THE CENTRE OF A SIMPLE P-TYPE LIE SUPERALGEBRA.

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ABSTRACT. We describe the centre of a simple Lie superalgebra of type $P(n)$. The description is based on the notion of anticentre.

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1. Introduction

1.1. The Lie superalgebra $\mathfrak{g} := P(n - 1)$ is described in [K1]. It consists of the matrices of the block form

$$
\begin{pmatrix}
    a & b \\
    -c & -a^T
\end{pmatrix}
$$

where $a, b, c$ are $n \times n$-matrices over a base field $k$ of characteristic zero such that $a$ is traceless, $b$ is symmetric and $c$ is skew-symmetric. The even part $\mathfrak{g}_{\text{ev}}$ consists of the matrices with $b = c = 0$ and the odd part $\mathfrak{g}_{\text{od}}$ consists of the matrices with $a = 0$. The Lie bracket on $\mathfrak{g}$ is given by the formula $[x, y] = xy - yx$ if $x$ or $y$ is even and $[x, y] = xy + yx$ if both $x$ and $y$ are odd. The Lie superalgebra $P(n - 1)$ is simple for $n \geq 3$; its even part $\mathfrak{g}_{\text{ev}}$ is a simple Lie algebra $\mathfrak{sl}(n)$. The Lie superalgebra $\mathfrak{g}$ admits also a $\mathbb{Z}$-grading $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$ where $\mathfrak{g}_0$ coincides with $\mathfrak{g}_{\text{ev}}$, $\mathfrak{g}_1$ consists of the matrices with $a = b = 0$ and $\mathfrak{g}_{-1}$ consists of the matrices with $a = c = 0$. The last grading induces a $\mathbb{Z}$-grading on the universal enveloping superalgebra $U(\mathfrak{g})$.

The central elements of the $P$-type Lie superalgebras were investigated by Scheunert in [Sch]. It was shown that any central element without constant term is of degree $-n$ (with respect to the $\mathbb{Z}$-grading above) and its order is at least $\frac{1}{2} n(n + 1)$. The first statement has the following important consequences. First, it shows that the centre $\mathcal{Z}(\mathfrak{g})$ of $U(\mathfrak{g})$ is highly degenerate: the product of any two central elements without constant term vanishes. Second, this implies that such central elements annihilate all completely reducible representations. For $n = 3$ Scheunert constructed the lowest-order central element (of order 6).

1.2. Our goal is to determine the centre $\mathcal{Z}(\mathfrak{g})$. It turns out that for $P$-type Lie superalgebras the structure of the centre $\mathcal{Z}(\mathfrak{g})$ is similar to the structure of the anticentre $\mathcal{A}(\mathfrak{g})$. Recall that the even elements of the centre $\mathcal{Z}(\mathfrak{g})$ commute with all element of $U(\mathfrak{g})$ and the odd elements of $\mathcal{Z}(\mathfrak{g})$ commute with the even elements of $U(\mathfrak{g})$ and anticommute with the odd ones. By contrast, the odd elements of the anticentre $\mathcal{A}(\mathfrak{g})$ commute with all element of $U(\mathfrak{g})$ and the even elements of $\mathcal{A}(\mathfrak{g})$ commute with the even elements of $U(\mathfrak{g})$ and anticommute with the odd ones. In the situation when any even element of the Lie superalgebra $\mathfrak{p} = \mathfrak{p}_{\text{ev}} \oplus \mathfrak{p}_{\text{od}}$ annihilates the one dimensional module $\Lambda^\text{top} \mathfrak{p}_{\text{od}}$, the anticentre $\mathcal{A}(\mathfrak{p})$ can be easily determined — see [G1]. Namely, there is an explicit construction of a linear isomorphism from the centre $\mathcal{Z}(\mathfrak{p}_{\text{ev}})$ to the anticentre $\mathcal{A}(\mathfrak{p})$ and the image of $\mathcal{A}(\mathfrak{p})$ in the symmetric algebra $S(\mathfrak{p})$ is equal to $\Lambda^\text{top} \mathfrak{p}_{\text{od}} S(\mathfrak{p}_{\text{od}})^{\text{top}}$.

Since $\mathfrak{g}_{\text{ev}}$ is a simple Lie algebra $\mathfrak{sl}(n)$, $\Lambda^\text{top} \mathfrak{p}_{\text{od}}$ meets the above condition and the isomorphism $\phi^\prime : \mathcal{Z}(\mathfrak{g}_{\text{ev}}) \rightarrow \mathcal{A}(\mathfrak{g})$ can be easily written down. Note that $\mathcal{A}(\mathfrak{g})$ lies in the homogeneous component $U(\mathfrak{g})_{-n}$ since the image of $\mathcal{A}(\mathfrak{g})$ in the symmetric algebra $S(\mathfrak{g})$ is equal to $\Lambda^\text{top} \mathfrak{g}_{\text{ev}} S(\mathfrak{g}_{\text{ev}})^{\text{top}} = \Lambda^\text{top} \mathfrak{g}_1 \Lambda^\text{top} \mathfrak{g}_{-1} S(\mathfrak{g}_{\text{ev}})^{\text{top}}$ and $\dim \mathfrak{g}_1 - \dim \mathfrak{g}_{-1} = -n$. Using the isomorphism $\phi^\prime$, we construct a linear isomorphism $\phi : \mathcal{Z}(\mathfrak{g}_{\text{ev}}) \rightarrow \mathcal{Z}(\mathfrak{g}) \cap U(\mathfrak{g})_{-n}$. This
provides a full description of $Z(g)$ since, due to Scheunert, $Z(g) = k \oplus Z(g)_{-n}$ where $Z(g)_{-n} := Z(g) \cap U(g)_{-n}$.

1.3. Remark that for other Lie superalgebras the structures of the centre and the anticentre are not so similar. However one might notice a certain connection. For instance, for a non-simple $P$-type Lie superalgebra $GP(n-1)$ ($n \geq 3$) (consisting of the block matrices of the same type as above but with an arbitrary matrix $a$) and for a Cartan type superalgebra $W(n)$ ($n \geq 3$) both centre and anticentre are trivial: the centre coincides with the base field— see [Sch], [Sh] and the anticentre is equal to zero ([G1]).

