ON COMPLETE REDUCIBILITY FOR INFINITE-DIMENSIONAL LIE ALGEBRAS

MARIA GORELIK [†] AND VICTOR KAC [‡]

ABSTRACT. In this paper we study the complete reducibility of representations of infinite-dimensional Lie algebras from the perspective of representation theory of vertex algebras.

0. INTRODUCTION

The first part of the paper is concerned with restricted representations of an arbitrary Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, associated to a symmetrizable generalized Cartan matrix A ([K2], Chapter 1). Let $\mathfrak{h}$ be the Cartan subalgebra of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha)$ be the root space decomposition of $\mathfrak{g}$ with respect to $\mathfrak{h}$. A $\mathfrak{g}$-module M is called *restricted* if for any $v \in M$, we have $\mathfrak{g}_\alpha v = 0$ for all but finitely many positive roots α. This condition allows the action of the Casimir operator on M, which is the basic tool of representation theory of Kac-Moody algebras [K2] and also, in the case of an affine matrix A, allows the basic constructions of the vertex algebra theory, like normally ordered product of quantum fields.

In representation theory of Kac-Moody algebras one usually considers $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-modules $M = \oplus_{\mu \in \Omega(M)} M_\mu$, where

$$M_\mu := \{v \in M \mid hv = \mu(h)v, h \in \mathfrak{h}\} \neq 0,$$

is the weight space, attached to a weight μ, and $\Omega(M)$ is the set of weights. The $\mathfrak{h}$-diagonalizable module M is called *bounded* if the set $\Omega(M)$ is bounded by a finite set of elements $\lambda_1, \ldots, \lambda_s \in \mathfrak{h}^*$, i.e. for any $\mu \in \Omega(M)$ one has $\mu \leq \lambda_i$ for some $1 \leq i \leq s$.

The category $\mathcal{R}$ of restricted $\mathfrak{g}$-modules contains the extensively studied category $\mathcal{O}$, whose objects are $\mathfrak{h}$-diagonalizable bounded $\mathfrak{g}$-modules. Recall that all irreducibles in the category $\mathcal{O}$ are the irreducible highest weight modules $L(\lambda)$ with highest weight $\lambda \in \mathfrak{h}^*$.

One of the basic facts of representation theory of Kac-Moody algebras is that the subcategory $\mathcal{O}_{int}$ of $\mathcal{O}$, which consists of modules, all of whose irreducible subquotients are integrable (i.e. are isomorphic to $L(\lambda)$ with λ dominant integral), is semisimple ([K2], Chapter 10). This generalization of Weyl's complete reducibility theorem can be generalized further as follows.

Let $(.,.)$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$. When restricted to $\mathfrak{h}$ it is still non-degenerate, and one can normalize it, such that $(\alpha_i, \alpha_i) \in \mathbb{Q}_{>0}$ for all simple roots α_i ([K2], Chapter 2). Then $\Delta = \Delta^{re} \coprod \Delta^{im}$, where Δ^{re} (resp.Δ^{im}) $= \{\alpha \in \Delta \mid (\alpha, \alpha) > 0$ (resp.$(\alpha, \alpha) \leq 0)\}$; for $\alpha \in \Delta^{re}$ we let $\alpha^\vee \in \mathfrak{h}$ be such that $\langle \lambda, \alpha^\vee \rangle = \frac{2(\lambda, \alpha)}{(\alpha, \alpha)}$. The Weyl group W of $\mathfrak{g}$ is the subgroup of $GL(\mathfrak{h}^*)$, generated by orthogonal reflections r_α in the roots $\alpha \in \Delta^{re}$ ([K2], Chapter 3). Given $\lambda \in \mathfrak{h}^*$, let

$$\Delta(\lambda) := \{\alpha \mid \alpha \in \Delta^{re} \ \& \ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}, \quad \Delta(\lambda)^\vee := \{\alpha^\vee \mid \alpha \in \Delta(\lambda)\}, \quad \Delta_+(\lambda) := \Delta(\lambda) \cap \Delta_+^{re},$$

and let $W(\lambda)$ be the subgroup of W, generated by $\{r_\alpha \mid \alpha \in \Delta(\lambda)\}$.

Set

$$\mathfrak{h}' := \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}].$$

Notice that for any λ one has $\Delta(\lambda)^\vee \subset \mathfrak{h}'$. An element $\lambda \in \mathfrak{h}^*$, is called *rational* if

$$\mathbb{C}\Delta(\lambda)^\vee = \mathfrak{h}'.$$

An element $\lambda \in \mathfrak{h}^*$, is called *non-critical* if

$$2(\lambda + \rho, \alpha) \notin \mathbb{Z}_{>0}(\alpha, \alpha) \text{ for all } \alpha \in \Delta_+^{im}.$$

By [KK], Thm. 2, all simple subquotients of a Verma module $M(\mu)$ with μ non-critical are of the form $L(w(\mu + \rho) - \rho)$ for some $w \in W(\mu)$. We call $\lambda \in \mathfrak{h}^*$ and the corresponding $L(\lambda)$ *weakly admissible* if it is non-critical and for any non-critical $\mu \neq \lambda$, $L(\lambda)$ is not a subquotient of $M(\mu)$. By [KK], Thm. 2, a non-critical λ is weakly admissible iff

$$\langle \lambda + \rho, \alpha^\vee \rangle \geq 0 \ \text{ for all } \alpha \in \Delta_+(\lambda).$$

We denote by $\mathcal{O}_{w.adm}$ the subcategory of $\mathcal{O}$, which consists of modules, all of whose simple subquotients are weakly admissible. Clearly, any integrable module is weakly admissible, hence the category $\mathcal{O}_{int}$ is a subcategory of $\mathcal{O}_{w.adm}$. It follows from [DGK] that the category $\mathcal{O}_{w.adm}$ is semisimple as well.

From the vertex algebra theory viewpoint it is important to study modules, which are not necessarily $\mathfrak{h}$-diagonalizable, but are $\mathfrak{h}$-locally finite (see [M]). In this case a $\mathfrak{g}$-module M decomposes into a direct sum of *generalized weight spaces*:

$$M_\mu := \{v \in M \mid (h - \mu(h))^N v = 0, h \in \mathfrak{h}, \ N >> 0\},$$

and we still denote by $\Omega(M)$ the set of generalized weights. We still call M *bounded* if $\Omega(M)$ is bounded by a finite set of elements of $\mathfrak{h}^*$. We denote by $\tilde{\mathcal{O}}$ the category of all $\mathfrak{h}$-locally finite bounded $\mathfrak{g}$-modules. While in the category $\mathcal{O}$, any self-extension of $L(\lambda)$ is trivial, this is not the case for the category $\tilde{\mathcal{O}}$. In fact, if $\mathfrak{h}' \neq \mathfrak{h}$ (which is the case when $\mathfrak{g}$ is affine), then any $L(\lambda)$ has obvious non-trivial self-extensions. Hence we may expect triviality of self-extensions of $L(\lambda)$ in $\tilde{\mathcal{O}}$ only if $L(\lambda)$ is viewed as a $[\mathfrak{g}, \mathfrak{g}]$-module (recall that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] + \mathfrak{h}$). Our main result in this direction is the following.

0.1. **Theorem.** *Let $\lambda \in \mathfrak{h}^*$ be a non-critical weight which is dominant, i.e.*

$$\langle \lambda + \rho, \alpha^\vee \rangle > 0 \ \ \text{for all } \alpha \in \Delta_+(\lambda).$$

Then $\mathrm{Ext}^1(L(\lambda), L(\lambda))$ is in canonical bijection with the annihilator $\Delta(\lambda)^\vee$ in $\mathfrak{h}^*$ $(= \{\mu \in \mathfrak{h}^* |\ \langle \mu, \Delta(\lambda)^\vee \rangle = 0\})$.

Consequently, if, in addition, λ is rational, then any self-extension of $L(\lambda)$, viewed as a $[\mathfrak{g}, \mathfrak{g}]$-module, is trivial.

A $\mathfrak{g}$-module $L(\lambda)$ and the corresponding λ are called *admissible* if it is weakly admissible and, viewed as a $[\mathfrak{g}, \mathfrak{g}]$-module, has no non-trivial self-extensions. Denote by $\tilde{\mathcal{O}}_{adm}$ the subcategory of $\tilde{\mathcal{O}}$, consisting of $\mathfrak{g}$-modules, all of whose simple subquotients are admissible. Theorem 0.1 implies that $\mathcal{O}_{int}$ is a subcategory of $\tilde{\mathcal{O}}_{adm}$. Again, the following result is easily derived from [DGK].

0.2. **Theorem.** *Any $\mathfrak{g}$-module from the category $\tilde{\mathcal{O}}_{adm}$, viewed as a $[\mathfrak{g}, \mathfrak{g}]$-module, is completely reducible.*

A $\mathfrak{g}$-module $L(\lambda)$ and the corresponding λ are called *KW-admissible* if λ is rational non-critical and dominant. By Theorem 0.1, the KW-admissibility is a slightly stronger condition than the admissibility, namely, an admissible $\lambda \in \mathfrak{h}^*$ is KW-admissible iff $\lambda + \rho$ is regular, i.e. $(\lambda + \rho, \alpha) \neq 0$ for all $\alpha \in \Delta^{re}$.

KW-admissible modules have been introduced in [KW1], where their characters were computed. The importance of this notion comes from the fact that if $\mathfrak{g}$ is an affine Kac-Moody algebra ([K2], Chapter 6), then the (normalized) complete character of a KW-admissible $\mathfrak{g}$-module $L(\lambda)$ is a modular function [KW1],[KW2] (and conjecturally these are all $\mathfrak{g}$-modules $L(\lambda)$ with this property, which is known only for $\mathfrak{g} = \hat{\mathfrak{sl}}_2$ [KW1]). This fact, in turn, implies modular invariance of characters of modules over the associated W-algebras [KRW],[KW3].

Observe that rationality, weak admissibility, KW-admissibility, and admissibility of $\lambda \in \mathfrak{h}^*$ are, in fact, the properties that depend only on the restriction of λ to $\mathfrak{h}'$. Indeed, take $\nu \in \mathfrak{h}^*$ such that $\langle \nu, \mathfrak{h}' \rangle = 0$. Since any root is a linear combination of simple roots, $\lambda \in \mathfrak{h}^*$ is non-critical iff $\lambda + \nu$ is non-critical. Therefore λ is weakly admissible (resp., KW-admissible) iff $\lambda + \nu$ is weakly admissible (resp., KW-admissible). Since $L(\nu)$ is trivial as a $[\mathfrak{g}, \mathfrak{g}]$-module, and $L(\lambda + \nu) \cong L(\lambda) \otimes L(\nu)$, λ is admissible iff $\lambda + \nu$ is admissible.

The KW-admissibility and admissibility of a $\mathfrak{g}$-module $L(\lambda)$ can be understood in geometric terms as follows. Given a rational $\Lambda \in (\mathfrak{h}')^*$, we associate to it the following (infinite) polyhedron:

$$\mathcal{P}(\Lambda) := \{\lambda \in (\mathfrak{h}')^* |\ \langle \lambda + \rho, \alpha^\vee \rangle \geq 0 \ \text{ for all } \alpha \in \Delta_+(\Lambda)\}.$$

An element $\lambda \in \mathcal{P}(\Lambda)$ is called *integral* if $\Delta(\lambda) = \Delta(\Lambda)$. The set of KW-admissible weights is the union of the sets of all interior integral points of all the polyhedra $\mathcal{P}(\Lambda)$, whereas the set of admissible weights is obtained by adding some integral points of the boundary

of $\mathcal{P}(\Lambda)$'s, and that of weakly admissible rational weights by adding all the integral points of the boundary.

It would be important to replace in Theorem 0.2 the boundedness condition by the restrictness condition, but for general Kac-Moody algebras we can do it only in the integrable case:

0.3.   **Theorem.**   *Let M be a restricted $\mathfrak{h}$-locally finite $\mathfrak{g}$-module with finite-dimensional generalized weight spaces. If any irreducible subquotient of M is isomorphic to an integrable $\mathfrak{g}$-module $L(\lambda)$, $\lambda \in P_+$, then, viewed as a $[\mathfrak{g}, \mathfrak{g}]$-module, M is completely reducible.*

Another version of this theorem is

0.4.   **Theorem.**   *The category of all restricted integrable $[\mathfrak{g}, \mathfrak{g}]$-modules is semisimple (recall that a $[\mathfrak{g}, \mathfrak{g}]$-module M is called integrable if all Chevalley generators e_i and f_i are locally nilpotent on M).*

If $\mathfrak{g}$ is an affine Kac-Moody algebra we can treat the admissible case as well. Recall that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}D$ for some $D \in \mathfrak{h}$ and $[\mathfrak{g}, \mathfrak{g}]$ contains the canonical central element K, such that its eigenvalue k in all integrable $\mathfrak{g}$-module $L(\lambda)$ runs over all non-negative integers ([K2], Chapters 6 and 12). It follows from [KK], Thm. 2 that a $\mathfrak{g}$-module $L(\lambda)$ is non-critical iff $k \neq -h^\vee$, where $h^\vee$ is the dual Coxeter number, which is a positive integer ([K2], Chapter 6). By [KW2], if $L(\lambda)$ is KW-admissible, then $k + h^\vee \in \mathbb{Q}_{>0}$.

Furthermore, if M is a restricted $[\mathfrak{g}, \mathfrak{g}]$-module on which $K = k \operatorname{id}_M$ and $k \neq -h^\vee$, then, by the Sugawara construction ([K2], Chapter 12), the action of $[\mathfrak{g}, \mathfrak{g}]$ can be extended to that of $\mathfrak{g}$ by letting $D = -L_0$. We call a simple $[\mathfrak{g}, \mathfrak{g}]$-module $L(\lambda)$ ($\lambda \in (\mathfrak{h}')^*$) k-*admissible*, if $K = k \operatorname{id}_M$, if $L(\lambda)$ is weakly admissible (i.e. a Verma $[\mathfrak{g}, \mathfrak{g}]$-module $M(\mu)$ with $\mu \neq \lambda$ has no subquotients isomorphic to $L(\lambda)$), and if $L(\lambda)$ has no non-trivial self-extensions N with $K = k \operatorname{id}_N$.

Given $k \neq -h^\vee$, denote by $\mathcal{A}\mathrm{dm}_k$ the category of restricted $[\mathfrak{g}, \mathfrak{g}]$-modules M with $K = k \operatorname{id}_M$, which are locally finite over $\mathfrak{h}' + \mathbb{C}L_0$, and such that any simple subquotient of M is k-admissible. It is easy to see that $\mathcal{A}\mathrm{dm}_k$ is empty if $k + h^\vee \in \mathbb{Q}_{\leq 0}$.

0.5.   **Theorem.**   *For $k \in \mathbb{Q}$ the category $\mathcal{A}\mathrm{dm}_k$ is semisimple with finitely many simple objects. A weight λ, such that $\lambda + \rho$ is regular, is k-admissible iff it is KW-admissible.*

From the vertex algebra theory viewpoint the most important modules over a (non-twisted) affine Lie algebra $\mathfrak{g}$, associated to a finite-dimensional simple Lie algebra $\dot{\mathfrak{g}}$, are irreducible vacuum modules $L(k\Lambda_0)$ of level k, where $\Lambda_0|_{\dot{\mathfrak{h}}} = 0$, $\dot{\mathfrak{h}}$ being a Cartan subalgebra of $\dot{\mathfrak{g}}$. This $\mathfrak{g}$-module is KW-admissible iff $k + h^\vee = \frac{p}{q}$, where p, q are coprime positive integers, $p \geq h^\vee$ (resp. $p \geq h$, the Coxeter number), if q and the "lacety" $l = 1, 2, 3$ of $\dot{\mathfrak{g}}$ are coprime (resp. l divides q) [KW4]. However, the $\mathfrak{g}$-module $L(k\Lambda_0)$ for rational k is k-admissible for a slightly larger set of levels, namely, we add the "boundary"

values $p = h^\vee - 1$ (resp. $p = h - 1$) and arbitrary positive integer values of q, provided that $\gcd(q, l) = 1$ (resp. $\gcd(q, l) = l$).

Similar picture persists in the description of all k-admissible weights vs all KW-admissible weights of level k for rational k. Namely, there is a finite number of finite polyhedra in $(\mathfrak{h}')^*$, such that the set of KW-admissible weights is the set of interior integral points of these polyhedra , whereas the set of k-admissible weights is obtained by adding some boundary integral points; the set of admissible weights is obtained by removing from the latter some of the boundary points. For example, for $\dot{\mathfrak{g}} = \mathfrak{sl}_2$, all KW-admissible modules of level $k = -2 + \frac{p}{q}$ are of the form $L(\lambda_{r,s})$, where $\langle \lambda_{r,s}, K \rangle = k, (\lambda_{r,s}, \alpha) = (s-1) - (r-1)\frac{p}{q}$ and α is a simple root of $\mathfrak{sl}_2$, p and q are positive coprime integers, $p \geq 2$, r and s are integers, $1 \leq r \leq q, 1 \leq s \leq p - 1$ [KW2]. On the other hand, the category $\mathcal{A}\mathrm{dm}_k$ for rational k is non-empty for $\dot{\mathfrak{g}} = \mathfrak{sl}_2$ iff $k = -2 + \frac{p}{q}$ for all positive coprime integers p, q and all simple objects of this category are $L(\lambda_{r,s})$, where $r, s \in \mathbb{Z}, 1 \leq r \leq q, 0 \leq s \leq p$. The set of admissible weights in this case coincides with the set of KW-admissible weights.

Theorem 0.5 explains why the simple affine vertex algebra V_k, associated to $\mathfrak{g}$, with non-negative integral level k, is *regular*, i.e. any (restricted) V_k-module is completely reducible [DLM]. Indeed, V_k satisfies Zhu's C_2 condition ([Zh],[FZ],[KWa]), hence any V_k-module M is locally finite over $\mathfrak{h}' \oplus \mathbb{C}L_0$ (Proposition 7.3.1). Moreover, any irreducible V_k-module is one of the $L(\lambda)$ (cf. Subsection 9.2), and it is easy to see that $L(\lambda)$ must be integrable. Thus, M is in the category $\mathcal{A}\mathrm{dm}_k$, hence is admissible. If $k \notin \mathbb{Z}_{\geq 0}$, the C_2 condition obviously fails, hence V_k is not regular.

Along the same lines the complete reducibility can be studied over the Virasoro, Neveu-Schwarz, and other W-algebras.

For example, in the case of the Virasoro algebra $\mathcal{V}ir = (\sum_j \mathbb{C}L_j) + \mathbb{C}C$, we denote by $\mathcal{A}\mathrm{dm}_c$ the category of all restricted $\mathcal{V}ir$-modules M, for which $C = c\,\mathrm{id}_M$, L_0 is locally finite on M, and every irreducible subquotient of M is a c-admissible highest weight $\mathcal{V}ir$-module $L(h, c)$, where h is the lowest eigenvalue of L_0. As before, $L(h, c)$ is called *c-admissible* if it is a subquotient of a Verma module $M(h', c)$ only for $h' = h$, and it has no non-trivial self-extensions, where C acts by $c\,\mathrm{id}$. Of course, "restricted" means that for any $v \in M$ one has $L_n v = 0$ for $n >> 0$.

0.6. **Theorem.**   *Set $c(k) := 1 - \frac{6(k+1)^2}{k+2}$, where $k \in \mathbb{Q} \setminus \{-2\}$. The category $\mathcal{A}\mathrm{dm}_{c(k)}$ of $\mathcal{V}ir$-modules is semisimple with finitely many simple objects if $k > -2$ and is empty if $k < -2$.*

Recall that $c(k)$ with $k + 2 = p/q$, where $p, q \in \mathbb{Z}_{\geq 2}$ are coprime, is called a *minimal series central charge* [BPZ] and denoted by $c^{p,q} = 1 - \frac{6(p-q)^2}{pq}$. In this case all simple objects of $\mathcal{A}\mathrm{dm}_{c(k)}$ are $L(h_{r,s}^{p,q}, c^{p,q})$, where

$$h_{r,s}^{p,q} = \{ \frac{(rp - sq)^2 - (p - q)^2}{4pq} \mid r = 0, \ldots, q, \ s = 0, \ldots, p, \ (r, s) \neq (0, p) \text{ or } (q, 0) \}.$$

(Note that $h_{r,s}^{p,q} = h_{q-r,p-s}^{p,q}$). The $\mathcal{V}ir$-modules $L(h_{r,s}^{p,q}, c^{p,q})$ with $0 < r < q, 0 < s < p$ are called the *minimal series modules* [BPZ]. Since the simple vertex algebra V_c, associated with the Virasoro algebra, with the minimal series central charge $c = c^{p,q}$, satisfies the C_2 condition (see e.g. [GK]), it follows from [M] that L_0 is locally finite on any (restricted) V_c-module M. It is also well known that all irreducible V_c-modules are the minimal series modules, hence the category $\mathcal{A}\mathrm{dm}_{c^{p,q}}$ contains the category of $V_{c^{p,q}}$-modules, as a (strictly smaller) subcategory. Hence, by Theorem 0.6, M is completely reducible, proving regularity of V_c [DLM].

On the contrary, since the vertex algebra V_c with c not a minimal series central charge has infinitely many irreducible modules, the category of its modules is much larger than the category $\mathcal{A}\mathrm{dm}_{c(k)}$ with $k \in \mathbb{Z}\backslash\{-2\}$. Indeed, letting $p = k+2$, it follows from Theorem 0.6 that the category $\mathcal{A}\mathrm{dm}_{c^{p,1}}$ is semisimple with finitely many simple objects if p is a positive integer, and is empty if p is a negative integer. All simple modules of this category are $L(h_{1,s}^{p,1}, c^{p,1})$, with $s = 1, 2, \ldots, p$, provided that p is a positive integer.

Since the minimal series modules are precisely the irreducible $\mathcal{V}ir$-modules, for which the (normalized) characters are modular invariant functions [KW1], there is an obvious analogy between them and the KW-admissible modules over affine Lie algebras. In fact, conjecturally, the latter are precisely the simple highest weight modules over the corresponding simple vertex algebras. This is known only for $\mathfrak{g} = \hat{\mathfrak{sl}}_2$ [AM].

0.7. The contents of the paper is as follows. In Section 1 we prove a general lemma on complete reducibility of a category of modules over an associative algebra $\mathcal{U}$, satisfying certain conditions, see Lemma 1.3.1. We introduce the notions of highest weight modules, Jantzen-type filtrations and Shapovalov-type determinants in this general setup. This is used to find the conditions, under which any extension between two isomorphic (resp. non-isomorphic) highest weight modules over $\mathcal{U}$ splits, see Proposition 1.6.5 (resp. Corollary 1.8.2).

In Sections 2 and 3 the general approach, developed in Section 1, is used to establish the complete reducibility results for symmetrizable Kac-Moody algebras. In Section 4 some stronger results for affine Lie algebras are obtained (see Theorems 0.1–0.5 above). Here we also classify k-admissible irreducible vacuum modules and give a geometric description of all k-admissible vs KW-admissible highest weights. In Sections 5 and 6 we classify admissible modules for the Virasoro and the Neveu-Schwarz algebra, using Proposition 1.6.5.

After recalling some background material on vertex algebras in Section 7, we study the admissible modules over minimal W-algebras in Section 8. In particular, we classify the admissible irreducible vacuum modules over those W-algebras (see Corollary 8.8.3) and describe all admissible weights vs KW-admissible highest weights via the quantum Hamiltonian reduction (Subsection 8.8.2).

In conclusion, in Chapter 9 we state a conjecture, which is supposed to imply that all simple W-algebras $\mathcal{W}_k(\mathfrak{g}, f)$ satisfying the C_2-condition, are regular.

Our ground field is $\mathbb{C}$. All tensor products are considered over $\mathbb{C}$, unless otherwise specified.

1. Verma modules and Jantzen-type filtrations

In this section we introduce, for a natural class of associative superalgebras, the notions of Verma module, Shapovalov form and Jantzen filtration. For a simple highest weight module L we introduce a linear map $\Upsilon_L : \text{Ext}^1(L, L) \to \mathfrak{h}^*$ and describe its image in terms of a Jantzen-type filtration. We prove Lemma 1.3.1, Proposition 1.6.5, and Lemma 1.8.1.

1.1. **Notation.** For a commutative Lie algebra $\mathfrak{h}$ and an $\mathfrak{h}$-module N, we still denote by N_ν the generalized weight space:

$$N_\nu := \{v \in N| \; \forall h \in \mathfrak{h} \; \exists r \text{ s.t. } (h - \langle \nu, h \rangle)^r v = 0\},$$

and by $\Omega(N)$ the set of generalized weights. We call N_ν the generalized weight space of weight ν and we say that a non-zero v is a *generalized weight vector* if $v \in N_\nu$ for some ν. Recall that, if N is a $\mathfrak{g}$-module with a locally finite action of $\mathfrak{h}$, then N is a direct sum of its generalized weight spaces and

$$(1) \qquad\qquad \Omega(N) = \cup_{L \in \text{supp}(N)} \Omega(L),$$

where $\text{supp}(N)$ stands for the set of irreducible subquotients of N.

1.2. **Assumptions.** Let $\mathcal{U}$ be a unital associative superalgebra and let $\mathfrak{h}$ be an even finite-dimensional subspace of $\mathcal{U}$. We will assume some of the following properties:

(U1) the map $\mathfrak{h} \hookrightarrow \mathcal{U}$ induces an injective map of associative algebras $\mathcal{S}(\mathfrak{h}) \hookrightarrow \mathcal{U}$;

(U2) $\mathcal{U}$ contains two unital subalgebras $\mathcal{U}_\pm$ such that the multiplication map gives the bijection $\mathcal{U} = \mathcal{U}_- \otimes \mathcal{U}(\mathfrak{h}) \otimes \mathcal{U}_+$;

(U3) with respect to the adjoint action $(adh)(u) := hu - uh$, $\mathfrak{h}$ acts diagonally on $\mathcal{U}_\pm$ and the weight spaces $\mathcal{U}_{\pm;\nu}$ in $\mathcal{U}_\pm$ are finite-dimensional, the 0th weight space $\mathcal{U}_{\pm;0}$ being $\mathbb{C} \cdot 1$;

(U4) there exists $h \in \mathfrak{h}$ such that $\langle \nu, h \rangle < 0$ for any $\nu \in \Omega(\mathcal{U}_-) \setminus \{0\}$;

(U4') $\exists h \in \mathfrak{h}^*$ s.t. $\forall \nu \in \Omega(\mathcal{U}_+) \setminus \{0\}$ (resp. $\forall \nu \in \Omega(\mathcal{U}_-) \setminus \{0\}$) $\langle \nu, h \rangle > 0$ (resp. $\langle \nu, h \rangle < 0$);

(U5) $\forall \nu \quad \dim \mathcal{U}_{+;\nu} = \dim \mathcal{U}_{-;\nu}$;

(U6) $\mathcal{U}$ admits an antiautomorphism σ, which interchanges $\mathcal{U}_+$ with $\mathcal{U}_-$ and fixes the elements of $\mathfrak{h}$.

The assumption (U4') is a strong form of (U4). Note that (U1)–(U5) imply (U4'), and that (U1)–(U4) and (U6) imply (U5).

In this section $\mathcal{U}$ satisfies the assumptions (U1)–(U4).

1.2.1.   For a contragredient Lie superalgebra $\mathfrak{g}(A, \tau)$, the Virasoro algebra or the Neveu-Schwarz superalgebra we have the natural triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. In these cases we let $\mathcal{U}$ be the universal enveloping algebra of $\mathfrak{g}$ and let $\mathcal{U}_\pm := \mathcal{U}(\mathfrak{n}_\pm)$. Then $\mathcal{U}$ satisfies the assumptions (U1)–(U6). For minimal $\mathcal{W}$-algebras, $\mathcal{U}$ is the universal enveloping algebra of the vertex algebra $\mathcal{W}$; this is a topological algebra, satisfying assumptions (U1)–(U5), where (U2) is substituted by its topological version (28).

1.2.2.   Let $Q_\pm$ be the $\mathbb{Z}_{\geq 0}$-span of $\Omega(\mathcal{U}_\pm)$:

$$Q_\pm = \sum_{\alpha \in \Omega(\mathcal{U}_\pm)} \mathbb{Z}_{\geq 0}\alpha.$$

Introduce the standard partial order on $\mathfrak{h}^*$: $\nu \leq \nu'$ if $\nu - \nu' \in Q_-$. If $\mathcal{U}$ satisfies (U5) then $Q_+ = -Q_-$.

1.3. **Verma module.** Set

$$\mathcal{U}'_\pm := \sum_{\nu \neq 0} \mathcal{U}_{\pm;\nu}.$$

By (U4), $\mathcal{U}'_-$ is a two-sided ideal of $\mathcal{U}_-$.

We view $\mathcal{M} := \mathcal{U}/\mathcal{U}\mathcal{U}'_+$ as a left $\mathcal{U}$-module and a right $\mathcal{S}(\mathfrak{h})$-module; we call it a *universal Verma module*. We consider the adjoint action of $\mathcal{S}(\mathfrak{h})$ on $\mathcal{M}$ ($(\mathrm{ad}\,h)v := hv - vh$ for $h \in \mathfrak{h}, v \in \mathcal{M}$) and define the weight spaces $\mathcal{M}_\nu$ with respect to this action. For each $\lambda \in \mathfrak{h}^*$ the *Verma module* $M(\lambda)$ is the evaluation of $\mathcal{M}$ at λ, that is the $\mathcal{U}$-module $\mathcal{U}/\mathcal{U}(\mathcal{U}'_+ + \sum_{h \in \mathfrak{h}}(h - \langle\lambda, h\rangle))$. Observe that $\mathfrak{h}$ acts diagonally on $M(\lambda)$, a weight space $M(\lambda)_{\lambda+\nu}$ being the image of $\mathcal{M}_\nu$, hence $\Omega(M(\lambda)) = \lambda + Q_-$, and $\dim M(\lambda)_{\lambda+\nu} = \dim \mathcal{U}_{-;\nu} < \infty$.

Any quotient N of the $\mathcal{U}$-module $M(\lambda)$ is called a *module with highest weight* λ. In other words, N is a module with highest weight λ if $\dim N_\lambda = 1$, $\mathcal{U}'_+ N_\lambda = 0$ and N_λ generates N. Note that any highest weight module is indecomposable.

Each Verma module has a unique maximal proper submodule $M'(\lambda)$; the irreducible module $L(\lambda) := M(\lambda)/M'(\lambda)$ is called the irreducible module with highest weight λ.

Our study of complete reducibility in various categories will be based on the following lemma.

1.3.1. **Lemma.**   *Assume that $\mathcal{U}$ satisfies the assumptions (U1)-(U4). Assume, in addition, that $\mathcal{U}_+$ is finitely generated (as an associative algebra).*

Let $\mathcal{A}$ be a category of $\mathcal{U}$-modules with the following properties:

(i) $\mathcal{A}$ is closed under taking subquotients;

(ii) $\operatorname{Ext}^1_{\mathcal{A}}(L', L) = 0$ for any irreducible modules L, L' in $\mathcal{A}$;

(iii) each irreducible module in $\mathcal{A}$ is of the form $L(\lambda), \lambda \in \mathfrak{h}^*$,

and, in addition, one of the following conditions hold:

(iv) each module contains an irreducible submodule;

or

(iv') $\mathcal{U}$ satisfies (U4'), each module in $\mathcal{A}$ is $\mathfrak{h}$-locally finite and $\mathcal{A}$ has finitely many irreducible modules.

Then $\mathcal{A}$ is a semisimple category.

Proof. By (ii), $\mathcal{A}$ does not contain indecomposable modules of length two so any highest weight module in $\mathcal{A}$ is irreducible.

Let $\mathcal{A}$ satisfy (i)-(iv). Let N be a module in $\mathcal{A}$. Assume that N is not completely reducible. Let N' be the socle of N, that is the sum of all irreducible submodules. By (iv), $N' \neq 0$. By [L], Ch. XVII, a sum of irreducible submodules can be decomposed in a direct sum: $N' = \oplus_{i \in I} L_i$, where L_i are irreducible. Let L be an irreducible submodule of N/N' and $N'' \subset N$ be the preimage of L under the map $N \to N/N'$. We obtain the exact sequence

(2) $0 \to \oplus_{i \in I} L_i \to N'' \to^{\phi} L \to 0,$

which does not split, because N' is the socle of N''. By (iii), L_i, L are irreducible highest weight modules, so both N', L are $\mathfrak{h}$-diagonalizable. Thus N'' is $\mathfrak{h}$-locally finite.

Let ν be the highest weight of L: $L = L(\nu)$. Let $v \in N''_\nu$ be a preimage of the highest weight vector in $L(\nu)$. Since $\phi(\mathcal{U}'_+ v) = 0$ and $\mathcal{U}_+$ is finitely generated, there exists a finite set $J \subset I$ such that $\mathcal{U}'_+ v \subset \oplus_{j \in J} L_j$. Let ψ be the natural map $\psi : N'' \to N''/\oplus_{j \in J} L_j$. Since any highest weight module in $\mathcal{A}$ is irreducible, the equality $\mathcal{U}'_+ \psi(v) = 0$ implies that $\psi(v)$ generates a module, which admits a finite filtration with the subquotients isomorphic to $L(\nu)$. Thus, by (ii), $\mathcal{U}(\psi(v)) \cong L(\nu)$. The exact sequence

$$0 \to \oplus_{j \in J} L_j \to \psi^{-1}(\mathcal{U}(\psi(v))) \to^{\psi} L(\nu) \to 0$$

splits since J is finite and $\operatorname{Ext}^1_{\mathcal{A}}(L(\nu), L_i) = 0$ by (iii). Let $L' \subset \psi^{-1}(\mathcal{U}(\psi(v))) \subset N''$ be such that $\psi^{-1}(\mathcal{U}(\psi(v))) = \oplus_{j \in J} L_j \oplus L'$. Since $v \in \oplus_{j \in J} L_j \oplus L'$, one has $\phi(L') \neq 0$. Since $L' \cong L(\nu)$, we obtain $\phi(L') = L$ so the exact sequence (2) splits, a contradiction. Hence (i)–(iv) imply semisimplicity of $\mathcal{A}$.

Let $\mathcal{A}$ satisfy the assumptions (i)–(iv)'. It is enough to verify that $\mathcal{A}$ satisfies (iv). Let N be a module in $\mathcal{A}$. By (iv)', $\mathrm{supp}(N)$ is finite: $\mathrm{supp}(N) = \{L(\lambda_i)\}_{i=1}^s$. Let λ_1 be such that, for $h \in \mathfrak{h}$ introduced in (U4'), the value $\langle \lambda_1, h \rangle$ is maximal, that is $\langle \lambda_1 - \lambda_i, h \rangle \not< 0$. Then, by (1) and (U4'), for any $\nu \in \Omega(N)$ one has $\langle \lambda_1 - \nu, h \rangle \not< 0$, so $\mathcal{U}'_+ N_{\lambda_1} = 0$, by (U4'). Let $v \in N_{\lambda_1}$ be an $\mathfrak{h}$-eigenvector. Since $\mathcal{U}'_+ v = 0$, the submodule generated by v is a highest weight module and it belongs to $\mathcal{A}$ by (i). By above, this submodule is irreducible. This establishes (iv) and completes the proof. $\square$

1.3.2. *Remark.*   The assumption that $\mathcal{U}_+$ is finitely generated was used only for the claim that $\mathcal{U}'_+ v \subset \oplus_{j \in J} L_j$ for a finite set $J \subset I$. As a result, this assumption can be weakened as follows: for any $N \in \mathcal{A}$ and each $v \in N$ the left $\mathcal{U}_+$-module $\mathcal{U}'_+ / \mathrm{Ann}_{\mathcal{U}'_+} v$ is finitely generated. This holds for the continuous representations of finitely generated vertex algebras.

1.3.3. *Remark.*   The following example shows why (i), (ii), (iv) do not imply semisimplicity. Let $A = \mathbb{C}[x_1, x_2, \ldots]$ be a polynomial ring in infinitely many variables, let W be a one-dimensional A-module $\mathbb{C}w$, such that $x_i w = 0$ for all i, and let, for $i = 1, 2, \ldots, V_i$ be the following one-dimensional A-module: $x_j|_{V_i} = \delta_{ij} \mathrm{Id}_{V_i}$. Then $\mathrm{Ext}^1(W, V_i) = 0$ for all i, but $\mathrm{Ext}^1(W, \oplus_i V_i) \neq 0$. Namely, there exists a non-splitting extension of $\oplus_i V_i$ by W with the basis $w, v_1, v_2, \ldots$, and the action of A given by $x_i w = v_i$, $x_i v_j = \delta_{ij} v_j$.

1.4. **Shapovalov-type map.** In this section we introduce in our setup a generalization of Shapovalov's bilinear form and Shapovalov's determinant [Sh].

