\Big(\textstyle\prod_{n-r-1\leq i<j\leq n} x_{-\varepsilon_i-\varepsilon_j} \prod_{n-r-1\leq i<j\leq n} y_{\varepsilon_i+\varepsilon_j}\Big) \\ &= \pm P\Big(X_{(r)} \textstyle\prod_{n-r\leq i<j\leq n} x_{-\varepsilon_i-\varepsilon_j} \prod_{n-r\leq i<j\leq n} y_{\varepsilon_i+\varepsilon_j} \prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) \\ &= \pm P\Big(X_{(r)}(t_r + \textstyle\sum_s u_s^- u_s) \prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) \end{aligned} \tag{8}$$

where each term u_s^- belongs to $\mathfrak{n}^-_{(r)}$ and u_s are some elements of $\mathcal{U}(\mathfrak{g})$. As we have shown above $(\operatorname{ad}\mathfrak{n}^-_{(r)})X_{(r)}$ lies in $\mathfrak{n}^-\mathcal{U}(\mathfrak{g})$ and thus $P\Big(X_{(r)}u_s^-u_s\prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) = 0$ for any index s. Therefore

$$\begin{aligned} t_{r+1} &= \pm P\Big(X_{(r)} t_r \textstyle\prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) \\ &= \pm P\Big(X_{(r)} \textstyle\prod_{n-r\leq i<j\leq n}\Big((\varepsilon_i-\varepsilon_j)^\vee + (\varepsilon_i-\varepsilon_j,\rho)-1\Big) \prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) \\ &= \pm P\Big(\textstyle\prod_{n-r\leq j\leq n} x_{-\varepsilon_{n-r-1}-\varepsilon_j} \prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) \prod_{n-r\leq i<j\leq n}\Big((\varepsilon_i-\varepsilon_j)^\vee + (\varepsilon_i-\varepsilon_j,\rho)-1\Big). \end{aligned}$$

The formula (5) implies that

$$\begin{aligned} P\Big(\textstyle\prod_{n-r\leq j\leq n} x_{-\varepsilon_{n-r-1}-\varepsilon_j} \prod_{n-r\leq j\leq n} y_{\varepsilon_{n-r-1}+\varepsilon_j}\Big) &= \textstyle\prod_{n-r\leq j\leq n}\Big((\varepsilon_{n-r-1}-\varepsilon_j)^\vee + j-(n-r-1)-1\Big) \\ &= \textstyle\prod_{n-r\leq j\leq n}\Big((\varepsilon_{n-r-1}-\varepsilon_j)^\vee + (\varepsilon_{n-r-1}-\varepsilon_j,\rho)-1\Big). \end{aligned}$$

Hence

$$t_{r+1} = \pm \prod_{n-r-1\leq i<j\leq n}\Big((\varepsilon_i-\varepsilon_j)^\vee + (\varepsilon_i-\varepsilon_j,\rho)-1\Big)$$

as required. Finally observing that $t=\pm t_{n-1}$ one completes the proof. $\square$

3.2. **Proposition.** *For any Zariski dense subset Ω of $\mathfrak{h}^*$ one has*

$$\bigcap_{\lambda\in\Omega} \operatorname{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\lambda) = 0.$$

Proof. Let Ω be a Zariski dense subset of $\mathfrak{h}^*$. By Proposition 3.1, $P(x_J y_J)$ is a non-zero polynomial and so the set $\Omega' := \Omega \cap \{\lambda \in \mathfrak{h}^* |\ P(x_J y_J)(\lambda) \neq 0\}$ is also Zariski dense in $\mathfrak{h}^*$. Assume that $N := \bigcap_{\lambda\in\Omega'} \operatorname{Ann} \widetilde{M}(\lambda)$ is non-zero. One has $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_0 + \mathfrak{n}_1^-)\mathcal{U}(\mathfrak{n}_1^+) = \mathcal{U}(\mathfrak{g}_0 + \mathfrak{n}_1^-)\Lambda\mathfrak{n}_1^+$. Since N is a right ideal, it contains a non-zero element of the form ux_J where $u \in \mathcal{U}(\mathfrak{g}_0 + \mathfrak{n}_1^-)$. Let λ be an element of Ω' and v be a canonical generator of $\widetilde{M}(\lambda)$. Then

$$0 = ux_J(\mathcal{U}(\mathfrak{g}_0)y_J v) = u\mathcal{U}(\mathfrak{g}_0)x_J y_J v = u\mathcal{U}(\mathfrak{g}_0)P(x_J y_J)(\lambda)v.$$