The sum of the centre and the anticentre is a subalgebra of $U(g)$ which we call ghost centre. Contrary to the case of basic classical Lie superalgebras where all central and anticantral elements are non-zero divisors, for $g = P(n-1)$ ($n \geq 3$) one has $Z(g)_{-n}A(g) = A(g) = 0$. Thus the ghost centre $\tilde{Z}(g) := Z(g) + A(g) = k \oplus Z(g)_{-n} \oplus A(g)$ is an algebra with a trivial multiplication.

1.4. As in the cases of basic classical Lie superalgebras (that are the general linear, the special linear, and the orthosymplectic Lie superalgebras) we denote a lowest-order anticantral element by $T$ (this is an element of $A(g)$ whose image in the symmetric algebra belongs to $\Lambda^{\text{top}}_{\mathfrak{h}}$, the above condition determines $T$ up to a scalar). For a basic classical Lie superalgebra $p$ the restrictions of Harish-Chandra projection $P$ to the centre $Z(p)$ and to the anticantral $A(p)$ are injections. The image of $Z(p)$ is a subalgebra of the algebra of $W$-invariant polynomials $S(h)^W$ described in [K2], [S], [BZV]. The image of $A(p)$ is simply $tS(h)^W$ where $t := P(T)$ takes the form

$$t = \prod_{\alpha \in \Delta^+_1} (\alpha^\vee + (\alpha, \rho)),$$

see [G1]. The element $t$ is “in charge” of strong typicality. This means that for $\lambda \in \mathfrak{h}^*$ satisfying $t(\lambda) \neq 0$ the category of $p$ representations whose central character coincides with the one of a simple module of the highest weight $\lambda$ is equivalent to the the category of $p_{\mathfrak{T}}$ representations with a certain central character (see [PS],[P],[G2]).

A specific feature of $g = P(n-1)$ is the lack of symmetry: the fact that the anticantral elements as well central elements without constant terms are homogeneous of degree $-n$ reflects the fact that the dimensions of $g_1$ and $g_{-1}$ are not equal (the difference is exactly $-n$). Since the Harish-Chandra projection of an element of non-zero degree vanishes, we substitute the Harish-Chandra projection by another map $P_{-n} : U(g)_{-n} \to S(h)$. The restrictions of this map to the centre $Z(g) \cap U(g)_{-n}$ and to the anticantral $A(g)$ are again injections. Moreover both images are equal to $tS(h)^W$ where $t := P_{-n}(T)$ can be written in the form

$$t = \prod_{\alpha \in \Delta^+_0} (\alpha^\vee + (\alpha, \rho) - 1).$$
Observe that the linear factors of $t$ correspond to the odd coroots in the formula (1) and to the even coroots in the formula (2). This difference is connected to the following fact: if $x, y$ are odd elements of a basic classical Lie superalgebra of the opposite weights $\beta$ and $-\beta$ respectively then $[x, y]$ is proportional to the odd coroot $\beta^\vee$; by contrast, if $x, y$ are odd elements of $g = P(n - 1)$ meeting the same condition then $[x, y]$ is proportional to a certain even coroot (see (3)).

1.5. For a basic classical Lie superalgebra $p$ a central (or anticentral) element $z$ annihilates a Verma module of the highest weight $\lambda$ iff $P(z)(\lambda) = 0$. Since $P(A(p)) = P(T)S(h)^W$; it follows that a Verma module annihilated by $T$ is annihilated by any anticentral element. This last property remains true for $g = P(n - 1)$; moreover, if a Verma module is annihilated by $T$ then it is annihilated not only by all anticentral elements but also by all central elements without constant term (this immediately follows from Theorem 4.1 (iii)). However, the equality $P_{-n}(z)(\lambda) = 0$ does not force that $z \in Z(g)_n \cup A(g)$ annihilates a Verma module of the highest weight $\lambda$ (see 4.3).

2. Preliminaries

2.1. Notation. Let $g = g_0 \oplus g_1$ be a Lie superalgebra $P(n - 1)$ endowed with the $\mathbb{Z}$-grading described above. Extend this $\mathbb{Z}$-grading to the universal enveloping algebra $U(g)$ and denote by $U(g)_r$ ($r \in \mathbb{Z}$) the corresponding graded component. For any subspace $N$ of $U(g)$ set $N_r := N \cap U(g)_r$. Denote by ad the adjoint action of $U(g)$ on itself.

For a superalgebra $p$ denote its universal enveloping algebra by $U(p)$. Since $g_{\pm 1}$ are supercommutative pure odd Lie superalgebras, $U(g_{\pm 1})$ is canonically isomorphic to the exterior algebra $\Lambda g_{\pm 1}$.

2.1.1. Retain notation of 1.1. Denote by $h$ the set of diagonal matrices belonging to $g_0$, by $n_0$ (resp., $n_0^+$) the set of matrices whose upper-left block $a$ is lower (resp., upper) triangular and both blocks $b$ and $c$ are equal to zero. Then $g_0 := n_0^0 \oplus h \oplus n_0^+$ is a “standard” triangular decomposition of $g_0 \cong sl(n)$. As usual, it is convenient to present $h^*$ as the quotient of the $n$ dimensional vector space with a basis $\{\varepsilon_i\}_1^n$ by the one-dimensional subspace spanned by $\sum_i \varepsilon_i$. Denote by $W$ the Weyl group of $g_0$; it acts on the set $\{\varepsilon_i\}_1^n$ by the permutations. Denote by $(\cdot, \cdot)$ the canonical $W$-invariant bilinear form on $h^*$.

Set $n_1^+ := g_1$, $n_1^- := g_{-1}$ and $n^+ := n_0^+ + n_1^+$.

With this notation one has

$$\Delta_0^+ = \Omega(n_0^+) = \{\varepsilon_i - \varepsilon_j\}_{1 \leq i < j \leq n},$$

$$\Delta_1^+ = \Omega(n_1^+) = \{-\varepsilon_i - \varepsilon_j\}_{1 \leq i < j \leq n},$$

$$\Delta_1^- = \Omega(n_1^-) = \{2\varepsilon_i; \varepsilon_i + \varepsilon_j\}_{1 \leq i < j \leq n},$$

where $\Omega(N)$ stands for the multiset of $h$-weights of $N$. Set $\rho := \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha$. 