1.4.1.   By (U2), $\mathcal{U} = \mathcal{U}'_- \mathcal{S}(\mathfrak{h}) \mathcal{U}'_+ \oplus \mathcal{U}'_- \mathcal{S}(\mathfrak{h}) \oplus \mathcal{S}(\mathfrak{h})$; let

$$\mathrm{HC} : \mathcal{U} \to \mathcal{S}(\mathfrak{h})$$

be the projection with respect to this decomposition. Identify $\mathcal{M}$ with $\mathcal{U}_- \otimes \mathcal{S}(\mathfrak{h})$ (as $\mathcal{U}_-$-$\mathcal{S}(\mathfrak{h})$ bimodules) and introduce a bilinear map $\mathcal{U} \otimes \mathcal{M} \to \mathcal{S}(\mathfrak{h})$ by

$$B(u, v) := \mathrm{HC}(uv).$$

Notice that for $u \in \mathcal{U}, v \in \mathcal{M}$ one has $B(u, v) = P(uv)$, where P is the projection $\mathcal{M} \to \mathcal{S}(\mathfrak{h})$ with the kernel $\mathcal{U}'_- \otimes \mathcal{S}(\mathfrak{h})$. One has

$$(3) \qquad B(au, v) = B(u, va) = B(u, v)a \ \text{ for any } a \in \mathcal{S}(\mathfrak{h}), \ u \in \mathcal{U}, \ v \in \mathcal{M}.$$

Let v_0 be the canonical generator of the module $\mathcal{M}$ (i.e., the image of $1 \in \mathcal{U}$). Denote by v_λ the image of v_0 in $M(\lambda)$ and by $B(\lambda)$ the evaluation of the map B in $M(\lambda)$, i.e. $B(\lambda) : \mathcal{U} \otimes M(\lambda) \to \mathbb{C}$ ((3) ensures that the evaluation is well defined).

Denote by $P_\lambda : M(\lambda) \to M(\lambda)_\lambda$ the projection with the kernel $\mathcal{U}'_- v_\lambda = \sum_{\nu \neq \lambda} M(\lambda)_\nu$. Then, for each $v \in M(\lambda)$, one has

$$(4) \qquad\qquad B(\lambda)(u, v) v_\lambda = P_\lambda(uv).$$

By (U4), $\mathcal{U}'_-$ is a two-sided ideal of $\mathcal{U}_-$, so $B(\mathcal{U}'_-\mathcal{U}, \mathcal{M}) = 0$. Using (3) we get

(5) $$B(\lambda)(\mathcal{U}, v) = B(\lambda)(\mathcal{U}_{+,\nu}, v) \text{ for each } v \in M(\lambda)_{\lambda-\nu}.$$

1.4.2. We claim that

(6) $$\{v \in M(\lambda)|\ B(\lambda)(\mathcal{U}_+, v) = 0\} = M'(\lambda).$$

Indeed, by (5), $M' := \{v|\ B(\lambda)(\mathcal{U}_+, v) = 0\}$ is a submodule, and $M'_\lambda = 0$, because $B(\lambda)(1, v_\lambda) = 1$. Thus $M' \subset M'(\lambda)$. On the other hand, using (4), we get

$$v \in M'(\lambda) \implies \mathcal{U}v \cap M(\lambda)_\lambda = 0 \implies P_\lambda(\mathcal{U}v) = 0 \implies B(\lambda)(\mathcal{U}, v) = 0.$$

So $M'(\lambda) \subset M'$, as required.

1.4.3. Assume that $\mathcal{U}$ satisfies (U1)–(U5).

One has $B(\mathcal{U}_\mu, \mathcal{M}_\nu) = 0$ if $\mu + \nu \neq 0$. For $\nu \in Q_+$ let $B_\nu : \mathcal{U}_{+;\nu} \otimes \mathcal{M}_{-\nu} \to \mathcal{S}(\mathfrak{h})$ be the restriction of B. A matrix of the bilinear map B_ν is called a *Shapovalov matrix*. By (U5), $\mathcal{U}_{+;\nu}$ and $\mathcal{M}_{-\nu}$ have the same dimension, so this is a square matrix. The determinant of B_ν is the determinant of this matrix; this is an element in $\mathcal{S}(\mathfrak{h})$ defined up to multiplication by an invertible scalar. This is called a *Shapovalov determinant*. Define $B_\nu(\lambda)$ as the evaluation of B_ν at λ; clearly, this coincides with the restriction of $B(\lambda)$ to $\mathcal{U}_{+;\nu} \otimes M(\lambda)_{\lambda-\nu}$. Since $M'(\lambda) = \{v|\ B(\lambda)(\mathcal{U}_+, v) = 0\}$ one has

$$\det B_\nu(\lambda) = 0 \iff M'(\lambda)_{\lambda-\nu} \neq 0.$$

1.5. **Jantzen-type filtrations.** The Jantzen filtration and the sum formula were described by Jantzen in [Jan] for a Verma module over a semisimple Lie algebra. We describe this construction in our setup, i.e. for $\mathcal{U}$ satisfying (U1)–(U5).

Assume that $\mathcal{U}$ satisfies (U1)–(U4). Let R be the localization of $\mathbb{C}[t]$ by the maximal ideal (t). We shall extend the scalars from $\mathbb{C}$ to R. For a $\mathbb{C}$-vector superspace V denote by V_R the R-module $V \otimes R$. Then $\mathcal{U}_R, \mathcal{U}_{\pm;R}$ are algebras and $\mathcal{U}_R = \mathcal{U}_{-;R} \otimes \mathcal{S}(\mathfrak{h})_R \otimes \mathcal{U}_{+;R}$. Identify $\mathcal{S}(\mathfrak{h})_R$ with $\mathcal{S}_R(\mathfrak{h}_R)$. For any $\tilde{\lambda} \in \mathfrak{h}^*_R = \mathrm{Hom}_R(\mathfrak{h}_R, R)$ denote by $M_R(\tilde{\lambda})$ the corresponding Verma module over $\mathcal{U}_R$, and denote by $v_{\tilde{\lambda}}$ the canonical generator of $M_R(\tilde{\lambda})$. Define a filtration on $M_R(\tilde{\lambda})$ as follows: for $m \in \mathbb{Z}_{\geq 0}$

$$M_R(\tilde{\lambda})^m := \{v \in M_R(\tilde{\lambda})|\ \mathcal{U}_R v \cap R v_{\tilde{\lambda}} \subset R t^m v_{\tilde{\lambda}}\}.$$

Clearly, $\{M_R(\tilde{\lambda})^m\}$ is a decreasing filtration, $M_R(\tilde{\lambda})^0 = M_R(\tilde{\lambda})$. Let $\lambda \in \mathfrak{h}^*$ be the evaluation of $\tilde{\lambda}$ at $t = 0$ (the composition of $\tilde{\lambda} : \mathfrak{h}^*_R \to R$ and $R \to R/tR = A$). One has $M(\lambda) = M_R(\tilde{\lambda})/tM_R(\tilde{\lambda})$. Denote by $\{M(\lambda)^m\}$ the image of $\{M_R(\tilde{\lambda})^m\}$ in $M(\lambda)$. Then $\{M(\lambda)^m\}$ is a decreasing filtration of $M(\lambda)$ by $\mathcal{U}$-submodules, and, by (6), one has

(7) $$M(\lambda)^0 = M(\lambda), \quad M(\lambda)^1 = M'(\lambda).$$

We call the filtration $\{M(\lambda)^m\}$ a *Jantzen-type filtration*.

Define the Shapovalov map $\mathcal{U}_R \otimes \mathcal{M}_R \to \mathcal{S}(\mathfrak{h})_R$ and its evaluation $B(\tilde{\lambda}) : \mathcal{U} \otimes_R M_R(\tilde{\lambda}) \to R$ as above. Then $B(\lambda)$ is the evaluation of $B(\tilde{\lambda})$ at $t = 0$. One readily sees that for $m \in \mathbb{Z}_{\geq 0}$ one has

$$(8) \qquad M_R(\tilde{\lambda})^m = \{v \in M_R(\tilde{\lambda}) |\ B(\tilde{\lambda})(\mathcal{U}_{+;R}, v) \subset Rt^m\}.$$

Now assume that $\mathcal{U}$ satisfies (U1)–(U5). Observe that the Shapovalov matrix for $\mathcal{U}_R$ written with respect to bases lying in $\mathcal{U}$ coincide with the Shapovalov matrix for $\mathcal{U}$ written with respect to the same bases. Consequently, the Shapovalov determinants $\det B_\nu \in \mathcal{S}(\mathfrak{h})$ viewed as elements of the algebra $\mathcal{S}(\mathfrak{h})_R$ coincide with the Shapovalov determinants $\det B_\nu$ constructed for $\mathcal{U}_R$.

Recall that the matrix of a bilinear form on a free module of finite rank over a local ring with values in this ring is diagonalizable. This implies that, for each $\nu \in Q_+$, $\dim M(\lambda)^r_{\lambda - \nu}$ is equal to the corank of the map $B_\nu(\tilde{\lambda})$ modulo Rt^r. Using the standard reasoning of Jantzen [Jan],[J] we obtain the following sum formula

$$(9) \qquad \sum_{r=1}^{\infty} \dim M(\lambda)^r_{\lambda - \nu} = \upsilon(\det B_\nu(\tilde{\lambda})) \text{ for } \nu \in Q_+,$$

where $\upsilon : R \to \mathbb{Z}_{\geq 0}$ is given by $\upsilon(a) = r$ if $a \in Rt^r \setminus (Rt^{r+1})$. In particular, the above sum is finite iff $\det B_\nu(\tilde{\lambda}) \neq 0$. Thus

$$(10) \qquad \exists r :\ M(\lambda)^r_{\lambda - \nu} = 0 \iff \det B_\nu(\tilde{\lambda}) \neq 0.$$

We call a Jantzen-type filtration $\{M(\lambda)^m\}$ *non-degenerate* if for each ν there exists r such that $M(\lambda)^r_{\lambda - \nu} = 0$.

We say that $\mu \in \mathfrak{h}^*$ is λ-*generic* if μ is transversal to all hypersurfaces $\det B_\nu = 0$ at point λ, i.e. transversal to all irreducible components of $\det B_\nu = 0$ passing through λ for each $\nu \in Q_+$, where $\det B_\nu$ is non-zero. For example, for symmetrizable Kac-Moody algebras, $\det B_\nu \neq 0$ for all ν and ρ is λ-generic for each $\lambda \in \mathfrak{h}^*$.

The *Jantzen filtration* on the $\mathfrak{g}$-module $M(\lambda)$ is the filtration $\{M(\lambda)^m\}$ for $\tilde{\lambda} = \lambda + t\nu$, where ν is λ-generic. We do not know whether the Jantzen filtration depends on ν (for semisimple Lie algebras the fact that the Jantzen filtration does not depend on ν follows from [BB]). Below we will use more general filtrations, taking $\tilde{\lambda} = \lambda + t\lambda_1 + t^2\lambda_2$, where $\lambda, \lambda_1, \lambda_2 \in \mathfrak{h}^*$. In our setup λ_1 will be fixed and not always λ-generic, but we will take λ_2 to be λ-generic; then, the Jantzen-type filtration $\{M(\lambda)^m\}$ is non-degenerate by (10).

1.6. **The map $\Upsilon : \operatorname{Ext}^1(L(\lambda), L(\lambda)) \to \mathfrak{h}^*$.** In this subsection we introduce for a highest weight module M a map $\Upsilon_M : \operatorname{Ext}^1_\mathfrak{g}(M, M) \to \mathfrak{h}^*$, and establish some useful properties of this map (see Corollary 1.6.5).

1.6.1. *Definition of* Υ. Let M be a module with the highest weight λ (i.e. a quotient of $M(\lambda)$), and let $v_\lambda \in M$ be the highest weight vector, i.e. the image of the canonical generator of $M(\lambda)$. Introduce the natural map

$$\Upsilon_M : \operatorname{Ext}^1_{\mathfrak{g}}(M, M) \to \mathfrak{h}^*$$

as follows. Let $0 \to M \xrightarrow{\phi_1} N \xrightarrow{\phi_2} M \to 0$ be an exact sequence. Let $v := \phi_1(v_\lambda)$ and fix $v' \in N_\lambda$ such that $\phi_2(v') = v_\lambda$. Observe that v, v' is a basis of N_λ and so there exists $\mu \in \mathfrak{h}^*$ such that for any $h \in \mathfrak{h}$ one has $h(v') = \lambda(h)v' + \mu(h)v$ (i.e., the representation $\mathfrak{h} \to \operatorname{End}(N_\lambda)$ is $h \mapsto \begin{pmatrix} \lambda(h) & \mu(h) \\ 0 & \lambda(h) \end{pmatrix}$). The map Υ_M assigns μ to the exact sequence.

Notice that if $0 \to M \to N_1 \to M \to 0$ and $0 \to M \to N_2 \to M \to 0$ are two exact sequences then

$$N_1 \cong N_2 \iff \Upsilon_M(N_1) = c\Upsilon_M(N_2) \text{ for some } c \in \mathbb{C} \setminus \{0\}.$$

If N is an extension of M by M (i.e., $N/M \cong M$) we denote by $\Upsilon_M(N)$ the corresponding one-dimensional subspace of $\mathfrak{h}^*$, i.e. $\Upsilon_M(N) = \mathbb{C}\mu$, where μ is the image of the exact sequence $0 \to M \to N \to M \to 0$.

1.6.2. **Lemma.** *The map Υ_M has the following properties:*

(Υ1) *$\Upsilon_M : \operatorname{Ext}^1(M, M) \to \mathfrak{h}^*$ is injective;*

(Υ2) *$\Upsilon_{M(\lambda)} : \operatorname{Ext}^1(M(\lambda), M(\lambda)) \to \mathfrak{h}^*$ is bijective;*

(Υ3) *if $\Upsilon_{M(\lambda)}(N) = \Upsilon_M(N')$, then N' is isomorphic to a quotient of N;*

(Υ4) *if $\Upsilon_{M(\lambda)}(N) = \Upsilon_{L(\lambda)}(N')$, then $N' \cong N/N''$, where N'' is the maximal submodule of N which intersects trivially the highest weight space N_λ.*

Proof. Property (Υ1) is clear. For (Υ2) let us construct a preimage of a non-zero element $\mu \in \mathfrak{h}^*$. Take a two-dimensional $\mathfrak{h}$-module E spanned by v, v' such that $h(v) = \lambda(h)v$, $h(v') = \lambda(h)v' + \mu(h)v$, view E as a trivial $\mathcal{U}_+$-module (i.e., $\mathcal{U}'_+ E = 0$), and consider the $\mathcal{U}$-module $N := \mathcal{U} \otimes_{\mathcal{S}(\mathfrak{h}) \otimes \mathcal{U}_+} E$. Clearly, $\Upsilon_{M(\lambda)}(N) = \mu$, proving ($\Upsilon$2). Observe that N is universal in the following sense: any module N' satisfying (i) $N'_\lambda \cong E$ as $\mathfrak{h}$-module, (ii) $\mathcal{U}'_+ N'_\lambda = 0$, (iii) N' is generated by N'_λ, is a quotient of N. In particular, if M is a quotient of $M(\lambda)$ and $\Upsilon_{M(\lambda)}(N) = \Upsilon_M(N')$, then N' is a quotient of N. This proves (Υ3).

For (Υ4) suppose that $\Upsilon_{M(\lambda)}(N) = \Upsilon_{L(\lambda)}(N')$. We claim that $N' \cong N/N''$, where N'' is the maximal submodule of N which intersects N_λ trivially. Indeed, by above, N' is a quotient of N: $N' \cong N/X$. Since $\dim N'_\lambda = \dim N_\lambda = 2$, $X_\lambda = 0$, hence $X \subset N''$. Then N''/X is isomorphic to a submodule of N' and $(N''/X)_\lambda = 0$. Hence $N''/X = 0$ as required. $\square$

1.6.3. *The map* $\Upsilon_{L(\lambda)}$. Let us describe the image $\operatorname{Im} \Upsilon_{L(\lambda)}$ in terms of the Jantzen-type filtration. For $\mu, \mu' \in \mathfrak{h}^*$ let $\{M(\lambda)^i\}$ be the non-degenerate Jantzen-type filtration described in Subsection 1.5, which is the image of the filtration $\{M_R(\lambda + t\mu + t^2\mu')^i\}$ in $M(\lambda) = M_R(\lambda + t\mu + t^2\mu')/tM_R(\lambda + t\mu + t^2\mu')$.

Set $\tilde{M} := M_R(\lambda + t\mu + t^2\mu')$, $\tilde{N} := \tilde{M}/t^2\tilde{M}$ and view $\tilde{N}$ as a $\mathfrak{g}$-module. Observe that $\tilde{N}$ has a submodule $t\tilde{N} \cong M(\lambda)$ and that $\Upsilon_{M(\lambda)}$ maps the exact sequence $0 \to t\tilde{N} \to \tilde{N} \to M(\lambda) \to 0$ to μ.

1.6.4. **Corollary.** *If* $\Upsilon_{L(\lambda)} : \operatorname{Ext}^1(L(\lambda), L(\lambda)) \to \mathfrak{h}^*$ *maps* $0 \to L(\lambda) \to N \to L(\lambda) \to 0$ *to* μ, *then* $N \cong M_R(\lambda + t\mu + t^2\mu')/M_R(\lambda + t\mu + t^2\mu')^2$.

Proof. Retain notation of 1.4 and 1.5. Let $\{\tilde{M}^i\}$ be the Jantzen filtration of $\tilde{M} = M_R(\lambda + t\mu + t^2\mu')$ and $\{\tilde{N}^i\}$ be its image in $\tilde{N} = \tilde{M}/t^2\tilde{M}$. In light of ($\Upsilon 4$), it is enough to show that $\tilde{N}^2$ is the maximal submodule of $\tilde{N}$ which intersects $\tilde{N}_\lambda$ trivially. Let $v \in \tilde{M}$ be the canonical generator of the Verma module $\tilde{M}$. Clearly, $\tilde{M}^2 \cap Rv = Rt^2v$, so $\tilde{N}_\lambda^2 = 0$. If $u_- \in \mathcal{U}_{-,R}$ is a weight element such that $u_-v \notin \tilde{M}^2$ then, by (8), there exists a weight element $u_+ \in \mathcal{U}(\mathfrak{n}_+)$ such that $\operatorname{HC}(u_+ u_-)(\lambda + t\mu + t^2\mu') \notin Rt^2$ so $u_+(u_-v) \in Rv \setminus Rt^2v$. As a result, the submodule spanned by the image of u_-v in $\tilde{N}$ intersects $\tilde{N}_\lambda$ non-trivially. The assertion follows. $\square$

1.6.5. **Proposition.** *Let* $\lambda, \mu, \mu' \in \mathfrak{h}^*$, *and let* $\{M(\lambda)^i\}$ *be the image in* $M(\lambda)$ *of the Jantzen filtration* $\{M_R(\lambda + t\mu + t^2\mu')^i\}$. *Then*

$$\mu \in \operatorname{Im} \Upsilon_{L(\lambda)} \iff M(\lambda)^1 = M(\lambda)^2.$$

Proof. Recall that, by (7), $M(\lambda)^1$ is a maximal proper submodule of $M(\lambda)$. Retain notation of the proof of Corollary 1.6.4. In order to deduce the assertion from Corollary 1.6.4 we have to show that $\tilde{M}/\tilde{M}^2$ is an extension of $L(\lambda)$ by $L(\lambda)$ iff $M(\lambda)^1 = M(\lambda)^2$. Recall that $M(\lambda)^i = \phi_t(\tilde{M}^i)$, where $\phi_t : \tilde{M} \to \tilde{M}/t\tilde{M} = M(\lambda)$. Thus $M(\lambda)/M(\lambda)^1$ is isomorphic to $\tilde{M}/(\tilde{M}^1 + t\tilde{M}) = \tilde{M}/\tilde{M}^1$ and so $\tilde{M}/\tilde{M}^1 \cong L(\lambda)$ by (i). One has the exact sequence of $\mathcal{U}$-modules

$$0 \to \tilde{M}^1/\tilde{M}^2 \to \tilde{M}/\tilde{M}^2 \to \tilde{M}/\tilde{M}^1 \cong L(\lambda) \to 0.$$

As a result, $\tilde{M}/\tilde{M}^2$ is an extension of $L(\lambda)$ by $L(\lambda)$ iff $\tilde{M}^1/\tilde{M}^2 \cong L(\lambda)$. By above, $M(\lambda)^1/M(\lambda)^2$ is isomorphic to the quotient of $\tilde{M}^1/\tilde{M}^2$ by $(t\tilde{M} + \tilde{M}^2)/\tilde{M}^2$. One has

$$(t\tilde{M} + \tilde{M}^2)/\tilde{M}^2 \cong t\tilde{M}/(t\tilde{M} \cap \tilde{M}^2) = t\tilde{M}/t\tilde{M}^1 \cong \tilde{M}/\tilde{M}^1 = L(\lambda),$$

because $t\tilde{M} \cap \tilde{M}^2 = t\tilde{M}^1$. Hence $\tilde{M}^1/\tilde{M}^2 \cong L(\lambda)$ iff $M(\lambda)^1/M(\lambda)^2 = 0$; this completes the proof. $\square$

1.7. **Duality.** Assume that $\mathcal{U}$ satisfies (U1)–(U6).

For a $\mathcal{U}$-module N we view N^* as a left $\mathcal{U}$-module via the antiautomorphism σ (see (U6)): $af(n) = f(\sigma(a)n)$, $a \in \mathcal{U}, f \in N^*, n \in N$. If N is a locally finite $\mathfrak{h}$-module with finite-dimensional generalized weight spaces, N^* has a submodule $N^{\#} := \oplus_{\nu \in \Omega(N)} N_\nu^*$. One has $(N^{\#})^{\#} \cong N$. Clearly, $\Omega(N^{\#}) = \Omega(N)$. This implies $L(\nu)^{\#} \cong L(\nu)$ for any $\nu \in \mathfrak{h}^*$.

Retain notation of 1.6. The following criterion will be used in Subsection 8.7.

1.7.1. **Lemma.** *Let* $0 \to M(\lambda) \to N \to M(\lambda) \to 0$ *be a preimage of* $\mu \in \mathfrak{h}^*$ *under* $\Upsilon_{M(\lambda)}$ *in* $\mathrm{Ext}^1(M(\lambda), M(\lambda))$. *Then* $\mu \notin \mathrm{Im}\, \Upsilon_{L(\lambda)}$ *iff* N *has a subquotient which is isomorphic to a submodule of* $M(\lambda)^{\#}$ *and is not isomorphic to* $L(\lambda)$.

Proof. Set $M := M(\lambda)$, $L := L(\lambda)$. Let N' be the maximal submodule of N which intersects N_λ trivially. By ($\Upsilon 3$) of Lemma 1.6.2, $\mu \in \mathrm{Im}\, \Upsilon_L$ iff N/N' is an extension of L by L.

Assume that $\mu \in \mathrm{Im}\, \Upsilon_L$. Suppose that $N_1 \subset N_2$ are submodules of N and N_2/N_1 is isomorphic to a submodule of $M^{\#}$. Since M is generated by its highest weight space, any submodule of $M^{\#}$ intersects its highest weight space $M_\lambda^{\#}$ non-trivially. For any $v \in N_2 \cap N'$ a submodule generated by v intersects the highest weight space N_λ trivially, so $v \in N_1$. Thus $N_2 \cap N' \subset N_1$ and so N_2/N_1 is a quotient of $N_2/(N_2 \cap N')$. In its turn, $N_2/(N_2 \cap N')$ is a submodule of N/N' so N_2/N_1 is a subquotient of N/N'. Since N/N' is an extension of L by L, $N_2/N_1 \cong L$ or $N_2/N_1 \cong N/N'$. Since $\dim M_\lambda^{\#} = 1$, $M^{\#}$ does not have a submodule isomorphic to an extension of L by L. Hence $N_2/N_1 \cong L$.

Now assume that $\mu \notin \mathrm{Im}\, \Upsilon_L$ so, by ($\Upsilon 3$), N/N' is not an extension of L by L. Write

$$0 \to M \to^\iota N \to^\phi M \to 0.$$

Let M' be a maximal proper submodule of M. Note that $N' \subset \phi^{-1}(M')$. Let us show that $E := \phi^{-1}(M')/N'$ is the required subquotient of N. Indeed, since E is a submodule of N/N', any submodule of E intersects E_λ non-trivially. Clearly, E_λ is one-dimensional. As a result, $E^{\#}$ is a quotient of M, so E is a submodule of $M^{\#}$. One has

$$(N/N')/E = (N/N')/(\phi^{-1}(M')/N') \cong N/\phi^{-1}(M') \cong L.$$

By above, N/N' is not an extension of L by L. Hence $E \not\cong L$ as required. $\qquad\square$

1.8. **Weakly admissible modules.** It is well known that representation theory of an affine Lie algebra at the critical level $k = -h^\vee$ is much more complicated than for a non-critical level $k \neq -h^\vee$. For any Kac-Moody algebra one has a similar set of critical weights

$$C = \{\nu \in \mathfrak{h}^*|\ 2(\nu + \rho, \alpha) \in \mathbb{Z}_{>0}(\alpha, \alpha)\ \text{ for some } \alpha \in \Delta_+^{im}\}.$$

In our general setup, introduced in Subsection 1.2, fix a subset $C \subset \mathfrak{h}^*$, called the subset of *critical weights*. We call $\lambda \in \mathfrak{h}^*$ and the corresponding irreducible $\mathcal{U}$-module

$L(\lambda)$ *weakly admissible* if $\lambda \notin C$ and for any $\nu \neq \lambda$, such that $\nu \notin C$, $L(\lambda)$ is not a subquotient of the Verma module $M(\nu)$.

For example, for Kac-Moody algebras C is as above, for the Virasoro and for the Neveu-Schwartz algebras, C is empty. For minimal $\mathcal{W}$-algebras $W^k(\mathfrak{g}, e_{-\theta})$, C is empty if $k \neq -h^\vee$.

The following lemma is similar to the one proven in [DGK].

1.8.1.   **Lemma.**   *Let $\lambda, \lambda' \in \mathfrak{h}^*$ be two distinct elements. If the exact sequence $0 \to L(\lambda') \to N \to L(\lambda) \to 0$ is non-splitting, then either N is a quotient of $M(\lambda)$ or $N^\#$ is a quotient of $M(\lambda')$.*

Proof. Consider the case $\lambda' - \lambda \notin Q_+$. Then λ is a maximal element in $\Omega(N)$ and N_λ is one-dimensional. Therefore N_λ generates a submodule N' which is isomorphic to a quotient of $M(\lambda)$. Since N is indecomposable, $N' = N$ and so N is a quotient of $M(\lambda)$.

Let now $\lambda' - \lambda \in Q_+$. Recall that $L(\nu)^\# = L(\nu)$, hence we have the dual exact sequence $0 \to L(\lambda) \to N^\# \to L(\lambda') \to 0$. By the first case, $N^\#$ is a quotient of $M(\lambda')$. $\square$

1.8.2.   **Corollary.**   *If $\lambda \neq \lambda'$ are weakly admissible weights, then $\mathrm{Ext}^1(L(\lambda), L(\lambda')) = 0$.*

We will use the following criterion of weakly admissibility.

1.8.3.   **Lemma.**   *The weight $\lambda \in \mathfrak{h}^*$ is not weakly admissible iff there exists $\lambda' \in \lambda + Q_+$ such that $\det B_{\lambda' - \lambda}(\lambda') = 0$.*   The proof is similar to the proof of Thm. 2 in [KK].

1.9.   **Admissible modules.** Let $\mathcal{H}$ be a category of $\mathcal{U}$-modules. We say that an irreducible module $L(\lambda)$ in $\mathcal{H}$ is $\mathcal{H}$-*admissible* if it is weakly admissible and $\mathrm{Ext}^1_\mathcal{H}(L(\lambda), L(\lambda)) = 0$.

If $\mathcal{U} = \mathcal{U}(\mathfrak{g})$, where $\mathfrak{g}$ is a Kac-Moody algebra, we call $L(\lambda)$ and the corresponding highest weight λ *admissible* if it is admissible in the category of $\mathfrak{g}$-modules with a diagonal action of $\mathfrak{h}''$, where $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{h}''$, $\mathfrak{h}'' \subset \mathfrak{h}$. For an affine Lie algebra $\mathfrak{g}$ (in this case $\mathfrak{h}'' = \mathbb{C}D$), we call $L(\lambda)$ and the corresponding highest weight λ k-*admissible* ($k \in \mathbb{C}$) if it is admissible in the category of $\mathfrak{g}$-modules with a diagonal action of D, which are annihilated by $K - k$.

We call a $\mathcal{U}$-module M *bounded* if $\mathfrak{h}$ acts locally finitely on M and $\Omega(M)$ is bounded by a finite set of elements of $\mathfrak{h}^*$. Denote by $\tilde{\mathcal{O}}$ the category of all bounded $\mathcal{U}$-modules, and by $\tilde{\mathcal{O}}_{adm}$ its subcategory, consisting of all bounded modules lying in $\mathcal{H}$, all of whose irreducible subquotients are $\mathcal{H}$-admissible.

1.9.1.   **Corollary.**   *Any $\mathcal{U}$-module from the category $\tilde{\mathcal{O}}_{adm}$ is completely reducible.*

Proof. By Lemma 1.3.1, it is enough to show that any module N in $\tilde{\mathcal{O}}_{adm}$ has an irreducible submodule. Let λ be a maximal element in $\Omega(N)$ (this exists since $\Omega(N)$ is bounded). Let $v \in N_\lambda$ be an $\mathfrak{h}$-eigenvector. Then a submodule N', generated by v, is a quotient of $M(\lambda)$. A quotient of $M(\lambda)$, which is not irreducible, has a non-splitting quotient of length two. However, N' does not have such a quotient, since $\mathrm{Ext}^1(L, L') = 0$ for irreducible modules $L \not\cong L'$ in $\tilde{\mathcal{O}}_{adm}$. Hence N' is irreducible as required. $\qquad\square$

2. $\mathrm{Ext}^1(L(\lambda), L(\lambda))$ for the Lie superalgebra $\mathfrak{g}(A, \tau)$

In this section we prove Propositions 2.2 and 2.3.5.

2.1. **The construction of $\mathfrak{g}(A, \tau)$.** Let $A = (a_{ij})$ be an $(n \times n)$-matrix over $\mathbb{C}$ and let τ be a subset of $I := \{1, \ldots, n\}$. Let $\mathfrak{g} = \mathfrak{g}(A, \tau) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ be the associated Lie superalgebra constructed as in [K1],[K2]. Recall that, in order to construct $\mathfrak{g}(A, \tau)$, one considers a realization of A, i.e. a triple $(\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a vector space of dimension $n + corank A$, $\Pi \subset \mathfrak{h}^*$ (resp. $\Pi^\vee \subset \mathfrak{h}$) is a linearly independent set of vectors $\{\alpha_i\}_{i \in I}$ (resp. $\{\alpha_i^\vee\}_{i \in I}$), such that $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ji}$, and constructs a Lie superalgebra $\tilde{\mathfrak{g}}(A, \tau)$ on generators $e_i, f_i, \mathfrak{h}$, subject to relations:

$$[\mathfrak{h}, \mathfrak{h}] = 0, \quad [h, e_i] = \langle \alpha_i, h \rangle e_i, \quad [h, f_i] = -\langle \alpha_i, h \rangle f_i, \quad \text{for } i \in I, h \in \mathfrak{h}, \quad [e_i, f_j] = \delta_{ij} \alpha_i^\vee,$$
$$p(e_i) = p(f_i) = \overline{1} \text{ if } i \in \tau, \quad p(e_i) = p(f_i) = \overline{0} \text{ if } i \notin \tau, \quad p(\mathfrak{h}) = \overline{0}.$$

Then $\mathfrak{g}(A, \tau) = \tilde{\mathfrak{g}}(A, \tau)/J = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$, where J is the maximal ideal of $\tilde{\mathfrak{g}}(A, \tau)$, intersecting $\mathfrak{h}$ trivially, and $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is the subalgebra generated by the images of the e_i's (resp. f_i's). One readily sees that if $a_{ii} = 0$ and $i \in \tau$ (i.e., e_i, f_i are odd), then $e_i^2, f_i^2 \in J$ so $e_i^2 = f_i^2 = 0$ in $\mathfrak{g}(A, \tau)$.

We say that a simple root α_i is *even* (resp., *odd*) if $i \notin \tau$ (resp., $i \in \tau$) and that α_i is *isotropic* if $a_{ii} = 0$.

Let $\mathcal{U}$ (resp. $\mathcal{U}_\pm$) be the universal enveloping algebra of $\mathfrak{g}$ (resp. $\mathfrak{n}_\pm$). Observe that $(\mathcal{U}, \mathfrak{h}, \mathcal{U}_\pm)$ satisfies the assumptions (U1)–(U6) of Subsection 1.2 with σ identical on $\mathfrak{h}$ and $\sigma(e_i) = f_i$. As before, we let $\mathfrak{h}' = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$. By the construction, the simple coroots $\alpha_i^\vee, i \in I$, form a basis of $\mathfrak{h}'$.

Note that, multiplying the ith row of the matrix A by a non-zero number corresponds to multiplying e_i and $\alpha_i^\vee$ by this number, thus giving an isomorphic Lie superalgebra. Hence we may assume from now on that $a_{ii} = 2$ or 0 for all $i \in I$.

The above construction includes all Kac-Moody algebras, all basic finite-dimensional Lie superalgebras and the associated affine superalgebras, and the above assumption holds for all of them (but it does not hold for generalized Kac-Moody Lie algebras).

2.2. **Proposition.** *Let $\mathfrak{g} = \mathfrak{g}(A, \tau)$ and assume that $a_{ii} \neq 0$ for all $i \notin \tau$. Then:*

(i) For any $\lambda \in \mathfrak{h}^*$ the image of $\Upsilon_{L(\lambda)}$ contains the subspace $\{\mu \in \mathfrak{h}^* | \langle \mu, \mathfrak{h}' \rangle = 0\}$.

(ii) Let $\lambda \in \mathfrak{h}^*$ be such that $\langle \lambda, \alpha^\vee \rangle$ is a non-negative integer (resp. an even non-negative integer, resp. zero) for any even (resp. odd non-isotropic, resp. odd isotropic) simple root α. Then

$$\operatorname{Im} \Upsilon_{L(\lambda)} = \{\mu \in \mathfrak{h}^* |\ \langle \mu, \mathfrak{h}' \rangle = 0\}.$$

In other words, under the above condition on λ, $\mathfrak{h}'$ acts semisimply on any extension of $L(\lambda)$ by $L(\lambda)$. Writing $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$, we obtain that the extension splits if $\mathfrak{h}''$ acts semisimply.

Proof. For (i) fix μ such that $\langle \mu, \mathfrak{h}' \rangle = 0$, set $\mu' = 0$ and retain notation of 1.6.3. In the light of Proposition 1.6.5, it is enough to verify that $M(\lambda)^1 = M(\lambda)^2$. Denote the highest weight vector of $M_R(\lambda + t\mu)$ by $v_{\lambda+t\mu}$ and its image, the highest weight vector of $M(\lambda)$, by v_λ. Let $u \in \mathcal{U}(\mathfrak{n}_-)$ be such that $u v_\lambda \in M(\lambda)^1$. Then for any $u_+ \in \mathcal{U}(\mathfrak{n}_+)$ one has $\operatorname{HC}(u_+ u)(\lambda) = 0$. Since $\operatorname{HC}(u_+ u) = \operatorname{HC}([u_+, u]) \in \mathcal{U}([\mathfrak{g}, \mathfrak{g}])$, we conclude that $\operatorname{HC}(u_+ u) \in \mathcal{S}(\mathfrak{h}')$ and so $\operatorname{HC}(u_+ u)(\lambda + t\mu) = \operatorname{HC}(u_+ u)(\lambda) = 0$. Hence $u v_{\lambda+t\mu} \in M_R(\lambda + t\mu)^i$ for all i, and thus $u v_\lambda \in M(\lambda)^2$, as required.

(ii) Assume that N is an extension of $L(\lambda)$ by $L(\lambda)$. Fix a simple root α and let $e, h := \alpha^\vee, f$ be the Chevalley generators of $\mathfrak{g}$ corresponding to α. Recall that $e^2 = f^2 = 0$ if α is odd and isotropic. Otherwise one has

$$(11) \quad \begin{array}{ll} \operatorname{HC}(e^k f^k) = k! h(h-1) \ldots (h - (k-1)), & \text{if } \alpha \text{ is even,} \\ \operatorname{HC}(e^{2k} f^{2k}) = (-1)^k 2^k k! h(h-2) \ldots (h - 2(k-1)), & \text{if } \alpha \text{ is odd non-isotropic,} \\ \operatorname{HC}(e^{2k+1} f^{2k+1}) = (-1)^k 2^k k! h(h-2) \ldots (h - 2k), & \text{if } \alpha \text{ is odd non-isotropic,} \\ \operatorname{HC}(ef) = h, & \text{if } \alpha \text{ is odd isotropic,} \end{array}$$

where HC stands for the Harish-Chandra projection defined in Subsection 1.4.1. Set $k := \langle \lambda, \alpha^\vee \rangle + 1$. From (11) it follows that $L(\lambda)_{\lambda - k\alpha} = 0$ so $N_{\lambda - k\alpha} = 0$. For $v \in N_\lambda$ one has $f^k v \in N_{\lambda - k\alpha} = 0$ so $e^k f^k v = 0$; however, $\mathfrak{n}_+ v = 0$ so $e^k f^k v = \operatorname{HC}(e^k f^k) v$. Therefore the polynomial $P(h) := \operatorname{HC}(e^k f^k) \in \mathbb{C}[h]$ annihilates N_λ. From (11) one sees that $P(h)$ does not have multiple roots. Since $P(h) N_\lambda = 0$, we conclude that $h = \alpha^\vee$ acts diagonally on N_λ. Since $\mathfrak{h}'$ is spanned by the simple coroots, $\mathfrak{h}'$ acts diagonally on N_λ. This means that $\langle \mu, \mathfrak{h}' \rangle = 0$ for any $\mu \in \operatorname{Im} \Upsilon_{L(\lambda)}$. $\qquad \square$

2.2.1. *Remark.* The exact sequence $0 \to L(\lambda) \to N \to L(\lambda) \to 0$ splits over $[\mathfrak{g}, \mathfrak{g}]$ iff $\Upsilon_{L(\lambda)}$ maps this sequence to $\mu \in \mathfrak{h}^*$ such that $\langle \mu, \mathfrak{h}' \rangle = 0$.