Note that $P(x_J y_J)(\lambda)v \in k^* v$ since $\lambda \in \Omega'$ and thus the space $\mathcal{U}(\mathfrak{g}_0)P(x_J y_J)(\lambda)v$ is isomorphic to a Verma $\mathfrak{g}_0$-module $M(\lambda)$. Writing $u = \sum_{S\subseteq I} y_S u_S$ where $u_S \in \mathcal{U}(\mathfrak{g}_0)$, one concludes that each u_S annihilates $M(\lambda)$. However, by [D],

$$\bigcap_{\lambda\in\Omega'} \operatorname{Ann}_{\mathcal{U}(\mathfrak{g}_0)} M(\lambda) = 0$$

and thus all terms u_S are equal to zero. This gives the required contradiction. □

3.3. **The map P_{-n}.** The Harish-Chandra projection P annihilates the homogeneous component $\mathcal{U}(\mathfrak{g})_r$ for any $r \neq 0$. In particular, $P(\mathcal{Z}(\mathfrak{g})_{-n}) = 0$ and thus P itself is useless for a description of the centre $\mathcal{Z}(\mathfrak{g})$. For this purpose it is convenient to use a map $P_{-n} : \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}_{-n} \to \mathcal{U}(\mathfrak{h})$ constructed below. For $a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}_{-n}$ one has $y_J a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g})_{\#\Delta_1^-} = y_I \mathcal{U}(\mathfrak{g}_0)^{\mathfrak{h}}$. This allows us to define the linear map $P_{-n} : \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}_{-n} \to \mathcal{U}(\mathfrak{h})$ by the condition

$$P_+(y_J a) = y_I P_{-n}(a)$$

for any $a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}_{-n}$.

3.3.1. **Lemma.** *The restrictions of P_{-n} to $\mathcal{Z}(\mathfrak{g})_{-n}$ and to $\mathcal{A}(\mathfrak{g})$ are (vector space) monomorphisms.*

Proof. Take a non-zero element $a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})_{-n}$. Combining Proposition 3.1 and Proposition 3.2 one concludes the existence of $\lambda \in \mathfrak{h}^*$ such that $P(x_J y_J)(\lambda) \neq 0$ and $a\widetilde{M}(\lambda) \neq 0$. Let v be a canonical generator of $\widetilde{M}(\lambda)$; the condition $P(x_J y_J)(\lambda) \neq 0$ implies that $x_J y_J v \in k^* v$ and so the vector $y_J v$ generates $\widetilde{M}(\lambda)$. Since a is either central or anti-central, $a\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})a$ and so the condition $a\widetilde{M}(\lambda) \neq 0$ forces $a(y_J v) \neq 0$. One has $a(y_J v) = P_+(a y_J)v = \pm y_I P_{-n}(a)v$. Hence $P_{-n}(a) \neq 0$ as required. □

3.3.2. Set

$$T := (\mathrm{ad}'\, x_J)y_I.$$

Lemma. *For any $a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})$ one has*

$$P_+(a)t = P_+(T)P_{-n}(a). \tag{9}$$

Proof. Any $a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})$ commutes with the term $x_J y_J$ and so $P_+(x_J y_J a) = P_+(a)P(x_J y_J)$ by (4). On the other hand,

$$P_+(x_J y_J a) = P_+(x_J P_+(y_J a)) = P_+(x_J y_I P_{-n}(a)) = P_+(x_J y_I)P_{-n}(a).$$

Therefore $P_+(x_J y_I)P_{-n}(a) = P_+(a)P(x_J y_J) = P_+(a)t$ for any $a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})$. It is easy to see from the definition of ad' that $P_+(T) = P_+(x_J y_I)$. The assertion follows. $\square$

4. Main result

Recall that $\mathcal{Z}(\mathfrak{g}) = k \oplus \mathcal{Z}(\mathfrak{g})_{-n}$ (see [Sch]). In this section we prove the following theorem which describes $\mathcal{Z}(\mathfrak{g})_{-n}$ and $\mathcal{A}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})_{-n}$.