For \( r, s \in \{1, \ldots, 2n\} \) let \( E_{r,s} \) be the \( 2n \times 2n \) matrix whose only non-zero entry is 1 at the place \((r, s)\). For each pair \((i, j)\) with \(1 \leq i < j \leq n\) set

\[
\begin{align*}
\varepsilon_{e_i-e_j} &:= E_{i,j} - E_{n+i,n+i}, \\
\varepsilon_{f_i-f_j} &:= E_{j,i} - E_{n+i,n+j}, \\
(\varepsilon_i - \varepsilon_j)^\vee &:= E_{i,i} - E_{n+j,n+i} + E_{n+i,n+j}, \\
x_{e_i-e_j} &:= E_{n+j,i} - E_{n+i,j}, \\
y_{e_i+e_j} &:= E_{i,n+j} + E_{j,n+i}, \\
y_{2e_i} &:= E_{i,n+i}.
\end{align*}
\]

The set \( \{e_\alpha\}_{\alpha \in \Delta_0^+} \) (resp., \( \{f_\alpha\}_{\alpha \in \Delta_0^+} \)) forms a basis of \( \mathfrak{n}_0^+ \) (resp., \( \mathfrak{n}_0^- \)) and the set \( \{x_\alpha\}_{\alpha \in \Delta_1^+} \) (resp., \( \{y_\alpha\}_{\alpha \in \Delta_1^-} \)) forms a basis of \( \mathfrak{n}_1^+ \) (resp., \( \mathfrak{n}_1^- \)). As always \( (\varepsilon_i - \varepsilon_j)^\vee \) is the coroot corresponding to \((\varepsilon_i - \varepsilon_j)\) that is given by the formula \((\varepsilon_i - \varepsilon_j)^\vee(\mu) = (\varepsilon_i - \varepsilon_j, \mu)\) for any \( \mu \in \mathfrak{h}^* \). One has

\[
[e_{e_i-e_j}, f_{e_i+f_j}] = [x_{e_i+e_j}, y_{e_i-e_j}] = (\varepsilon_i - \varepsilon_j)^\vee. \tag{3}
\]

Denote by \( J \) the set of odd positive roots (that is \( \Delta_1^+ \)) with a fixed total order. For a subset \( J' \subseteq J \) denote by \( x_{J'} \) and \( y_{J'} \) respectively the products \( \prod_{\beta \in J'} x_{\beta}, \prod_{\beta \in J'} y_{-\beta} \) taken with respect to the total order. Set also

\[
y_{I \setminus J} := \prod_{i=1}^n y_{-2e_i}, \\
y_I := y_{I \setminus J} y_J.
\]

Since \( \mathfrak{n}_1^\pm \) are supercommutative one has \( y_I = \pm y_J y_{I \setminus J}, \ x_{J'} x_{J''} = \pm x_{J' \cup J''}, \ y_{J'} y_{J''} = \pm y_{J' \cup J''} \) if \( J' \cap J'' = \emptyset \) and \( x_{J'} x_{J''} = y_{J'} y_{J''} = 0 \) if \( J' \cap J'' \neq \emptyset \). Note that the elements \( y_I, y_J, y_{I \setminus J}, x_J \) lie in \( \mathcal{U}(\mathfrak{g})^\mathfrak{h} \). Moreover \( x_J \in \Lambda^{\top \alpha_+} \mathfrak{n}_1^+, y_I \in \Lambda^{\top \alpha_-} \mathfrak{n}_1^- \) and, in particular, \( x_J, y_I \) are \( \mathfrak{g}_0 \)-invariant.

2.1.2. For \( \mu \in \mathfrak{h}^* \) and a vector subspace \( N \subset \mathcal{U}(\mathfrak{g}) \) denote by \( N|_\mu \) the corresponding \( \mathfrak{h} \)-weight subspace of \( N \).

We identify \( \mathcal{U}(\mathfrak{h}) \) with \( \mathcal{S}(\mathfrak{h}) \). Define a twisted action of the Weyl group \( W \) on \( \mathcal{S}(\mathfrak{h}) \) by setting

\[
w.p(\lambda) = p(w^{-1}(\lambda + \rho) - \rho)
\]

for any \( w \in W, p \in \mathcal{S}(\mathfrak{h}), \lambda \in \mathfrak{h}^* \).

Recall that the elements \( y_\beta \ (\beta \in \Delta^-) \) have degree \(-1\) and the elements \( x_\beta \ (\beta \in \Delta_1^+) \) have degree \(1\); thus one has \( \mathcal{U}(\mathfrak{g}), r = 0 \) if \( r < \# \Delta^- = -\frac{n(n+1)}{2} \) or \( r > \# \Delta_1^+ = \frac{n(n-1)}{2} \) (where \# stands for the cardinality).

The universal enveloping algebra \( \mathcal{U}(\mathfrak{g}) \) admits the canonical filtration given by \( \mathcal{F}^k(\mathcal{U}(\mathfrak{g})) = \mathfrak{g}^k \); the associated graded algebra is the symmetric algebra \( \mathcal{S}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g}_0) \Lambda_{\mathfrak{g}_T} \). For any \( u \in \mathcal{U}(\mathfrak{g}) \) denote by \( \text{gr} u \) the image of \( u \) in the symmetric algebra \( \mathcal{S}(\mathfrak{g}) \).
2.1.3. Verma modules. For $\lambda \in \mathfrak{h}^*$ denote by $k_\lambda$ a one-dimensional $(\mathfrak{h} + \mathfrak{n}^+)$-module such that $n^+ v = 0$ and $hn = \lambda(h)v$ for any $h \in \mathfrak{h}, v \in k_\lambda$. Define a Verma module $\tilde{M}(\lambda)$ by setting $\tilde{M}(\lambda) := \mathcal{U}(g) \otimes_{\mathcal{U}(\mathfrak{b})} k_\lambda$. Call the image of a fixed non-zero element of $k_\lambda$ in $\tilde{M}(\lambda)$ a canonical generator of $\tilde{M}(\lambda)$. Similarly, denote by $M(\lambda)$ a Verma $g_0$-module of the highest weight $\lambda$.