2.3. **Symmetrizable case.** Assume that the matrix A is symmetrizable, i.e. for some invertible diagonal $n \times n$-matrix D the matrix DA is symmetric. In this case the Lie superalgebra $\mathfrak{g} = \mathfrak{g}(A, \tau)$ admits a non-degenerate invariant bilinear form $(-, -)$ [K2], Chapter 2. We denote by $(-, -)$ also the induced non-degenerate bilinear form on $\mathfrak{h}^*$. For $X \subset \mathfrak{h}^*$ we set $X^\perp := \{\mu \in \mathfrak{h}^* | (\mu, X) = 0\}$.

We denote by Δ_+ (resp. Δ_-) the set of weights of ad $\mathfrak{h}$ on $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$). Introduce $\rho \in \mathfrak{h}^*$ by the conditions $\langle \rho, \alpha_i^\vee \rangle = a_{ii}/2$ for each simple coroot $\alpha_i^\vee$. Notice that for any

simple non-isotropic root α and any $\mu \in \mathfrak{h}^*$ one has $\langle \mu, \alpha^\vee \rangle = 2(\alpha, \mu)/(\alpha, \alpha)$. For any non-isotropic $\alpha \in \Delta_+$ introduce $\alpha^\vee \in \mathfrak{h}$ by the condition $\forall \mu \in \mathfrak{h}^*$ $\langle \mu, \alpha^\vee \rangle = \frac{2(\mu, \alpha)}{(\alpha, \alpha)}$.

Recall that a $\mathfrak{g}$-module (or $[\mathfrak{g}, \mathfrak{g}]$-module) N is called *restricted* if for every $v \in N$, we have $\mathfrak{g}_\alpha v = 0$ for all but a finite number of positive roots α.

Recall the standard construction of the Casimir operator ([K2], Chapter 2). Choose the dual bases $\{h_i\}, \{h^i\}$ in $\mathfrak{h}$ (i.e., $(h_i, h^j) = \delta_{ij}$) and for each positive root $\alpha \in \Delta_+$ choose the dual bases $\{a_i\}$ in $\mathfrak{g}_\alpha$ and $\{b_i\} \in \mathfrak{g}_{-\alpha}$ (i.e., $(a_i, b_j) = \delta_{ij}$). Let $\rho^* \in \mathfrak{h}$ be such that $\langle \mu, \rho^* \rangle = (\mu, \rho)$ for each $\mu \in \mathfrak{h}^*$. Define the Casimir operator by the formula

$$Cas := 2\rho^* + \sum_i h_i h^i + 2 \sum_{\alpha \in \Delta_+} \sum_i b_i a_i$$

Note that the Casimir operator acts on any restricted $\mathfrak{g}$-module by a $\mathfrak{g}$-endomorphism.

2.3.1. Recall the formula for Shapovalov determinant for an arbitrary symmetrizable $\mathfrak{g}(A, \tau)$ [GK] (in the non-super case it was proven in [KK]):

$$(12) \qquad det_\nu(\lambda) = \prod_{\alpha \in \Delta_+} \prod_{n=1}^\infty \left(2(\lambda + \rho, \alpha) - n(\alpha, \alpha)\right)^{(-1)^{p(\alpha)(n+1)} P(\nu - n\alpha)},$$

where $p(\alpha)$ is the parity of α and P is the partition function for $\mathfrak{g}$, i.e. $P(\mu)$ is the number of partitions of μ in a sum of positive roots, where each odd root occurs at most once, counting their multiplicities. Observe that, if $\alpha, 2\alpha \in \Delta_+$ and $p(\alpha) = \overline{1}$, then the factor corresponding to $(\alpha, 2n)$ cancels with the factor corresponding to $(2\alpha, n)$.

2.3.2. Introduce $S \subset \mathfrak{h}^*$ by the condition: $\lambda \in S$ iff

(S1) if $\alpha \in \Delta_+$ is even non-isotropic and $\alpha/2$ is not an odd root, then $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{<0}$;

(S2) if $\alpha \in \Delta_+$ is odd non-isotropic then $\langle \lambda + \rho, \alpha^\vee \rangle$ is not a negative odd integer;

(S3) $(\lambda + \rho, \beta) \neq 0$ for every isotropic root β.

2.3.3. **Lemma.** *(i) If $\lambda, \lambda' \in S$ are such that $[M(\lambda) : L(\lambda')] \neq 0$ then $\lambda = \lambda'$.*

(ii) $\mathrm{Ext}^1(L(\lambda'), L(\lambda)) = 0$ for $\lambda, \lambda' \in S$, $\lambda \neq \lambda'$.

Proof. For $\lambda \in S$ one has $(\lambda + \rho, \beta) \neq 0$ for any isotropic root β. Using (12) and the standard argument [Jan],[KK], we conclude that if $[M(\lambda) : L(\nu)] \neq 0$ and $\nu \neq \lambda$, then either $\nu = \nu' - \alpha$, where α is isotropic and $(\nu' + \rho, \alpha) = 0$, or $\nu = \nu' - k\alpha$ for some non-isotropic root $\alpha \in \Delta^+$ such that $k := \langle \nu' + \rho, \alpha^\vee \rangle$ is a positive (resp. odd positive) integer if α is even such that $\alpha/2$ is not an odd root (resp. α is odd). In the first case $(\nu + \rho, \alpha) = 0$ and in the second case $\langle \nu + \rho, \alpha^\vee \rangle = -k$. This proves (i). Combining Lemma 1.8.1 and (i) we get (ii). $\qquad \square$

2.3.4. **Definition.**   For $\lambda \in \mathfrak{h}^*$ let $C'(\lambda) \subset \Delta_+ \times \mathbb{Z}_{>0}$ consist of the pairs (α, m), such that $2(\lambda + \rho, \alpha) = m(\alpha, \alpha)$ and

(i) if α is even, then $\alpha/2$ is not an odd root;

(ii) if α is odd and non-isotropic, then m is odd.

By the argument of [KK] Thm. 2, if $(\alpha, m) \in C'(\lambda)$ and α is not isotropic (resp. α is isotropic), then $M(\lambda)$ has a singular vector of weight $\lambda - m\alpha$ (resp. $\lambda - \alpha$), and, moreover, if $[M(\lambda) : L(\lambda - \nu)] \neq 0$, then either $\nu = 0$ or $[M(\lambda - m\alpha) : L(\lambda - \nu)] \neq 0$ for some $(\alpha, m) \in C'(\lambda)$. In particular, $M(\lambda)$ is irreducible iff $C'(\lambda)$ is empty.

Let $C(\lambda)$ be the projection of $C'(\lambda)$ to Δ_+, i.e. $C(\lambda) := \{\alpha \in \Delta_+ | \exists m : (\alpha, m) \in C'(\lambda)\}$. We call $\alpha \in C(\lambda)$ λ-*minimal* if for some pair $(\alpha, m) \in C'(\lambda)$ one has

(13) $\forall (\beta, n) \in C'(\lambda) \setminus \{(\alpha, m)\}$ $[M(\lambda - n\beta) : L(\lambda - m\alpha)] = 0.$

Notice that, if an isotropic root α is λ-minimal, then formula (13) holds for the pair $(\alpha, 1)$. Observe that for a non-isotropic root α there exists at most one value of m such that $(\alpha, m) \in C'(\lambda)$.

Let α be such that $m\alpha$ is a minimal element of the set $\{n\beta | (\beta, n) \in C'(\lambda)\}$ with respect to the order introduced in 1.2.2. If α is such that $m\alpha = n\beta$ for $(\beta, n) \in C'(\lambda)$ forces $\alpha = \beta$, then α is λ-minimal. Notice that for isotropic α, if $m\alpha$ is a minimal element of the set $\{n\beta | (\beta, n) \in C'(\lambda)\}$, then $m = 1$ and $\alpha \neq n\beta$ for $(\beta, n) \in C'(\lambda) \setminus \{(\alpha, 1)\}$ so α is λ-minimal. If $\alpha \in C(\lambda)$ is a simple root, then α is λ-minimal, because $m\alpha = n\beta$ forces $\alpha = \beta$.

2.3.5. **Proposition.**   Let $\mathfrak{g} = \mathfrak{g}(A, \tau)$, where A is a symmetrizable matrix, such that $a_{ii} \neq 0$ for $i \notin \tau$. Then for any $\lambda \in \mathfrak{h}^*$ one has

$$C(\lambda)^\perp \subset \operatorname{Im} \Upsilon_{L(\lambda)} \subset \{\mu \in \mathfrak{h}^* | \ (\mu, \alpha) = 0 \ \text{ for all } \lambda\text{-minimal } \alpha\}.$$

Proof. Let μ' be such that $(\mu', \beta) \neq 0$ for all $\beta \in \Delta$. Using the formula (12) and the observation after the formula we obtain

$$det_\nu(\lambda + t\mu + t^2\mu') = a \prod_{(\alpha, m) \in C'(\lambda)} (t(\mu, \alpha) + t^2(\mu', \alpha))^{(-1)^{p(\alpha)}(n+1)P(\nu - n\alpha)},$$

where a is an invertible element of the local ring R. By (9) the following sum formula holds

$$\sum_{i=1}^{\infty} \operatorname{ch} M(\lambda)^i = \sum_{(\alpha, m) \in C'(\lambda)} (-1)^{(m-1)p(\alpha)} k(\alpha) \operatorname{ch} M(\lambda - m\alpha),$$

where $k(\alpha) = 1$ if $(\mu, \alpha) \neq 0$ and $k(\alpha) = 2$ if $(\mu, \alpha) = 0$. Note that $(-1)^{(m-1)p(\alpha)} = -1$ forces α to be odd isotropic.

To verify the second inclusion, assume that α is λ-minimal and $(\mu, \alpha) \neq 0$. The above formula implies

$$\sum_{i=1}^{\infty} [M(\lambda)^i : L(\lambda - m_\alpha \alpha)] = 1,$$

where $m_\alpha := 2\langle \lambda + \rho, \alpha^\vee \rangle$ if α is not isotropic and $m_\alpha := 1$ otherwise. Therefore $[M(\lambda)^1 : L(\lambda - m_\alpha \alpha)] = 1$, $[M(\lambda)^2 : L(\lambda - m_\alpha \alpha)] = 0$. Hence, by Proposition 1.6.5, $\mu \notin \operatorname{Im} \Upsilon_{L(\lambda)}$ as required.

Now take μ such that $(\mu, \alpha) = 0$ for all $\alpha \in C(\lambda)$. Retain notation of 1.6.3. Set

$$M := M(\lambda), \quad \tilde{M}(s) := M_R(\lambda + st\mu + t^2 \mu') \text{ for } s \in \mathbb{R},$$

and identify $\tilde{M}(s)/t\tilde{M}(s)$ with M. Let $\{\tilde{M}(s)^i\}$ be the Jantzen filtration for $\tilde{M}(s)$ and let $\{\mathcal{F}_s^i(M)\}$ be the image of this filtration in M. Then $M(\lambda)^i = \mathcal{F}_1^i(M)$. In the light of Proposition 1.6.5

$$\mu \in \operatorname{Im} \Upsilon_{L(\lambda)} \iff \mathcal{F}_1^1(M) = \mathcal{F}_1^2(M).$$

Clearly, $\mathcal{F}_0^1(M) = \mathcal{F}_0^2(M)$; below we will deduce from this that $\mathcal{F}_1^1(M) = \mathcal{F}_1^2(M)$.

Fix $\nu \in Q_+$ and set $d^j(s) := \dim \mathcal{F}_s^j(M)_{\lambda - \nu}$. Let us show that $d^j(s)$ are constant functions of s. Indeed, since $(\alpha, s\mu) = 0$ for all $\alpha \in C(\lambda)$, the above sum formula gives

$$\sum_{j=1}^{\infty} \operatorname{ch} \mathcal{F}_s^j(M) = \sum_{(\alpha, m) \in C'(\lambda)} (-1)^{(m-1)p(\alpha)} 2 \operatorname{ch} M(\lambda - m\alpha).$$

Since the right-hand side does not depend on s, the sum $\sum_{j=1}^{\infty} d^j(s)$ does not depend on s. Denote by $S_\nu(s)$ the Shapovalov matrix of $\mathcal{U}(\mathfrak{n}_-)_{-\nu}$ evaluated at $\lambda + st\mu + t^2\mu'$ (the entries of $S_\nu(s)$ lie in $R[s]$). Then for each value $s_0 \in \mathbb{R}$ one has: $d^j(s_0) = \dim \mathcal{F}_{s_0}^j(M)_{\lambda - \nu}$ is equal the corank of $S_\nu(s_0)$ modulo t^j. Set $m_j := \max_s d^j(s)$. Since the corank of matrix, depending on one real parameter, takes its maximal value for almost all values of the parameter, one has $d^j(s) = m_j$ for each $s \in \mathbb{R} \setminus J_j$, where $J_j \subset \mathbb{R}$ is a finite set. Thus for some $s \in \mathbb{R}$ one has $d^j(s) = m_j$ for all j. Since $\sum_{j=1}^{\infty} d^j(s) < \infty$ is a constant, $\sum_{j=1}^{\infty} d^j(s) = \sum_{j=1}^{\infty} m_j$ so $d^j(s) = m_j$ for any s. Hence $d^j(s)$ does not depend on s.

Now $\mathcal{F}_0^1(M) = \mathcal{F}_0^2(M)$ gives $d^1(s) = d^2(s)$, hence $\dim \mathcal{F}_s^1(M)_{\lambda - \nu} = \dim \mathcal{F}_s^2(M)_{\lambda - \nu}$ for all s. Therefore $\mathcal{F}_1^1(M) = \mathcal{F}_1^2(M)$ and so $\mu \in \operatorname{Im} \Upsilon_{L(\lambda)}$; this establishes the first inclusion. $\square$

2.3.6. *Remark.* Recall that $\mathfrak{h}' = \mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}]$ is spanned by $\Pi^\vee$. From Remark 2.2.1 we see that the exact sequence $0 \to L(\lambda) \to N \to L(\lambda) \to 0$ splits over $[\mathfrak{g}, \mathfrak{g}]$ iff $\Upsilon_{L(\Lambda)}$ maps this sequence to $\mu \in \Delta^\perp$. By Proposition 2.3.5, $\Delta^\perp \subset \operatorname{Im} \Upsilon_{L(\lambda)}$; by above, the equality means that any self-extension of $L(\lambda)$ splits over $[\mathfrak{g}, \mathfrak{g}]$.

2.3.7. **Example.**   Let $\mathfrak{g} = \mathfrak{sl}_3$. Take λ such that $m := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}, \langle \lambda + \rho, \beta^\vee \rangle = 0$. In this case, $C'(\lambda) = \{(\alpha, m), (\alpha + \beta, m)\}$ and α is the only λ-minimal root. Proposition 2.3.5 gives $0 \subset \Upsilon_{L(\Lambda)} \subset \{\alpha\}^\perp$. One can deduce from Proposition 1.6.5 that, in, fact one has $\Upsilon_{L(\Lambda)} = \{\alpha\}^\perp$, see Example 3.5.3.

2.4. **Generalized Verma modules.** Let $\mathfrak{g} = \mathfrak{g}(A, \tau)$, where A is a symmetrizable matrix, such that $a_{ii} \neq 0$ for $i \notin \tau$. This assumption means that every triple of Chevalley generators $e_i, f_i, \alpha_i^\vee$ with $i \notin \tau$ (resp. $i \in \tau, a_{ii} \neq 0$; resp. $i \in \tau, a_{ii} = 0$) spans $\mathfrak{sl}_2$ (resp. $\mathfrak{osp}(1,2)$; resp. $\mathfrak{sl}(1,1)$). Retain notation of 2.1. For $J \subset \Pi$ let Q_J be the $\mathbb{Z}$-span of J. Set

$$\mathfrak{n}_{\pm, J} := \oplus_{\alpha \in Q_J} \mathfrak{g}_{\pm \alpha}, \quad \mathfrak{h}_J := \sum_{\alpha \in J} \mathbb{C}\alpha^\vee.$$

Note that $\mathfrak{h}_J = [\mathfrak{n}_+, \mathfrak{n}_{-,J}] \cap \mathfrak{h} = [\mathfrak{n}_{+,J}, \mathfrak{n}_{-,J}] \cap \mathfrak{h}$.

Fix $\lambda \in \mathfrak{h}^*$ such that $\langle \lambda, \mathfrak{h}_J \rangle = 0$. Let $\mathbb{C}_\lambda$ be a one-dimensional $\mathfrak{h}$-module corresponding to λ, endowed by the trivial action of $\mathfrak{n}_+ + \mathfrak{h}_J + \mathfrak{n}_{-,J}$. The *generalized Verma module* $M_J(\lambda)$ is

$$M_J(\lambda) := \mathrm{Ind}_{\mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_{-,J}}^{\mathfrak{g}} \mathbb{C}_\lambda.$$

Retain notation of 2.3.4 and observe that $J \subset C(\lambda)$ and that the elements of J are λ-minimal. Therefore, $\Delta^\perp \subset \mathrm{Im}\,\Upsilon_{L(\lambda)} \subset J^\perp$, by Proposition 2.3.5. Fix $\mu, \mu' \in J^\perp$. The Jantzen-type filtrations on the generalized Verma module $M_J(\lambda)$ can be defined as in 1.5. Namely, we define the generalized Verma module $N := M_{J,R}(\lambda + t\mu + t^2\mu')$, denote by v_0 its canonical highest-weight generator, and set $N^i := \{v \in N| \mathcal{U}(\mathfrak{n}_+)v \cap Rv_0 \subset Rt^i v_0\}$. Then $\{N^i\}$ is a decreasing filtration of N and its image in $M_J(\lambda) = N/tN$ is the Jantzen-type filtration $\{M_J(\lambda)^i\}$. Repeating the reasonings of Subsection 1.6 we obtain that M_J^1 is the maximal proper submodule of $M_J(\lambda)$, that is $L(\lambda) = M_J/M_J^1$, and that

$$\mu \in \mathrm{Im}\,\Upsilon_{L(\lambda)} \iff M_J(\lambda)^1 = M_J(\lambda)^2,$$

where the Jantzen-type filtration $\{M_J(\lambda)^i\}$ is induced from $M_{J,R}(\lambda + t\mu + t^2\mu')$ (for any $\mu' \in J^\perp$). Furthermore, Proposition 2.3.5 gives

(14) $$\Delta^\perp \subset \mathrm{Im}\,\Upsilon_{L(\lambda)} \subset J^\perp.$$

Similar facts hold for other generalized Verma modules (that is the modules induced from a finite-dimensional representation of $\mathfrak{n}_+ + \mathfrak{h}_J + \mathfrak{n}_{-,J}$).

3. Complete reducibility for a symmetrizable Kac-Moody algebra

In this section we prove Theorems 0.1, 0.3, 0.4 (see 3.4, 3.2, 3.2.1 respectively).

In this section $\mathfrak{g} = \mathfrak{g}(A)$ is a symmetrizable Kac-Moody algebra [K2]. In this case $\tau = \emptyset$ and $a_{ii} = 2, a_{ij} \in \mathbb{Z}_{\leq 0}$ for each $i, j \in I$. We can (and will) normalize $(-, -)$ in such a way that $(\alpha_i, \alpha_i) \in \mathbb{Z}_{>0}$ for all i, so that $(\rho, \alpha_i) > 0$.

The results of this section extend to symmetrizable Kac-Moody superalgebras $\mathfrak{g}(A, \tau)$, such that A has the same properties as in the non-super case, and, in addition, $a_{ij} \in 2\mathbb{Z}$ if $i \in \tau$. This includes $\mathfrak{osp}(1, n)$ and $\mathfrak{osp}(1, n)\hat{}$.

3.1. We retain notation of 1.6.1. Recall that a $[\mathfrak{g}, \mathfrak{g}]$-module N is called *integrable* if, for each $i \in I$, the Chevalley generators e_i, f_i act locally nilpotently on N. This condition implies that N is $\mathfrak{h}'$-diagonalizable.

Recall ([K2], Chapter 10) that $L(\lambda)$ is integrable iff $\lambda \in P_+$, where

$$P_+ := \{\lambda \in \mathfrak{h}^* |\ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}\ \text{ for each simple root } \alpha\}.$$

3.2. **Proposition.** *Let N be a restricted $\mathfrak{g}$-module such that $\mathfrak{h}$ and the Casimir operator act locally finitely and such that any irreducible subquotient of N is of the form $L(\lambda)$ with $\lambda \in P_+$. Then N is completely reducible over $[\mathfrak{g}, \mathfrak{g}]$.*

Proof. In view of Lemma 1.3.1, it is enough to show that

(i) $\operatorname{Ext}^1_{[\mathfrak{g},\mathfrak{g}]}(L(\lambda), L(\lambda')) = 0$ for $\lambda, \lambda' \in P_+$,

(ii) N contains an irreducible submodule.

Recall that the Casimir operator acts on $L(\mu)$ by the scalar $c_\mu := (\mu + \rho, \mu + \rho) - (\rho, \rho)$ ([K2], Chapter 2). For any $\lambda, \lambda' \in P_+$ such that $\lambda' > \lambda$ one has $c_{\lambda'} - c_\lambda = (\lambda' - \lambda, \lambda' + \lambda + 2\rho) > 0$. Thus $\operatorname{Ext}^1(L(\lambda), L(\lambda')) = 0$ for $\lambda, \lambda' \in P_+, \lambda \neq \lambda'$. Moreover, $\operatorname{Ext}^1_{[\mathfrak{g},\mathfrak{g}]}(L(\lambda), L(\lambda)) = 0$ for $\lambda \in P_+$, by Proposition 2.2 (ii). Hence (i) holds.

For (ii) we may (and will) assume that N is indecomposable. Since the Casimir operator of $\mathfrak{g}$ acts on N locally finitely, N admits a decomposition into generalized eigenspaces with respect to this action. Since N is indecomposable, the Casimir operator has a unique eigenvalue on N. In particular, the Casimir operator acts on all irreducible subquotients of N by the same scalar. Let $\operatorname{supp}(N)$ be the set of irreducible subquotients of N. By the assumption, $L(\lambda') \in \operatorname{supp}(N)$ forces $\lambda' \in P_+$. Take λ such that $L(\lambda) \in \operatorname{supp}(N)$. By above, $L(\lambda') \notin \operatorname{supp}(N)$ if $\lambda' > \lambda$ or $\lambda' < \lambda$. Thus by (1), $\Omega(N) \cap (\lambda + Q_+) = \{\lambda\}$ and so $\mathfrak{n}_+ N_\lambda = 0$. Let $v \in N_\lambda$ be an eigenvector of $\mathfrak{h}$. The submodule generated by v is a quotient of $M(\lambda)$ and so, by above, it is isomorphic to $L(\lambda)$. Hence N contains an irreducible submodule. The assertion follows. $\square$

3.2.1. **Theorem.** *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Any restricted integrable $[\mathfrak{g}, \mathfrak{g}]$-module is completely reducible and its irreducible submodules are of the form $L(\lambda)$ with $\lambda \in P_+$.*

Proof. By above, $\operatorname{Ext}^1_{[\mathfrak{g},\mathfrak{g}]}(L(\lambda), L(\lambda')) = 0$ for $\lambda, \lambda' \in P_+$. In the light of Lemma 1.3.1, it is enough to show that each restricted integrable module contains an irreducible submodule of the form $L(\lambda)$ with $\lambda \in P_+$.

Let $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots. Write $\beta \in \Delta_+$ as $\beta = \sum_{i=1}^{l} m_i \alpha_i$ and set $\operatorname{ht} \beta = \sum m_i$. For $m \in \mathbb{Z}_{\geq 0}$ set $\mathfrak{n}_{>m} := \sum_{\beta: \operatorname{ht} \beta > m} \mathfrak{n}_\beta$ and note that $\mathfrak{n}_{>m}$ is an ideal of $\mathfrak{n}$. Take $v \in N$. Since N is restricted, $\mathfrak{n}_{>m} v = 0$ for some $m \in \mathbb{Z}_{\geq 0}$. Set $N' := \mathcal{U}(\mathfrak{n})v$. By above, $\mathfrak{n}_{>m} N' = 0$ so N' is a module over $\mathfrak{m} := \mathfrak{n}/\mathfrak{n}_{>m}$ which is a finite-dimensional nilpotent Lie algebra.

Recall the following proposition [K2], 3.8. If $\mathfrak{p}$ is a Lie algebra and M is an $\mathfrak{p}$-module, satisfying the following condition: $\mathfrak{p}$ is generated by ad-locally finite elements which are also locally finite on M, then $\mathfrak{p}$ is a span of elements which act locally finitely on M. Since N is integrable, $\{e_{\alpha_i}\}_{i=1}^{l}$ act locally finitely on N' and thus there exists a basis $\{u_j\}_{j=1}^{s}$ of $\mathfrak{m}$, where each u_j acts locally finitely on N'. Since N' is a cyclic $\mathfrak{m}$-module (generated by v), this implies that it is finite-dimensional. Then, by the Lie Theorem, $\mathfrak{m}$ has an eigenvector $v' \in N'$. Since $\mathfrak{m}$ is generated by $e_i, i \in I$ and these elements act locally nilpotently on N', one has $\mathfrak{m}v' = 0$. Thus $\mathfrak{n}v' = 0$ that is $N^{\mathfrak{n}} \neq 0$. Recall that $\mathfrak{h}'$ acts diagonally on N so $N^{\mathfrak{n}}$ contains an eigenvector v'' for $\mathfrak{h}'$. The vector v'' generates a submodule which is a quotient of a Verma module. Since this submodule is integrable, it is isomorphic to $L(\lambda)$ with $\lambda \in P_+$. $\qquad\square$

3.3. **The set $\Delta(\lambda)$.** Retain notation of 2.3. Let Π be the set of simple roots and let W be the Weyl group of $\mathfrak{g}$. Recall that a root α is *real* if $W\alpha \cap \Pi \neq \emptyset$; a root is *imaginary* if it is not real. One has: α is real iff $(\alpha, \alpha) > 0$.

3.3.1. **Definition.**   For $\lambda \in \mathfrak{h}^*$ let $\Delta(\lambda)$ to be the set of real roots α such that $m_\alpha := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}$, and set $\Delta(\lambda)^\vee := \{\alpha^\vee | \ \alpha \in \Delta(\lambda)\}$.

Notice that $\Delta(\lambda)^\vee \subset \mathfrak{h}'$. Recall that λ is called rational iff $\Delta(\lambda)^\vee$ spans $\mathfrak{h}'$, which is equivalent to $\mathbb{C}\Delta(\lambda) = \mathbb{C}\Delta$.

Set $\Delta_+(\lambda) := \{\alpha \in \Delta(\lambda) | \ \alpha \in \Delta_+\}$.

3.3.2. Let $W(\lambda)$ be the subgroup of the Weyl group W of $\mathfrak{g}$ generated by the reflections s_α with $\alpha^\vee \in \Delta(\lambda)$. Introduce the dot action by the usual formula $w.\lambda := w(\lambda + \rho) - \rho$ for any $w \in W$. Notice that $s_\alpha(\Delta(\lambda)) = \Delta(\lambda)$ for any $\alpha \in \Delta(\lambda)$. As a result, $\Delta(\lambda)$ is $W(\lambda)$-invariant and $\Delta(w.\lambda) = \Delta(\lambda)$ for any $w \in W(\lambda)$. Thus $\mathfrak{h}^*$ is a disjoint union of $W(\lambda)$-orbits (with respect to the dot action) and $\Delta(\lambda)$ is the same for every λ in a given orbit.

Recall that a weight $\lambda \in \mathfrak{h}^*$ is regular if $(\lambda, \alpha) \neq 0$ for all $\alpha \in \Delta^{re}$. We call $\lambda \in \mathfrak{h}^*$ *shifted-regular* if $\lambda + \rho$ is regular. Observe that if λ is shifted-regular, then $w.\lambda$ is shifted-regular for any $w \in W$.

3.3.3. *Remark.*   For $\alpha \in \Delta_+(\lambda)$ one has $s_\alpha.\lambda > \lambda$ iff $m_\alpha = \langle \lambda + \rho, \alpha^\vee \rangle < 0$. In particular, a shifted-regular weight λ is maximal in its $W(\lambda)$.-orbit iff $m_\alpha > 0$ for all $\alpha \in \Delta_+(\lambda)$. If λ is maximal in its $W(\lambda)$.-orbit and is shifted-regular, then $\operatorname{Stab}_{W(\lambda).}\lambda = \{\operatorname{id}\}$.

3.3.4. Recall that a non-empty subset Δ' of a root system is called a *root subsystem* if $s_\alpha\beta \in \Delta'$ for any $\alpha, \beta \in \Delta'$. One readily sees that $\Delta(\lambda)$ is a root subsystem of Δ. Let $\Pi(\lambda)$ be the set of indecomposable elements of $\Delta_+(\lambda)$: $\alpha \in \Pi(\lambda)$ iff $\alpha \notin \sum_{\beta\in\Delta(\lambda)_+\setminus\{\alpha\}} \mathbb{Z}_{\geq 0}\beta$. By [MP], 5.7 the following properties hold:

(Π1) $W(\lambda)$ is generated by s_α with $\alpha \in \Pi(\lambda)$ and $\Delta(\lambda) = W(\lambda)\big(\Pi(\lambda)\big)$;

(Π2) $\Delta_+(\lambda) \subset \sum_{\alpha\in\Pi(\lambda)} \mathbb{Z}_{\geq 0}\alpha$;

(Π3) for any $\alpha \in \Pi(\lambda)$ the set $\Delta(\lambda)_+ \setminus \{\alpha\}$ is invariant under the reflection s_α.

Note that the elements of $\Pi(\lambda)$ can be linearly dependent.

3.3.5. We recall that $\lambda \in \mathfrak{h}^*$ is *non-critical* if for any positive imaginary root α one has $2(\lambda + \rho, \alpha) \notin \mathbb{Z}_{>0}(\alpha, \alpha)$. Remark that, for a non-critical weight λ, the orbit $W.\lambda$ consists of non-critical weights, since the set of positive imaginary roots is W-invariant ([K2], Chapter 5).

By [KK], Thm. 2 if λ is non-critical then all irreducible subquotients of $M(\lambda)$ are of the form $L(w.\lambda)$ for $w \in W(\lambda)$. In particular, a non-critical weight λ is weakly admissible (see Subsection 1.9) iff λ is maximal in its $W(\lambda).$-orbit, or, equivalently, iff $\langle\lambda + \rho, \beta^\vee\rangle \in \mathbb{Z}_{\geq 0}$ for all $\beta \in \Pi(\lambda)$.

3.4. **Theorem.**   *Let $\mathfrak{g}$ be a symmetrizable Kac-Moody algebra. Let $\lambda \in \mathfrak{h}^*$ be a non-critical shifted-regular weight, maximal in its $W(\lambda).$-orbit. Then $\Upsilon_{L(\lambda)}$ induces a bijection*

$$\mathrm{Ext}^1(L(\lambda), L(\lambda)) \xrightarrow{\sim} \Delta(\lambda)^\perp.$$

3.4.1. **Corollary.**   *(i) If λ is admissible, then λ is rational.*

(ii) A non-critical shifted-regular weight is admissible iff it is rational and maximal in its $W(\lambda).$-orbit.

3.4.2. *Proof of Theorem 3.4.* Retain notation of 2.3.4. Since λ is non-critical and maximal in its $W(\lambda).$-orbit, Remark 3.3.3 implies that $C(\lambda) = \Delta_+(\lambda)$, so, by (Π2), $C(\lambda)$ and $\Pi(\lambda)$ span the same subspace of $\mathfrak{h}^*$. In the light of Proposition 2.3.5, it is enough to verify that each $\alpha \in \Pi(\lambda)$ is λ-minimal in the sense of Definition 2.3.4. This follows from the following lemma.

3.4.3. **Lemma.**   *Let $\lambda \in \mathfrak{h}^*$ be non-critical. Each $\alpha \in \Pi(\lambda)$ such that $\langle\lambda + \rho, \alpha^\vee\rangle > 0$ is λ-minimal in the sense of Definition 2.3.4.*

Proof. For $\beta \in \Delta_+(\lambda)$ set $m_\beta := \langle\lambda+\rho, \beta^\vee\rangle$. By Remark 3.3.3, $C(\lambda) = \{\beta \in \Delta_+(\lambda)|\ m_\beta > 0\}$. It is enough to verify that for any $\beta \in \Delta_+(\lambda) \setminus \{\alpha\}$ such that $m_\beta > 0$ one has $[M(s_\beta.\lambda) : L(s_\alpha.\lambda)] = 0$. Take $\beta \in \Delta_+(\lambda) \setminus \{\alpha\}$ such that $m_\beta > 0$ and assume that

$[M(s_\beta.\lambda) : L(s_\alpha.\lambda)] > 0$. By [KK], Thm. 2 we have a chain $\nu_1 = s_\beta.\lambda$, $\nu_2, \ldots, \nu_n = s_\alpha.\lambda$ such that $\nu_{i+1} = s_{\beta_i}.\nu_i = \nu_i - k_i\beta_i$, where $\beta_i \in \Delta_+(\nu_i)$ and $k_i > 0$. Since ν_i, λ lie in the same $W(\lambda)$-orbit, one has $\Delta_+(\nu_i) = \Delta_+(\lambda)$. Therefore

$$m_\alpha\alpha - m_\beta\beta = s_\beta.\lambda - s_\alpha.\lambda = \sum_i k_i\beta_i = \sum_{\gamma \in \Pi(\lambda)} k_\gamma\gamma \quad \text{for some } k_\gamma \geq 0.$$

Writing $\beta = \sum_{\gamma \in \Pi(\lambda)} n_\gamma\gamma$, we obtain $(m_\alpha - k_\alpha - m_\beta n_\alpha)\alpha = \sum_{\gamma \in \Pi(\lambda) \setminus \{\alpha\}} (k_\gamma + m_\beta n_\gamma)\gamma$. Since $\alpha \in \Pi(\lambda)$, this implies $k_\gamma = n_\gamma = 0$ for $\gamma \neq \alpha$, so β is proportional to α, which is impossible, since α is a real root. $\qquad\square$

3.4.4. *Proof of Corollary 3.4.1.* Since $C(\lambda) \subset \Delta(\lambda)$ for non-critical λ, Proposition 2.3.5 implies that $\Delta(\lambda)^\perp \subset \operatorname{Im} \Upsilon_{L(\lambda)}$. If λ is admissible, then $\operatorname{Im} \Upsilon_{L(\lambda)} = \Delta^\perp$ so $\Delta(\lambda)^\perp \subset \Delta^\perp$ that is $\mathbb{C}\Delta(\lambda) = \mathbb{C}\Delta$; this gives (i). Now (ii) follows from (i) and Theorem 3.4. $\qquad\square$

3.4.5. *Example.* For the affine Lie algebra $\hat{\mathfrak{sl}}_2$ the weight λ is k-admissible for $k := \langle \lambda, K \rangle$ iff $k \neq -2$, λ is maximal in its $W(\lambda)$.-orbit and $W(\lambda).\lambda \neq \lambda$. Indeed, $W(\lambda).\lambda \neq \lambda$ means that $M(\lambda)$ is not irreducible. If λ is k-admissible, then $k \neq -2$, λ is maximal in its $W(\lambda)$.-orbit and $M(\lambda)$ is not irreducible. Assume that $k \neq -2$, λ is maximal in its $W(\lambda)$.-orbit and $M(\lambda)$ is not irreducible. Combining Proposition 2.3.5 and Lemma 3.4.3, we conclude that $\Upsilon_{L(\lambda)} \subset \alpha^\perp$ for some real root α. Then $\Upsilon_{L(\lambda)} \cap \{\mu| \langle \mu, K \rangle = 0\} \subset \{\mu| (\mu, \alpha) = (\mu, \delta) = 0\} = \mathbb{C}\delta$. Hence λ is k-admissible.