4.1. **Theorem.** *i) The map $\phi : \mathcal{Z}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g})$ given by $z \mapsto (\mathrm{ad}\, x_J)(y_I z)$ induces a linear isomorphism $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{Z}(\mathfrak{g})_{-n}$.*

ii) The map $\phi' : \mathcal{Z}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g})$ given by $z \mapsto (\mathrm{ad}'\, x_J)(y_I z)$ induces a linear isomorphism $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{A}(\mathfrak{g})$.

iii) One has

$$P_+(\phi(z)) = P_+(\phi'(z)) = P_+(T)P(z),$$
$$P_{-n}(\phi(z)) = P_{-n}(\phi'(z)) = tP(z).$$

iv) The restriction of P_{-n} to $\mathcal{Z}(\mathfrak{g})_{-n}$ and to $\mathcal{A}(\mathfrak{g})$ induces linear isomorphisms

$$\mathcal{Z}(\mathfrak{g})_{-n} \xrightarrow{\sim} t\mathcal{S}(\mathfrak{h})^{W\cdot}, \quad \mathcal{A}(\mathfrak{g}) \xrightarrow{\sim} t\mathcal{S}(\mathfrak{h})^{W\cdot}.$$

4.2. **Proof of Theorem 4.1.**

4.2.1. *The image of ϕ lies in $\mathcal{Z}(\mathfrak{g})$.* To show that $\operatorname{Im}\phi \subseteq \mathcal{Z}(\mathfrak{g})$ fix $z \in \mathcal{Z}(\mathfrak{g}_0)$. First, let us check that $(\operatorname{ad} y_{2\varepsilon_1})\phi(z) = 0$. One has $(\operatorname{ad} y_{2\varepsilon_1})\phi(z) = (\operatorname{ad} y_{2\varepsilon_1} x_J)(y_I z)$. The element $y_{2\varepsilon_1} x_J$ belongs to $\mathcal{U}(\mathfrak{g})(\mathfrak{g}_0 + \mathfrak{n}_1^-)$ because $\Lambda\mathfrak{n}_1^+$ does not contain a non-zero element whose weight and degree coincide respectively with the weight and the degree of $y_{2\varepsilon_1} x_J$. Since the element $y_I z$ is $\operatorname{ad}(\mathfrak{g}_0 + \mathfrak{n}_1^-)$-invariant one obtains $(\operatorname{ad} y_{2\varepsilon_1} x_J)(y_I z) = 0$.

Since x_J, y_I and z are $\operatorname{ad}\mathfrak{g}_0$-invariant, $\phi(z) = (\operatorname{ad} x_J)(y_I z)$ is $\operatorname{ad}\mathfrak{g}_0$-invariant. Combining the equalities $(\operatorname{ad}\mathfrak{g}_0)\phi(z) = (\operatorname{ad} y_{2\varepsilon_1})\phi(z) = 0$ and $\mathfrak{n}_1^- = [\mathfrak{g}_0, y_{2\varepsilon_1}]$, one concludes $(\operatorname{ad}\mathfrak{n}_1^-)\phi(z) = 0$. The remaining equality $(\operatorname{ad}\mathfrak{n}_1^+)\phi(z) = 0$ immediately follows from the supercommutativity of $\mathfrak{n}_1^+$. Hence $\phi(z) \in \mathcal{Z}(\mathfrak{g})$.

4.2.2. *Proof of (ii).* Replacing the adjoint action ad by the twisted adjoint action ad' and repeating the above reasoning one concludes that $\operatorname{Im}\phi' \subseteq \mathcal{U}(\mathfrak{g})^{\operatorname{ad}'\mathfrak{g}} = \mathcal{A}(\mathfrak{g})$.

Remark that $(\operatorname{ad}' g)u = 2gu - (\operatorname{ad} g)u$ for all $g \in \mathfrak{g}_1, u \in \mathcal{U}(\mathfrak{g})$ and that $\operatorname{gr} u$ and $\operatorname{gr}((\operatorname{ad} g)u)$ have the same degree in the symmetric algebra $\mathcal{S}(\mathfrak{g})$. Therefore

$$\operatorname{gr}((\operatorname{ad}' g)u) = 2(\operatorname{gr} g)(\operatorname{gr} u),\ \forall g \in \mathfrak{g}_1, u \in \mathcal{U}(\mathfrak{g}) \text{ s.t. } \operatorname{gr}(gu) = (\operatorname{gr} g)(\operatorname{gr} u).$$