Suppose that $\lambda \in \mathfrak{h}^*$ is such that $(\lambda + \rho, \alpha)$ is a positive integer for some $\alpha \in \Delta_0^+$. Let $v$ be a canonical generator of $\tilde{M}(\lambda)$; then $\mathcal{U}(\mathfrak{g}_0)v \cong M(\lambda)$ contains a $n_0$-invariant vector $uv$ (with $u \in \mathcal{U}(\mathfrak{g}_0)$) of the weight $s_\alpha \lambda$ (here $s_\alpha \in W$ is the reflection corresponding to the root $\alpha$). The vector $uv$ is $n$-invariant because $n_\lambda^+ \in \mathfrak{g}_0$-invariant. Since $u \in \mathcal{U}(\mathfrak{g}_0)$ is a non-zero divisor in $\mathcal{U}(\mathfrak{g})$, the vector $uv$ generates a submodule isomorphic to $\tilde{M}(s_\alpha \lambda)$. Hence $\tilde{M}(s_\alpha \lambda) \subset \tilde{M}(\lambda)$.

**Caution:** The module $\tilde{M}(\lambda)$ is never simple because $[y_{2\alpha}, n^+] \subset n^+$ and so for a canonical generator $v \in \tilde{M}(\lambda)$ the subspace $\mathcal{U}(\mathfrak{g})(y_{2\alpha}, v)$ is a proper submodule.

2.1.4. Projections $P_+, P$. Denote by $P_+$ the projection $\mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(n^- + \mathfrak{h})$ with respect to the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(n^- + \mathfrak{h}) \oplus \mathcal{U}(\mathfrak{g})n^+$ and by $P$ the Harish-Chandra projection $P : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h})$ with respect to the decomposition $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (n^- \mathcal{U}(\mathfrak{g}) + \mathcal{U}(\mathfrak{g})n^+)$. The restrictions of $P_+$ and $P$ to the subalgebra $\mathcal{U}(\mathfrak{g})^h_0$ coincide and give an algebra homomorphism. Note that the restrictions of $P_+$ and $P$ to the subalgebra $\mathcal{U}(\mathfrak{g})^h$ do not coincide and are not algebra homomorphisms — for instance, both $y_{ij}, x_{ij}$ lie in $\mathcal{U}(\mathfrak{g})^h$ and $P(y_{ij}) = 0$, but $P_+(y_{ij}) = y_{ij}$ and $P(x_{ij}y_{ij}) \neq 0$ by Lemma 3.1 below.

The inclusion $\mathcal{U}(\mathfrak{g})n^+U(\mathfrak{h}) \subset \mathcal{U}(\mathfrak{g})n^+$ implies that

$$P_+(ab) = P_+(a)P(b), \quad \forall a \in \mathcal{U}(\mathfrak{g}), b \in \mathcal{U}(\mathfrak{g})^h_0.$$  \hfill (4)

2.2. Anticentre $\mathcal{A}(\mathfrak{g})$. The anticentre $\mathcal{A}(\mathfrak{g})$ can be defined as the set of invariants of $\mathcal{U}(\mathfrak{g})$ with respect to a twisted adjoint action: $\mathcal{A}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^{ad' \mathfrak{g}}$ where $ad'$ is given by the formula

$$(ad' \mathfrak{g})u = gu - (-1)^{d(g)(d(u)+1)}ug$$

for all homogeneous $g \in \mathfrak{g}, u \in \mathcal{U}(\mathfrak{g})$ (here $d(\cdot)$ stands for the $\mathbb{Z}_2$-degree of the element that is $d(u) = 0$ for $u \in \mathcal{U}(\mathfrak{g})^F$ and $d(u) = 1$ for $u \in \mathcal{U}(\mathfrak{g})^T$). Note that the odd elements of the anticentre $\mathcal{A}(\mathfrak{g})$ commute with all element of $\mathcal{U}(\mathfrak{g})$ and the even elements of $\mathcal{A}(\mathfrak{g})$ commute with the even elements of $\mathcal{U}(\mathfrak{g})$ and anticommute with the odd ones.

By Theorem 3.3 of [G1], $\text{gr} \mathcal{A}(\mathfrak{g}) = \Lambda^{\text{top}} \mathfrak{g}_T \text{gr}(\mathcal{Z}(\mathfrak{g}_0))$. This can be rewritten as

$$\text{gr} \mathcal{A}(\mathfrak{g}) = \text{gr}(x_{ij}y_{ij} \mathcal{Z}(\mathfrak{g}_0))$$

because $\text{gr} x_{ij}y_{ij}$ spans $\Lambda^{\text{top}} \mathfrak{g}_T$. Since $\mathcal{A}(\mathfrak{g})$ is a graded subspace of $\mathcal{U}(\mathfrak{g})$, the elements of $\mathcal{A}(\mathfrak{g})$ have degree equal to $\# J - \# I = -n$. Therefore,

$$\mathcal{A}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})_{-n}.$$
3. Useful assertions.

The element
\[ t := P(x_1 y_1) \]
plays an important role in the description of the centre and seems to be instrumental in the study of representations of \( \mathfrak{g} \).