3.5. **KW-admissible modules.** We call $\lambda \in \mathfrak{h}^*$ *KW-admissible* if it is a non-critical rational shifted-regular weakly admissible weight; this means that

(A0) for any positive imaginary root α one has $2(\lambda + \rho, \alpha) \notin \mathbb{Z}_{>0}(\alpha, \alpha)$ (i.e. λ is non-critical);

(A1) for any $\alpha \in \Delta_+$ one has $\langle \lambda + \rho, \alpha^\vee \rangle \notin \mathbb{Z}_{\leq 0}$ (i.e. λ is dominant), or, equivalently: $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}$ for each $\alpha \in \Pi(\lambda)$;

(A2) $\mathbb{C}\Delta(\lambda)^\vee = \mathfrak{h}'$ (i.e. λ is rational).

The set of KW-admissible weights for an affine Lie algebra $\mathfrak{g}$ was described in [KW2]. From Corollary 3.4.1 one obtains

3.5.1. **Corollary.** *The set of KW-admissible weights coincides with the set of shifted-regular admissible weights.*

3.5.2. *Remark.* We will use the following fact: if λ is a shifted-regular admissible weight and α is a simple root such that $\alpha \notin \Delta(\lambda)$, then $s_\alpha.\lambda$ is a shifted-regular admissible weight. This follows from the above corollary and the equality $s_\alpha(\Delta_+ \setminus \{\alpha\}) = \Delta_+ \setminus \{\alpha\}$.

3.5.3. *Example:* $\hat{\mathfrak{sl}}_2, \mathfrak{sl}_3$. In these cases any admissible weight is shifted-regular (hence KW-admissible); this follows from the fact that if λ is not shifted-regular (and non-critical for $\hat{\mathfrak{sl}}_2$), then the maximal proper submodule of $M(\lambda)$ is either zero or isomorphic to a Verma module. Indeed, assume that $\lambda \in Adm$ is not shifted-regular. It is easy to see that in this case $\Pi(\lambda) = \{\alpha, \beta\}$, where $m := \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{>0}, \langle \lambda + \rho, \beta^\vee \rangle = 0$. The Shapovalov determinant $\det S_{m\alpha}$ evaluated at $\lambda + t\mu + t^2\mu'$ is proportional to $t(\mu, \alpha) + t^2(\mu', \alpha)$. If $(\mu, \alpha) = 0$ the sum formula gives $\sum_{i=1}^\infty \dim M(\lambda)^i_{\lambda - m\alpha} = 2$. Since $M(\lambda)^1 = M(\lambda - m\alpha)$, one has $M(\lambda)^2 = M(\lambda - m\alpha)$. Thus, by Proposition 1.6.5, $\mu \in \mathrm{Im}\, \Upsilon_{L(\lambda)}$ if $(\mu, \alpha) = 0$. Hence $L(\lambda)$ admits self-extensions over $[\mathfrak{g}, \mathfrak{g}]$ and thus $\lambda \notin Adm$, a contradiction.

3.5.4. *Example:* $\mathfrak{sl}_4, \hat{\mathfrak{sl}}_3$. In these cases there are admissible weights, which are not shifted-regular; for instance, λ satisfying $\langle \lambda + \rho, \alpha_1^\vee \rangle = \langle \lambda + \rho, \alpha_3^\vee \rangle = 1, \ \langle \lambda + \rho, \alpha_2^\vee \rangle = 0$, where $\alpha_1, \alpha_2, \alpha_3$ are the simple roots. In order to verify that $L(\lambda)$ does not have self-extensions over $[\mathfrak{g}, \mathfrak{g}]$, note that, by Proposition 2.3.5, $(\mu, \alpha_1) = (\mu, \alpha_3)$ for $\mu \in \mathrm{Im}\, \Upsilon_{L(\lambda)}$. The module $M(\lambda)$ contains a subsingular vector v of weight $\lambda - (\alpha_1 + 2\alpha_2 + \alpha_3)$. It is not hard to show $v \notin M(\lambda)^2$ if $(\mu, \alpha_2) \neq 0$, hence $\mu \in \mathbb{C}\delta$ as required.

For $\hat{\mathfrak{sl}}_3$ such λ is a particular case of admissible weights described in Remark 4.5.3.

4. FINITENESS OF $\mathcal{A}\mathrm{dm}_k$ FOR AFFINE LIE ALGEBRAS

In this section $\mathfrak{g}$ is an affine Lie algebra. (However all results can be extended to $\mathfrak{osp}(1, 2l)\hat{\ }$). Recall that $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathbb{C}D$. We denote by K ($\in [\mathfrak{g}, \mathfrak{g}]$) the canonical central element of $\mathfrak{g}$. One says that a $\mathfrak{g}$ (or $[\mathfrak{g}, \mathfrak{g}]$)-module N has *level* $k \in \mathbb{C}$ if $K|_N = k \cdot \mathrm{id}$, and that $\lambda \in \mathfrak{h}^*$ has level k if $\langle \lambda, K \rangle = k$.

Recall that a simple $[\mathfrak{g}, \mathfrak{g}]$-module $L(\lambda)$ (and its highest weight λ) is k-admissible if it is weakly admissible of level k, and each self-extension N of $L(\lambda)$ satisfying $K|_N = k \cdot \mathrm{id}$ splits over $[\mathfrak{g}, \mathfrak{g}]$.

4.1. **Main results.** In this section we deduce Theorem 0.5 from Theorem 0.1. A key fact is that for rational k the category $\mathcal{A}\mathrm{dm}_k$ has finitely many irreducibles, see Corollary 4.4.1. The semisimplicity follows from this fact and Lemma 1.3.1. Indeed, extend the action of $[\mathfrak{g}, \mathfrak{g}]$ on modules in $\mathcal{A}\mathrm{dm}_k$ to that of $\mathfrak{g}$ by letting $D = -L_0$, where L_0 is a Virasoro operator, see 4.2.2. The category $\mathcal{A}\mathrm{dm}_k$, viewed as a category of $\mathfrak{g}$-modules, satisfies the assumptions of Lemma 1.3.1 and so it is semisimple.

The fact that for rational k three sets: shifted-regular k-admissible weights, shifted-regular admissible weights and KW-admissible weights coincide is proven in Corollary 4.3.1. In Subsection 4.4 we prove the description of k-admissible weights given in Subsection 0.5. In Subsection 4.5 we establish a criterion of k-admissibility for vacuum modules.

4.2. **Notation.** Let $\delta \in \mathfrak{h}^*$ be the minimal imaginary root, i.e. $\langle \delta, \mathfrak{h}' \rangle = 0$, $\langle \delta, D \rangle = 1$. One has $\Delta^\perp = \mathbb{C}\delta$.

Let r be the *tier number* of $\mathfrak{g}$ ($r = 1, 2, 3$ is such that $\mathfrak{g}$ is of the type $X_N^{(r)}$). Let A be the Cartan matrix of $\mathfrak{g}$ and let $r^\vee$ be the dual tier number of $\mathfrak{g}$, i.e. the tier number of the affine Lie algebra $\mathfrak{g}(A^t, \tau)$. Recall that Δ (resp., $\Delta^\vee$) is invariant under the shift by $r\delta$ (resp., $r^\vee K$)

4.2.1.   We normalize the invariant form $(-, -)$ by the condition that $(\alpha, \alpha) = 2r$, where r is the tier number of $\mathfrak{g}$, and α is a long simple root ([K2], Chapter 6); note that (β, β) is a positive rational number for any real root β. One has $\langle \mu, K \rangle = (\mu, \delta)$ for all $\mu \in \mathfrak{h}^*$. Recall that $(\delta, \delta) = 0$ and that any imaginary root is an integral multiple of δ. In particular, λ is non-critical iff $(\lambda + \rho, \delta) \neq 0$.

One has $(\rho, \delta) = h^\vee$, where $h^\vee$ is the dual Coxeter number. By above, λ is critical iff the level of $L(\lambda)$ is $-h^\vee$ (the *critical level*).

4.2.2.   A restricted $[\mathfrak{g}, \mathfrak{g}]$-module N (see Subsection 2.3) of a non-critical level $k \neq -h^\vee$ admits an action of Virasoro algebra given by Sugawara operators $L_n, n \in \mathbb{Z}$, see [K2], Section 12.8. In this case the Casimir operator $\hat{\Omega}$ takes the form $\hat{\Omega} = 2r(k + h^\vee)(D + L_0)$, and it acts on $L(\lambda)$ by $(\lambda + 2\rho, \lambda)\,\mathrm{id}$ (see [K2], Chapter 2). Notice that $\hat{\Omega}$ acts by different scalars on $L(\nu)$ and on $L(\nu + s\delta)$, if ν is non-critical and $s \neq 0$.

4.2.3.   Recall that $L(\lambda) \cong L(\lambda')$ as $[\mathfrak{g}, \mathfrak{g}]$-modules iff $\lambda|_{\mathfrak{h}'} = \lambda'|_{\mathfrak{h}'}$ that is $\lambda' - \lambda \in \mathbb{C}\delta$. Moreover, if $\lambda' - \lambda = s\delta$, then $\Delta(\lambda) = \Delta(\lambda')$, and, taking tensor products by the one-dimensional module $L(s\delta)$, we obtain isomorphisms $\mathrm{Ext}^1(L(\lambda), L(\nu)) \xrightarrow{\sim} \mathrm{Ext}^1(L(\lambda + s\delta), L(\nu + s\delta))$ for any $\nu \in \mathfrak{h}^*$. In this subsection we consider non-critical weights $\lambda \in (\mathfrak{h}')^*$, and denote by $L(\lambda)$ the corresponding $[\mathfrak{g}, \mathfrak{g}]$-module. By above, $\Delta(\lambda), \Upsilon_{L(\lambda)}$ is well defined for $\lambda \in (\mathfrak{h}')^*$. Notice that $\mathrm{Ext}^1_{[\mathfrak{g},\mathfrak{g}]}(L(\lambda), L(\nu)) \neq 0$ implies that for each $\lambda' \in \mathfrak{h}^*$ satisfying $\lambda'|_{\mathfrak{h}'} = \lambda$ there exists a unique $\nu' \in \mathfrak{h}^*$ satisfying $\nu'|_{\mathfrak{h}'} = \nu$ such that $\mathrm{Ext}^1_{\mathfrak{g}}(L(\lambda'), L(\nu')) \neq 0$; such ν' is determined by the condition $(\lambda' + 2\rho, \lambda') = (\nu' + 2\rho, \nu')$. It is easy to see that $\mathrm{Ext}^1_{\mathfrak{g}}(L(\lambda'), L(\nu')) = \mathrm{Ext}^1_{[\mathfrak{g},\mathfrak{g}]}(L(\lambda), L(\nu))$ if $\lambda \neq \nu$.

4.2.4.   *Remark.*   Let $\lambda \in \mathfrak{h}^*$ be such that $k := \langle \lambda, K \rangle \in \mathbb{Q} \setminus \{-h^\vee\}$. Write $k + h^\vee = \frac{p}{q}$ with coprime $p, q \in \mathbb{Z}, p \neq 0, q > 0$.

Since $\Delta^\vee$ is invariant under the shift by $r^\vee K$, $\Delta(\lambda)^\vee$ is invariant under the shift by $qr^\vee K$. In particular, $\mathbb{Q}\Delta(\lambda)^\vee + \mathbb{Q}K = \mathbb{Q}\Delta(\lambda)^\vee$ so $\mathbb{Q}\Delta(\lambda) + \mathbb{Q}\delta = \mathbb{Q}\Delta(\lambda)$ if λ is non-critical and has rational level.

4.3.   Recall that (see the introduction) a weight $\lambda \in (\mathfrak{h}')^*$ of non-critical level k is weakly admissible iff

$$(15) \qquad\qquad \langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \ \text{ for all } \alpha \in \Delta_+(\lambda).$$

Denote the set of such weights by $wAdm_k$.

Retain notation of Subsection 3.3.4. By Remark 4.2.4, for $\beta \in \Pi(\lambda)$ one has $\beta \pm qr\delta \in \Delta(\lambda)$. Moreover, $\beta + qr\delta \in \Delta(\lambda)_+$, so $qr\delta - \beta = s_\beta(\beta + qr\delta) \in \Delta(\lambda)_+$, by (Π3) of 3.3.4. Hence $\beta \in \Pi(\lambda)$ forces $qr\delta - \beta \in \Delta(\lambda)_+$. If λ is weakly admissible, then, by Remark 4.2.4, $(\lambda + \rho, \beta)$, $(\lambda + \rho, qr\delta - \beta) \geq 0$, so $k + h^\vee = (\lambda + \rho, \delta) \geq 0$. Hence there are no weakly admissible weights of level k for $k + h^\vee \in \mathbb{Q}_{<0}$.

Theorem 3.4 implies the following corollary.

4.3.1. **Corollary.** *(i) If any self-extension of $L(\lambda)$ with a diagonal action of K splits, then $\mathbb{C}\Delta(\lambda) + \mathbb{C}\delta = \mathbb{C}\Delta$; if, in addition, $k \in \mathbb{Q} \setminus \{-h^\vee\}$, then λ is rational.*

(ii) A shifted-regular weight λ is k-admissible iff it is maximal in its $W(\lambda)$.-orbit and $\mathbb{C}\Delta(\lambda) + \mathbb{C}\delta = \mathbb{C}\Delta$.

(iii) For rational k three sets: shifted-regular k-admissible weights, shifted-regular admissible weights and KW-admissible weights coincide.

Proof. Recall that $\mathfrak{h}'$ is spanned by $\{\alpha^\vee | \alpha \in \Pi\}$, and that λ is rational iff $\mathbb{C}\Delta(\lambda)^\vee = \mathfrak{h}'$, that is $\mathbb{C}\Delta(\lambda) = \mathbb{C}\Delta$. Observe that $\xi \in \mathfrak{h}^*$ vanishes on $\mathfrak{h}'$ iff $\xi \in \Delta^\perp = \mathbb{C}\delta$.

For (i) assume that any self-extension of $L(\lambda)$ with a diagonal action of K splits over $[\mathfrak{g}, \mathfrak{g}]$. In the light of Remark 2.3.6, this means that

$$(16) \qquad \operatorname{Im} \Upsilon_{L(\lambda)} \cap \{\mu | \langle \mu, K \rangle = 0\} \subset \Delta^\perp = \mathbb{C}\delta,$$

By Proposition 2.3.5, $\delta \in \operatorname{Im} \Upsilon_{L(\lambda)}$. Thus (16) is equivalent to $\operatorname{Im} \Upsilon_{L(\lambda)} \cap \{\mu | \langle \mu, K \rangle = 0\} = \mathbb{C}\delta$, which can be rewritten as $(\operatorname{Im} \Upsilon_{L(\lambda)})^\perp + \mathbb{C}\delta = \mathbb{C}\Delta$. Hence any self-extension of $L(\lambda)$ with a diagonal action of K splits over $[\mathfrak{g}, \mathfrak{g}]$ iff

$$(17) \qquad (\operatorname{Im} \Upsilon_{L(\lambda)})^\perp + \mathbb{C}\delta = \mathbb{C}\Delta.$$

By Proposition 2.3.5, $(\operatorname{Im} \Upsilon_{L(\lambda)})^\perp \subset \mathbb{C}\Delta(\lambda)$. This gives (i).

Recall that k-admissibility of λ means that $\langle \lambda, K \rangle = k$, that λ is weakly admissible, that is λ is maximal in its $W(\lambda)$.-orbit, and that any self-extension of $L(\lambda)$ with a diagonal action of K splits over $[\mathfrak{g}, \mathfrak{g}]$. By Theorem 3.4, if λ is shifted-regular and maximal in its $W(\lambda)$.-orbit, then $(\operatorname{Im} \Upsilon_{L(\lambda)})^\perp = \mathbb{C}\Delta(\lambda)$. Now (17) implies (ii) and (iii) follows from (ii), Remark 4.2.4, and Corollary 3.5.1. $\square$

4.4. **Case of rational level.** By Remark 4.2.4, the set of weakly admissible weight of level k is empty, if $k + h^\vee \in \mathbb{Q}_{<0}$. Take $k + h^\vee \in \mathbb{Q}_{>0}$ and write $k + h^\vee = p/q$ for coprime $p, q \in \mathbb{Z}_{>0}$. Set

$$X_k := \{\lambda \in (\mathfrak{h}')^* | \lambda \in wAdm_k \ \& \ \mathbb{C}\Delta(\lambda)^\vee = \mathfrak{h}'\}.$$

By Corollary 4.3.1, the set of k-admissible weights is a subset of X_k.

Recall that $\Delta(\lambda)^\vee_+ := \{\beta^\vee \in \Delta^\vee_+ | \langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{Z}\}$. Set

$$X_k(\Gamma) := \{\lambda \in wAdm_k | \Delta(\lambda)^\vee_+ = \Gamma\} \ \text{ for } \Gamma \subset \Delta^\vee_+, \ B_k := \{\Gamma \subset \Delta^\vee_+ | \mathbb{C}\Gamma = \mathfrak{h}' \ \& \ X(\Gamma) \neq \emptyset\}.$$

One has

$$X_k = \coprod_{\Gamma \in B_k} X_k(\Gamma).$$

By Remark 4.2.4, if λ has level k, then $\Delta(\lambda)$ is invariant under the shift by $rq\delta$ ($r \in \{1, 2, 3\}$). Since Δ has finitely many orbits modulo $\mathbb{Z}rq\delta$, Δ has finitely many subsets which are stable under the shift by $rq\delta$. Hence there are finitely many possibilities for $\Delta(\lambda)$, so the set B_k is finite.

For each $\Gamma \in B_k$ define the polyhedron $P(\Gamma) \subset (\mathfrak{h}')^*$ by

$$\mathcal{P}^k(\Gamma) := \{\lambda \in (\mathfrak{h}')^* | \ \langle \lambda, K \rangle = k \ \text{ and } \ \forall \beta^\vee \in \Gamma, \ \langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{R}_{\geq 0}\}.$$

We will show that each $\mathcal{P}^k(\Gamma)$ is compact.

We call $\lambda \in \mathcal{P}^k(\Gamma)$ *integral* if $\Gamma = \Delta(\lambda)^\vee_+$. Clearly, if $\lambda \in \mathcal{P}^k(\Gamma)$ is integral, then $\langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for each $\beta^\vee \in \Gamma$. Moreover, if $\lambda \in X_k(\Gamma)$, then $\lambda \in \mathcal{P}^k(\Gamma)$ and is integral. Thus $X_k(\Gamma)$ is the set of integral points of the polyhedron $\mathcal{P}^k(\Gamma)$. If $\lambda \in X_k(\Gamma)$ is shifted-regular, then $\langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{Z}_{>0}$ for each $\beta^\vee \in \Gamma$. Thus the set of KW-admissible weights of a level k is a finite union of the interior integral points of the polyhedra $\mathcal{P}^k(\Gamma)$, $\Gamma \in B_k$. The set of k-admissible weights lies in the union of integral points of these polyhedra and contains all their interior integral points (see Corollary 4.3.1).

For each $\Gamma \in B_k$ the set $\Gamma \coprod (-\Gamma) \subset \Delta^\vee$ is a root subsystem. By Remark 4.2.4, $\Gamma \coprod (-\Gamma)$ is invariant under the shift by $qr^\vee K$. We denote by $\Pi(\Gamma)$ the corresponding set of simple roots, i.e. $\Gamma = \Delta(\lambda)^\vee_+, \Pi(\Gamma) = \Pi(\lambda)$ for $\lambda \in X_k(\Gamma)$, see 3.3.4 for the notation. One has $\mathcal{P}^k(\Gamma) = \{\lambda \in (\mathfrak{h}')^* | \ \langle \lambda, K \rangle = k \ \& \ \forall \beta^\vee \in \Pi(\Gamma) \ \langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{R}_{\geq 0}\}$.

Let us show that $\mathcal{P}^k(\Gamma)$ is compact for any $\Gamma \in B_k$. Indeed, by Subsection 4.3, $qr^\vee K - \beta^\vee \in \Gamma$ if $\beta \in \Pi(\Gamma)$ (because $\Gamma = \Delta(\lambda)^\vee, \Pi(\Gamma) = \Pi(\lambda)$ for $\lambda \in X_k(\Gamma)$). Then for each $\beta \in \Pi(\Gamma)$ and any $\lambda' \in \mathcal{P}^k(\Gamma)$ one has $0 \leq \langle \lambda' + \rho, qr^\vee K - \beta^\vee \rangle = -\langle \lambda' + \rho, \beta^\vee \rangle + pr^\vee$, that is $\langle \lambda' + \rho, \beta^\vee \rangle \in [0; pr^\vee]$. Since $\mathbb{C}\Pi(\Gamma) = \mathfrak{h}'$, $\mathcal{P}^k(\Gamma)$ is compact. For any $\lambda \in X_k(\Gamma)$ and for each $\beta \in \Pi(\Gamma)$ the value $\langle \lambda + \rho, \beta^\vee \rangle$ is an integer in the interval $[0; pr^\vee]$. Thus $X_k(\Gamma)$ is a finite set.

4.4.1. **Corollary.**   *For $k + h^\vee \in \mathbb{Q}_{\leq 0}$, the category $\mathcal{A}\text{dm}_k$ is empty. For $k + h^\vee \in \mathbb{Q}_{>0}$, the category $\mathcal{A}\text{dm}_k$ contains finitely many irreducibles.*

4.4.2. Consider the case, when $\Gamma \in B_k$ corresponds to an irreducible root subsystem ($\Pi(\Gamma)$ is the set of simple roots of an irreducible root system). We claim that the interior integral points of each face of the polyhedron corresponding to $X_k(\Gamma)$ are k-admissible weights (by interior points of a face we mean the ones that do not lie on the faces of codimension 2).

Indeed, the polyhedron corresponding to $X_k(\Gamma)$ is given by $\langle \lambda + \rho, \beta^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for each $\beta \in \Pi(\Gamma)$. The faces are parameterized by the elements of $\Pi(\Gamma)$. Consider the face corresponding to some $\beta \in \Pi(\Gamma)$. The interior integral points of this face are λs such that

$\langle\lambda+\rho,\beta^\vee\rangle=0$, $\langle\lambda+\rho,\gamma^\vee\rangle\in\mathbb{Z}_{>0}$ for each $\gamma\in\Pi(\Gamma)\setminus\{\beta\}$. By Lemma 3.4.3, $\gamma\in\Pi(\Gamma)$ is λ-minimal for $\lambda\in X_k(\Gamma)$ in the sense of 2.3.4 iff $\langle\lambda+\rho,\gamma^\vee\rangle\in\mathbb{Z}_{>0}$. In particular, if λ is an interior integral point of the face corresponding to β, then each $\gamma\in\Pi(\Gamma)\setminus\{\beta\}$ is λ-minimal. By Proposition 2.3.5, $\mathrm{Im}\,\Upsilon_{L(\lambda)}\subset(\Pi(\Gamma)\setminus\{\beta\})^\perp$. By (16), in order to show that λ is k-admissible it is enough to verify that $((\Pi(\Gamma)\setminus\{\beta\})^\perp\cap\delta^\perp)\subset\mathbb{C}\delta$. One has $(\Pi(\Gamma)\setminus\{\beta\})^\perp\cap\delta^\perp=(\Pi(\Gamma)\setminus\{\beta\}\cup\{\delta\})^\perp$, so it suffices to show that the span of $(\Pi(\Gamma)\setminus\{\beta\})\cup\{\delta\}$ coincides with the span of Δ. By above, $qr\delta=\sum_{\gamma\in\Pi(\Gamma)}a_\gamma\gamma$; since $\Pi(\Gamma)$ is the set of simple roots of an irreducible root subsystem, $a_\gamma\neq0$ for all γ. Hence $a_\beta\neq0$, so the span of $(\Pi(\Gamma)\setminus\{\beta\})\cup\{\delta\}$ coincides with the span of Γ, which is equal to the span of Δ as required.

4.4.3. Let us describe $X_k(\Gamma)$, where $\Gamma\in B_k$ is such that the root system $\Gamma\cup(-\Gamma)$ is isomorphic to $\Delta^\vee$. Note that for a type $A_n^{(1)}$ all $\Gamma\in B_k$ have this property. Let $\Pi=\{\alpha_i\}_{i=0}^n$, where α_0 is the affine root. Let

$$M:=\{\nu\in\sum_{i=1}^n\mathbb{Q}\alpha_i|\ (\nu,\alpha_i)\in\mathbb{Z}\ \forall i=1,\ldots,n\}$$

and let $\tilde{W}$ be the semidirect product of the finite Weyl group (generated by $\{s_{\alpha_i}\}_{i=1}^n$) and translations $t_\alpha,\alpha\in M$ (see [K2], Chapter 6 or [KW2] for notation). One has: $\tilde{W}=\tilde{W}^+\ltimes W$, where W is the Weyl group of $\mathfrak{g}$ and $\tilde{W}^+(\Pi)=\Pi$.

Let $\Lambda_0\in\mathfrak{h}^*$ be such that $\langle\Lambda_0,\alpha_0^\vee\rangle=1,\langle\Lambda_0,\alpha_i^\vee\rangle=0$ for $i>0$. For $u\in\mathbb{Z}_{>0}$ set

$$S_{(u)}:=\{(u-1)K+\alpha_0^\vee,\alpha_1^\vee,\ldots,\alpha_n^\vee\}.$$

The proof of the following result (and, moreover, a complete description of B_k and $X_k(\Gamma)$ for $\Gamma\in B_k$) is similar to that Thm. 2.1 (resp. Thm. 2.2) of [KW2].

4.4.4. **Theorem.** *Let $k+h^\vee=\frac{p}{q}$, where p,q are coprime positive integers. Let $\Gamma\in B_k$ be such that the root system $\Gamma\cup(-\Gamma)$ is isomorphic to $\Delta^\vee$. Then*

(i) $\gcd(q,r^\vee)=1$;

(ii) $\Pi(\Gamma)^\vee=y(S_{(q)})$ for some $y\in\tilde{W}$ such that $y(S_{(q)})\subset\Delta_+^\vee$;

(iii) $X_k(\Gamma)=\{y.(\lambda-\frac{p(q-1)}{q}\Lambda_0)|\ \langle\lambda+\rho,K\rangle=p\ \&\ \langle\lambda+\rho,\alpha^\vee\rangle\in\mathbb{Z}_{\geq0}\ for\ all\ \alpha\in\Pi\}$.

4.4.5. *Remark.* If $0<p<h^\vee$, then the polyhedra $\mathcal{P}^k(\Gamma)$ do not have interior integral points, since, by Thm. 2.1 of [KW2], there are no KW-admissible weights of level $\frac{p}{q}-h^\vee$ for such p.

4.4.6. *Example.* Consider the example $\mathfrak{g} = \hat{\mathfrak{sl}}_2$, $k = -2 + \frac{p}{q}$, where p, q are coprime positive integers. Let $\alpha, \delta - \alpha$ be the simple roots. One has

$$B_k = \{\Gamma_r\}_{r=1}^{q}, \quad \text{where } \Gamma_r = \{(r-1)K + \alpha^{\vee}; (q-r+1)K - \alpha^{\vee}\}; \ X_k(\Gamma_r) = \{\lambda_{r,s}\}_{s=0}^{p},$$

where $\langle \lambda_{r,s} + \rho, (r-1)K + \alpha^{\vee} \rangle = s$, $\langle \lambda_{r,s} + \rho, (q-r+1)K - \alpha^{\vee} \rangle = p - s$. The interior integral points of Γ_r are $\{\lambda_{r,s}\}_{s=1}^{p-1}$, so the KW-admissible weights are $\lambda_{r,s}, r = 1, \dots, q; s = 1, \dots, p-1$ (this set is empty for $p = 1$). The face of Γ_r corresponding to $(r-1)\delta + \alpha$ (resp. to $(q-r+1)\delta - \alpha$) is $\lambda_{r,0}$ (resp. $\lambda_{r,p}$), and these points are interior in this face. Thus $\lambda_{r,0}, \lambda_{r,p}$ are k-admissible, and the k-admissible weights are $\lambda_{r,s}, r = 1, \dots, q; s = 0, \dots, p$.

4.5. **Vacuum modules.** Retain notation of Subsection 2.4. Let $\dot{\mathfrak{g}}$ be a simple finite-dimensional Lie algebra, and let $\mathfrak{g}$ be its (non-twisted) affinization. Let l be the lacety of $\dot{\mathfrak{g}}$, i.e., the ratio of the lengths squared of a long and a short root of $\dot{\mathfrak{g}}$. Let $\dot{\Pi}$ be the set of simple roots of $\dot{\mathfrak{g}}$, and let θ be its highest root and θ' its highest short root. The set of simple roots of $\mathfrak{g}$ is $\{\alpha_0\} \cup \dot{\Pi}$, where $\alpha_0 = \delta - \theta$. We normalize the form $(-, -)$ in such a way that $(\alpha_0, \alpha_0) = (\theta, \theta) = 2$. Let Δ (resp. $\dot{\Delta}$) be the set of roots of $\dot{\mathfrak{g}}$ (resp. of $\mathfrak{g}$). Let ρ (resp. $\dot{\rho}$) be the Weyl vector for $\mathfrak{g}$ (resp. for $\dot{\mathfrak{g}}$). Recall that $h^{\vee} = (\rho, \theta) + 1$ is the dual Coxeter number and $h = (\rho, \theta') + 1$ is the Coxeter number of $\dot{\mathfrak{g}}$, and that both numbers are positive integers.

Retain notation of Subsections 2.4 and 4.4.3. The space $\dot{\Pi}^{\perp}$ is the span of Λ_0, δ. The generalized Verma module $M_J(k\Lambda_0)$, where $J = \dot{\Pi}$ is called a *vacuum module* and is denoted by V^k; the irreducible *vacuum module* V_k is its quotient, i.e. $V_k = L(k\Lambda_0)$.

By [GK], V^k is not irreducible iff $l(k + h^{\vee})$ is a non-negative rational number, which is not the inverse of an integer. In particular, if k is irrational, then V^k is irreducible. If k, hence $k + h^{\vee}$, is rational, we write $k + h^{\vee}$ in minimal terms:

$$k + h^{\vee} = \frac{p}{q}, \quad \text{where } p, q \in \mathbb{Z}, \ q > 0, \ \gcd(p, q) = 1.$$

4.5.1. **Proposition.** *(i) $L(k\Lambda_0)$ is not weakly admissible iff $k + h^{\vee} = \frac{p}{q}$ is rational with $p < h^{\vee} - 1$ and $\gcd(q, l) = 1$, or $p < h - 1$ and $\gcd(q, l) = l$.*

(ii) $L(k\Lambda_0)$ is KW-admissible iff k is rational with $p \geq h^{\vee}$ and $\gcd(q, l) = 1$ or $p \geq h$ and $\gcd(q, l) = l$.

(iii) Any self-extension of $L(k\Lambda_0)$ with a diagonal action of K splits over $[\mathfrak{g}, \mathfrak{g}]$;

(iv) $L(k\Lambda_0)$ is not admissible for all irrational k and for $k + h^{\vee} = \frac{p}{q}$ rational with $p < h^{\vee} - 1$, $\gcd(q, l) = 1$, or $p < h - 1$, $\gcd(q, l) = l$.

Proof. Retain notation of 3.3.4. One has $\Pi(k\Lambda_0) = \dot{\Pi}$ for irrational k. For $k + h^{\vee} = \frac{p}{q}$ one has $\Pi(k\Lambda_0) = \{q\delta - \theta\} \cup \dot{\Pi}$, if $\gcd(q, l) = 1$, and $\Pi(k\Lambda_0) = \{\frac{q}{l}\delta - \theta'\} \cup \dot{\Pi}$ if $\gcd(q, l) = l$. One has $\langle k\Lambda_0 + \rho, \alpha^{\vee} \rangle = 1$ for $\alpha \in \dot{\Pi}$, $\langle k\Lambda_0 + \rho, (q\delta - \theta)^{\vee} \rangle = p + 1 - h^{\vee}$, and $\langle k\Lambda_0 + \rho, (\frac{q}{l}\delta - \theta')^{\vee} \rangle = p + 1 - h$ if $\gcd(q, l) = l$. Now (i), (ii) follow from Subsection 3.3.5, 3.5 respectively.

Since the elements of $\dot{\Pi}$ are $k\Lambda_0$-minimal in the sense of 2.3.4, Proposition 2.3.5 gives

(18) $$\operatorname{Im} \Upsilon_{L(k\Lambda_0)} \subset \dot{\Pi}^{\perp} = \mathbb{C}\Lambda_0 + \mathbb{C}\delta$$

for any k, and the above inclusion is equality for $k \notin \mathbb{Q}$. By Remark 2.3.6, (iii) is equivalent to $\operatorname{Im} \Upsilon_{L(k\Lambda_0)} \cap \{\mu \in \mathfrak{h}^* | \langle \mu, K \rangle = 0\} = \mathbb{C}\delta$, which follows from the above inclusion.

For irrational k, from Remark 2.3.6 and the above equality $\operatorname{Im} \Upsilon_{L(k\Lambda_0)} = \mathbb{C}\Lambda_0 + \mathbb{C}\delta$, we conclude that $L(k\Lambda_0)$ admits self-extensions which do not split over $[\mathfrak{g}, \mathfrak{g}]$, so it is not admissible. Combining with (i) we obtain (iv). $\qquad\square$

4.5.2. Corollary. $L(k\Lambda_0)$ is k-admissible iff k is irrational or $k + h^{\vee} = \frac{p}{q}$ is rational with $p \geq h^{\vee} - 1$, $\gcd(q, l) = 1$, or $p \geq h - 1$, $\gcd(q, l) = l$.

4.5.3. *Remark.* By Remark 2.3.6, Υ maps the self-extensions splitting over $[\mathfrak{g}, \mathfrak{g}]$ to $\mathbb{C}\delta$, so $L(k\Lambda_0)$ admits a non-splitting extension over $[\mathfrak{g}, \mathfrak{g}]$ iff $\operatorname{Im} \Upsilon_{L(k\Lambda_0)} \neq \mathbb{C}\delta$ which is equivalent to $\Lambda_0 \in \operatorname{Im} \Upsilon_{L(k\Lambda_0)}$, because $\operatorname{Im} \Upsilon_{L(k\Lambda_0)} \subset \mathbb{C}\delta + \mathbb{C}\Lambda_0$ by (18). From Subsection 2.4, $\Lambda_0 \in \operatorname{Im} \Upsilon_{L(k\Lambda_0)}$ iff $M_J(k\Lambda_0)^1 = M_J(k\Lambda_0)^2$, where $\{M_J(k\Lambda_0)^i\}$ is the Jantzen-type filtration constructed for $\mu = \Lambda_0, \mu' = 0$. This filtration is, by definition, the Jantzen filtration $\{(V^k)^j\}$. We conclude that

(19) $$L(k\Lambda_0) \ \text{ admits a non-splitting extension over } [\mathfrak{g}, \mathfrak{g}] \iff (V^k)^1 = (V^k)^2.$$

For $\dot{\mathfrak{g}} = \mathfrak{sl}_2$ the module $L(k\Lambda_0)$ is admissible iff $p \geq 2$, since in the sets of admissible weights and KW-admissible weights coincide (see Example 3.5.3). We conjecture that, for $\dot{\mathfrak{g}} \neq \mathfrak{sl}_2$, $L(k\Lambda_0)$ is admissible iff $k + h^{\vee} = \frac{p}{q}$, where $p \geq h^{\vee} - 1$, $\gcd(q, l) = 1$, $p \geq h - 1$, $\gcd(q, l) = l$. We verified this conjecture for simply-laced $\dot{\mathfrak{g}}$ using formula (19) and the sum formula for the Jantzen filtration given in [GK].

5. Admissible modules for the Virasoro algebra

In this section we prove Theorem 0.6 and other results, stated in Subsection 0.6.

In this section $\mathcal{U}$ is the universal enveloping algebra of the Virasoro algebra. Recall that $\mathfrak{h}$ is spanned by the central element C and L_0; $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is spanned by L_j (resp. L_{-j}) with $j \in \mathbb{Z}_{>0}$. Note that the eigenvalues of $\operatorname{ad} L_0$ on $\mathfrak{n}_+$ are *negative*.

We consider only modules with a diagonal action of C. We write $\mu \in \mathfrak{h}^*$ as $\mu = (h, c)$, where $\langle \mu, L_0 \rangle = h$, $\langle \mu, C \rangle = c$ and we write a Verma module $M(\mu)$ as $M(h, c)$. If v is a weight vector of $M(h, c)$, its weight is of the form $(h + j, c)$ for $j \in \mathbb{Z}_{\geq 0}$; the integer j is called the *level* of v; we denote the corresponding weight space by $M(h, c)_j$ (instead of $M(h, c)_{(h+j,c)}$).

Let $\mathcal{H}_c$ the category of $\mathcal{U}$-modules with a fixed central charge c, i.e. the category of $\mathcal{U}/(C - c)$-modules. We write $\operatorname{Ext}^1_c(N, N') := \operatorname{Ext}^1_{\mathcal{H}_c}(N, N')$.