This implies $\operatorname{gr}\phi'(z) = \operatorname{gr}((\operatorname{ad}' x_J)(y_I z)) = 2^{\#J}\operatorname{gr}(x_J y_I z)$ for any $z \in \mathcal{Z}(\mathfrak{g}_0)$. In particular, $\operatorname{gr}\phi'(z) \neq 0$ and so ϕ' is a monomorphism. Moreover, by 2.2, $\operatorname{gr}\mathcal{A}(\mathfrak{g}) = \operatorname{gr}(x_J y_I \mathcal{Z}(\mathfrak{g}_0))$ that is $\operatorname{gr}\mathcal{A}(\mathfrak{g}) = \operatorname{gr}(\operatorname{Im}\phi')$. This proves that ϕ' is an isomorphism.

4.2.3. *Proof of (iii).* Combining Lemma 3.3.1 and the definition of P_{-n}, one concludes that $P_+(a) \neq 0$ for any non-zero $a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})$. Recall that $T = (\operatorname{ad}' x_J)y_I = \phi'(1)$. Applying the formula (4) one obtains

$$P_+(\phi'(z)) = P_+((\operatorname{ad}' x_J)(y_I z)) = P_+(x_J y_I z) = P_+(x_J y_I)P(z) = P_+(T)P(z).$$

Similarly $P_+(\phi(z)) = P_+(T)P(z)$. Taking $a := \phi(z)$ in the formula (9) one gets

$$P_+(T)P(z)t = P_+(T)P_{-n}(\phi(z)) = P_+(T)P_{-n}(\phi'(z)).$$

Using the fact that the non-zero elements of $\mathcal{U}(\mathfrak{h})$ are non-zero divisors in $\mathcal{U}(\mathfrak{g})$ and the inequality $P_+(T) \neq 0$, one obtains

$$P_{-n}(\phi(z)) = P_{-n}(\phi'(z)) = tP(z).$$

This completes the proof of (iii).

4.2.4. *Proof of (iv).* It is well-known that the restriction of P to $\mathcal{Z}(\mathfrak{g}_0)$ induces the Harish-Chandra (algebra) isomorphism $\mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{S}(\mathfrak{h})^{W.}$. Combining already proven assertions (ii) and (iii) of Theorem 4.1, one concludes that the restriction of P_{-n} to $\mathcal{A}(\mathfrak{g})$ induces a linear isomorphism $\mathcal{A}(\mathfrak{g}) \xrightarrow{\sim} t\mathcal{S}(\mathfrak{h})^{W.}$.

Combining Lemma 3.3.1 and (iii) one concludes that the restriction of P_{-n} to $\mathcal{Z}(\mathfrak{g})_{-n}$ is an injective map whose image contains $t\mathcal{S}(\mathfrak{h})^{W.}$. Thus to show that the restriction of P_{-n} to $\mathcal{Z}(\mathfrak{g})$ induces a linear isomorphism $\mathcal{Z}(\mathfrak{g}) \xrightarrow{\sim} t\mathcal{S}(\mathfrak{h})^{W.}$, it remains to check that $P_{-n}(a) \in t\mathcal{S}(\mathfrak{h})^{W.}$ for any $a \in \mathcal{Z}(\mathfrak{g})_{-n}$. We proceed in two steps. First, we verify that $P_{-n}(a) \in$

$t\mathcal{S}(\mathfrak{h})$. By (9) $P_+(a) = P_+(T)(P_{-n}(a)/t)$. Write $P_+(T) = \sum u_r s_r$ where u_r are elements of a basis of $\mathcal{U}(\mathfrak{n}^-)$ and s_r are elements of $\mathcal{S}(\mathfrak{h})$. Then $\sum u_r s_r(P_{-n}(a)/t) = P_+(a) \in \mathcal{U}(\mathfrak{n}^- + \mathfrak{h})$ and so $s_r(P_{-n}(a)/t) \in \mathcal{S}(\mathfrak{h})$ for all k. Lemma 4.2.5 asserts that $P_+(T) \notin \mathcal{U}(\mathfrak{b}^-)t$ or, in other words, that $s_r \notin \mathcal{S}(\mathfrak{h})t$ for some r. This gives that $P_{-n}(a)/t \in \mathcal{S}(\mathfrak{h})$ and completes the first step. In the second step (Lemma 4.2.6) we show that the fraction $P_{-n}(a)/t$ is W.-invariant.