3.1. Proposition. One has
\[ t = \pm \prod_{\alpha \in \Delta^+_i} (\alpha^\vee + (\alpha, \rho) - 1). \]

Proof. The proof has two steps. As a first step, let us prove by induction that for all \( r = 2, \ldots, n \) one has
\[ P\left( \prod_{1 < j \leq r} x_{-\varepsilon_1 - \varepsilon_j} \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j} \right) = \pm \prod_{1 < j \leq r} ((\varepsilon_1 - \varepsilon_j)^\vee + j - 2). \tag{5} \]
For \( r = 2 \) the assertion immediately follows from the equality (3). For the induction step observe that
\[ P\left( \prod_{1 < j \leq r+1} x_{-\varepsilon_1 - \varepsilon_j} \prod_{1 < j \leq r+1} y_{\varepsilon_1 + \varepsilon_j} \right) = \pm P\left( \prod_{1 < j \leq r} x_{-\varepsilon_1 - \varepsilon_j} \cdot x_{-\varepsilon_1 - \varepsilon_{r+1}} \cdot y_{\varepsilon_1 + \varepsilon_{r+1}} \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j} \right) \]
\[ = \pm P\left( \prod_{1 < j \leq r} x_{-\varepsilon_1 - \varepsilon_j} ((\varepsilon_1 - \varepsilon_{r+1})^\vee - y_{\varepsilon_1 + \varepsilon_{r+1}} x_{-\varepsilon_1 - \varepsilon_{r+1}}) \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j} \right) \tag{6} \]
It is easy to see that \( U(\mathfrak{n}^-)|_{\mu} \neq 0 \) for \( \mu = \sum c_i \varepsilon_i \) only if \( c_s + c_{s+1} + \ldots + c_n \geq 0 \) for all \( s = 1, \ldots, n \). In particular, \( U(\mathfrak{n}^- + \mathfrak{h}) \) does not contain a non-zero element of weight \((r-2)\varepsilon_1 + \varepsilon_2 + \ldots + \varepsilon_r - \varepsilon_{r+1} \). Therefore the element \( x_{-\varepsilon_1 - \varepsilon_{r+1}} \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j} \) belongs to \( U(\mathfrak{g})\mathfrak{n}^+ \) and thus (6) implies
\[ P\left( \prod_{1 < j \leq r+1} x_{-\varepsilon_1 - \varepsilon_j} \prod_{1 < j \leq r+1} y_{\varepsilon_1 + \varepsilon_j} \right) = \pm P\left( \prod_{1 < j \leq r} x_{-\varepsilon_1 - \varepsilon_j} (\varepsilon_1 - \varepsilon_{r+1})^\vee \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j} \right). \]
Taking into account that
\[ (\varepsilon_1 - \varepsilon_{r+1})^\vee \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j} = \prod_{1 < j \leq r} y_{\varepsilon_1 + \varepsilon_j}((\varepsilon_1 - \varepsilon_{r+1})^\vee + r - 1), \]
on one obtains the required equality (5) by induction.

The second step of the proof is to show that for \( r = 1, \ldots, n - 1 \) the term
\[ t_r := P\left( \prod_{n-r \leq i < j \leq n} x_{-\varepsilon_i - \varepsilon_j} \prod_{n-r \leq i < j \leq n} y_{\varepsilon_i + \varepsilon_j} \right) \]
is given by the formula
\[ t_r = \pm \prod_{n-r \leq i < j \leq n} ((\varepsilon_i - \varepsilon_j)^\vee + (\varepsilon_i - \varepsilon_j, \rho) - 1). \tag{7} \]
We again proceed by induction. For \( r = 1 \) the equality (3) gives

\[
t_1 = P(x_{-\varepsilon_{n-1} - \varepsilon_n} y_{\varepsilon_{n-1} + \varepsilon_n}) = (\varepsilon_{n-1} - \varepsilon_n)^\vee = (\varepsilon_{n-1} - \varepsilon_n)^\vee + (\varepsilon_{n-1} - \varepsilon_n, \rho) - 1 \]

as required.

Assume that (7) holds for some \( r < n \). For \( s = 1, \ldots, n-1 \) let \( n^- (\rho) \) (resp., \( n^+ (\rho) \)) be the Lie subalgebra of \( n^- \) (resp., \( n^+ \)) spanned by the elements \( f_{-\varepsilon_i, -\varepsilon_j}, y_{2\varepsilon_i}, y_{\varepsilon_i + \varepsilon_j} \) (resp., \( e_{\varepsilon_i, -\varepsilon_j}, x_{-\varepsilon_i, -\varepsilon_j} \)) with \( n - s \leq i < j \leq n \). Set \( X_r := \prod_{n-r \leq j \leq n} x_{-\varepsilon_{n-r-1} - \varepsilon_j} \). Let us show that \( (\text{ad} n^-)_r) X_r \in n^- \mathcal{U}(\mathfrak{g}) \). Indeed, fix a pair \((i, j)\) with \( n-r \leq i < j \leq n \). The equality \( (\text{ad} f_{-\varepsilon_i, -\varepsilon_j}) X_r = 0 \) immediately follows from the supercommutativity of \( n^+ \). Furthermore it is easy to see that a homogeneous element of degree \( m \) belonging to the algebra \( \mathcal{U}(\mathfrak{h} + n^+_{(r+1)}) \) has a weight of the form \( \sum_{j=n-r-1}^n c_j \varepsilon_j \) with \( c_{n-r-1} \geq -m \). Combining the facts that the terms \( (\text{ad} y_{\varepsilon_i + \varepsilon_j}) X_r \), \( (\text{ad} y_{2\varepsilon_i}) X_r \) lie in \( \mathcal{U}' := \mathcal{U}(n^-_{(r+1)} + \mathfrak{h} + n^+_{(r+1)}) \), have degree \( r \) and weights of the form \( -(r+1)\varepsilon_{n-r-1} + \ldots \), one concludes that these terms lie in \( n^- \mathcal{U}(\mathfrak{g}) \) because \( \mathcal{U}' = n^-_{(r+1)} \mathcal{U}' \oplus \mathcal{U}(n^+_{(r+1)} + \mathfrak{h}) \) and the \( r \)th graded component of the algebra \( \mathcal{U}(n^+_{(r+1)} + \mathfrak{h}) \) does not contain non-zero elements of the weights of the above form.

One has

\[
t_{r+1} = P\left( \prod_{n-r \leq j \leq n} x_{-\varepsilon_{n-r-1} - \varepsilon_j} \prod_{n-r \leq j \leq n} y_{\varepsilon_{n-r-1} + \varepsilon_j} \right) \]

where each term \( u_s \) belongs to \( n^- (\rho) \) and \( u_s \) are some elements of \( \mathcal{U}(\mathfrak{g}) \). As we have shown above \( (\text{ad} n^-) X_r \) lies in \( n^- \mathcal{U}(\mathfrak{g}) \) and thus \( P\left( X_r u_s u_s \prod_{n-r \leq j \leq n} y_{\varepsilon_{n-r-1} + \varepsilon_j} \right) = 0 \) for any index \( s \). Therefore

\[
t_{r+1} = \pm P\left( X_r \prod_{n-r \leq j \leq n} y_{\varepsilon_{n-r-1} + \varepsilon_j} \right) \]