We use the following parametrization of c:

$$(20) \qquad c(k) = 1 - \frac{6(k+1)^2}{k+2}, \quad k \in \mathbb{C} \setminus \{-2\}.$$

5.1. **Kac determinant.** Recall that

$$\dim M(h,c)_m = P(m), \quad \text{where} \quad \sum_m P(m)q^m := \prod_{j=1}^{\infty}(1-q^j)^{-1}.$$

The Kac determinant $\det B_n$ is the Shapovalov determinant, described in 1.4.3, for the weight space $\mathcal{U}(\mathfrak{n}_-)_n$. The Kac determinant formula (see [KR],[KW3]) can be written as follows

$$\det B_N(h, c(k)) = \prod_{m,n \in \mathbb{Z}_{>0}} (h - h_{m,n}(k))^{P(N-mn)},$$

where $h_{m,n}(k)$ for $m, n, k \in \mathbb{C}, k \neq -2$ is given by

$$h_{m,n}(k) = \frac{1}{4(k+2)}\big((m(k+2) - n)^2 - (k+1)^2\big).$$

Let $b(h,k) \in \mathbb{C}$ be such that $b(h,k)^2 := 4(k+2)h + (k+1)^2$. Observe that for $h = h_{m,n}(k)$, the straight line $x(k+2) - y = b(h,k)$ contains a point (m,n) or $(-m,-n)$. Conversely, if the straight line $x(k+2) - y = b(h,k)$ contains an integral point (m,n), then $h = h_{m,n}(k) = h_{-m,-n}(k)$.

5.1.1. **Lemma.** *(i) The Verma module $M(h, c(k))$ is irreducible iff the straight line $x(k+2) - y = b(h,k)$ does not contain integral points (m,n) with $mn > 0$.*

(ii) A weight $(h, c(k))$ is weakly admissible iff the straight line $x(k+2) - y = b(h,k)$ does not contain integral points (m,n) with $mn < 0$.

Proof. The first statement follows from the fact that $h = h_{m,n}(k)$ iff one of the points $(m,n), (-m,-n)$ lies on the straight line $x(k+2) - y = b(h,k)$. For (ii), observe that, by Lemma 1.8.3, $(h, c(k))$ is not weakly admissible iff there exist $m, n \in \mathbb{Z}_{>0}$ such that $(m(k+2) - n)^2 - (k+1)^2 = 4(k+2)(h - mn)$, which is equivalent to $m(k+2) + n = \pm b(h,k)$. $\qquad\square$

5.1.2. If the straight line $x(k+2) - y = b$ does not contain integral points (m,n) with $mn \neq 0$, then $M(h, c(k))$ is irreducible and weakly admissible.

If k is irrational, the line $x(k+2) - y = b$ contains at most one integral point. Thus $M(h, c(k))$ is irreducible or weakly admissible.

If $k+2 \in \mathbb{Q}_{<0}$, there are two cases: the line does not contain integral points or contains infinitely many integral points (m,n) with $mn < 0$, so $M(h, c(k))$ is not weakly admissible.

If $k + 2 \in \mathbb{Q}_{>0}$, there are two cases: the line does not contain integral points or contains infinitely many integral points (m, n) with $mn > 0$; in this case $M(h, c(k))$ has an infinite length.

5.1.3. Assume that $M(h, c(k))$ is reducible and the weight $(h, c(k))$ is weakly admissible. By 5.1.2 , this happens if $h = h_{m,n}(k)$ for some $m, n \in \mathbb{Z}_{>0}$ and either k is irrational, or $k + 2 \in \mathbb{Q}_{>0}$ and the line $x(k+2) - y = b(h, k)$ does not contain integral points (m', n') with $m'n' < 0$.

Take $k + 2 \in \mathbb{Q}_{>0}$ and write $k + 2 = \frac{p}{q}$ in minimal terms. Take $h = h_{m,n}(k)$ for $m, n \in \mathbb{Z}_{>0}$. The equation $x(k+2) - y = b(h, k)$ can be rewritten as $px - qy = pm - qn$. By above, the line $px - qy = pm - qn$ does not contain integral points (m', n') with $m'n' < 0$. Since the integral points of this line are of the form $(m + qj; n + pj), j \in \mathbb{Z}$, this holds iff the line contains an integral point (r, s) in the rectangle $0 \le r \le q, 0 \le s \le p$. We conclude that the set of weakly admissible weights $(h, c(k))$, such that $M(h, c(k))$ is reducible, is $\{(h_{r,s}(k), c(k))\}$, where $r, s \in \mathbb{Z}$, $0 \le r \le q, 0 \le s \le p$. Notice that $h_{r,s}(k) = h_{q-r,p-s}(k)$.

5.2. Jantzen-type filtration.
Define the map $\Upsilon_{L(\lambda)}$ as in Subsection 1.6. The equality $\mathrm{Ext}^1_c(L(\lambda), L(\lambda)) = 0$ is equivalent to $\mu \notin \mathrm{Im}\, \Upsilon_{L(\lambda)}$, where $\mu \in \mathfrak{h}^*$ is defined by $\mu(L_0) = 1$, $\mu(C) = 0$. Denote by $\{M(h, c)^i\}$ the Jantzen-type filtration introduced in 1.5, which is the image of the Jantzen filtration on $M(\lambda + t\mu)$. We have $\langle C, \lambda + t\mu \rangle = c$, $\langle L_0, \lambda + t\mu \rangle = h + t$, hence

$$\det B_N(h + t, c) = \prod_{m,n\in\mathbb{Z}_{>0}} (h + t - h_{m,n}(k))^{P(N-mn)}.$$

5.2.1. We call an integral point (m, n) on the line $x(k+2) - y = b(h, k)$ *minimal for* $(h, c(k))$ if the product mn is positive and minimal among the positive products: $mn > 0$ and for any integral point (m', n') on the line $x(k+2) - y = b(h, k)$ one has $m'n' \le 0$ or $m'n' \ge mn$.

5.2.2. **Lemma.** *If for a weight* $\lambda = (h, c)$ *there is exactly one minimal point or exactly two minimal points of the form* (m, n), $(-m, -n)$, *then* $\mathrm{Ext}^1_c(L(\lambda), L(\lambda)) = 0$.

Proof. In both cases the sum formula (9) gives $\sum_{j=1}^{\infty} \dim M(h, c)^j_{mn} = 1$. Therefore $M(h, c)^1 \ne M(h, c)^2$, and, by Proposition 1.6.5, $\mu \notin \mathrm{Im}\, \Upsilon_{L(\lambda)}$ as required. $\square$

5.3. Admissible weights.
Let us describe the c-admissible weights.

If the weight (h, c) is c-admissible, then it is weakly admissible and $M(h, c)$ is reducible. The weights with these properties are described in 5.1.3.

If k is irrational, these weights are of the form $h = h_{m,n}(k)$ for $m, n \in \mathbb{Z}_{>0}$. The line $x(k+2) - y = b(h,k)$ contains a unique integral point $((m,n)$ or $(-m,-n))$, so, by Lemma 5.2.2, this weight is c-admissible.

Consider the case $k + 2 \in \mathbb{Q}_{>0}$. By 5.1.3, the weights with the above properties are of the form $(h_{m,n}(k), c(k))$ with $0 \leq n \leq p, 0 \leq m \leq q$.

If $0 < n < p, 0 < m < q$ and (r,s) is minimal for $(h_{m,n}(k), c(k))$, then $(r,s) \in \{(m,n), (m-q, n-p)\}$. Since $mn \neq (m-q)(n-p)$, this minimal point is unique. For $0 < m < q$ (resp. for $0 < n < p$) the minimal point for $(h_{m,p}(k), c(k)) = (h_{q-m,0}(k), c(k))$ (resp. for $(h_{q,n}(k), c(k)) = (h_{0,q-n}(k), c(k)))$ is (m, p) (resp. (q,n)), and it is unique. For $(h_{0,0}(k), c(k)) = (h_{q,p}(k), c(k))$ there are two minimal points: (q, p) and $(-q, -p)$. Hence, by Lemma 5.2.2, $(h_{m,n}(k), c(k))$ is c-admissible for $0 \leq n \leq p, 0 \leq m \leq q, (m,n) \neq (0,p), (q,0)$.

For $(h_{0,p}(k), c(k)) = (h_{q,0}(k), c(k))$ there are two minimal points: $(q, 2p)$ and $(-2q, -p)$. By [Ast], in this case the maximal proper submodule of $M(h,c)$ is generated by a singular vector v of level $2pq$. Since $h_{2p,q}(k) = h_{p,2q}(k)$, the sum formula implies that $\sum_{j=1}^{\infty} \dim M(h,c)^j_{2pq} = 2$. By above, $\dim M(h,c)^1_{2pq} = 1$ so $\dim M(h,c)^2_{2pq} = 1$, that is $v \in M(h,c)^2$. Thus $M(h,c)^1 = M(h,c)^2$, so $(h_{2p,q}(k), c(k))$ is not c-admissible.

5.3.1.   For $k = \frac{p}{q} - 2$, where $p, q \in \mathbb{Z}_{>0}$, $gcd(p,q) = 1$, write

$$c^{p,q} := c(k) = 1 - \frac{6(p-q)^2}{pq}, \quad h_{r,s}^{p,q} := h_{r,s}(k) = \frac{(pr - qs)^2 - (p-q)^2}{4pq}.$$

5.3.2.   **Corollary.**   (i) The weakly admissible weights $(h, c(k))$ with the property that $M(h, c(k))$ is reducible are of the form $h = h_{r,s}(k)$, where r, s are integers, and

   if k is not rational, then $r, s > 0$;

   if $k + 2 = \frac{p}{q}$ is rational, then $0 \leq r \leq q, 0 \leq s \leq p$.

   There are no such weights, if $k + 2 \in \mathbb{Q}_{<0}$.

   (ii) All these weakly admissible weights, except for the weights $(h_{p,0}^{p,q}, c^{p,q}) = (h_{0,q}^{p,q}, c^{p,q})$ are c-admissible.

5.3.3.   *Remark.*   Notice that $h_{1,1}(k) = 0$ for all k. Hence the weight $(0, c(k))$ is c-admissible iff $k + 2 \notin \mathbb{Q}_{<0}$.

5.3.4. *Minimal models.* Recall that $L(h,c)$ is called a *minimal model* if $c = c^{p,q}$ and $h = h_{r,s}^{p,q}$, where p, q are coprime positive integers ≥ 2, and r, s are positive integers with $r < q$, $s < p$.

Consider the set of integral points of the rectangle $0 \leq r \leq q, 0 \leq s \leq p$. Recall that $h_{r,s}^{p,q} = h_{q-r,p-s}^{p,q}$ so the symmetrical points of this rectangle with respect to the rotation by

$180°$ give the same value of h. We see that the $h_{r,s}^{p,q}$, corresponding to weakly admissible weights with $c = c^{p,q}$, where p and q are coprime positive integers, are parameterized by the integral points in the rectangle; the $h_{r,s}^{p;q}$, corresponding to c-admissible weights, are parameterized by the integral points of this rectangle, except for $(0, p), (q, 0)$ (which give the same h); the $h_{r,s}^{p,q}$, corresponding to minimal models, are parameterized by the inner integral points of this rectangle, namely $1 \leq r \leq q - 1, 1 \leq s \leq p - 1$.

5.4. **Self-extensions of $L(h, c)$.** Using Lemma 5.2.2 and the description of the Jantzen filtration given in [Ast], it is not hard to show that $\mathrm{Ext}_c^1(L(h, c), L(h, c)) = \mathbb{C}$ in the following cases:

(a) $M(h, c)$ is irreducible;

(b) $b(h, k), \frac{b(h,k)}{k+2} \in \mathbb{Z}$ and $k + 2 \neq \pm 1, b(h, k) \neq 0$.

(c) $c = 1, 25$ and $h \neq 0$.

In all other cases $\mathrm{Ext}_c^1(L(h, c), L(h, c)) = 0$.

6. Admissible modules for the Neveu-Schwarz superalgebra

In this section $\mathcal{U}$ is the universal enveloping algebra of the Neveu-Schwarz superalgebra. Recall that $\mathfrak{h}$ is spanned by the central element C and $h := L_0$; $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) is spanned by L_j (resp. L_{-j}) with $j \in \frac{1}{2}\mathbb{Z}_{>0}$. Note that the eigenvalues of L_0 on $\mathfrak{n}_+$ are *negative*.

We consider only modules with a diagonal action of C. We use the same notation for Verma modules as in Sect. 5. Let $\mathcal{H}_c$ be the category of $\mathcal{U}$-modules with a fixed central charge c, i.e. the category of $\mathcal{U}/(C - c)$-modules. We write $\mathrm{Ext}_c^1(N, N') := \mathrm{Ext}_{\mathcal{H}_c}^1(N, N')$.

We use the following parameterization of c:

$$c(k) = \frac{3}{2} - \frac{12(k + 1)^2}{2k + 3}, \quad k \in \mathbb{C} \setminus \{-\frac{3}{2}\}.$$

6.1. **Kac determinant.** One has $\dim M(h, c)_N = P(N)$, where the partition function $P(N)$ is given by $\sum_{n \in \frac{1}{2}\mathbb{Z}} P(N)q^N = \prod_{j=1}^{\infty}(1 - q^j)^{-1}(1 + q^{j-1/2})$.

The Kac determinant formula for $N \in \frac{1}{2}\mathbb{Z}_{\geq 0}$ can be written as follows (see [KW3]):

$$\det B_N(h, c(k)) = \prod_{\substack{m, n \in \mathbb{Z}_{>0} \\ m \equiv n \mod 2}} (h - h_{m,n}(k))^{P(N - \frac{mn}{2})},$$

where $h_{m,n}(k)$ for $m, n, k \in \mathbb{C}, k \neq -\frac{3}{2}$ is given by

$$h_{m,n}(k) = \frac{1}{2(2k + 3)}\left((m(k + \frac{3}{2}) - \frac{n}{2})^2 - (k + 1)^2\right).$$

We call a point $(m, n) \in \mathbb{C}^2$ *nice* if $m, n \in \mathbb{Z}$, $m \equiv n \mod 2$.

Let $b(h, k) \in \mathbb{C}$ be such that $b(h, k)^2 := 4(2(2k + 3)h + (k + 1)^2)$. For $h = h_{m,n}(k)$, the straight line $x(2k+3) - y = b(h, k)$ contains a point (m, n) or $(-m, -n)$. Conversely, if the straight line $x(2k+3) - y = b(h, k)$ contains a point (m, n), then $h = h_{m,n}(k) = h_{-m,-n}(k)$. The following lemma is similar to Lemma 5.1.1.

6.1.1. **Lemma.**   (i) *The Verma module* $M(h, c(k))$ *is irreducible iff the straight line* $x(2k + 3) - y = b(h, k)$ *does not contain nice points with* $mn > 0$.

(ii) *A weight* $(h, c(k))$ *is weakly admissible iff the straight line* $x(2k + 3) - y = b(h, k)$ *does not contain nice points* (m, n) *with* $mn < 0$.

6.1.2.   If the straight line $x(2k + 3) - y = b$ does not contain nice points (m, n) with $mn \neq 0$, then $M(h, c(k))$ is irreducible and weakly admissible.

If k is irrational, the line $x(2k + 3) - y = b$ contains at most one integral point. Thus $M(h, c(k))$ is irreducible or weakly admissible.

If $k + \frac{3}{2} \in \mathbb{Q}_{<0}$, there are two cases: the line does not contain nice points or contains infinitely many nice points (m, n) with $mn < 0$, so $M(h, c(k))$ is not weakly admissible.

If $k + \frac{3}{2} \in \mathbb{Q}_{>0}$, there are two cases: the line does not contain integral points or contains infinitely many nice points (m, n) with $mn > 0$; in this case $M(h, c(k))$ has an infinite length.

6.1.3.   Assume that $M(h, c(k))$ is reducible and the weight $(h, c(k))$ is weakly admissible. By 6.1.2 , this happens if $h = h_{m,n}(k)$ for some nice (m, n) with $mn > 0$, and either k is irrational, or $k + \frac{3}{2} \in \mathbb{Q}_{>0}$ and the line $x(2k + 3) - y = b(h, k)$ does not contain nice points (m', n') with $m'n' < 0$.

Take $k + \frac{3}{2} \in \mathbb{Q}_{>0}, h = h_{m,n}(k)$ and write

$$(21) \qquad 2k + 3 = \frac{p}{q}, \;\; p, q \in \mathbb{Z} \setminus \{0\}, \; q > 0, \; p \equiv q \mod 2, \; gcd(\frac{p - q}{2}, p) = 1.$$

Take $h = h_{m,n}(k)$ for nice (m, n) with $mn > 0$. The equation $x(2k + 3) - y = b(h, k)$ can be rewritten as $px - qy = pm - qn$. The nice points of this line are of the form $(m + qj; n + pj)$, $j \in \mathbb{Z}$. By above, the line $px - qy = pm - qn$ does not contain nice points (m', n') with $m'n' < 0$. This is equivalent to the condition that this line contains a nice point (r, s) in the rectangle $0 \leq r \leq q, 0 \leq s \leq p$. We conclude that the set of weakly admissible weights $(h, c(k))$, such that $M(h, c(k))$ is reducible, is $\{(h_{r,s}(k), c(k))\}$, where $0 \leq r \leq q, 0 \leq s \leq p, \; r \equiv s \mod 2$. Notice that $h_{r,s}(k) = h_{q-r,p-s}(k)$.

6.2. **Jantzen-type filtration.** Define the map $\Upsilon_{L(\lambda)}$ as in Subsection 1.6. The formula $\mathrm{Ext}^1_c(L(\lambda), L(\lambda)) = 0$ means that $\mu \notin \mathrm{Im}\, \Upsilon_{L(\lambda)}$, where $\mu \in \mathfrak{h}^*$ is defined by $\mu(L_0) = 1$, $\mu(C) = 0$. Denote by $\{M(h, c)^i\}$ the Jantzen-type filtration introduced in 1.5, which is

the image of the Jantzen filtration on $M(\lambda + t\mu)$. We have $\langle C, \lambda + t\mu \rangle = c$, $\langle L_0, \lambda + t\mu \rangle = h + t$, hence

$$\det B_N(h + t, c) = \prod_{\substack{m, n \in \mathbb{Z}_{>0} \\ m \equiv n \mod 2}} (h + t - h_{m,n}(k))^{P(N - \frac{mn}{2})}.$$

6.2.1. Call a nice point (m, n) belonging to one of the lines $x(2k + 3) - y = \pm b(h, k)$ *minimal for* $(h, c(k))$ if the product mn is positive and minimal among the positive products: $mn > 0$ and for any nice point (m', n') on the line $x(2k + 3) - y = b(h, k)$ one has $m'n' \leq 0$ or $m'n' \geq mn$.

6.2.2. **Lemma.**   *If for a weight $\lambda = (h, c)$ there is exactly one minimal point or exactly two minimal points of the form (m, n), $(-m, -n)$, then $\mathrm{Ext}^1_c(L(\lambda), L(\lambda)) = 0$.*

The proof is the same as in Lemma 5.2.2.

6.3. **Admissible weights.** Let us describe the c-admissible weights.

If the weight (h, c) is c-admissible, then it is weakly admissible and $M(h, c)$ is reducible. The weights with these properties are described in 6.1.3.

If k is irrational, these weights are of the form $h = h_{m,n}(k)$ for nice (m, n) with $m, n > 0$. The line $x(2k + 3) - y = b(h, k)$ contains a unique integral point $((m, n)$ or $(-m, -n))$, so, by Lemma 6.2.2, this weight is c-admissible.

Consider the case $k + \frac{3}{2} \in \mathbb{Q}_{>0}$. By 6.1.3, the weights with the above properties are of the form $(h_{m,n}(k), c(k))$ with $0 \leq n \leq p, 0 \leq m \leq q, m \equiv n \mod 2$. If $0 < m < q, 0 < n < p$ and (r, s) is minimal for $(h_{m,n}(k), c(k))$, then $(r, s) \in \{(m, n), (m - q, n - p)\}$. One has $mn = (m - q)(n - p)$ iff $(m, n) = (\frac{q}{2}, \frac{p}{2})$, which is impossible, since $(\frac{q}{2}, \frac{p}{2})$ is not nice. Hence $(h_{m,n}(k), c(k))$ has a unique minimal point. For $0 < m < q$ (resp. for $0 < n < p$) the minimal point for $(h_{m,p}(k), c(k)) = (h_{q-m,0}(k), c(k))$ (resp. for $(h_{q,n}(k), c(k)) = (h_{0,q-n}(k), c(k)))$ is (m, p) (resp. (q, n)), and it is unique. For $(h_{0,0}(k), c(k)) = (h_{q,p}(k), c(k))$ there are two minimal points: (q, p) and $(-q, -p)$. Hence, by Lemma 6.2.2, $(h_{m,n}(k), c(k))$ is c-admissible for nice (m, n), where $0 \leq n \leq p, 0 \leq m \leq q, \ (m, n) \neq (0, p), (q, 0)$.

By [Ast],[IK2],[IK1], the Verma modules over the Neveu-Schwarz algebra have no sub-singular vectors. For the weight $(h_{q,0}^{(p,q)}, c^{(p,q)}) = (h_{0,p}^{(p,q)}, c^{(p,q)})$, where p, q are even, there are two minimal points: $(q, 2p)$ and $(-2q, -p)$. The absence of subsingular vectors implies that the maximal proper submodule of $M(h_{q,0}^{(p,q)}, c^{(p,q)})$ is generated by a singular vector of level pq and the sum formula implies that this vector belongs to $M(h_{q,0}^{(p,q)}, c^{(p,q)})^2$. Thus $M(h_{q,0}^{(p,q)}, c^{(p,q)})^1 = M(h_{q,0}^{(p,q)}, c^{(p,q)})^2$ and so $L(h_{p,0}^{(p,q)}, c^{(p,q)})$ is not c-admissible.

6.4. **Corollary.** *(i) The weakly admissible weights $(h, c(k))$ such that $M(h, c(k))$ is reducible are of the form $h = h_{r,s}(k)$, where r and s are integers and*

if k is not rational, then $r, s > 0, r \equiv s \mod 2$;

if $k + \frac{3}{2} \in \mathbb{Q}_{>0}$ and p, q are chosen as in (21), then $0 \le r \le q, 0 \le s \le p$ and $r \equiv s \mod 2$.

There are no such weights, if $k + \frac{3}{2} \in \mathbb{Q}_{<0}$.

(ii) All these weakly admissible weights, except for the weights $h_{q,0}(k) = h_{0,p}(k)$ (when p, q are even) are c-admissible.

6.4.1. *Remark.* Notice that $h_{1,1}(k) = 0$ for all k. Hence the weight $(0, c(k))$ is c-admissible iff $k + \frac{3}{2} \notin \mathbb{Q}_{<0}$.

6.4.2. *Remark.* If k is rational and p, q are chosen as in (21), then

$$c^{(p,q)} := c(k) = \frac{3}{2}\left(1 - \frac{2(p-q)^2}{pq}\right), \quad h_{r,s}^{(p,q)} := h_{r,s}(k) = \frac{(pr - qs)^2 - (p-q)^2}{8pq}.$$

Notice that $c^{(p,q)} < 3/2$ and that $h_{r,s}^{(p,q)} = h_{q-r,p-s}^{(p,q)}$.

6.5. **Minimal models.** Recall that the minimal series are the modules $L(h_{r,s}^{(p,q)}, c^{(p,q)})$, where $p, q \in \mathbb{Z}_{\ge 2}$, $p \equiv q \mod 2$, $\gcd(\frac{p-q}{2}, p) = 1$, and r, s are integers, $0 < r < q, 0 < s < p$, $r \equiv s \mod 2$.

Consider the rectangle $0 \le r \le q, 0 \le s \le p$. Recall that $h_{r,s}^{(p,q)} = h_{q-r,p-s}^{(p,q)}$ so the symmetrical points of this rectangle with respect to the rotation by $180°$ give the same value of h. We see that the $h_{r,s}^{(p,q)}$, corresponding to weakly admissible weights with $c = c^{(p,q)}$, where p and q are positive integers of the same parity and $\gcd(\frac{p-q}{2}, p) = 1$, are parameterized by the integral points (r, s) such that $r \equiv s \mod 2$ in this rectangle; the $h_{r,s}^{(p,q)}$, corresponding to c-admissible weights, are parameterized by these points except for $(0, p), (q, 0)$ (which give the same h); the $h_{r,s}^{(p,q)}$, corresponding to minimal models, are parameterized by these points except for the ones on the boundary of the rectangle.

7. Vertex algebras

Recall that a vertex algebra V is a vector space with a vacuum vector $|0\rangle$, and a linear map $V \to (\text{End } V)[[z, z^{-1}]]$, $a \mapsto a(z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}$, such that $a(z)v \in V((z))$ for any $v \in V$, subject to the vacuum axiom $|0\rangle(z) = \text{id}_V$, and the Borcherds identity [K3].

A vertex algebra V is called *graded* if it is endowed with a diagonalizable operator L_0 with non-negative real eigenvalues such that $\Delta(a_{(j)}b) = \Delta(a) + \Delta(b) - j - 1$, where $\Delta(a)$ stands for the L_0-eigenvalue of a. If Γ is the additive subgroup of $\mathbb{R}$ generated by the eigenvalues of L_0, we call V Γ-*graded*. Given an L_0-eigenvector $a \in V$ we write

the corresponding field as $a(z) = \sum_{j \in \mathbb{Z} - \Delta(a)} a_j z^{-j - \Delta(a)}$. Then one has the following commutator formula [K3], 4.6.3

$$(22) \qquad [a_m, b_n] = \sum_{j \in \mathbb{Z}_{\geq 0}} \binom{m + \Delta(a)}{j} (a_{(j)}b)_{m+n}.$$

One also has

$$(23) \qquad (Ta)_n = -(n + \Delta(a))a_n,$$

where $T \in \mathrm{End}(V)$ is the translation operator defined by $Ta := a_{(-2)}|0\rangle$, and one has the following identity

$$(24) \qquad (a_{(-1)}b)_n = \sum_{j=-1}^{-\infty} a_{j-\Delta(a)+1} b_{n-j-\Delta(b)} + (-1)^{p(a)p(b)} \sum_{j=0}^{\infty} b_{n-j-\Delta(b)} a_{j-\Delta(a)+1},$$

where $p(a)$ stands for the parity of a. Recall that the identities (22) and (24) together are equivalent to the Borcherds identity ([K3] Section 4.8).

Recall that V is called C_2-*cofinite* if $V_{(-2)}V$ has a finite codimension in V.

7.1. Universal enveloping algebra.
The universal enveloping algebra of V, introduced in [FZ], can be defined as follows.

Let $LieV$ be the quotient of the vector space with the basis consisting of the formal symbols a_m, where $a \in V_{\Delta(a)}, m \in \mathbb{Z} - \Delta(a)$ subject to the following linearity relations $(a + \gamma b)_m = a_m + \gamma b_m$ for $a \in V_{\Delta(a)}, b \in V_{\Delta(b)}, \gamma \in \mathbb{C}$ and relation (23). Then the commutator formula (22) induces on $LieV$ a well defined Lie superalgebra structure. Note that $LieV$ is a Γ-graded Lie superalgebra, where $\deg a_j = j$.

Let $\tilde{\mathcal{U}}$ be the universal enveloping algebra of the Lie superalgebra $LieV$ subject to the relation $|0\rangle_{(-1)} = 1$. Extend the Γ-grading to $\tilde{\mathcal{U}}$: $\tilde{\mathcal{U}} = \oplus_N \tilde{\mathcal{U}}_N$, and define a system of fundamental neighborhoods of zero in $\tilde{\mathcal{U}}_N$ by $\tilde{\mathcal{U}}_N^s := \{\sum_{\gamma \geq s} u_{-\gamma} u_{N+\gamma} | u_\gamma \in \tilde{\mathcal{U}}_\gamma\}$. Denote by $\hat{\tilde{\mathcal{U}}}_N$ the completion of $\tilde{\mathcal{U}}_N$; the direct sum $\hat{\tilde{\mathcal{U}}} := \oplus_{N \in \Gamma} \hat{\tilde{\mathcal{U}}}_N$ is a Γ-graded complete topological algebra. The *universal enveloping algebra* $\mathcal{U}(V)$ of V is the quotient of $\hat{\tilde{\mathcal{U}}}$ by the relations (24). Since these relations are homogeneous, $\mathcal{U}(V)$ inherits the Γ-grading.

7.1.1. For example, if $V = V^k$ (resp. $V = \mathcal{V}ir^c$ or $V = \mathcal{N}S^c$) is the affine vertex algebra (resp. Virasoro or Neveu-Schwarz vertex algebras), then $\mathcal{U}(V)$ is a completion of the universal enveloping algebra $\mathcal{U}(\hat{\mathfrak{g}})/(K - k)$ (resp. of $\mathcal{U}(\mathcal{V}ir)/(C - c)$ or of $\mathcal{U}(\mathcal{N}S)/(C - c)$).

7.2. Modules over vertex algebras.
Recall that a representation of a vertex algebra V in a vector space M is a linear map $V \to (\mathrm{End}M)[[z, z^{-1}]]$, $a \mapsto a^M(z) = \sum_{n \in -\Delta(a)+\mathbb{Z}} a_n^M z^{-n-\Delta(a)}$, such that $a(z)v \in M((z))$ for each $a \in V, v \in M$, $|0\rangle^M(z) = \mathrm{id}_M$, and the Borcherds identity holds. Note that this is the same as a continuous representation of the topological algebra $\mathcal{U}(V)$ in M, endowed with the discrete topology.

7.3. **Toric subalgebras.** For an associative superalgebra B we call $\mathfrak{h} \subset B$ a *toric (Lie) subalgebra* if

(T1) $\mathfrak{h}$ is an even commutative Lie algebra: $\forall a, b \in \mathfrak{h}$ $p(a) = 0$ & $[a, b] := ab - ba = 0$;

(T2) for any $a \in \mathfrak{h}$ the map $b \mapsto [a, b]$ is a diagonalizable endomorphism of B.

The following proposition is similar to a part of Thm. 2.6 in [M].

7.3.1. **Proposition.**   *Let V be a graded vertex algebra such that its vertex subalgebra $W := \oplus_{x \in \mathbb{Z}} V_x$ is C_2-cofinite. Let $\mathfrak{h}$ be a toric subalgebra of $\mathcal{U}(W)$ such that*

(T3) $\mathfrak{h}$ is spanned by elements of the form a_0 with $a \in W$.

Then every V-module is $\mathfrak{h}$-locally finite.

Proof. Notice that any continuous $\mathcal{U}(V)$-module, viewed as a $\mathcal{U}(W)$-module, is continuous. Thus it is enough to prove that any $\mathcal{U}(W)$-module is $\mathfrak{h}$-locally finite. Hence we may (and will) assume that $W = V$.

Denote by U' the unital commutative associative subalgebra of $\mathcal{U}(V)$ generated by $\mathfrak{h}$. Let M' be the $\mathfrak{h}$-locally finite part of M, i.e. $M' := \{v \in M \,|\, \dim(U'v) < \infty\}$. Since $\mathfrak{h}$ is commutative, M' is the direct sum of generalized $\mathfrak{h}$-weight spaces:

$$M' = \oplus_{\nu \in \mathfrak{h}^*} M'_\nu, \quad M'_\nu := \{v \in M \,|\, \forall h \in \mathfrak{h}\ \exists N : (h - \langle \nu, h \rangle)^N v = 0\}.$$

From (T2) it follows that M' is a $\mathcal{U}(V)$-submodule of M. Clearly, M/M' is a continuous $\mathcal{U}(V)$-module and its $\mathfrak{h}$-locally finite part is trivial: for any non-zero $v \in M/M'$ the space $U'v$ is infinite-dimensional.

Suppose that $M/M' \neq 0$. Fix a non-zero $v \in M/M'$ and let M'' be a cyclic $\mathcal{U}(V)$-submodule generated by v. For $m \in \mathbb{Z}$ introduce

$$M''(m) := \mathcal{U}(V)_m v,$$

where $\mathcal{U}(V)_m$ is the subspace of elements of degree m in $\mathcal{U}(V)$. By Lemma 2.4 of [M], C_2-cofiniteness of V forces $M''(m) = 0$ for $m << 0$. Let $m \in \mathbb{Z}$ be minimal such that $M''(m) \neq 0$. Then for any $a \in V$ one has $a_j M''(m) = 0$ for $j < 0$. By [Zh], there is a well defined action of the Zhu algebra $A(V)$ on $M''(m)$: for $a \in V$ its image in $A(V)$ acts on $M''(m)$ as a_0. The C_2-cofiniteness of V means that $A(V)$ is finite-dimensional so the subalgebra of $\mathrm{End}(M''(m))$ generated by $a_0, a \in V$, is finite-dimensional. By (T3), $\mathfrak{h}$ is spanned by elements of the form a_0 for some $a \in V$. Hence $\dim(U'v) < \infty$, a contradiction. As a result, $M/M' = 0$ as required. $\square$

7.3.2. *Remark.*   Recall that $\omega \in W$ is called a *conformal vector*, if the field $\omega(z)$ is the Virasoro field $L(z) = \sum_{n \in \mathbb{Z}} L_n z^{-n-2}$, (i.e. the L_n satisfy the Virasoro commutation relations), where $L_{-1} = T$ and L_0 is a diagonalisable operator (in this case L_0 defines a grading of V).

The examples of toric subalgebras of $\mathcal{U}(W)$ satisfying (T3) can be constructed as follows. First, if W contains a conformal vector and $\sum_{n \in \mathbb{Z}} L_n z^{-n-2}$ is the corresponding Virasoro field, then $\mathbb{C}L_0 \subset \mathcal{U}(W)$ is toric.

Second, let $a^1, \ldots, a^r \in W_1 = V_1$ be even vectors such that every a_0^j is a diagonalizable endomorphism of W and $a_0^j a^i = 0$ for all $i, j = 1, \ldots, r$. Then $\{a_0^j\}_{j=1}^r$ span a toric subalgebra of $\mathcal{U}(W)$ satisfying (T3). These follow from the formula $[a_0, b_n] = (a_0 b)_n$.

Finally, if $\omega \in W$ is a conformal vector and $a^1, \ldots, a^r \in V_1$ are as in the second example, then $\{L_0; a_0^j\}_{j=1}^r$ span a toric subalgebra of $\mathcal{U}(W)$ satisfying (T3).

8. Admissible modules for the minimal W-algebras

In this section $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ is a finite-dimensional simple Lie algebra with a triangular decomposition, θ is its maximal root, $h^\vee$ is its dual Coxeter number and $k \neq -h^\vee$ is a scalar. In this section we study self-extensions of irreducible representations $\mathbf{L}(\nu)$ of the vertex algebra $\mathcal{W} := \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$. Recall that $\mathcal{W}$ is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex algebra, called a *minimal W-algebra*, constructed in [KRW],[KW3]. The results of this section extend without difficulty to the case of the Lie superalgebra $\mathfrak{g} = \mathfrak{osp}(1, n)$. The Virasoro and Neveu-Schwarz vertex algebras are particular cases of $W^k(\mathfrak{g}, e_{-\theta})$: Virasoro case corresponds to $\mathfrak{g} = \mathfrak{sl}_2$ and Neveu-Schwarz case to $\mathfrak{g} = \mathfrak{osp}(1, 2)$.

The main results of this section are Theorems 8.5.1, 8.7.1.

8.1. **Structure of $\mathcal{W}$.** Denote by Δ_+ the set of positive roots of $\mathfrak{g}$. Define ρ and the bilinear form $(-, -)$ on $\mathfrak{h}^*$ as in 2.3; we normalize the form by the condition $(\theta, \theta) = 2$.

8.1.1. *Notation.* Let $e \in \mathfrak{g}_\theta, h \in \mathfrak{h}, f \in \mathfrak{g}_{-\theta}$ be an $\mathfrak{sl}_2$-triple, and let $x := h/2$. Let $\mathfrak{g}_j := \{a \in \mathfrak{g} | [x, a] = ja\}$ be the jth eigenspace of $\mathrm{ad}\, x$. One has

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \quad \mathfrak{g}_1 = \mathbb{C}e, \quad \mathfrak{g}_{-1} = \mathbb{C}f, \quad \mathfrak{g}_{\pm 1/2} \subset \mathfrak{n}_\pm.$$

For a subspace $\mathfrak{m} \subset \mathfrak{g}$ we set $\mathfrak{m}^f := \{a \in \mathfrak{m} | [f, a] = 0\}$. The centralizer of the triple $\{e, x, f\}$ is $\mathfrak{g}^\natural := \{a \in \mathfrak{g}_0 | (x|a) = 0\} = \mathfrak{g}_0^f$. One has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^f = \mathbb{C}f \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}^\natural, \quad \mathfrak{h} = \mathfrak{h}^f \oplus \mathbb{C}x, \quad \mathfrak{h}^f = \mathfrak{g}^\natural \cap \mathfrak{h}.$$

Then $\mathfrak{g}^\natural$ is a reductive Lie algebra, $\mathfrak{h}^f$ is a Cartan subalgebra of $\mathfrak{g}^\natural$ and $\mathfrak{g}^\natural = (\mathfrak{g}_0 \cap \mathfrak{n}_-) \oplus \mathfrak{h}^f \oplus (\mathfrak{g}_0 \cap \mathfrak{n}_+)$ is a triangular decomposition. Let $P^\natural$ be a weight lattice for $\mathfrak{g}^\natural$ and $Q_+^\natural$ be the positive part of its root lattice with respect to the above triangular decomposition. Note that $\mathfrak{h}^\natural$ acts diagonally on $\mathfrak{g}^f$ and the weights lie in $P^\natural$.