4.2.5. **Lemma.** *The element $P_+(T)$ does not belong to $\mathcal{U}(\mathfrak{b}^-)t$.*

Proof. This follows from 2.1.3 and the fact that t is not W.-invariant. Indeed, if v is a canonical generator of a Verma module $\widetilde{M}(\lambda)$ then

$$y_J T v = P_+(y_J T)v = y_I P_{-n}(T)v = y_I t(\lambda)v \tag{10}$$

and thus $T\widetilde{M}(\lambda) \neq 0$ provided $t(\lambda) \neq 0$. By 3.1, $t(\lambda) = 0$ iff $(\lambda + \rho, \alpha) = 1$ for some $\alpha \in \Delta_0^+$. Take μ such that $(\mu + \rho, \varepsilon_1 - \varepsilon_2) = 1$ and $(\mu + \rho, \alpha) \notin \mathbb{Z}$ for the other roots $\alpha \in \Delta_0^+$. A Verma module $\widetilde{M} := \widetilde{M}(\mu)$ contains a submodule isomorphic to a Verma module $\widetilde{M}' := \widetilde{M}(\mu - (\varepsilon_1 - \varepsilon_2))$, see 2.1.3. Since $(\mu - (\varepsilon_1 - \varepsilon_2) + \rho, \alpha) \neq 1$ for all $\alpha \in \Delta_0^+$, one has $T\widetilde{M}' \neq 0$ and, consequently, $T\widetilde{M} \neq 0$. Since T is anticentral, this implies that T does not annihilate a canonical generator of $\widetilde{M}$ that is $P_+(T)(\mu) \neq 0$. Taking into account that $t(\mu) = 0$ one obtains the required assertion. □

As we explained in 4.2.4, the above lemma implies that $P_{-n}(\mathcal{Z}(\mathfrak{g})) \subseteq t\mathcal{S}(\mathfrak{h})$. The following lemma demonstrates that $P_{-n}(\mathcal{Z}(\mathfrak{g})) \subseteq t\mathcal{S}(\mathfrak{h})^{W.}$.

4.2.6. **Lemma.** *For any $a \in \mathcal{Z}(\mathfrak{g})_{-n}$ the fraction $P_{-n}(a)/t$ is W.-invariant.*

Proof. Fix $\alpha \in \Delta_0^+$ and let $s \in W$ be the corresponding reflection. Let $\lambda \in \mathfrak{h}^*$ be such that $t(\lambda) \neq 0, t(s.\lambda) \neq 0$ and that $(\lambda + \rho, \alpha)$ is a positive integer. Observe that the set of suitable λ's is Zariski dense in $\mathfrak{h}^*$. Let v be a canonical generator of $\widetilde{M}(\lambda)$ and $v' = uv$ $(u \in \mathcal{U}(\mathfrak{g}_0))$ be a canonical generator of $\widetilde{M}(s.\lambda) \subset \widetilde{M}(\lambda)$ (see 2.1.3). Take $a \in \mathcal{Z}(\mathfrak{g})_{-n}$. One has $av = P_+(a)v,\ av' = P_+(a)v'$. Applying (9) one obtains

$$\begin{aligned} av &= cTv, &\quad &\text{where } c := P_{-n}(a)(\lambda)/t(\lambda),\\ av' &= c'Tv', &\quad &\text{where } c' := P_{-n}(a)(s.\lambda)/t(s.\lambda). \end{aligned}$$

On the other hand,

$$av' = auv = uav = cuTv = cTuv = cTv'$$

By (10), the inequality $t(\lambda) \neq 0$ implies $Tv \neq 0$. Thus $c = c'$ and the assertion follows. □

4.2.7. Now (iv) follows from 4.2.4. Combining 4.2.1 with (iii) and (iv) one concludes (i). Theorem 4.1 is proven.

4.3. *Remark.* Lemma 4.2.6 might let one think that P_{-n} plays for $\mathcal{Z}(\mathfrak{g})_{-n}$ a role similar to the one played by the Harish-Chandra projection for the centre of the enveloping algebra of semisimple Lie algebra. However, Lemma 4.2.5 shows that $t(\lambda) = 0$ does not imply $\phi(1)\widetilde{M}(\lambda) = 0$ even though $P_{-n}(\phi(1)) = t$.

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