The formula (5) implies that

\[
P\left( \prod_{n-r \leq j \leq n} x_{-\varepsilon_{n-r-1} - \varepsilon_j} \prod_{n-r \leq j \leq n} y_{\varepsilon_{n-r-1} + \varepsilon_j} \right) = \prod_{n-r \leq j \leq n} \left( (\varepsilon_{n-r-1} - \varepsilon_j)^\vee + j - (n - r - 1) - 1 \right)
\]

Hence

\[
t_{r+1} = \pm \prod_{n-r-1 \leq i < j \leq n} \left( (\varepsilon_i - \varepsilon_j)^\vee + (\varepsilon_i - \varepsilon_j, \rho) - 1 \right)
\]

as required. Finally observing that \( t = \pm t_{n-1} \) one completes the proof. \( \square \)
3.2. Proposition. For any Zariski dense subset \( \Omega \) of \( \mathfrak{h}^* \) one has
\[
\bigcap_{\lambda \in \Omega} \text{Ann}_{\mathcal{U}(\mathfrak{g})} \widetilde{M}(\lambda) = 0.
\]

Proof. Let \( \Omega \) be a Zariski dense subset of \( \mathfrak{h}^* \). By Proposition 3.1, \( P(x_J y_J) \) is a non-zero polynomial and so the set \( \Omega' := \Omega \cap \{ \lambda \in \mathfrak{h}^* \mid P(x_J y_J)(\lambda) \neq 0 \} \) is also Zariski dense in \( \mathfrak{h}^* \). Assume that \( N := \bigcap_{\lambda \in \Omega'} \text{Ann} \widetilde{M}(\lambda) \) is non-zero. One has \( \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_0 + \mathfrak{n}_1^+) \mathcal{U}(\mathfrak{n}_1^+) = \mathcal{U}(\mathfrak{g}_0 + \mathfrak{n}_1^+) \Lambda \mathfrak{n}_1^+ \). Since \( N \) is a right ideal, it contains a non-zero element of the form \( u x_J \) where \( u \in \mathcal{U}(\mathfrak{g}_0 + \mathfrak{n}_1^+) \). Let \( \lambda \) be an element of \( \Omega' \) and \( v \) be a canonical generator of \( \widetilde{M}(\lambda) \). Then
\[
0 = u x_J (\mathcal{U}(\mathfrak{g}_0) y_J v) = u \mathcal{U}(\mathfrak{g}_0) x_J y_J v = u \mathcal{U}(\mathfrak{g}_0) P(x_J y_J)(\lambda) v.
\]
Note that \( P(x_J y_J)(\lambda) v \in k^* v \) since \( \lambda \in \Omega' \) and thus the space \( \mathcal{U}(\mathfrak{g}_0) P(x_J y_J)(\lambda) v \) is isomorphic to a Verma \( \mathfrak{g}_0 \)-module \( M(\lambda) \). Writing \( u = \sum_{S \subseteq J} y_S u_S \) where \( u_S \in \mathcal{U}(\mathfrak{g}_0) \), one concludes that each \( u_S \) annihilates \( M(\lambda) \). However, by [D],
\[
\bigcap_{\lambda \in \Omega'} \text{Ann}_{\mathcal{U}(\mathfrak{g}_0)} M(\lambda) = 0
\]
and thus all terms \( u_S \) are equal to zero. This gives the required contradiction. \( \square \)

3.3. The map \( P_{-n} \). The Harish-Chandra projection \( P \) annihilates the homogeneous component \( \mathcal{U}(\mathfrak{g})_r \) for any \( r \neq 0 \). In particular, \( P(\mathcal{Z}(\mathfrak{g})_{-n}) = 0 \) and thus \( P \) itself is useless for a description of the centre \( \mathcal{Z}(\mathfrak{g}) \). For this purpose it is convenient to use a map \( P_{-n} : \mathcal{U}(\mathfrak{g})_{-n}^{\mathfrak{h}} \rightarrow \mathcal{U}(\mathfrak{h}) \) constructed below. For \( a \in \mathcal{U}(\mathfrak{g})_{-n}^{\mathfrak{h}} \) one has \( y_J a \in \mathcal{U}(\mathfrak{g})_{-n}^{\mathfrak{h}} \cap \mathcal{U}(\mathfrak{g})_{\# \Delta_1} = y_J \mathcal{U}(\mathfrak{g}_0)^{\mathfrak{h}} \). This allows us to define the linear map \( P_{-n} : \mathcal{U}(\mathfrak{g})_{-n}^{\mathfrak{h}} \rightarrow \mathcal{U}(\mathfrak{h}) \) by the condition
\[
P_{+}(y_J a) = y_I P_{-n}(a)
\]
for any \( a \in \mathcal{U}(\mathfrak{g})_{-n}^{\mathfrak{h}} \).

3.3.1. Lemma. The restrictions of \( P_{-n} \) to \( \mathcal{Z}(\mathfrak{g})_{-n} \) and to \( \mathcal{A}(\mathfrak{g}) \) are (vector space) monomorphisms.

Proof. Take a non-zero element \( a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})_{-n} \). Combining Proposition 3.1 and Proposition 3.2 one concludes the existence of \( \lambda \in \mathfrak{h}^* \) such that \( P(x_J y_J)(\lambda) \neq 0 \) and \( a \widetilde{M}(\lambda) \neq 0 \). Let \( v \) be a canonical generator of \( \widetilde{M}(\lambda) \); the condition \( P(x_J y_J)(\lambda) \neq 0 \) implies that \( x_J y_J v \in k^* v \) and so the vector \( y_J v \) generates \( \widetilde{M}(\lambda) \). Since \( a \) is either central or anticyclic, \( a \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}) a \) and so the condition \( a \widetilde{M}(\lambda) \neq 0 \) forces \( a(y_J v) \neq 0 \). One has \( a(y_J v) = P_{+}(a y_J) v = \pm y_I P_{-n}(a) v \). Hence \( P_{-n}(a) \neq 0 \) as required. \( \square \)
3.3.2. Set

\[ T := (ad' x_J)y_I. \]

Lemma. For any \( a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g}) \) one has

\[ P_+(a)t = P_+(T)P_{-n}(a). \quad (9) \]

Proof. Any \( a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g}) \) commutes with the term \( x_Jy_I \) and so \( P_+(x_Jy_Ia) = P_+(a)P(x_Jy_I) \) by (4). On the other hand,

\[ P_+(x_Jy_Ia) = P_+(x_JP_+(y_Ifa)) = P_+(x_Jy_IP_{-n}(a)) = P_+(x_Jy_I)P_{-n}(a). \]

Therefore \( P_+(x_Jy_I)P_{-n}(a) = P_+(a)P(x_Jy_I) = P_+(a)t \) for any \( a \in \mathcal{Z}(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g}) \). It is easy to see from the definition of \( ad' \) that \( P_+(T) = P_+(x_Jy_I) \). The assertion follows. \( \square \)

4. Main result

Recall that \( \mathcal{Z}(\mathfrak{g}) = k \oplus \mathcal{Z}(\mathfrak{g})_{-n} \) (see [Sch]). In this section we prove the following theorem which describes \( \mathcal{Z}(\mathfrak{g})_{-n} \) and \( \mathcal{A}(\mathfrak{g}) = \mathcal{A}(\mathfrak{g})_{-n} \).