8.1.2. *Generators of $\mathcal{W}$.* The structure of the vertex algebra $\mathcal{W}$ is explicitly described in [KW3]. Recall that $\mathcal{W}$ contains a conformal vector ω, for which $\omega(z)$ is a Virasoro field $L(z) = \sum_n L_n z^{-n-2}$, such that the action of L_0 endows $\mathcal{W}$ with a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-grading $\mathcal{W} = \oplus_{i\in\mathbb{Z}_{\geq 0}}\mathcal{W}_{i/2}$. Recall that a vector $a \in \mathcal{W}$ and the corresponding field $a(z)$ is called *primary of conformal weight* Δ if $L_0 a = \Delta a$ and $L_j a = 0$ for $j > 0$.

The vertex algebra $\mathcal{W}$ has primary fields $J^a(z)$ of conformal weight 1 (resp. 3/2), labeled by $a \in \mathfrak{g}^\natural$ (resp. $a \in \mathfrak{g}_{-1/2}$). We view the Fourier coefficients J_n^a as elements of the enveloping algebra $\mathcal{U}(\mathcal{W})$, see Subsection 7.1. One has

(25) $$[J_0^a, J_n^b] = J_n^{[a,b]} \quad \text{for } a \in \mathfrak{g}^\natural, b \in \mathfrak{g}^\natural \cup \mathfrak{g}_{-1/2}.$$

8.1.3. *The algebra $\mathcal{U}(\mathcal{W})$.* Let $\{a_1, \ldots, a_s\}$ be a basis of $\mathfrak{g}_{-1/2} \oplus \mathfrak{g}^\natural$ consisting of root vectors; set $J^i(z) := J^{a_i}(z)$ for $i = 1, \ldots, s$ and $J^0(z) := L(z)$. Let $I := \{(i,n)|\ i = 0, \ldots, s;\ n \in \mathbb{Z} - \Delta(J^i)\}$ and fix a total order on I in such a way that $(i,m) < (j,n)$ for $m < n$.

By [KW3], Thm. 5.1, the vertex algebra $\mathcal{W}$ is strongly generated by the fields $J^i(z), i = 0, \ldots, s$. This means that the universal enveloping algebra $\mathcal{U}(\mathcal{W})$, has a topological PBW-basis, which consists of the monomials of the form $\prod_{(i,m)}(J_m^i)^{k_{m,i}}$, where $(i,m) \in I, k_{m,i} \in \mathbb{Z}_{>0}$ with $k_{m,i} = 1$ if a_i is odd, and the factors are ordered with respect to the total order on I.

8.1.4. *The algebra $\mathfrak{h}_\mathcal{W}$.* Write

$$I = I_- \coprod I_0 \coprod I_+, \quad \text{where}$$
$$I_- := \{(i,n) \in I|\ n < 0 \text{ or }\ n = 0 \ \& \ a_i \in \mathfrak{n}_- \cap \mathfrak{g}_0\},$$
$$I_+ := \{(i,n) \in I|\ n > 0 \text{ or }\ n = 0 \ \& \ a_i \in \mathfrak{n}_+ \cap \mathfrak{g}_0\},$$
$$I_0 := \{(i,0)|\ i = 0 \ \text{ or } a_i \in \mathfrak{h}^f\}.$$

Set

$$\mathfrak{h}_\mathcal{W} := \text{span}\{J_m^i, (i,m) \in I_0\} = \mathbb{C}L_0 \oplus \{J_0^a|\ a \in \mathfrak{h}^f\} \subset \mathcal{U}(\mathcal{W}).$$

By (25), the elements of $\mathfrak{h}_\mathcal{W}$ commute and

(26) $$[L_0, J_n^a] = -nJ_n^a, \quad [J_0^a, J_n^b] = \langle a, \text{wt } b\rangle J_n^b \quad \text{if } a \in \mathfrak{h}^f, \ b \in \mathfrak{g}^\natural \cup \mathfrak{g}_{-1/2},$$

where $\text{wt } b \in \mathfrak{h}^*$ stands for the weight of b. We identify $\mathfrak{h}_\mathcal{W}$ with $\mathbb{C}L_0 \oplus \mathfrak{h}^f$ via the map $J_0^a \mapsto a$. By above, $\mathfrak{h}_\mathcal{W}$ acts semisimply on $\mathcal{W}$: for every $a \in \mathfrak{g}^\natural + \mathfrak{g}_{-1/2}$ of weight $\beta \in \mathfrak{h}^*$ the weight of J_n^a is

(27) $$\text{wt } J_n^a = \mu \in \mathfrak{h}_\mathcal{W}^* = (\mathbb{C}L_0 \oplus \mathfrak{h}^f)^*, \quad \text{where } \langle \mu, L_0 \rangle = n, \quad \mu|_{\mathfrak{h}^f} = \beta.$$

We consider the adjoint action of $\mathfrak{h}_\mathcal{W}$ on $\mathcal{U}(\mathcal{W})$ given by $h.u := hu - uh, \ h \in \mathfrak{h}_\mathcal{W}, u \in \mathcal{U}(\mathcal{W})$; this action is semisimple and for each $\nu \in \mathfrak{h}_\mathcal{W}^*$ we denote by $\mathcal{U}_\nu$ the corresponding weight space. Note that $\mathfrak{h}_\mathcal{W}$ is a toric Lie subalgebra of $\mathcal{U}(\mathcal{W})$, see Subsection 7.3.

8.1.5.   Let $Q_+ \subset \mathfrak{h}_{\mathcal{W}}$ be the semigroup generated by the weights of J_n^i with $(i, n) \in I_+$. One has

$$\nu \in Q_+ \ \Rightarrow \ \langle \nu, L_0 \rangle \in \frac{1}{2}\mathbb{Z}_{<0}, \ \nu|_{\mathfrak{h}^f} \in P^\natural \ \text{ or } \ \langle \nu, L_0 \rangle = 0, \ \nu|_{\mathfrak{h}^f} \in Q_+^\natural.$$

Note that $-Q_+ \cap Q_+ = \{0\}$. Introduce a partial order on $\mathfrak{h}_{\mathcal{W}}^*$ by setting $\nu' \geq \nu$ iff $\nu' - \nu \in Q_+$.

8.1.6. *Triangular decomposition.* Since $\mathcal{W}$ is strongly generated by $J^i(z)$, the subalgebra generated by $\{J_m^i, (m, i) \in I_0\}$ is the symmetric algebra $\mathcal{S}(\mathfrak{h}_{\mathcal{W}})$.

Fix a PBW-basis in $\mathcal{U}(\mathcal{W})$ as in Subsection 8.1.3. Denote by $U_\pm$ the span of elements of the PBW-basis which are product of elements $\{J_m^i, (m, i) \in I_\pm\}$. Notice that $U_\pm$ are closed subspaces of the topological algebra $\mathcal{U}(\mathcal{W})$ and $U_+ \cap U_- = \mathbb{C}$.

Denote by U_+' (resp. U_-') the closure of the left (resp. right) ideal generated by the monomials $\{J_m^i, (m, i) \in I_+\}$ (resp. $\{J_m^i, (m, i) \in I_-\}$). From the existence of PBW-basis, we conclude that the multiplication map induces embeddings $U_- \otimes \mathcal{S}(\mathfrak{h}_{\mathcal{W}}) \to \mathcal{U}(\mathcal{W})$, $\mathcal{S}(\mathfrak{h}_{\mathcal{W}}) \otimes U_+ \to \mathcal{U}(\mathcal{W})$ and that

(28) $$\mathcal{U}(\mathcal{W}) = (U_- \otimes \mathcal{S}(\mathfrak{h}_{\mathcal{W}})) \oplus U_+' = U_-' \oplus (\mathcal{S}(\mathfrak{h}_{\mathcal{W}}) \otimes U_+).$$

Clearly, $\mathcal{S}(\mathfrak{h}_{\mathcal{W}}), U_\pm$ are $\mathfrak{h}_{\mathcal{W}}$-submodules of $\mathcal{U}(\mathcal{W})$; we set $U_{\pm;\nu} := \mathcal{U}(\mathcal{W})_\nu \cap U_\pm$. One has

$$\Omega(U_+) \subset Q_+, \quad \Omega(U_-) \subset -Q_+, \quad \Omega(\mathcal{S}(\mathfrak{h}_{\mathcal{W}})) = \{0\}.$$

8.1.7.   **Lemma.**   *One has* $\dim U_{+;\nu} = \dim U_{-;-\nu} < \infty$ *for any* $\nu \in \mathfrak{h}_{\mathcal{W}}^*$.

Proof. Retain notation of 8.1.3. The weight spaces $U_{\pm;\nu}$ are spanned by subsets of the PBW basis. In order to show that $\dim U_{+;\nu} = \dim U_{-;-\nu}$ we construct an involution on the PBW-basis which interchanges the vectors of weight ν with the vectors of weight $-\nu$ and the vectors lying in U_+ with the vectors lying in U_-. Introduce an involution on the set I as follows: for $a_i \in \mathfrak{g}_\mu$, where $\mu \in \mathfrak{h}^* \setminus \{0\}$ set $(i, n) \mapsto (i', -n)$, where $a_{i'} \in \mathfrak{g}_{-\mu}$ (such i' is unique, since $\mathfrak{g}_\mu$ is one-dimensional for $\mu \neq 0$); otherwise (if $i = 0$ or $a_i \in \mathfrak{h}$) set $(i, n) \mapsto (i, -n)$. Then the involution maps I_+ onto I_-. Define the corresponding involution σ on the set $\{J_n^i, \ (i, n) \in I\}$ and extend σ to the PBW basis of $\mathcal{U}(\mathcal{W})$. Then σ is a required involution and this establishes the equality $\dim U_{+;\nu} = \dim U_{-;-\nu}$.

It remains to verify that $\dim U_{+;\nu} < \infty$. A monomial of the form $\prod (J_m^i)^{k_{m,i}}$ lying in U_+ can be written as $y_0 y_+$, where y_0 is of the form $\prod (J_0^i)^{k_i}$, $a_i \in \mathfrak{g}_0 \cap \mathfrak{n}_+$ and y_+ is of the form $\prod (J_m^i)^{k_{m,i}}$ with all $m > 0$. Suppose that the monomial $y_0 y_+$ has weight ν. Then $\sum m k_{m,i} = -\langle \nu, L_0 \rangle$. Thus there are finitely many possibilities for y_+. For a given monomial y_+ the weight of y_0 is fixed (it is $\nu - \text{wt}\, y$). Assign to the monomial $y_0 = \prod (J_0^i)^{k_i}$ the element $\prod a_i^{k_i} \in \mathcal{U}(\mathfrak{g}_0 \cap \mathfrak{n}_+)$. Clearly, the images of the monomials of weight $\mu \in \mathfrak{h}_{\mathcal{W}}^*$ form a PBW basis in $\mathcal{U}(\mathfrak{g}_0 \cap \mathfrak{n}_+)_{\mu'}$, where $\mu' \in (\mathfrak{h}^f)^*$ is the restriction of $\mu \in \mathfrak{h}_{\mathcal{W}}^* = (\mathfrak{h}^f \oplus \mathbb{C}L_0)^*$ to $\mathfrak{h}^f$. By 8.1.1, $(\mathfrak{g}_0 \cap \mathfrak{n}_-) \oplus \mathfrak{h}^f \oplus (\mathfrak{g}_0 \cap \mathfrak{n}_+)$ is a triangular decomposition of a reductive Lie algebra. As a result, $\mathcal{U}(\mathfrak{g}_0 \cap \mathfrak{n}_+)_{\mu'}$ is finite-dimensional

for any $\mu' \in (\mathfrak{h}^f)^*$. Thus for a given y_+ there are finitely many possibilities for y_0. Hence $U_{+;\nu}$ is finite-dimensional. $\qquad\square$

Thus, the pair $(\mathcal{U}(\mathcal{W}), \mathfrak{h}_{\mathcal{W}})$ satisfies the conditions (U1)–(U5) of Subsection 1.2 with (U2) replaced by (28). In fact, it follows from [KW2] that (U6) holds as well, but we will not need it.

8.2. The category $\tilde{\mathcal{O}}(\mathcal{W})$.
Let $\mathbf{N}$ be a $\mathcal{U}(\mathcal{W})$-module. For $\nu \in \mathfrak{h}_{\mathcal{W}}^*$ define $\mathbf{N}_\nu$ and the set of generalized weights as usual:

$$(29) \qquad \mathbf{N}_\nu := \{w \in \mathbf{N}|\ \forall u \in \mathfrak{h}_{\mathcal{W}}\ (u - \langle \nu, u \rangle)^n w = 0 \text{ for } n >> 0\},$$

$\Omega(\mathbf{N}) := \{\nu \in \mathfrak{h}_{\mathcal{W}}^*|\ \mathbf{N}_\nu \neq 0\}$. Clearly, $\mathcal{U}(\mathcal{W})_\mu \mathbf{N}_\nu \subset \mathbf{N}_{\mu+\nu}$.

Let $\tilde{\mathcal{O}}(\mathcal{W})$ be the full subcategory of the category of $\mathcal{U}(\mathcal{W})$-modules such that

(O1) $\mathbf{N} = \oplus_{\nu \in \Omega(\mathbf{N})} \mathbf{N}_\nu$;

(O2) $\exists \nu_1, \ldots, \nu_m \in \mathfrak{h}_{\mathcal{W}}^*$ s.t. $\forall \nu \in \Omega(\mathbf{N})\ \exists i$ for which $\nu \leq \nu_i$.

Notice that any module in $\tilde{\mathcal{O}}(\mathcal{W})$ is a continuous $\mathcal{U}(\mathcal{W})$-module and that $\tilde{\mathcal{O}}(\mathcal{W})$ is closed with respect to the extensions. This category is an analogue of the category $\tilde{\mathcal{O}}$ of modules over a Kac-Moody algebra.

8.2.1. *Verma modules over $\mathcal{W}$.* For $\nu \in \mathfrak{h}_{\mathcal{W}}^*$ a Verma module $\mathbf{M}(\nu)$ can be defined as follows: extend ν to an algebra homomorphism $\nu : \mathcal{S}(\mathfrak{h}_{\mathcal{W}}) \to \mathbb{C}$ and let $\operatorname{Ker} \nu$ be its kernel; then

$$\mathbf{M}(\nu) = \mathcal{U}(\mathcal{W})/\big(\mathcal{U}(\mathcal{W})\operatorname{Ker}\nu + U_+'\big),$$

where U_+' is the left ideal introduced in Subsection 8.1.6. A Verma module is a cyclic $\mathcal{U}(\mathcal{W})$-module generated by v of weight ν such that $U_+' v = 0$. Using (28) we can identify $\mathbf{M}(\nu)$ with U_- as vector spaces: the preimage of the weight space $U_{-,\mu}$ is the weight space $\mathbf{M}(\nu)_{\nu+\mu}$; in particular,

$$\Omega(\mathbf{M}(\nu)) \subset (\nu - Q_+), \quad \dim \mathbf{M}(\nu)_\nu = 1.$$

By Lemma 8.1.7, the weight spaces of $\mathbf{M}(\nu)$ are finite-dimensional so $\mathbf{M}(\nu)$ lies in $\tilde{\mathcal{O}}(\mathcal{W})$. The module $\mathbf{M}(\nu)$ has a unique irreducible quotient which we denote by $\mathbf{L}(\nu)$; this is an irreducible module with highest weight ν. Any irreducible module in $\tilde{\mathcal{O}}(\mathcal{W})$ is an irreducible highest module.

8.3. Functor H.
Let $\hat{\mathfrak{g}}$ be the affinization of $\mathfrak{g}$, $\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D$ be the Cartan subalgebra of $\hat{\mathfrak{g}}$; define $\hat{\rho} \in \hat{\mathfrak{h}}^*$ for $\hat{\mathfrak{g}}$ as usual (see Subsection 2.3). Let δ be the minimal imaginary root and let $\alpha_0 = \delta - \theta$. We denote by s_0 the reflection with respect to α_0 ($s_0 \in W \subset GL(\hat{\mathfrak{h}}^*)$). For $\lambda \in \hat{\mathfrak{h}}^*$ we denote by $M(\lambda)$ the $\hat{\mathfrak{g}}$-module with highest weight λ and by $L(\lambda)$ its irreducible quotient. Consider the category $\mathcal{O}$ for the affine Kac-Moody

algebra $\hat{\mathfrak{g}}$ and let $\mathcal{O}_k$ be the full subcategory of the category $\mathcal{O}$ with the objects V such that $K|_V = k \cdot \mathrm{id}$. Set

$$\hat{\mathfrak{h}}_k^* := \{\lambda \in \hat{\mathfrak{h}}^* | \langle \lambda, K \rangle = k\}.$$

For $N \in \mathrm{Obj}\,\mathcal{O}$ we denote by $N^\#$ its graded dual, see 1.7; one has $L(\lambda)^\# \cong L(\lambda)$.

8.3.1. Reduction functors $V \mapsto H^i(V), i \in \mathbb{Z}$, from the category $\mathcal{O}_k$ to the category of continuous $\mathcal{W}$-modules were introduced in [KRW]; in Subsection 8.9.1 we describe these functor for $\mathfrak{g} = \mathfrak{sl}_2$. The following properties are proven in [KRW],[KW3],[Ar]:

(H1) for any $V \in \mathrm{Obj}\,\mathcal{O}_k$ one has $H^i(V) = 0$ for $i \neq 0$; the functor $V \mapsto H^0(V)$ is exact;

(H2) for $\lambda \in \hat{\mathfrak{h}}_k^*$ one has $H^0(M(\lambda)) = \mathbf{M}(\lambda_{\mathcal{W}})$, where $\lambda_{\mathcal{W}} \in \mathfrak{h}_{\mathcal{W}}^*$ is given by

$$(30) \qquad \lambda_{\mathcal{W}}|_{\mathfrak{h}^f} = \lambda|_{\mathfrak{h}^f}, \quad \lambda_{\mathcal{W}}(L_0) = \frac{(\lambda + 2\hat{\rho}, \lambda)}{2(k + h^\vee)} - \langle \lambda, x + D \rangle;$$

(H3) for $\lambda \in \hat{\mathfrak{h}}_k^*$ the continuous $\mathcal{W}$-module $H^0(L(\lambda))$ is irreducible ($\cong \mathbf{L}(\lambda_{\mathcal{W}})$) if $(\lambda, \alpha_0) \notin \mathbb{Z}_{\geq 0}$ and $H^0(L(\lambda)) = 0$ if $(\lambda, \alpha_0) \in \mathbb{Z}_{\geq 0}$;

(H4) for any $N \in \mathrm{Obj}\,\mathcal{O}_k$ and any irreducible module $L \in \mathcal{O}_k$ such that $H^0(L) \neq 0$ one has

$$[H^0(N) : H^0(L)] = \sum_{\lambda : H^0(L(\lambda)) = H^0(L)} [N : L(\lambda)];$$

(H5) for $\lambda \in \hat{\mathfrak{h}}_k^*$ any non-zero submodule of $H^0(M(\lambda)^\#)$ intersects non-trivially the weight space $H^0(M(\lambda)^\#)_{\lambda_{\mathcal{W}}}$ ([Ar], Thm. 6.6.2).

From (H2) it follows that any Verma $\mathcal{W}$-module is the image of $M(\lambda)$ with $\lambda \in \hat{\mathfrak{h}}_k^*$. From this and Lemma 8.3.3 below, all simple objects in $\tilde{\mathcal{O}}(\mathcal{W})$ are of the form $H^0(L(\lambda))$ for $\lambda \in \mathfrak{h}_k^*$.

8.3.2. Let $\mathcal{O}_k'$ be the full category of $\hat{\mathfrak{g}}$-modules N admitting finite filtrations with the subquotients belonging to the category $\mathcal{O}_k$ and with the condition $K|_N = k \cdot \mathrm{id}$.

Extend the functors $N \mapsto H^i(N)$ from $\mathcal{O}_k$ to $\mathcal{O}_k'$ (define the differential d by the same formula). The property (H1) ensures that $H^i(N) = 0$ for $i \neq 0$ and that the functor $N \mapsto H^0(N)$ is exact.

8.3.3. **Lemma.**   *For $\lambda, \lambda' \in \hat{\mathfrak{h}}_k^*$ one has*

$$H^0(M(\lambda)) \cong H^0(M(\lambda')) \iff \lambda' - \lambda \in \mathbb{C}\delta \ \ or \ \lambda' - s_0.\lambda \in \mathbb{C}\delta.$$

Proof. From (H2) one sees that $H^0(M(\lambda)) \cong H^0(M(\lambda'))$ is equivalent to the conditions

(31) $(\lambda' - \lambda)|_{\mathfrak{h}^f} = 0$ and $\dfrac{(\lambda' - \lambda, \lambda' + \lambda + 2\hat{\rho})}{2(k + h^\vee)} - \langle \lambda' - \lambda, x + D \rangle = 0.$

Write $\lambda' - \lambda$ in the form

$\lambda' - \lambda = a\delta + a_0\alpha_0 + \mu,$ where $a, a_0 \in \mathbb{C},$ and $\langle \mu, K \rangle = \langle \mu, D \rangle = \langle \mu, x \rangle = 0.$

Then $(\lambda' - \lambda)|_{\mathfrak{h}^f} = 0$ means that $\mu|_{\mathfrak{h}^f} = 0$, hence $\mu = 0$. Now the second condition of (31) takes the form

$$\frac{(2\lambda + 2\hat{\rho} + a\delta + a_0\alpha_0, a\delta + a_0\alpha_0)}{2(k + h^\vee)} - \langle a\delta + a_0\alpha_0, x + D \rangle = 0,$$

which is equivalent to $(2\lambda + 2\hat{\rho} + a_0\alpha_0, a_0\alpha_0) = 0$. The assertion follows.   □

8.3.4.   Define $W(\lambda) \subset \hat{W}$ as in Subsection 3.3.2.

Corollary.   *If N is a subquotient of $M(\lambda)$, then any irreducible subquotient of $H^0(N)$ is isomorphic to $H^0(L(w.\lambda))$ for some $w \in W(\lambda)$, and $[H^0(N) : H^0(L(w.\lambda))] = [N : L(w\lambda)]$, provided that $H^0(L(w.\lambda)) \neq 0$.*

Proof. By (H4), $[H^0(N) : H^0(L)] = \sum_{\lambda':H^0(L(\lambda'))=H^0(L)}[N : L(\lambda')]$. Since N is a subquotient of $M(\lambda)$, $[N : L(\lambda')] \neq 0$ forces $\lambda' = w\lambda$ for some $w \in W(\lambda)$. One has

$$[H^0(N) : H^0(L(w.\lambda))] = \sum_{y \in W(\lambda):H^0(L(y.\lambda))=H^0(L(w.\lambda))} [N : L(y.\lambda)].$$

By Lemma 8.3.3, $H^0(L(y.\lambda)) = H^0(L(w.\lambda))$ forces $y.\lambda = w.(\lambda + a\delta)$ or $y.\lambda = s_0 w.(\lambda + a\delta)$ for some $a \in \mathbb{C}$. Recall that the value $(\lambda' + \hat{\rho}, \lambda' + \hat{\rho})$ is invariant of a W.-orbit. Notice that $(\lambda + \hat{\rho} + a\delta, \lambda + \hat{\rho} + a\delta) = (\lambda + \hat{\rho}, \lambda + \hat{\rho})$ forces $a = 0$ or $(\lambda + \hat{\rho}, \delta) = 0$. Therefore $H^0(L(y.\lambda)) = H^0(L(w.\lambda))$ forces $y.\lambda = w.\lambda$ or $y.\lambda = s_0 w.\lambda$. If $(w.\lambda, \alpha_0) \notin \mathbb{Z}$, then $s_0 w \notin W(\lambda)$. Since $H^0(L(w.\lambda)) \neq 0$, $(w.\lambda, \alpha_0) \notin \mathbb{Z}_{\geq 0}$. If $(w.\lambda, \alpha_0) \in \mathbb{Z}_{\leq -2}$, then $(s_0 w.\lambda, \alpha_0) \in \mathbb{Z}_{\geq 0}$, so $H^0(L(s_0 w.\lambda)) = 0$. Finally, if $(w.\lambda, \alpha_0) = -1$, then $s_0 w.\lambda = w.\lambda$. Hence $H^0(L(y.\lambda)) = H^0(L(w.\lambda))$ forces $y.\lambda = w.\lambda$ and so $[H^0(N) : H^0(L(w.\lambda))] = [N : L(w\lambda)]$ as required.   □

8.4. **Admissible modules.** Define the admissible weights for $\mathcal{W}$ in the same way as before. That is $\lambda \in \mathfrak{h}^*_{\mathcal{W}}$ is called *weakly admissible* if $[\mathbf{M}(\lambda') : \mathbf{L}(\lambda)] \neq 0$ forces $\lambda' = \lambda$, and λ is called *admissible* if it is weakly admissible and $\mathrm{Ext}^1(\mathbf{L}(\lambda), \mathbf{L}(\lambda)) = 0$.

8.4.1. **Definition.**   Let $\mathcal{A}\mathrm{dm}$ be the full category of $\mathcal{W}$-modules $\mathbf{N}$ which are locally finite $\mathfrak{h}_{\mathcal{W}}$-modules and are such that every irreducible subquotient of $\mathbf{N}$ is admissible.

8.5. **Main results.** One of our main results is Theorem 8.7.1, which relates self-extensions of a $\mathcal{W}$-module $\mathbf{L}(\lambda_{\mathcal{W}})$ to self-extensions of the $\hat{\mathfrak{g}}$-module $L(\lambda)$. Using this theorem we obtain important information on admissible weights for $\mathcal{W}$ for rational k in Subsection 8.8. The main results here are the following.

Recall that the set of rational weakly admissible weights X_k for $\hat{\mathfrak{g}}$ is the union of the sets of all integral points of all the polyhedra $\mathcal{P}^k(\Gamma)$, $\Gamma \in B_k$, whereas that of KW-admissible weights is the union of the sets of all integral points in the interiors of all the polyhedra $\mathcal{P}^k(\Gamma)$, $\Gamma \in B_k$. In Proposition 8.8.1 (iii) we show that the set of admissible weights in $\mathfrak{h}_{\mathcal{W}}^*$ lies in the image of X_k under the map $\lambda \mapsto \lambda_{\mathcal{W}}$. On the other hand, by Corollary 8.8.9, the set of admissible weights in $\mathfrak{h}_{\mathcal{W}}^*$ contains the image of the set of KW-admissible weights.

Another main result is the following theorem.

8.5.1. **Theorem.** *(i) If $k + h^\vee \in \mathbb{Q}_{>0}$ the category $\mathcal{A}$dm is semisimple with finitely many irreducibles and, in particular, is a subcategory of $\tilde{\mathcal{O}}(\mathcal{W})$.*

(ii) If $k + h^\vee \in \mathbb{Q}_{<0}$ the category $\mathcal{A}$dm is empty.

Our strategy of the proof of Theorem 8.5.1 is as follows. In Proposition 8.8.1 (iii) we show that if a $\mathcal{W}$-module $\mathbf{L}(\lambda')$ is admissible for $\lambda' \in \mathfrak{h}_{\mathcal{W}}^*$, then there exists weakly admissible rational $\lambda \in \hat{\mathfrak{h}}_k^*$ such that $\lambda' = \lambda_{\mathcal{W}}$. The set of weakly admissible rational weights of level k is described in Subsection 4.4 (it is denoted by X_k); this set is finite for $k + h^\vee \in \mathbb{Q}_{>0}$ and is empty for $k + h^\vee \in \mathbb{Q}_{<0}$. Therefore the category $\mathcal{A}$dm is empty for $k + h^\vee \in \mathbb{Q}_{<0}$ and has finitely many irreducibles for $k + h^\vee \in \mathbb{Q}_{>0}$. The formula $\text{Ext}^1(\mathbf{L}', \mathbf{L}) = 0$ for non-isomorphic weakly admissible $\mathcal{W}$-modules $\mathbf{L}$ and $\mathbf{L}'$ is established in Corollary 8.6.2. Finally, using Lemma 1.3.1, we obtain the semisimplicity.

8.6. $\text{Ext}^1(\mathbf{L}, \mathbf{L}') = 0$. In this subsection we show that $\text{Ext}^1(\mathbf{L}, \mathbf{L}') = 0$ if $\mathbf{L} \not\cong \mathbf{L}'$ are irreducible weakly admissible $\mathcal{W}$-modules.

8.6.1. **Lemma.** *Let $\mathbf{N} \in \tilde{\mathcal{O}}(\mathcal{W})$ be a module with finite-dimensional generalized weight spaces $\mathbf{N}_\nu$. Suppose that for some $\nu \in \mathfrak{h}_{\mathcal{W}}^*$ one has*

(i) $\Omega(\mathbf{N}) \subset \{\nu' \in \mathfrak{h}_{\mathcal{W}}^*| \ \nu' \le \nu\}$;

(ii) each submodule of $\mathbf{N}$ intersects $\mathbf{N}_\nu$ non-trivially and $\dim \mathbf{N}_\nu = 1$.

Then $\mathbf{N}$ is isomorphic to a submodule of $H^0(M(\lambda)^\#)$ for $\lambda \in \mathfrak{h}_k^*$ such that $\lambda_{\mathcal{W}} = \nu$.

Proof. For a continuous $\mathcal{W}$-module $M = \oplus_{\nu \in \Omega(M)} M_\nu$ with finite-dimensional generalized weight spaces M_ν, we view $M^\# := \oplus M_\nu^*$ as a right $\mathcal{U}(\mathcal{W})$-module via the action $(fu)(v) := f(uv)$, $f \in M^\#$, $u \in \mathcal{U}$, $v \in M$. Take $\mathbf{N}$ satisfying the assumptions (i), (ii) and let $\lambda \in \mathfrak{h}_k^*$ be such that $\lambda_{\mathcal{W}} = \nu$. Set

$$\mathbf{M}' := H^0(M(\lambda)^\#).$$

View $\mathbf{N}^\#, (\mathbf{M}')^\#$ as right $\mathcal{U}(\mathcal{W})$-modules. Let us show that $\mathbf{N}^\#$ is a quotient of $(\mathbf{M}')^\#$.

View $\mathbf{N}^*_{\lambda_{\mathcal{W}}}$ as a subspace of $\mathbf{N}^{\#}$. The assumption (ii) implies that $\mathbf{N}^*_{\lambda_{\mathcal{W}}}$ is one-dimensional and it generates $\mathbf{N}^{\#}$. For each weight element $a \in \mathcal{U}(\mathcal{W})$ with $\mathrm{wt}\, a < 0$ one has $\mathbf{N}^*_{\lambda_{\mathcal{W}}} a = 0$, because $a\mathbf{N} \cap \mathbf{N}_{\lambda_{\mathcal{W}}} = 0$ by (i). Therefore $\mathbf{N}^*_{\lambda_{\mathcal{W}}} U'_{-} = 0$, where U'_{-} is the right ideal introduced in Subsection 8.1.6. Extend $\lambda_{\mathcal{W}} \in \mathfrak{h}^*_{\mathcal{W}}$ to an algebra homomorphism $\lambda_{\mathcal{W}} : \mathcal{S}(\mathfrak{h}_{\mathcal{W}}) \to \mathbb{C}$ and let $\mathrm{Ker}\,\lambda_{\mathcal{W}}$ be its kernel. By above, $\mathbf{N}^*_{\lambda_{\mathcal{W}}}$ is annihilated by a right ideal $J := U'_{-} + \mathrm{Ker}\,\lambda_{\mathcal{W}} \cdot \mathcal{U}(\mathcal{W})$. Thus $\mathbf{N}^{\#}$ is a quotient of a right cyclic $\mathcal{U}(\mathcal{W})$-module $\mathcal{U}(\mathcal{W})/J$. We will show below that

$$(32) \qquad (\mathbf{M}')^{\#} \cong \mathcal{U}(\mathcal{W})/J \quad \text{as right } \mathcal{U}(\mathcal{W})\text{-modules.}$$

This implies that $\mathbf{N}$ is isomorphic to a submodule of $\mathbf{M}'$, as required.

It remains to verify (32). By (28), we can identify $\mathcal{U}(\mathcal{W})/J$ with U_{+} as vector spaces; for every $\mu \in \mathfrak{h}^*_{\mathcal{W}}$ the preimage of a weight space $U_{+,\mu}$ is the weight space of weight $\lambda_{\mathcal{W}} - \mu$ (since the weight of $U_{+,\mu}$ with respect to the *right* adjoint action $u.h := uh - hu$ is $-\mu$). Thus

$$\dim(\mathcal{U}(\mathcal{W})/J)_{\lambda_{\mathcal{W}}-\mu} = \dim U_{+,\mu} \overset{8.1.7}{=} \dim U_{-,-\mu} = \dim \mathbf{M}(\lambda_{\mathcal{W}})_{\lambda_{\mathcal{W}}-\mu} < \infty.$$

As usual, define the character of a $\mathcal{W}$-module M as $\mathrm{ch}_{\mathcal{W}} M := \sum_{\mu \in \mathfrak{h}^*_{\mathcal{W}}} \dim M_{\mu} e^{\mu}$. From (H4) it follows that

$$(33) \qquad \mathrm{ch}_{\mathcal{W}} H^0(M(\lambda)) = \sum_{\lambda' \in \hat{\mathfrak{h}}^*} [M(\lambda) : L(\lambda')] \cdot \mathrm{ch}_{\mathcal{W}} H^0(L(\lambda')) = \mathrm{ch}_{\mathcal{W}} H^0(M(\lambda)^{\#}),$$

since $[M(\lambda) : L(\lambda')] = [M(\lambda)^{\#} : L(\lambda')]$ for any $\lambda' \in \hat{\mathfrak{h}}$. As a result, for each $\mu \in \mathfrak{h}^*_{\mathcal{W}}$

$$(34) \qquad \dim \mathbf{M}'_{\mu} = \dim \mathbf{M}(\lambda_{\mathcal{W}})_{\mu} = \dim(\mathcal{U}(\mathcal{W})/J)_{\mu} < \infty.$$

Observe that $\mathbf{M}'$ satisfies the assumption (i), by (33), and the assumption (ii), by (H5). Thus, by above, $(\mathbf{M}')^{\#}$ is isomorphic to a quotient of $\mathcal{U}(\mathcal{W})/J$, and, by (34), $(\mathbf{M}')^{\#} \cong \mathcal{U}(\mathcal{W})/J$ as required. $\qquad \square$

The following corollary of 8.2.1 and 8.6.1 is an analogue of Lemma 1.8.1.

8.6.2. Corollary. *(i) If $0 \to \mathbf{L}(\lambda_{\mathcal{W}}) \to \mathbf{N} \to \mathbf{L}(\lambda'_{\mathcal{W}}) \to 0$ is a non-splitting extension and $\lambda_{\mathcal{W}} \neq \lambda'_{\mathcal{W}}$, then either $\mathbf{N}$ is isomorphic to a quotient of $\mathbf{M}(\lambda'_{\mathcal{W}}) = H^0(M(\lambda'))$ or $\mathbf{N}$ is isomorphic to a submodule of $H^0(M(\lambda)^{\#})$.*

(ii) If $\mathbf{L} \ncong \mathbf{L}'$ are irreducible weakly admissible $\mathcal{W}$-modules, then $\mathrm{Ext}^1(\mathbf{L}, \mathbf{L}') = 0$.

8.7. Self-extensions of irreducible $\mathcal{W}$-modules. Retain notation of 1.6.1. For a quotient $\mathbf{M}'$ of $\mathbf{M}(\lambda)$ introduce the natural map $\Upsilon_{\mathbf{M}'} : \mathrm{Ext}^1_{\mathcal{W}}(\mathbf{M}', \mathbf{M}') \to \mathfrak{h}^*_{\mathcal{W}}$ similarly to $\Upsilon_{M'}$ in 1.6.1. As in 1.6.1, if N' is an extension of $\mathbf{M}'$ by $\mathbf{M}'$ (i.e., $\mathbf{N}'/\mathbf{M}' \cong \mathbf{M}'$) denote by $\Upsilon_{\mathbf{M}'}(\mathbf{N}')$ the one-dimensional subspace of $\mathfrak{h}^*_{\mathcal{W}}$ spanned by the image of an exact sequence $0 \to \mathbf{M}' \to \mathbf{N}' \to \mathbf{M}' \to 0$. For $\Upsilon_{\mathbf{M}'}$ the the properties $(\Upsilon 1)$-$(\Upsilon 4)$ of Subsection 1.6.2 can be deduces along the same lines, using (28).

Recall that $\hat{\mathfrak{h}}_0^* = \{\mu \in \hat{\mathfrak{h}}^* | \langle \mu, K \rangle = 0\}$. Note that if $M' \in \mathcal{O}_k$ is a quotient of $M(\lambda)$ and $N' \in \mathcal{O}_k'$ (see Subsection 8.3.2 for the notation) is an extension of M' by M', then $\Upsilon_{M'}(N') \subset \hat{\mathfrak{h}}_0^*$.

8.7.1. **Theorem.** *Fix $\lambda \in \hat{\mathfrak{h}}_k^*$ such that $H^0(L(\lambda)) \neq 0$. Introduce the linear map*

$$\phi_\lambda : \hat{\mathfrak{h}}_0^* \to \mathfrak{h}_{\mathcal{W}}^* : \quad \phi_\lambda(\mu)|_{\mathfrak{h}^f} := \mu|_{\mathfrak{h}^f}, \quad \langle \phi_\lambda(\mu), L_0 \rangle := \frac{(\mu, \lambda + \hat{\rho})}{k + h^\vee} - \langle \mu, x + D \rangle.$$

Then

(i) If $N \in \mathcal{O}_k'$ is a self-extension of the $\hat{\mathfrak{g}}$-module $L(\lambda)$, then

$$\Upsilon_{H^0(L(\lambda))}(H^0(N)) = \phi_\lambda\big(\Upsilon_{L(\lambda)}(N)\big).$$

In particular, $\phi_\lambda\big(\operatorname{Im} \Upsilon_{L(\lambda)} \cap \hat{\mathfrak{h}}_0^*\big) \subset \operatorname{Im} \Upsilon_{H^0(L(\lambda))}$.

(ii) If $(\lambda, \alpha_0) \notin \mathbb{Z}$, then $\phi_\lambda\big(\operatorname{Im} \Upsilon_{L(\lambda)} \cap \hat{\mathfrak{h}}_0^*\big) = \operatorname{Im} \Upsilon_{H^0(L(\lambda))}$.

(iii) $\begin{array}{lll} \textit{If } (\lambda + \hat{\rho}, \alpha_0) \neq 0, & \textit{then} & \ker \phi_\lambda = \mathbb{C}\delta, \ \operatorname{Im} \phi_\lambda = \mathfrak{h}_{\mathcal{W}}^*; \\ \textit{if } (\lambda + \hat{\rho}, \alpha_0) = 0, & \textit{then} & \ker \phi_\lambda = \mathbb{C}\delta + \mathbb{C}\alpha_0, \ \dim \operatorname{Im} \phi_\lambda = \dim \mathfrak{h}_{\mathcal{W}}^* - 1. \end{array}$

Remark. The assumption $(\lambda, \alpha_0) \notin \mathbb{Z}$ in (ii) gives $H^0(L) \neq 0$ for any subquotient L of $M(\lambda)$. Example 8.10 shows that $\operatorname{Im} \phi_\lambda \neq \operatorname{Im} \Upsilon_{H^0(L(\lambda))}$ in general.

Proof. Set

$$M := M(\lambda), \ L := L(\lambda), \ \mathbf{M} := H^0(M(\lambda)), \ \mathbf{L} := H^0(L(\lambda)), \ \mathbf{N} := H^0(N).$$

Consider an exact sequence $0 \to L \to N \to L \to 0$ and let $\Upsilon_L : \operatorname{Ext}^1(L, L) \to \hat{\mathfrak{h}}^*$ maps this exact sequence to μ. Let $0 \to \mathbf{L} \to \mathbf{N} \to \mathbf{L} \to 0$ be the image of this sequence under H^0. For (i) let us verify that

(35) $$\Upsilon_{\mathbf{L}}\big(0 \to \mathbf{L} \to \mathbf{N} \to \mathbf{L} \to 0\big) = \phi_\lambda(\mu)$$

The space N_λ has a basis v, v' such that

$$hv' = \langle \lambda, h \rangle v', \quad hv = \langle \lambda, h \rangle v + \langle \mu, h \rangle v' \ \text{ for any } h \in \hat{\mathfrak{h}}.$$

By the assumption $(\lambda, \alpha_0) \notin \mathbb{Z}_{\geq 0}$. Let $|0\rangle$ be the vacuum vector of $F^{ch} \otimes F^{ne}$, see [KW3] for notation. Using the fact that $\Omega(N) \subset \Omega(M(\lambda))$, it is easy to show by an explicit computation or to deduce from [Ar], 4.7.1, 4.8.1, that the images of $v \otimes |0\rangle$, $v' \otimes |0\rangle$ lie in $\mathbf{N}$ and are linearly independent. Since $\mathbf{N}/\mathbf{L} \cong \mathbf{L}$ one has $\dim \mathbf{N}_{\lambda_{\mathcal{W}}} = 2$ and thus $v \otimes |0\rangle$, $v' \otimes |0\rangle$ form a basis of $\mathbf{N}_{\lambda_{\mathcal{W}}}$.

The explicit formulas for J^a and $L(z)$ given in [KW3] imply that for $h \in \mathfrak{h}^f$ one has

$$J^h(0)(v' \otimes |0\rangle) = \langle \lambda, h \rangle v' \otimes |0\rangle, \quad J^h(0)(v \otimes |0\rangle) = \langle \lambda, h \rangle v \otimes |0\rangle + \langle \mu, h \rangle v' \otimes |0\rangle;$$
$$L_0(v'' \otimes |0\rangle) = \big(\tfrac{\hat{\Omega}}{2(k + h^\vee)} - (x + D)\big) v'' \otimes |0\rangle, \text{ for any } v'' \in N_\lambda,$$

where $\hat{\Omega}$ is the Casimir operator of $\hat{\mathfrak{g}}$. One has

$$\hat{\Omega}v' = (\lambda, \lambda + 2\hat{\rho})v', \quad \hat{\Omega}v = (\lambda, \lambda + 2\hat{\rho})v + 2(\mu, \lambda + \hat{\rho})v'$$

and thus

$$L_0(v' \otimes |0\rangle) = av' \otimes |0\rangle, \quad L_0(v \otimes |0\rangle) = av \otimes |0\rangle + bv' \otimes |0\rangle, \quad \text{where}$$
$$a := \frac{(\lambda, \lambda + 2\hat{\rho})}{2(k + h^\vee)} - \langle \lambda, x + D \rangle, \quad b := \frac{(\lambda + \hat{\rho}, \mu)}{k + h^\vee} - \langle \mu, x + D \rangle.$$

This establishes (35) and proves (i).

For (ii) fix $\lambda \in \hat{\mathfrak{h}}_k^*$ such that $(\lambda, \alpha_0) \notin \mathbb{Z}$. Observe that for any subquotient L' of $M^\#$ one has $H^0(L') \neq 0$ since, by 3.3.2, one has $(w.\lambda, \alpha_0) \notin \mathbb{Z}$ for any $w \in W(\lambda)$.

Suppose that $\mu \notin \operatorname{Im} \Upsilon_L$ and $\phi_\lambda(\mu) \in \operatorname{Im} \Upsilon_{\mathbf{L}}$. Let N be an extension of M by M such that $\Upsilon_M(N) = \mathbb{C}\mu$. Set $\mathbf{N} := H^0(N)$. By (i), $\Upsilon_{\mathbf{M}}(\mathbf{N}) = \mathbb{C}\phi_\lambda(\mu)$. By the property ($\Upsilon 3$), the assumption $\phi_\lambda(\mu) \in \operatorname{Im} \Upsilon_{\mathbf{L}}$ implies the existence of $\mathbf{N}' \subset \mathbf{N}$ such that $\mathbf{N}/\mathbf{N}'$ is an extension of $\mathbf{L}$ by $\mathbf{L}$. Notice that $\mathbf{N}'_{\lambda_{\mathcal{W}}} = 0$ since $\dim \mathbf{N}_{\lambda_{\mathcal{W}}} = 2$ and $\dim \mathbf{L}_{\lambda_{\mathcal{W}}} = 1$.

In the light of Lemma 1.7.1, N has a subquotient $E \not\cong L$ which is isomorphic to a submodule of $M^\#$. Clearly, $H^0(E)$ is isomorphic to a submodule of $H^0(M^\#)$ so, by 8.3.1, any submodule of $H^0(E)$ contains a vector of weight $\lambda_{\mathcal{W}}$. By the above observation, $H^0(E) \not\cong \mathbf{L}$. Since H^0 is exact, $\mathbf{N}$ has submodules $\mathbf{N}_1 \subset \mathbf{N}_2$ such that $\mathbf{N}_2/\mathbf{N}_1 \cong H^0(E)$.

Since $\mathbf{N}'_{\lambda_{\mathcal{W}}} = 0$, each $v \in \mathbf{N}_2 \cap \mathbf{N}'$ generates a submodule which intersects trivially $\mathbf{N}_{\lambda_{\mathcal{W}}}$. Thus the image of $\mathbf{N}_2 \cap \mathbf{N}'$ in $\mathbf{N}_2/\mathbf{N}_1 \cong H^0(E)$ is zero. Therefore $\mathbf{N}_2 \cap \mathbf{N}' \subset \mathbf{N}_1$ and so $\mathbf{N}_2/\mathbf{N}_1$ is a quotient of $\mathbf{N}_2/(\mathbf{N}_2 \cap \mathbf{N}'') \subset \mathbf{N}/\mathbf{N}'$. Hence $H^0(E)$ is a subquotient of $\mathbf{N}/\mathbf{N}'$. However, since $\mathbf{N}/\mathbf{N}'$ is an extension of $\mathbf{L}$ by $\mathbf{L}$, the only subquotients of $\mathbf{N}/\mathbf{N}'$ are $\mathbf{L}$ and $\mathbf{N}/\mathbf{N}'$ itself. By above, $H^0(E) \not\cong \mathbf{L}$. By (33) $\dim H^0(M^\#)_{\lambda_{\mathcal{W}}} = 1$ so $\dim H^0(E)_{\lambda_{\mathcal{W}}} = 1 = \dim \mathbf{L}_{\lambda_{\mathcal{W}}}$ and thus $H^0(E)$ is not an extension of $\mathbf{L}$ by $\mathbf{L}$, a contradiction. This establishes (ii). The proof of (iii) is straightforward. $\square$

8.8. **Admissible weights.** In this subsection we study the admissible weights in $\mathfrak{h}_{\mathcal{W}}^*$.

Denote by $\varphi_{\mathcal{W}} : \hat{\mathfrak{h}}_k^* \to \mathfrak{h}_{\mathcal{W}}^*$ the map given by $\lambda \mapsto \lambda_{\mathcal{W}}$. Recall that $\varphi_{\mathcal{W}}$ is surjective and that $\varphi_{\mathcal{W}}(\lambda) = \varphi_{\mathcal{W}}(\lambda')$ iff $\lambda' \in \{\lambda + \mathbb{C}\delta, s_0.\lambda + \mathbb{C}\delta\}$, by Lemma 8.3.3. Let $\hat{\Delta}$ be the set of roots of $\hat{\mathfrak{g}}$ and $wAdm_k$ be the set of weakly admissible weights in $\hat{\mathfrak{h}}_k^*$. By Corollary 4.3.1, $k\text{-}Adm \subset \{\lambda \in wAdm_k| \ \mathbb{C}\hat{\Delta}(\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}\}$; moreover, by Remark 4.2.4, the k-admissible weights for rational k are the rational weakly admissible weights in $\hat{\mathfrak{h}}_k^*$. In Subsection 8.8.7 we prove the following proposition.

8.8.1. **Proposition.** *(i) If $(\lambda, \alpha_0) \notin \mathbb{Z}$, then $\lambda_{\mathcal{W}} \in Adm_{\mathcal{W}}$ iff $\lambda \in k\text{-}Adm$.*

(ii) $Adm_{\mathcal{W}} \subset \varphi_{\mathcal{W}}\big(\{\lambda \in wAdm_k| \ \mathbb{C}\Delta(\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}\}\big)$.

(iii) If k is rational, then $Adm_{\mathcal{W}} \subset \varphi_{\mathcal{W}}\big(\{\lambda \in wAdm_k \ \& \ \lambda \text{ is rational}\}\big)$.

(iv) The set $Adm_{\mathcal{W}}$ is empty if $k + h^\vee \in \mathbb{Q}_{<0}$ and is finite if $k + h^\vee \in \mathbb{Q}_{>0}$.

8.8.2. Let k be rational. Retain notation of 4.4 and recall that the set of rational weakly admissible weights of level k is the set of integral points of a finite union of polyhedra X_Γ. The image of each polyhedron in $\hat{\mathfrak{h}}^*/\mathbb{C}\delta$ is finite. The set of k-admissible weights is a subset of integral points of these polyhedra; it contains the set of interior integral points of each polyhedron. We see that $Adm_\mathcal{W}$ lies in the image (under $\varphi_\mathcal{W}$) of the set of integral points of the polyhedra X_Γ.

Let Γ be such that $\alpha_0 \notin \Gamma$. Set $s_0\Gamma := \{(s_0\alpha)^\vee|\ \alpha^\vee \in \Gamma\}$. Then $X_{s_0\Gamma} = s_0 X_\Gamma$ and so $\varphi_\mathcal{W}(X_\Gamma) = \varphi_\mathcal{W}(X_{s_0\Gamma})$; each point from $\varphi_\mathcal{W}(X_\Gamma) \in \mathfrak{h}^*_\mathcal{W}$ has exactly two preimages in $\hat{\mathfrak{h}}^*_k/\mathbb{C}\delta$ (one in X_Γ and another one in $X_{s_0\Gamma}$). By Proposition 8.8.1 (i), admissible weights in $\varphi_\mathcal{W}(X_\Gamma)$ are the images of k-admissible weights in X_Γ.

Let $\alpha_0 \in \Gamma$. In Proposition 8.8.8 we will show that $Adm_\mathcal{W}$ contains the image of the set of interior integral points of the polyhedron $X(\Gamma)$ and the image of the set of interior points of the face $(\lambda + \hat{\rho}, \alpha_0) = 0$ of the polyhedron X_Γ.

8.8.3. **Corollary.** *The irreducible vacuum module over the vertex algebra $\mathcal{W} = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is admissible for the following values of k: $k \notin \mathbb{Q}$, or $k + h^\vee = \frac{p}{q} > 0, \gcd(p, q) = 1$, where $p \geq h^\vee - 1$ and $\gcd(q, l) = 1$, or $p \geq h - 1$ and $\gcd(q, l) = l$. The irreducible vacuum module is not admissible for other values of k, except for, possibly, the case $k = -2$ and $\mathfrak{g} \neq C_n$.*

Proof. The module $\mathbf{L}(0)$ is the irreducible vacuum module and $0 = \varphi_\mathcal{W}(k\Lambda_0)$.

Consider the case when k is an integer. If $k + h^\vee < 0$, then $Adm_\mathcal{W}$ is empty and so $0 \in \mathfrak{h}^*_\mathcal{W}$ is not admissible. One has $\langle k\Lambda_0 + \hat{\rho}, \alpha_0^\vee \rangle = k + 1$, so $k\Lambda_0$ is weakly admissible iff $k \geq -1$. In the light of Proposition 8.8.8, $0 = \varphi_\mathcal{W}(k\Lambda_0)$ is admissible for $k \geq -1$.

Consider the case $k \in \mathbb{Z}_{\leq -2}$. Assume that $0 = \varphi_\mathcal{W}(k\Lambda_0)$ is admissible. Then, by Proposition 8.8.1 (ii), $k\Lambda_0$ or $s_0.k\Lambda_0$ is weakly admissible. Since $k\Lambda_0$ is not weakly admissible, $s_0.k\Lambda_0$ is weakly admissible, that is for each $\beta \in \Pi$ one has

$$0 \leq \langle s_0.k\Lambda_0 + \hat{\rho}, \beta^\vee \rangle = 1 - (k+1)\langle \alpha_0, \beta^\vee \rangle.$$

For $k < -2$ the above inequality does not hold if $\langle \alpha_0, \beta^\vee \rangle \neq 0$, so $0 = \varphi_\mathcal{W}(k\Lambda_0)$ is not weakly admissible. Hence $0 = \varphi_\mathcal{W}(k\Lambda_0)$ is not admissible for $k < -2$. For $k = -2$ the above inequalities hold iff $\langle \alpha_0, \beta^\vee \rangle \geq -1$ for all $\beta \in \Pi$ and this holds iff the 0th column of Cartan matrix of $\hat{\mathfrak{g}}$ contains only $0, -1, 2$; this means that the corresponding Dynkin diagram does not have arrows going from the 0th vertex. Thus $s_0.(-2\Lambda_0)$ is weakly admissible iff $\mathfrak{g}$ is not of the type C_n. Hence $0 = \varphi_\mathcal{W}(-2\Lambda_0)$ is not admissible for C_n.

If $k \notin \mathbb{Z}$, then $0 \in \mathfrak{h}^*_\mathcal{W}$ is admissible iff $k\Lambda_0$ is k-admissible, that is iff k is irrational, or $k + h^\vee = \frac{p}{q} \in \mathbb{Q}_{>0} \setminus \mathbb{Z}$, where $p \geq h^\vee - 1$ and $\gcd(q, l) = 1$, or $p \geq h - 1$ and $\gcd(q, l) = l$, see Corollary 4.5.2. Note that the case $q = 1$, $p \geq h^\vee - 1$ produces $k \in \mathbb{Z}_{\geq -1}$. $\qquad\square$

8.8.4.   Set

$$Y := \{\lambda \in \hat{\mathfrak{h}}_k^* | \ (\lambda, \alpha_0) \in \mathbb{Z}\}, \quad Y_0 := \{\lambda \in \hat{\mathfrak{h}}_k^* | \ (\lambda, \alpha_0) = 0\},$$
$$\overline{Y} := \hat{\mathfrak{h}}_k^* \setminus Y, \quad \overline{\varphi_{\mathcal{W}}(Y)} := \mathfrak{h}_{\mathcal{W}}^* \setminus \varphi_{\mathcal{W}}(Y),$$

and observe that

$$\overline{\varphi_{\mathcal{W}}(Y)} = \varphi_{\mathcal{W}}(\overline{Y}), \quad \varphi_{\mathcal{W}}^{-1}\big(\varphi_{\mathcal{W}}(Y)\big) = Y, \quad \varphi_{\mathcal{W}}^{-1}\big(\varphi_{\mathcal{W}}(Y_0)\big) = Y_0.$$

We start the proof of Proposition 8.8.1 from the following description of the set of weakly admissible weights in $\mathfrak{h}_{\mathcal{W}}^*$, which we denote by $wAdm_{\mathcal{W}}$.

8.8.5.   **Lemma.**   *One has*

$$wAdm_{\mathcal{W}} = \varphi_{\mathcal{W}}(wAdm_k), \quad \varphi_{\mathcal{W}}^{-1}\big(wAdm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(\overline{Y})\big) \subset wAdm_k.$$

Proof. Consider the case when $\lambda \in \overline{Y}$. Let us show that $\lambda \in \hat{\mathfrak{h}}_k^*$ is weakly admissible iff $\lambda_{\mathcal{W}} \in \mathfrak{h}_{\mathcal{W}}^*$ is weakly admissible.

Take $\lambda \in \overline{Y}$, which is not weakly admissible. Then $\operatorname{Hom}_{\mathfrak{g}}(M(\lambda), M(\lambda + m\beta)) \neq 0$ for some real root β. Therefore $\operatorname{Hom}_{\mathcal{W}}(\mathbf{M}(\lambda_{\mathcal{W}}), \mathbf{M}(\lambda + m\beta)_{\mathcal{W}}) \neq 0$. Since $(\lambda, \alpha_0) \notin \mathbb{Z}$ one has $m\beta, m\beta + (\lambda - s_0.\lambda) \notin \mathbb{C}\delta$ so $\lambda_{\mathcal{W}} \neq (\lambda + m\beta)_{\mathcal{W}}$. Hence $\lambda_{\mathcal{W}}$ is not weakly admissible.

Assume that $\lambda_{\mathcal{W}}$ is not weakly admissible that is $[\mathbf{M}(\lambda_{\mathcal{W}}') : \mathbf{L}(\lambda_{\mathcal{W}})] \neq 0$ for some λ' with $\lambda_{\mathcal{W}}' \neq \lambda_{\mathcal{W}}$. Recall that $\mathbf{M}(\lambda_{\mathcal{W}}') = H^0(M(\lambda')) = H^0(M(s_0.\lambda'))$. From (H4) we obtain

$$\sum_{a \in \mathbb{C}}[M(\lambda') : L(\lambda - a\delta)] + [M(\lambda') : L(s_0.\lambda - a\delta)] \neq 0,$$
$$\sum_{a \in \mathbb{C}}[M(s_0.\lambda') : L(\lambda - a\delta)] + [M(s_0.\lambda') : L(s_0.\lambda - a\delta)] \neq 0.$$

If $[M(\lambda') : L(s_0.\lambda - a_1\delta)] \neq 0$ and $[M(s_0.\lambda') : L(s_0.\lambda - a_2\delta)] \neq 0$ for some a_1, a_2, then $\lambda' - s_0.\lambda, \ s_0.\lambda' - s_0.\lambda \in \hat{Q} + \mathbb{C}\delta$ so $\lambda - s_0.\lambda \in \hat{Q} + \mathbb{C}\delta$, which contradicts to $(\lambda, \alpha_0) \notin \mathbb{Z}$. As a result, $[M(\lambda') : L(\lambda - a\delta)] \neq 0$ or $[M(s_0.\lambda') : L(\lambda - a\delta)] \neq 0$ for some a, then $[M(\lambda'' + a\delta) : L(\lambda)] \neq 0$ for $\lambda'' \in \{\lambda', s_0.\lambda'\}$. Notice that $(\lambda'' + a\delta)_{\mathcal{W}} = \lambda_{\mathcal{W}}' \neq \lambda_{\mathcal{W}}$ so $\lambda'' + a\delta \neq \lambda$. Thus λ is not weakly admissible.

Consider the case when $\lambda \in Y$. Take a weakly admissible $\lambda \in \hat{\mathfrak{h}}_k^*$ with $\alpha_0 \in \Delta(\lambda)$. We claim that $\lambda_{\mathcal{W}}$ is weakly admissible. Indeed, otherwise $[\mathbf{M}(\lambda_{\mathcal{W}}') : \mathbf{L}(\lambda_{\mathcal{W}})] \neq 0$ for some λ' with $\lambda_{\mathcal{W}}' \neq \lambda_{\mathcal{W}}$. Since $H^0(L(\lambda + a\delta)) = 0$ for all $a \in \mathbb{C}$, (H4) give $[M(\lambda') : L(s_0.\lambda + a\delta)] > 0$ for some $a \in \mathbb{C}$ that is $[M(\lambda' - a\delta) : L(s_0.\lambda)] > 0$. Since λ is weakly admissible, it is maximal in its $W(\lambda)$.-orbit so $\lambda' - a\delta = \lambda$ or $\lambda' - a\delta = s_0.\lambda$. This contradicts to the assumption $\lambda_{\mathcal{W}}' \neq \lambda_{\mathcal{W}}$.

Now suppose that $\lambda_{\mathcal{W}}$ is weakly admissible and the orbit $W(\lambda).\lambda$ has a maximal element λ'. Then $\operatorname{Hom}_{\mathfrak{g}}(M(\lambda), M(\lambda')) \neq 0$ so $\operatorname{Hom}_{\mathcal{W}}(\mathbf{M}(\lambda_{\mathcal{W}}), \mathbf{M}(\lambda_{\mathcal{W}}')) \neq 0$ that is $\lambda_{\mathcal{W}} = \lambda_{\mathcal{W}}'$, because $\lambda_{\mathcal{W}}$ is weakly admissible. Since λ' is maximal in its $W(\lambda).\lambda$-orbit, λ' is weakly admissible and so the preimage of $\lambda_{\mathcal{W}}$ in $\hat{\mathfrak{h}}_k^*$ contains a weakly admissible element.

Finally, suppose $\lambda_{\mathcal{W}}$ is weakly admissible and the orbit $W(\lambda).\lambda$ does not have a maximal element. Since $\lambda_{\mathcal{W}} = (s_0.\lambda)_{\mathcal{W}}$, we may (and will) assume that $s_0.\lambda \leq \lambda$. By the assumption, $s_\beta.\lambda > \lambda$ for some real root β so $\mathrm{Hom}_{\mathfrak{g}}(M(\lambda), M(s_\beta.\lambda)) \neq 0$. Therefore $\mathrm{Hom}_{\mathcal{W}}(\mathbf{M}(\lambda_{\mathcal{W}}), \mathbf{M}((s_\beta.\lambda)_{\mathcal{W}})) \neq 0$ so $\lambda_{\mathcal{W}} = (s_\beta.\lambda)_{\mathcal{W}}$, because $\lambda_{\mathcal{W}}$ is weakly admissible. This implies $s_\beta.\lambda - \lambda \in \mathbb{C}\delta$, which is impossible, or $s_\beta.\lambda - s_0.\lambda \in \mathbb{C}\delta$. Since $(s_\beta.\lambda + \hat{\rho}, s_\beta.\lambda + \hat{\rho}) = (s_0.\lambda + \hat{\rho}, s_0.\lambda + \hat{\rho})$ and $(s_\beta.\lambda + \hat{\rho}, \delta) \neq 0$, the formula $s_\beta.\lambda - s_0.\lambda \in \mathbb{C}\delta$ forces $s_\beta.\lambda = s_0.\lambda$, which contradicts to $s_\beta.\lambda > \lambda \geq s_0.\lambda$. $\qquad \square$

8.8.6. Retain notation of 2.3.4.

Lemma.

$$Adm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(Y \setminus Y_0) \subset \varphi_{\mathcal{W}}\big(\{\lambda \in Y \setminus Y_0 \cap wAdm_k | \; \mathbb{C}\,C(\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}\}\big),$$
$$Adm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(Y_0) \subset \varphi_{\mathcal{W}}\big(\{\lambda \in Y_0 \cap wAdm_k | \; \mathbb{C}\,C(\lambda) + \mathbb{C}\delta \supset (\mathbb{C}\hat{\Delta} \cap \alpha_0^\perp)\}\big).$$

Proof. Take $\nu \in Adm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(Y)$. By Lemma 8.8.5, $\nu = \lambda_{\mathcal{W}}$ for a weakly admissible $\lambda \in Y$. Then $(\lambda + \hat{\rho}, \alpha_0) \in \mathbb{Z}_{\geq 0}$ and so $\mathbf{L}(\lambda_{\mathcal{W}}) = H^0(L(s_0.\lambda))$.

Consider the case when $\lambda \in Y \setminus Y_0$ that is $(\lambda + \hat{\rho}, \alpha_0) \in \mathbb{Z}_{>0}$. Since $H^0(L(s_0.\lambda))$ is admissible, this module does not admits self-extensions and so $\mathrm{Im}\,\Upsilon_{L(s_0.\lambda)} \cap \hat{\mathfrak{h}}_0^* = \mathbb{C}\delta$, by Theorem 8.7.1. By Proposition 2.3.5, $\mathrm{Im}\,\Upsilon_{L(s_0.\lambda)}$ contain $C(s_0.\lambda)^\perp$ so $C(s_0.\lambda)^\perp \cap \hat{\mathfrak{h}}_0^* = \mathbb{C}\delta$ that is $\mathbb{C}\,C(s_0.\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}$. It is easy to see that $C(s_0.\lambda') \setminus \{\alpha_0\} = s_0(C(\lambda') \setminus \{\alpha_0\})$ for any λ'. The condition $(\lambda + \hat{\rho}, \alpha_0) \in \mathbb{Z}_{>0}$ gives $\alpha_0 \in C(\lambda), \alpha_0 \notin C(s_0.\lambda)$ that is $C(\lambda) = s_0(C(s_0.\lambda)) \cup \{\alpha_0\}$. Since $\mathbb{C}\,C(s_0.\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}$ we get $\mathbb{C}\,C(\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}$ as required.

Consider the case when $\lambda \in Y_0$ that is $s_0.\lambda = \lambda$. Using Theorem 8.7.1 we obtain $\mathrm{Im}\,\Upsilon_{L(\lambda)} \cap \hat{\mathfrak{h}}_0^* \subset \mathrm{span}\{\delta, \alpha_0\}$. By Proposition 2.3.5, $\mathrm{Im}\,\Upsilon_{L(\lambda)}$ contains $C(\lambda)^\perp$ so $\mathrm{Im}\,\Upsilon_{L(\lambda)} \cap \hat{\mathfrak{h}}_0^*$ contains $\big(C(\lambda) \cup \{\delta\}\big)^\perp$. Thus $\mathbb{C}C(\lambda) + \mathbb{C}\delta \supset (\mathbb{C}\hat{\Delta} \cap \alpha_0^\perp)$ as required. $\qquad \square$

8.8.7. *Proof of Proposition 8.8.1.* We rewrite (i) as follows

$$(36) \qquad \lambda_{\mathcal{W}} \in Adm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(\overline{Y}) \iff \lambda \in k\text{-}Adm \cap \overline{Y}.$$

Take $\lambda \in \overline{Y}$. Combining Lemma 8.8.5 and Theorem 8.7.1 we conclude that $\lambda_{\mathcal{W}} \in Adm_{\mathcal{W}}$ iff λ is weakly admissible and $\mathrm{Im}\,\Upsilon_{L(\lambda)} \cap \hat{\mathfrak{h}}_0^* = \mathbb{C}\delta$. Thus $\lambda_{\mathcal{W}} \in Adm_{\mathcal{W}}$ iff $\lambda \in k\text{-}Adm$. This establishes (36) and (i).

Rewrite (ii) in the form

$$(37) \qquad \lambda' \in Adm_{\mathcal{W}} \implies \exists \lambda \in \varphi_{\mathcal{W}}^{-1}(\lambda') \text{ s.t. } \lambda \in wAdm_k \; \& \; \mathbb{C}\Delta(\lambda) + \mathbb{C}\delta = \mathbb{C}\hat{\Delta}.$$

If $\lambda' \in Adm_{\mathcal{W}} \setminus \varphi_{\mathcal{W}}(Y)$ the existence of λ follows from (i) and Corollary 4.3.1; if $\lambda' \in Adm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(Y \setminus Y_0)$ this follows from Lemma 8.8.6 (because $C(\lambda) \subset \Delta(\lambda)$). Take $\lambda' \in Adm_{\mathcal{W}} \cap \varphi_{\mathcal{W}}(Y_0)$. By Lemma 8.8.6, $\lambda' = \varphi_{\mathcal{W}}(\lambda)$, where $\lambda \in Y_0 \cap wAdm_k$ is such that

$\mathbb{C}\Delta(\lambda) + \mathbb{C}\delta$ contains $\mathbb{C}\hat{\Delta} \cap \alpha_0^{\perp}$. Note that for $\lambda \in Y_0$ one has $\alpha_0 \in \Delta(\lambda), \alpha_0 \notin C(\lambda)$. Thus $\mathbb{C}\Delta(\lambda) + \mathbb{C}\delta$ contains $(\mathbb{C}\hat{\Delta} \cap \alpha_0^{\perp}) + \mathbb{C}\alpha_0 = \mathbb{C}\hat{\Delta}$. This establishes (ii).

Finally, (iii) follows from (ii) and Remark 4.2.4, and (iv) follows from (ii), Remark 4.2.4, and Corollary 4.4.1. $\square$

This completes the proof of Theorem 8.5.1. Now let us continue our study of admissible weights in $\mathfrak{h}_{\mathcal{W}}^*$.

8.8.8. Proposition.   *If a rational weakly admissible $\lambda \in \hat{\mathfrak{h}}_k^*$ is such that $\langle \lambda + \hat{\rho}, \beta^{\vee} \rangle > 0$ for each $\beta \in \Pi(\lambda) \setminus \{\alpha_0\}$, then $\lambda_{\mathcal{W}}$ is admissible.*

Proof. Set $B := \hat{\Delta}_+^{re} \setminus \{\alpha_0\} \times \mathbb{Z}_{>0}$, and let $B_{>i} \subset B$ (resp., $B_{\geq i}$) be the set of pairs $(\alpha, m) \in B$ such that $\langle \alpha, x \rangle > i$ (resp., $\langle \alpha, x \rangle \geq i$). The determinant formula in [KW3] (Thm. 7.2) for $\mathfrak{g}$ being Lie algebra can be rewritten in the following form: for $\nu \in Q_+$ one has

$$\det_{\nu}(\lambda_{\mathcal{W}}) = (k + h^{\vee})^M \prod_{(\alpha,m) \in B_{\geq 0}} \phi_{\alpha,m}(\lambda_{\mathcal{W}})^{\dim \mathcal{M}((s_{\alpha}.\lambda)_{\mathcal{W}})_{\lambda_{\mathcal{W}} - \nu}} \prod_{(\alpha,m) \in B_{>0}} \phi'_{\alpha,m}(\lambda_{\mathcal{W}})^{\dim \mathcal{M}((s_{\alpha}s_0.\lambda)_{\mathcal{W}})_{\lambda_{\mathcal{W}} - \nu}},$$

if $s_0.\lambda \neq \lambda$, and

$$\det_{\nu}(\lambda_{\mathcal{W}}) = (k + h^{\vee})^M \prod_{(\alpha,m) \in B_{\geq 0}} \phi_{\alpha,m}(\lambda_{\mathcal{W}})^{\dim \mathcal{M}((s_{\alpha}.\lambda)_{\mathcal{W}})_{\lambda_{\mathcal{W}} - \nu}},$$

if $s_0.\lambda = \lambda$, where $M \geq 0$, and $\phi_{\alpha,m}(\lambda_{\mathcal{W}}) = 0$ iff $(\lambda + \hat{\rho}, \alpha) = \frac{m}{2}(\alpha, \alpha)$, $\phi'_{\alpha,m}(\lambda_{\mathcal{W}}) = 0$ iff $(s_0.\lambda + \hat{\rho}, \alpha) = \frac{m}{2}(\alpha, \alpha)$.

One has $\phi'_{\alpha,m}(\lambda_{\mathcal{W}}) = 0$ iff $(\lambda + \hat{\rho}, s_0\alpha) = \frac{m}{2}(s_0\alpha, s_0\alpha)$ that is $\phi_{s_0\alpha,m}(\lambda_{\mathcal{W}}) = 0$. Furthermore, $(s_{\alpha}s_0.\lambda)_{\mathcal{W}} = (s_0s_{\alpha}s_0.\lambda)_{\mathcal{W}} = (s_{s_0\alpha}\lambda)_{\mathcal{W}}$, and $(\alpha, m) \in B_{>0}$ iff $(s_0\alpha, m) \in B_{<0}$. Hence the determinant formula for $s_0.\lambda \neq \lambda$ can be rewritten as

$$\det_{\nu}(\lambda_{\mathcal{W}}) = (k + h^{\vee})^M \prod_{\alpha \in \hat{\Delta}_+^{re} \setminus \{\alpha_0\}} \prod_{m=1}^{\infty} \phi_{\alpha,m}(\lambda_{\mathcal{W}})^{\dim \mathcal{M}((s_{\alpha}.\lambda)_{\mathcal{W}})_{\lambda_{\mathcal{W}} - \nu}}.$$

Let us make the following identifications

$$\mathfrak{h}^* = \{\xi \in \hat{\mathfrak{h}}^* | \ (\xi, \delta) = (\xi, \Lambda_0) = 0\}, \quad (\mathfrak{h}^f)^* = \{\xi \in \hat{\mathfrak{h}}^* | \ (\xi, \delta) = (\xi, \Lambda_0) = (\xi, \theta) = 0\}.$$

For $\lambda \in \hat{\mathfrak{h}}_k^*$ we write

$$\lambda = k\Lambda_0 + \langle \lambda, x \rangle \theta + \lambda^{\#}.$$

From [KW3] (Thm. 7.2) one has

$$(38) \quad \begin{aligned} &\phi_{\alpha,m}(\lambda_{\mathcal{W}} + t\mu) = \phi_{\alpha,m}(\lambda_{\mathcal{W}}) + t(\mu|_{\mathfrak{h}^f}, \alpha), && \text{if } \langle \alpha, x \rangle = 0, \\ &\phi_{\alpha,m}(\lambda_{\mathcal{W}} + t\mu) = \phi_{\alpha,m}(\lambda_{\mathcal{W}}) + ta(\mu, \lambda, \alpha) \mod t^2, && \text{if } \langle \alpha, x \rangle = \pm\frac{1}{2}, \pm 1, \\ &a(\mu, \lambda, \alpha) := \langle \mu, L_0 \rangle + \tfrac{((\lambda + \rho, \theta) + k + h^{\vee})(\mu|_{\mathfrak{h}^f}, \alpha) + (\mu|_{\mathfrak{h}^f}, \lambda^{\#} + \rho^{\#})}{(k + h^{\vee})}, \end{aligned}$$

where we view $\mu|_{\mathfrak{h}^f} \in (\mathfrak{h}^f)^*$ as a vector in $\hat{\mathfrak{h}}^*$ using the above identification.

Let $\lambda \in \hat{\mathfrak{h}}_k^*$ satisfy the assumptions of our proposition. If $\alpha_0 \not\in \Pi(\lambda)$, then λ is weakly admissible, rational and regular, so λ is KW-admissible and $(\lambda, \alpha_0) \not\in \mathbb{Z}$. In this case $\lambda_{\mathcal{W}}$ is admissible by Proposition 8.8.1.

From now on we assume that $\alpha_0 \in \Pi(\lambda)$. By Lemma 8.8.5, $\lambda_{\mathcal{W}}$ is weakly admissible, so it is enough to verify that $\mathrm{Ext}^1(\mathbf{L}(\lambda_{\mathcal{W}}), \mathbf{L}(\lambda_{\mathcal{W}})) = 0$.