4.1. Theorem. i) The map \( \phi : \mathcal{Z}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g}) \) given by \( z \mapsto (ad x_J)(y_Iz) \) induces a linear isomorphism \( \mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{Z}(\mathfrak{g})_{-n} \).

ii) The map \( \phi' : \mathcal{Z}(\mathfrak{g}_0) \to \mathcal{U}(\mathfrak{g}) \) given by \( z \mapsto (ad' x_J)(y_Iz) \) induces a linear isomorphism \( \mathcal{Z}(\mathfrak{g}_0) \xrightarrow{\sim} \mathcal{A}(\mathfrak{g}) \).

iii) One has

\[ P_+(\phi(z)) = P_+(\phi'(z)) = P_+(T)P(z), \]
\[ P_{-n}(\phi(z)) = P_{-n}(\phi'(z)) = tP(z). \]

iv) The restriction of \( P_{-n} \) to \( \mathcal{Z}(\mathfrak{g})_{-n} \) and to \( \mathcal{A}(\mathfrak{g}) \) induces linear isomorphisms

\[ \mathcal{Z}(\mathfrak{g})_{-n} \xrightarrow{\sim} t\mathcal{S}(\mathfrak{h})^W, \quad \mathcal{A}(\mathfrak{g}) \xrightarrow{\sim} t\mathcal{S}(\mathfrak{h})^W. \]

4.2. Proof of Theorem 4.1.
4.2.1. The image of $\phi$ lies in $Z(\mathfrak{g})$. To show that $\text{Im} \, \phi \subseteq Z(\mathfrak{g})$ fix $z \in Z(\mathfrak{g}_0)$. First, let us check that $(\text{ad} \, y_{2z}) \phi(z) = 0$. One has $(\text{ad} \, y_{2z}) \phi(z) = (\text{ad} \, y_{2z}, x_J)(y_J z)$. The element $y_{2z}x_J$ belongs to $U(\mathfrak{g})(\mathfrak{g}_0 + \mathfrak{n}_1^{-})$ because $\Lambda \mathfrak{n}_1^+$ does not contain a non-zero element whose weight and degree coincide respectively with the weight and the degree of $y_{2z}, x_J$. Since the element $y_J z$ is $\text{ad} \, (\mathfrak{g}_0 + \mathfrak{n}_1^{-})$-invariant one obtains $(\text{ad} \, y_{2z}, x_J)(y_J z) = 0$.

Since $x_J, y_J$ and $z$ are $\text{ad} \, \mathfrak{g}_0$-invariant, $\phi(z) = (\text{ad} \, x_J)(y_J z)$ is $\text{ad} \, \mathfrak{g}_0$-invariant. Combining the equalities $(\text{ad} \, \mathfrak{g}_0) \phi(z) = (\text{ad} \, y_{2z}) \phi(z) = 0$ and $\mathfrak{n}_1^{-} = [\mathfrak{g}_0, y_{2z}]$, one concludes $(\text{ad} \, \mathfrak{n}_1^{-}) \phi(z) = 0$. The remaining equality $(\text{ad} \, \mathfrak{n}_1^{+}) \phi(z) = 0$ immediately follows from the supercommutativity of $\mathfrak{n}_1^{+}$. Hence $\phi(z) \in Z(\mathfrak{g})$.

4.2.2. Proof of (ii). Replacing the adjoint action $\text{ad}$ by the twisted adjoint action $\text{ad}'$ and repeating the above reasoning one concludes that $\text{Im} \, \phi' \subseteq U(\mathfrak{g}) \text{ad}' \theta = \mathcal{A}(\mathfrak{g})$.

Remark that $(\text{ad}' \, g)u = 2gu - (\text{ad} \, g)u$ for all $g \in \mathfrak{g}_1, u \in U(\mathfrak{g})$ and that $\text{gr} \, u$ and $\text{gr}((\text{ad} \, g)u)$ have the same degree in the symmetric algebra $S(\mathfrak{g})$. Therefore

$$\text{gr}((\text{ad}' \, g)u) = 2(\text{gr} \, g)(\text{gr} \, u), \forall g \in \mathfrak{g}_1, u \in U(\mathfrak{g}) \text{ s.t. } \text{gr} \, (gu) = (\text{gr} \, g)(\text{gr} \, u).$$

This implies $\text{gr} \, \phi'(z) = \text{gr}((\text{ad}' \, x_J)(y_J z)) = 2\#I \, \text{gr} \, (x_J y_J z)$ for any $z \in Z(\mathfrak{g}_0)$. In particular, $\text{gr} \, \phi'(z) \neq 0$ and so $\phi'$ is a monomorphism. Moreover, by 2.2, $\text{gr} \, \mathcal{A}(\mathfrak{g}) = \text{gr} \, (x_J y_J z)(\mathfrak{g}_0)$ that is $\text{gr} \, \mathcal{A}(\mathfrak{g}) = \text{gr}(\text{Im} \, \phi')$. This proves that $\phi'$ is an isomorphism.

4.2.3. Proof of (iii). Combining Lemma 3.3.1 and the definition of $P_{-n}$, one concludes that $P_{+}(a) \neq 0$ for any non-zero $a \in Z(\mathfrak{g})_{-n} \cup \mathcal{A}(\mathfrak{g})$. Recall that $T = (\text{ad}' \, x_J)y_J = \phi'(1)$. Applying the formula (4) one obtains

$$P_{+}(\phi(z)) = P_{+}(\text{ad}' \, x_J)(y_J z)) = P_{+}(x_J y_J z) = P_{+}(x_J y_J)P(z) = P_{+}(T)P(z).$$

Similarly $P_{+}(\phi(z)) = P_{+}(T)P(z)$. Taking $a := \phi(z)$ in the formula (9) one gets

$$P_{+}(T)P(z)t = P_{+}(T)P_{-n}(\phi(z)) = P_{+}(T)P_{-n}(\phi'(z)).$$

Using the fact that the non-zero elements of $\mathcal{U}(\mathfrak{h})$ are non-zero divisors in $\mathcal{U}(\mathfrak{g})$ and the inequality $P_{+}(T) \neq 0$, one obtains

$$P_{-n}(\phi(z)) = P_{-n}(\phi'(z)) = tP(z).$$

This completes the proof of (iii).

4.2.4. Proof of (iv). It is well-known that the restriction of $P$ to $Z(\mathfrak{g}_0)$ induces the Harish-Chandra (algebra) isomorphism $Z(\mathfrak{g}_0) \xrightarrow{\sim} S(\mathfrak{h})^{\mathcal{W}}$. Combining already proven assertions (ii) and (iii) of Theorem 4.1, one concludes that the restriction of $P_{-n}$ to $\mathcal{A}(\mathfrak{g})$ induces a linear isomorphism $\mathcal{A}(\mathfrak{g}) \xrightarrow{\sim} tS(\mathfrak{h})^{\mathcal{W}}$.

Combining Lemma 3.3.1 and (iii) one concludes that the restriction of $P_{-n}$ to $Z(\mathfrak{g})_{-n}$ is an injective map whose image contains $tS(\mathfrak{h})^{\mathcal{W}}$. Thus to show that the restriction of $P_{-n}$ to $Z(\mathfrak{g})$ induces a linear isomorphism $Z(\mathfrak{g}) \xrightarrow{\sim} tS(\mathfrak{h})^{\mathcal{W}}$, it remains to check that $P_{-n}(a) \in tS(\mathfrak{h})^{\mathcal{W}}$ for any $a \in Z(\mathfrak{g})_{-n}$. We proceed in two steps. First, we verify that $P_{-n}(a) \in$
Lemma. The element $P_+(T)$ does not belong to $\mathcal{U}(\mathfrak{b}^-)t$.

Proof. This follows from 2.1.3 and the fact that $t$ is not $W$-invariant. Indeed, if $v$ is a canonical generator of a Verma module $M(\lambda)$ then

$$y_1Tv = P_+(y_1T)v = y_1P_-(T)v = y_1t(\lambda)v$$

and thus $TM(\lambda) \neq 0$ provided $t(\lambda) \neq 0$. By 3.1, $t(\lambda) = 0$ iff $(\lambda + \rho, \alpha) = 1$ for some $\alpha \in \Delta^+_0$. Take $\mu$ such that $(\mu + \rho, \varepsilon_1 - \varepsilon_2) = 1$ and $(\mu + \rho, \alpha) \notin \mathbb{Z}$ for the other roots $\alpha \in \Delta^+_0$. A Verma module $M := M(\mu)$ contains a submodule isomorphic to a Verma module $M' := M(\mu - (\varepsilon_1 - \varepsilon_2))$, see 2.1.3. Since $(\mu - (\varepsilon_1 - \varepsilon_2) + \rho, \alpha) \neq 1$ for all $\alpha \in \Delta^+_0$, one has $TM' \neq 0$ and, consequently, $T \tilde{M} \neq 0$. Since $T$ is anticentral, this implies that $T$ does not annihilate a canonical generator of $\tilde{M}$ that is $P_+(T)(\mu) \neq 0$. Taking into account that $t(\mu) = 0$ one obtains the required assertion.

As we explained in 4.2.4, the above lemma implies that $P_{-n}(\mathcal{Z}(\mathfrak{g})) \subseteq t\mathcal{S}(\mathfrak{h})$. The following lemma demonstrates that $P_{-n}(\mathcal{Z}(\mathfrak{g})) \subseteq t\mathcal{S}(\mathfrak{h})^W$.

Lemma. For any $a \in \mathcal{Z}(\mathfrak{g})_{-n}$ the fraction $P_{-n}(a)/t$ is $W$-invariant.

Proof. Fix $\alpha \in \Delta^+_0$ and let $s \in W$ be the corresponding reflection. Let $\lambda \in \mathfrak{h}^*$ be such that $t(\lambda) \neq 0, t(s\lambda) \neq 0$ and that $(\lambda + \rho, \alpha)$ is a positive integer. Observe that the set of suitable $\lambda$'s is Zariski dense in $\mathfrak{h}^*$. Let $v$ be a canonical generator of $\tilde{M}(\lambda)$ and $v' = uv$ ($u \in \mathcal{U}(\mathfrak{g}_0)$) be a canonical generator of $\tilde{M}(s\lambda) \subset \tilde{M}(\lambda)$ (see 2.1.3). Take $a \in \mathcal{Z}(\mathfrak{g})_{-n}$. One has $av = P_+(a)v$, $av' = P_+(a)v'$. Applying (9) one obtains

$$(av = cTv, \quad \text{where } c := P_{-n}(a)(\lambda)/t(\lambda),$$

$$(av' = c'Tv', \quad \text{where } c' := P_{-n}(a)(s\lambda)/t(s\lambda).$$

On the other hand,

$$av' = av = uav = cuTv = cTuw = cTv'$$

By (10), the inequality $t(\lambda) \neq 0$ implies $Tv \neq 0$. Thus $c = c'$ and the assertion follows.

4.2.7. Now (iv) follows from 4.2.4. Combining 4.2.1 with (iii) and (iv) one concludes (i). Theorem 4.1 is proven.
4.3.  **Remark.**  Lemma 4.2.6 might let one think that $P_{-n}$ plays for $Z(g)_{-n}$ a role similar to the one played by the Harish-Chandra projection for the centre of the enveloping algebra of semisimple Lie algebra. However, Lemma 4.2.5 shows that $t(\lambda) = 0$ does not imply $\phi(1)\tilde{M}(\lambda) = 0$ even though $P_{-n}(\phi(1)) = t$.

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