For each $\beta \in \Pi(\lambda) \setminus \{\alpha_0\}$ one has $(\beta, \alpha_0) = -(\beta, \theta) \le 0$ (by 3.3.4) and so $\langle \beta, x \rangle \ge 0$ that is $\langle \beta, x \rangle = 0, \frac{1}{2}, 1$, by 8.1.1. Write

$$\Pi(\lambda) = \{\alpha_0\} \cup \Pi(\lambda)_0 \cup \Pi(\lambda)_{\frac{1}{2}} \cup \Pi(\lambda)_1, \text{ where } \Pi(\lambda)_j = \{\beta \in \Pi(\lambda) | \langle \beta, x \rangle = j\}.$$

We will use the following fact (see [KT]): if a non-critical weight $\lambda' \in' \hat{\mathfrak{h}}^*$ is maximal in its $W.$-orbit, then $\mathrm{Stab}_{W(\lambda')}(\lambda' + \hat{\rho}) = \mathrm{Stab}_{\hat{W}}(\lambda' + \hat{\rho})$ is a finite Coxeter group generated by the reflections $s_\alpha, \alpha \in \Pi(\lambda')$ such that $(\lambda' + \hat{\rho}, \alpha) = 0$, and $[M(y.\lambda') : L(w.\lambda')] \ne 0$ implies the existence of $z \in \mathrm{Stab}_W(\lambda' + \hat{\rho})$ such that $y \le_{\lambda'} wz$, where $\le_{\lambda'}$ stands for the Bruhat order in the Coxeter group $W(\lambda')$.

Assume that

(39) $\qquad \beta \in \Pi(\lambda) \setminus \{\alpha_0\}, \ \alpha \in \hat{\Delta}_+^{re} \setminus \{\alpha_0\}$ s.t. $[\mathbf{M}((s_\alpha.\lambda)_{\mathcal{W}}) : \mathbf{L}((s_\beta.\lambda)_{\mathcal{W}})] \ne 0$.

Then, by Corollary 8.3.4, $[M(s_\alpha.\lambda) : L(w.\lambda)] \ne 0$, where $w \in W(\lambda)$ is such that $(w.\lambda)_{\mathcal{W}} = (s_\beta.\lambda)_{\mathcal{W}}$, that is $w.\lambda = s_\beta.\lambda$ or $w.\lambda = s_0 s_\beta.\lambda$. By above, $s_\alpha \le_\lambda wz$, where $w \in \{s_\beta, s_0 s_\beta\}$, $z \in \mathrm{Stab}_{W(\lambda)}(\lambda + \hat{\rho})$, and $\mathrm{Stab}_{W(\lambda)}(\lambda + \hat{\rho})$ is generated by s_0 if $(\lambda + \hat{\rho}, \alpha_0) = 0$ and is trivial otherwise. Since $\beta, \alpha_0 \in \Pi(\lambda)$, $s_\alpha \le_\lambda s_\beta$ forces $\alpha = \beta$; $s_\alpha \le_\lambda s_\beta s_0$ or $s_\alpha \le_\lambda s_0 s_\beta$ forces $\alpha = \beta, \alpha_0$; $s_\alpha \le_\lambda s_0 s_\beta s_0$ forces $\alpha = \beta, \alpha_0, s_0\beta$. Therefore (39) implies $\alpha = \beta$ if $(\lambda + \hat{\rho}, \alpha_0) \ne 0$, and $\alpha \in \{\beta, s_0\beta\}$ if $(\lambda + \hat{\rho}, \alpha_0) = 0$. Since $\langle \beta, x \rangle \ge 0$, one has $\langle s_0\beta, x \rangle \ge 0$ iff $s_0\beta = \beta$.

Take $\mu \in \mathfrak{h}_{\mathcal{W}}^*$ and let $\{\mathbf{M}(\lambda_{\mathcal{W}})^{\mu;j}\}$ be the corresponding Jantzen-type filtration, see Subsection 1.5. Combining the above determinant formulas and the fact that (39) implies $\alpha = \beta$ if $(\lambda + \hat{\rho}, \alpha_0) \ne 0$ or $(\lambda + \hat{\rho}, \alpha_0) = 0$, $(\alpha, m) \in B_{\ge 0}$, we obtain that

(40) $$\sum_{j=1}^{\infty} [\mathbf{M}(\lambda_{\mathcal{W}})^{\mu;j} : \mathbf{L}((s_\beta.\lambda)_{\mathcal{W}})] = 1,$$

if $\beta \in \Pi(\lambda)_0, \ (\mu|_{\mathfrak{h}^f}, \beta) \ne 0$ or $\beta \in \Pi(\lambda)_j, \ a(\mu, \lambda, \beta) \ne 0, j > 0$.

Take $\mu \in \mathrm{Im}\, \Upsilon_{\mathbf{L}(\lambda_{\mathcal{W}})}$ and set $\xi := \mu|_{\mathfrak{h}^f}$. Let us show that $\xi = 0$. By Proposition 1.6.5, $\mathbf{M}(\lambda_{\mathcal{W}})^{\mu;1} = \mathbf{M}(\lambda_{\mathcal{W}})^{\mu;2}$ so $\sum_{j=1}^{\infty} [\mathbf{M}(\lambda_{\mathcal{W}})^{\mu;j} : \mathbf{L}((s_\beta.\lambda)_{\mathcal{W}})] \ne 1$. Using (40) and (38), we conclude that $(\xi, \beta) = 0$ for all $\beta \in \Pi(\lambda)_0$ and $a(\mu, \lambda, \beta) = 0$ for each $\beta \in \Pi(\lambda)_j, j > 0$. By (38),

$$a(\mu, \lambda, \alpha) = a(\mu, \lambda, \theta) + \left(\frac{(\lambda + \rho, \theta)}{k + h^\vee} + 1 \right)(\xi, \alpha)$$

Hence

$$(41) \qquad \begin{aligned} &\forall \beta \in \Pi(\lambda)_0 \quad (\xi, \beta) = 0 \\ &\forall \beta \in \Pi(\lambda)_j, j > 0 \quad a(\mu, \lambda, \theta) + (\tfrac{(\lambda+\rho,\theta)}{k+h^\vee} + 1)(\xi, \beta) = 0. \end{aligned}$$

We claim that $(\lambda + \rho, \theta) + k + h^\vee \neq 0$. Indeed,

$$(\lambda + \rho, \theta) + k + h^\vee = (\lambda + \hat{\rho}, \delta + \theta) = 2(k + h^\vee) - (\lambda + \hat{\rho}, \alpha_0).$$

If $(\lambda + \hat{\rho}, \alpha_0) \neq 0$, then, by above, $\lambda + \hat{\rho}$ has a trivial stabilizer in $\hat{W}$ so $(\lambda + \hat{\rho}, \delta + \theta) \neq 0$. On the other hand, if $(\lambda + \hat{\rho}, \alpha_0) = 0$, then $(\lambda + \rho, \theta) + k + h^\vee = 2(k + h^\vee) \neq 0$.

Combining the inequality $(\lambda + \rho, \theta) + k + h^\vee \neq 0$ and (41), we conclude that for $a := -\frac{a(\mu,\lambda,\theta)(k+h^\vee)}{(\lambda+\rho,\theta)+k+h^\vee}$ one has $(\xi, \beta) = a$ for all $\beta \in \Pi(\lambda)_{\frac{1}{2}}$. Let us show that $\xi - a\theta$ is orthogonal to $\Pi(\lambda) \setminus \{\alpha_0\}$. Indeed, since $(\xi, \beta) = a$ for all $\beta \in \Pi(\lambda)_{\frac{1}{2}}$, $\xi - a\theta$ is orthogonal to $\Pi(\lambda)_{\frac{1}{2}}$. Moreover, by (41), ξ is orthogonal to $\Pi(\lambda)_0$ so $\xi - a\theta$ is orthogonal to $\Pi(\lambda)_0$. Thus $\xi - a\theta$ is orthogonal to $\Pi(\lambda) \setminus \{\alpha_0\}$, provided that $\Pi(\lambda)_1$ is empty. Consider the case when $\Pi(\lambda)_1 \neq \emptyset$. The root $\beta \in \Pi(\lambda)_1$ is of the form $m\delta + \theta$; one has $(\xi, \beta) = 0$ since $\xi \in (\mathfrak{h}^f)^*$. Hence $a(\mu, \lambda, \beta) = a(\mu, \lambda, \theta) = 0$ so $a = 0$, and thus $\xi = \xi - a\theta$ is orthogonal to β. Hence $\xi - a\theta$ is orthogonal to $\Pi(\lambda) \setminus \{\alpha_0\}$ as required.

Since $\alpha_0 = \delta - \theta \in \Delta(\lambda)$ and $k + h^\vee = \frac{p}{q}$ is rational, $(q-1)\delta + \theta \in \Delta(\lambda)$ so

$$(42) \qquad (q - 1)\delta + \theta = x_0(\delta - \theta) + \sum_{\beta \in \Pi(\lambda)\setminus\{\alpha_0\}} x_\beta \beta$$

for some $x_0, x_\beta \in \mathbb{Z}_{\geq 0}$. Thus $(q - 1 - x_0)\delta + (1 + x_0)\theta = \sum x_\beta \beta$ and so, by above, $\xi - a\theta$ is orthogonal to $(q - 1 - x_0)\delta + (1 + x_0)\theta$. Since $\xi - a\theta$ lies in $\mathfrak{h}^*$, it is orthogonal to δ. Therefore $\xi - a\theta$ is orthogonal to θ (because $1 + x_0 \neq 0$) and thus is orthogonal to α_0. Hence $\xi - a\theta$ is orthogonal to $\Pi(\lambda)$. The rationality of λ gives $\mathbb{C}\Pi(\lambda) = \mathbb{C}\hat{\Delta}$ so $\xi - a\theta \in \mathbb{C}\delta$. Taking into account that $\xi \in (\mathfrak{h}^f)^*$ is orthogonal to θ and Λ_0, we conclude that $\mu|_{\mathfrak{h}^f} = \xi = 0$.

From (42) it follows that $\Pi(\lambda) \neq \Pi(\lambda)_0 \cup \{\alpha_0\}$, so, by (41), $a(\mu, \lambda, \beta) = 0$ for some β. Substituting $\xi = 0$ to (38), we obtain $\langle \mu, L_0 \rangle = 0$. By above, $\mu|_{\mathfrak{h}^f} = 0$, so $\mu = 0$, that is $\mathrm{Ext}^1(\mathbf{L}(\lambda_\mathcal{W}), \mathbf{L}(\lambda_\mathcal{W})) = 0$ as required. $\qquad\square$

8.8.9. **Corollary.** *If $\lambda \in \hat{\mathfrak{h}}_k^*$ is KW-admissible weight, then $\lambda_\mathcal{W}$ is admissible.*

8.8.10. Recall that the Virasoro vertex algebra is isomorphic to $\mathcal{W}^k(\mathfrak{sl}_2, f)$, where f is a non-zero nilpotent element of $\mathfrak{sl}_2$, and that its central charge $c(k)$ is given by formula (20). Here we show how to recover some results of Section 5 using the theory of W-algebras. Retain notations of 4.4.6. One has $\varphi(\lambda_{r,s}) = (h_{r,s}^{p,q}, c^{p,q})$. Recall that $B = \{\Gamma_r\}_{r=1}^q$, $X_{\Gamma_r} = \{\lambda_{r,s}\}_{s=0}^p$.

Note that $\alpha_0^\vee \in \Gamma_r$ only for $r = q$. Thus, by above, $\varphi(\lambda_{r,s})$ is admissible if $r = 1, \ldots, q - 1, s = 0, \ldots, p$. For $r = q$ the KW-admissible weights are $\lambda_{q,s}$ with $s = 1, \ldots, p - 1$.

The face $(\lambda + \rho, \alpha_0) = 0$ of the polyhedron X_{Γ_q} is $\lambda_{q,p}$. By Proposition 8.8.8, $\varphi(\lambda_{q,s})$ is admissible if $s = 1, \ldots, p - 1$ and if $(r, s) = (q, p)$.

Summarizing, we get that $\varphi(\lambda_{r,s}) = (h_{r,s}^{p,q}, c^{p,q})$ is admissible if $r = 1, \ldots, q - 1, s = 0, \ldots, p$ or $r = q, s = 1, \ldots, p$. Taking into account that $h_{r,s}^{p,q} = h_{q-r,p-s}^{p,q}$, we see that the admissible weights lie in the set $\{(h_{r,s}^{p,q}, c^{p,q})\}_{r=0,\ldots,q,s=0,\ldots,p}$ and that $(h_{r,s}^{p,q}, c^{p,q})$ is admissible if $r = 0, \ldots, q, s = 0, \ldots, p$, and $(r, s) \neq (0, p), (q, 0)$. By Corollary 5.3.2, the latter set is exactly the set of admissible weights. Thus, we recover this corollary, except for the proof that points $(0, p), (q, 0)$ are not admissible. One can treat similarly the Neveu-Schwarz algebra by taking $\mathfrak{osp}(1, 2)$, recovering thereby Corollary 6.4.

8.9. Example. We give an example of $N \in \mathcal{O}_k$ with the image $H^0(N)$ which does not lie in $\mathcal{O}$-category for $\mathcal{W}$; in fact the image $H^0(N)$ is a non-splitting extension of a Verma $\mathcal{W}$-module by itself.

Take $\mathfrak{g} := \mathfrak{sl}_2$ and $k \neq -2$. In this case $\mathcal{W}$ is a Virasoro algebra $\mathcal{V}ir^c = \mathcal{V}ir/(C - c)$ for $c = 1 - \frac{6(k+1)^2}{k+2}$. Let $N \in \mathcal{O}_k$ be a non-splitting extension of $M(k\omega)$ by $M(k\omega - \alpha_0)$, where $\omega \in \hat{\mathfrak{h}}$ is given by

$$\langle \omega, D \rangle = (\omega, \alpha_0) = 0, \quad (\omega, \alpha) = 1;$$

it is easy to see that such extension exists if $M(k\omega - \alpha_0)$ is irreducible: it is a submodule of $M(k\omega - \Lambda_0) \otimes L(\Lambda_0)$ generated by $v \otimes w$, where v is a highest weight vector of $M(k\omega - \Lambda_0)$ and $w \in L(\Lambda_0)_{s_0\Lambda_0}$.

We show below that $H^0(N)$ is a non-splitting extension of a Verma module $\mathbf{M}((k\omega)_{\mathcal{W}})$ over $\mathcal{V}ir^c$ by itself (such extension is unique up to an isomorphism). Since L_0 does not act diagonally on $H^0(N)$, $H^0(N)$ does not lie in $\mathcal{O}$-category for the Virasoro algebra. Notice that $\mathbf{M}((k\omega)_{\mathcal{W}})$ is irreducible if k is not rational.

8.9.1. *Definition of H^i in the case $\mathfrak{g} = \mathfrak{sl}_2$.* Denote by f, h, e the standard basis of $\mathfrak{sl}_2$. Then $\hat{\mathfrak{sl}}_2$ is generated by $f, h = \alpha^\vee, e, f_0 := et^{-1}, e_0 := ft, \alpha_0^\vee = [e_0, f_0], K, D$.

Define a Clifford algebra generated by the odd elements ψ_n, ψ_n^* with $n \in \mathbb{Z}$ subject to the relations $[\psi_n, \psi_m] = [\psi_n^*, \psi_m^*] = 0, [\psi_n, \psi_m^*] = \delta_{m,n}$. Take a vacuum module F^{ch} over this algebra generated by $|0\rangle$ such that $\psi_n|0\rangle = 0$ for $n \geq 0, \psi_n^*|0\rangle = 0$ for $n > 0$. We define a $\mathbb{Z}$-grading on $F^{ch} = \sum (F^{ch})^i$ by the assignment $\deg |0\rangle = 0, \deg \psi_n = -1, \deg \psi_n^* = 1$. For a $\hat{\mathfrak{g}}$-module V we set $C(V) := V \otimes F^{ch}$ and define a $\mathbb{Z}$-grading by the assignment $C(V)^i := V \otimes (F^{ch})^i$. Finally, we define $d : C(V) \to C(V)$ by

$$d = \sum_{n \in \mathbb{Z}} et^{-n} \otimes \psi_n^* + \psi_1^*.$$

Then d is odd, $d^2 = 0$ and $d(C(V)^i) \subset C(V)^{i+1}$. By definition,

$$H^i(V) := \operatorname{Ker} d \cap C(V)^i / \operatorname{Im} d \cap C(V)^i.$$

Recall that for $V \in \mathcal{O}_k$ one has $H^i(V) = 0$ for $i \neq 0$.

Define a semisimple action of $\hat{\mathfrak{h}}$ on F^{ch} by assigning to ψ_n the weight $\alpha + n\delta$, to ψ_n^* the weight $-\alpha + n\delta$ and to $|0\rangle$ the zero weight. This induces an action of $\hat{\mathfrak{h}}$ on the tensor product $C(V) = V \otimes F^{ch}$. If $\hat{\mathfrak{h}}$ acts locally finitely on V, then $\hat{\mathfrak{h}}$ acts locally finitely on $C(V)$ and

$$d(C(V)_\nu) \subset C(V)_\nu + C(V)_{\nu+\alpha_0}.$$

Let $\hat{\Omega}$ be the Casimir operator on $V \in \mathcal{O}_k$; define the action of $\hat{\Omega}$ on $C(V)$ by

$$\hat{\Omega}(v \otimes u') := \hat{\Omega}v \otimes u, \ v \in V, u \in F^{ch}.$$

The action of L_0 on $H(V)$ is given by

$$L_0 = \frac{\hat{\Omega}}{2(k+2)} - (\frac{\alpha^\vee}{2} + D).$$

8.9.2. Set $\mathbf{N} := H^0(N)$. Since N is an extension of $M(k\omega)$ by $M(k\omega - \alpha_0)$, $\mathbf{N}$ is an extension of $M((k\omega)_{\mathcal{W}})$ by itself. Let us show that this extension is non-splitting.

Let E be the highest weight space of $\mathbf{N}$, i.e. $E := \mathbf{N}_{(k\omega)_{\mathcal{W}}}$. Then E is two-dimensional. Set $X := \sum_j C(N)_{k\omega - j\alpha_0}$ and note that $d(X) \subset X$. It is easy to see that E is the image of X in $\mathbf{N} = H^0(N) = \operatorname{Ker} d / \operatorname{Im} d$, i.e. $E = (\operatorname{Ker} d \cap X)/(\operatorname{Im} d \cap X)$.

Denote by v the highest weight generator of $M(k\omega) \subset N$, and by v' a vector in N satisfying $e_0 v' = v$. One readily sees that X is spanned by

$$x_n := f_0^n v \otimes |0\rangle, \quad x_n' := f_0^n v' \otimes |0\rangle, \quad y_n := f_0^n v \otimes \psi_{-1}|0\rangle, \quad y_n' := f_0^n v' \otimes \psi_{-1}|0\rangle, \ n \geq 0.$$

One has

$$d(x_n) = d(x_n') = 0, \quad d(y_n) = -x_n - x_{n+1}, \ d(y_n') = -x_n' - x_{n+1}'.$$

Thus $E = (\operatorname{Ker} d \cap X)/(\operatorname{Im} d \cap X)$ is spanned by the images $\overline{x}_0, \overline{x}_0'$.

The Casimir operator $\hat{\Omega}$ acts on $M(k\omega) \subset N$ by $a \cdot \operatorname{id}$ for some $a \in \mathbb{C}$ and one has $\hat{\Omega}v' = av' + f_0 v$. Therefore

$$L_0 \overline{x}_0 = b\overline{x}_0, \quad L_0 \overline{x}_0' = b\overline{x}_0' + \frac{\overline{x}_1}{2(k+2)} = b\overline{x}_0' - \frac{\overline{x}_0}{2(k+2)}, \text{ where } b := \frac{a}{2(k+2)} - \frac{1}{2}.$$

Hence L_0 does not act semisimply on the highest weight space of $\mathbf{N}$ and thus $\mathbf{N}$ is a non-splitting extension of $M((k\omega)_{\mathcal{W}})$ by itself.

8.10. **Example.** Below we give an example of $\mathbf{N} \in \tilde{\mathcal{O}}(\mathcal{W})$ which does not lie in $H^0(\mathcal{O}_k')$, see 8.3.2 for notation. This example is based on the following observation: if $M(\lambda)$ is an irreducible Verma module and $(\lambda + \hat{\rho}, \alpha_0) = 0$, then $\mathbf{M} := H^0(M(\lambda))$ is an irreducible Verma module over $\mathcal{W}$ and $Ext^1(\mathbf{M}, \mathbf{M}) \xrightarrow{\sim} \mathfrak{h}_{\mathcal{W}}^* \neq \operatorname{Im} \phi_\lambda$, by Theorem 8.7.1 (iii).

8.10.1. Fix $\lambda \in \hat{\mathfrak{h}}_k^*$ such that $(\lambda + \hat{\rho}, \alpha_0) = 0$ and $(\lambda + \hat{\rho} + \mu/2, \mu) \neq 0$ for all non-zero μ in the root lattice of $\hat{\mathfrak{g}}$ (such λ exists for $k \notin \mathbb{Q}$); this condition ensures that the Casimir operator have different eigenvalues on $M(\lambda)$ and on $M(\lambda + \mu)$. In particular, $M(\lambda + s\delta)$ is irreducible. Moreover, if an indecomposable module $N \in \mathcal{O}_k'$ has an irreducible subquotient isomorphic to $M(\lambda + s\delta)$, then all irreducible subquotients of N are isomorphic.

Set $\mathbf{M} := H^0(M(\lambda))$. Since $\mathbf{M}$ is a Verma module one has $\operatorname{Im} \Upsilon_{\mathbf{M}} = \mathfrak{h}_{\mathcal{W}}^*$. Notice that $\operatorname{Im} \phi_\lambda \neq \mathfrak{h}_{\mathcal{W}}^*$. We claim that if $\mathbf{N}$ is an extension of $\mathbf{M}$ by $\mathbf{M}$, then $\mathbf{N} \in H^0(\mathcal{O}_k')$ iff $\Upsilon_{\mathbf{M}}(\mathbf{N}) \subset \operatorname{Im} \phi_\lambda$. In particular, any extension which does not correspond to an element in $\operatorname{Im} \phi_\lambda$ does not lie in $H^0(\mathcal{O}_k')$.

Let $\mathbf{N}$ be a non-splitting extension of $\mathbf{M}$ by $\mathbf{M}$ and $\mathbf{N} \in H^0(\mathcal{O}_k')$. Then $\mathbf{N} = H^0(N)$ for some indecomposable module $N \in \mathcal{O}_k'$. Combining Lemma 8.3.3 and the assumption $(\lambda + \hat{\rho}, \alpha_0) = 0$, we conclude that $H^0(L(\lambda')) \cong \mathbf{M}$ iff $\lambda' = \lambda + s\delta$. Since $N \in \mathcal{O}_k'$, N has a finite filtration with the factors belonging to $\mathcal{O}_k$. Since $H^0(N) = \mathbf{N}$, N has a subquotient of the form $M(\lambda + s\delta)$. Then, by above, all irreducible subquotients of N are isomorphic to $M(\lambda + s\delta)$ and $[N : M(\lambda + s\delta)] = 2$. Hence N is an extension of $M(\lambda + s\delta)$ by itself and, by Theorem 8.7.1 (i), $\Upsilon_{\mathbf{M}}(\mathbf{N}) = \phi_{\lambda + s\delta}(\Upsilon_{M(\lambda + s\delta)}(N))$. One readily sees that $\phi_{\lambda + s\delta} = \phi_\lambda$ so $\Upsilon_{\mathbf{M}}(\mathbf{N}) \in \operatorname{Im} \phi_\lambda$. This establishes the claim.

8.11. **Example.** Let us give an example of an admissible $\mathcal{W}$-module $\mathbf{L}$, which does not admit a $\mathbb{R}_{\geq 0}$-grading $\mathbf{L} = \oplus_{j \geq 0} \mathbf{L}_j$, compatible with the grading on $\mathcal{W}$ (i.e., $\mathcal{W}_n \mathbf{L}_j \subset \mathbf{L}_{j-n}$) such that all $\mathbf{L}_j$ are finite-dimensional.

8.11.1. *Description of* $\mathbf{L}$. Let $\mathfrak{g} := \mathfrak{sl}(4)$. Set $\Pi = \{\alpha_1, \alpha_2, \alpha_3\}$, fix the standard bases $\{\alpha_i^\vee\}_{i=1}^3$ in $\mathfrak{h}$ and $\{e_\beta\}_{\beta \in \Delta}$ in $\mathfrak{n}_- \oplus \mathfrak{n}_+$. Then $\theta = \alpha_1 + \alpha_2 + \alpha_3$, $x = (\alpha_1^\vee + \alpha_2^\vee + \alpha_3^\vee)/2$ and $\mathfrak{g}^\natural \cong \mathfrak{gl}(3)$ has the following triangular decomposition $\mathbb{C}e_{-\alpha_2} \oplus (\mathbb{C}\alpha_2^\vee + \mathbb{C}(\alpha_1^\vee - \alpha_3^\vee)) \oplus \mathbb{C}e_{\alpha_2}$.

Set
$$\Pi' := \{2\delta - \theta; 3\delta + \alpha_1; 4\delta + \alpha_2; 4\delta + \alpha_3\} \subset \hat{\Delta}_+$$
and take $\lambda \in \hat{\mathfrak{h}}^*$ such that $(\lambda + \hat{\rho}, \beta) = 1$ for any $\beta \in \Pi'$. It is easy to check that $k := \langle \lambda, K \rangle = 4/13 - 4$, that λ is a KW-admissible weight with $\Pi(\lambda) = \Pi'$. Set $M := M(\lambda)$ and let M' be the maximal proper submodule of M. Note that

(43) $$\langle \beta, x + D \rangle > 0 \quad \text{for any } \beta \in \Pi'.$$

By Corollary 8.8.9, $\mathbf{L} := H^0(L(\lambda)) \cong H^0(M)/H^0(M')$ is an admissible $W^k(\mathfrak{g}, e_{-\theta})$-module.

8.11.2. Let $v \in \mathbf{M}$ (resp. $\overline{v} \in \mathbf{L}$) be a highest weight vector. Set
$$y := J_0^{e_{-\alpha_2}} \in \mathcal{U}(\mathcal{W}).$$
Observe that v generates a free module over $\mathbb{C}[y]$. Let us show that $\overline{v} \in \mathbf{L}$ also generates a free module over $\mathbb{C}[y]$. Indeed, for any $\lambda' \in \hat{\mathfrak{h}}^*$ one has

$[\mathbf{M}' : \mathbf{L}(\lambda'_{\mathcal{W}})] = [M' : L(\lambda')]$ by (H4); $\langle \lambda'_{\mathcal{W}} - \lambda_{\mathcal{W}}, L_0 \rangle = \langle \lambda - \lambda', x + D \rangle$, by (H2).

Note that $[M' : L(\lambda')] > 0$ implies that $\lambda - \lambda'$ is a non-negative integral linear combination of elements in Π' so, by (43), $\langle \lambda - \lambda', x + D \rangle > 0$. Therefore

$$[\mathbf{M}' : \mathbf{L}(\lambda'_{\mathcal{W}})] > 0 \implies \langle \lambda'_{\mathcal{W}} - \lambda_{\mathcal{W}}, L_0 \rangle > 0.$$

As a result, for any $\mu \in \Omega(\mathbf{M}')$ one has $\langle \mu - \lambda_{\mathcal{W}}, L_0 \rangle > 0$. Since $[L_0, y] = 0$, this means that $\mathbb{C}[y]v$ intersects $\mathbf{M}'$ trivially and thus $\overline{v} \in \mathbf{L}$ generates a free module over $\mathbb{C}[y]$.

8.11.3.   Suppose that $\mathbf{L}$ has such a grading. Let us show that $\dim \mathbf{L}_i = \infty$ for some i.

Write $\overline{v} = \sum_{i=1}^m v_i$, where v_i are homogeneous vectors. By above, $\overline{v}$ generates a free module over $\mathbb{C}[y]$. If some v_i generates a free module over $\mathbb{C}[y]$, then its homogeneous component is infinite-dimensional (since y has zero degree). If this is not the case, then for each v_i there exists a non-zero polynomial $P_i(y)$ such that $P_i(y)v_i = 0$. But then $\prod P_i(y)v = 0$, a contradiction.

9. A CONJECTURE ON SIMPLE W-ALGEBRAS

Recall that to any finite-dimensional simple Lie algebra $\mathfrak{g}$, a nilpotent element f of $\mathfrak{g}$ and $k \in \mathbb{C}$ one associates the W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$, which is a $\frac{1}{2}\mathbb{Z}_{\geq 0}$-graded vertex algebra [KRW], [KW3]. Provided that $k \neq -h^\vee$, the grading is the eigenspace decomposition with respect L_0, the 0th coefficient of a Virasoro filed $L(z)$, and the 0th eigenspace of L_0 is $\mathbb{C}|0\rangle$. It follows that the vertex algebra $\mathcal{W}^k(\mathfrak{g}, f)$ has a unique simple quotient, denoted by $\mathcal{W}_k(\mathfrak{g}, f)$.

One can define highest weight modules and Verma modules over $\mathcal{W}^k(\mathfrak{g}, f)$ [KW3] and, in the same way as in Section 8 for $\mathcal{W} = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$, one can define weakly admissible and admissible (irreducible highest weight) modules over $\mathcal{W}^k(\mathfrak{g}, f)$.

9.1. **Conjecture.** Suppose that the vertex algebra $\mathcal{W}_k(\mathfrak{g}, f)$ satisfies the C_2 condition. Then any irreducible $\mathcal{W}_k(\mathfrak{g}, f)$-module is obtained by pushing down from an admissible $\mathcal{W}^k(\mathfrak{g}, f)$-module.

9.2.   Suppose that $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ satisfies the C_2 condition. Let $\mathbf{L}$ be an irreducible $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$-module; then it is, of course, a $\mathcal{W}$-module. It follows from Lemma 2.4 of [M] that the eigenvalues of L_0 in $\mathbf{L}$ are bounded from the below and each eigenspace is finite-dimensional. Retain notation of Subsection 8.1.3. The elements $\{J_0^a, a \in ((\mathfrak{n}_+ \cap \mathfrak{g}_0) + \mathfrak{h}^f)\}$ span a Lie subalgebra $\mathfrak{p} \subset \mathcal{U}(\mathcal{W})$ which is isomorphic to $(\mathfrak{n}_+ \cap \mathfrak{g}_0) + \mathfrak{h}^f$. These elements preserve the eigenspaces of L_0 in $\mathbf{L}$. Since the Lie algebra $(\mathfrak{n}_+ \cap \mathfrak{g}_0) + \mathfrak{h}^f$ is solvable, each eigenspace contains a $\mathfrak{p}$-eigenvector. Let v be a $\mathfrak{p}$-eigenvector of the minimal L_0-eigenvalue. Then $J_i^b v = 0$ for each b and $i > 0$, since v has the minimal L_0-eigenvalue, $J_0^a v = 0$ if $a \in (\mathfrak{n}_+ \cap \mathfrak{g}_0)$, and $\mathfrak{h}_{\mathcal{W}} = \text{span}\{J_0^a, a \in \mathfrak{h}^f\} + \mathbb{C}L_0$ acts diagonally on v. Hence v is a highest weight vector and so $\mathbf{L}$ is a highest weight module. So, if $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ satisfies the C_2 condition, then any its irreducible module is a highest weight module. If, in addition,

this module is admissible (i.e. Conjecture 9.1 holds), it follows from Theorem 8.5.1 that $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ is a regular vertex algebra.

We conjecture that a theorem, similar to Theorem 8.5.1 can be established for any W-algebra $\mathcal{W}^k(\mathfrak{g}, f)$. Then Conjecture 9.1 would imply that any W-algebra $\mathcal{W}_k(\mathfrak{g}, f)$ satisfying C_2 condition, is regular.

References

[AM] D. Adamović, A. Milas *Vertex operator algebras associated to modular invariant representations for $A_1^{(1)}$*, Math. Res. Lett. **2** (1995), 563–575.

[Ar] T. Arakawa, *Representation theory of superconformal algebras and the Kac-Roan-Wakimoto conjecture*, Duke Math. J., **130** (2005), 435–478.

[Ast] A. Astashkevich, *On the structure of Verma modules over Virasoro and Neveu-Schwarz algebras*, Commun. in Math. physics, **186** (1997), 531–562.

[BB] A. Beilinson, J. Bernstein, *A proof of Jantzen conjectures*, I. M. Gelfand Seminar, 1–50, Adv. Soviet Math. 16, Part 1, AMS, Providence, RI, 1993.

[BPZ] A. A. Belavin, A. M. Polyakov, A. B. Zamolodchikov, *Infinite conformal symmetry in two-dimensional quantum field theory*, Nuclear Physics B **241** (1984), 333-380.

[DGK] V. V. Deodhar, O. Gabber, V. G. Kac, *Structures of some categories of representations of infinite-dimensional Lie algebras*, Adv. Math., **45** (1982), 92–116.

[DLM] Ch. Dong. H. Li, G. Mason, *Regularity of rational vertex operator algebras*, Adv. Math., **132** (1997), 148-166.

[FZ] I. B. Frenkel, Y. Zhu, *Vertex operator algebras associated to representations of affine and Virasoro algebras*, Duke Math. J. **66** (1992), 123–168.

[GK] M. Gorelik, V. G. Kac, *On simplicity of vacuum modules*, Adv. Math. **211**, (2007), 621–677.

[IK1] K. Iohara, Y. Koga, *Fusion algebras for N=1 superconformal field theories through coinvariants I: $\mathfrak{osp}(1|2)$-symmetry*, Journal für die reine und angewandte Mathematik **531**, (2001), 1–34.

[IK2] K. Iohara, Y. Koga, *Representation Theory of Neveu-Schwarz and Ramond algebra I: Verma modules*, Adv. in Math. **178** (2003), 1–65.

[Jan] J.-C. Jantzen, *Kontravariante Formen auf induzierten Darstellungen halbeinfacher Lie-Algebren*, Math. Ann. **226** (1977), 53–65.

[J] A. Joseph, *Sur l'annulateur d'un module de Verma*, in NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., **514** Representation Theories and Algebraic Geometry, (Montreal, PQ, 1997), 237–300, Kluwer Acad. Publ., Dordrecht, 1998.

[K1] V. G. Kac, *Lie superalgebras*, Adv. in Math., **26** (1977), 8–96.

[K2] V. G. Kac, *Infinite-dimensional Lie algebras*, Cambridge University Press, 1990.

[K3] V. G. Kac, *Vertex algebras for beginners*, University Lecture Series, Vol. 10, AMS, 1998.

[KK] V. G. Kac, D. Kazhdan, *Structure of representations with highest weight of infinite-dimensional Lie algebras*, Adv. Math., **34** (1979), 97-108.

[KR] V. G. Kac, A. K. Raina, *Bombay lectures on highest weight representations of infinite dimensional Lie algebras*, Singapore: World Sci., 1987.

[KRW] V. G. Kac, S.-S. Roan and M. Wakimoto, *Quantum reduction for affine superalgebras*, Commun. Math. Phys. **241** (2003), 307–342.

[KW1] V. G. Kac, M. Wakimoto, *Modular invariant representations of infinite-dimensional Lie algebras and superalgebras*, Proc. Natl. Acad. Sci., USA, **85** (1998), 4956–4960.

[KW2] V. G. Kac, M. Wakimoto, *Classification of modular invariant representations of affine Lie algebras*, Adv. Series in Math. Physics, **7**, World Scientific, 1989.

[KW3]   V. G. Kac, M. Wakimoto, *Quantum reduction and representation theory of superconformal algebras*, Adv. in Math., **185**, (2004), 400–458.

[KW4]   V. G. Kac, M. Wakimoto, *On rationality of $\mathcal{W}$-algebras*, Transform. Groups, **13**, (2008), 671–713.

[KWa]   V. G. Kac, W. Wang, *Vertex operator superalgebras and their representations*, in Mathematical Aspects of Conformal and Topological Field Theories and Quantum Groups, Contemp. Math. **175**, AMS, Providence, Rhode Island, 1994, 161–192.

[KT]   M. Kashiwara, T. Tanisaki, *Characters of irreducible modules with non-critical highest weights over affine Lie algebras*, Representations and Quantizations, Shanghai (1998), 275–296, China High. Educ. Press, Beijing, 2000.

[L]   S. Lang, *Algebra*, Addison-Wesley, 1965.

[M]   M. Miyamoto, *Modular invariance of vertex operator algebras satisfying C_2-cofiniteness*, Duke Math. J., **122**, (2004), 51–91.

[MP]   R. V. Moody, A. Pianzola, *Lie algebras with triangular decompositions*, Canadian math. Soc. series of monographs and advanced texts, A Wiley-Interscience Publication, John Wiley & Sons, 1995.

[Sh]   N. Shapovalov, *On a bilinear form on the universal enveloping algebra of a complex semisimple Lie algebra*, Functional Anal. Appl. **6** (1972), p. 307–312 (in Russian).

[Zh]   Y. Zhu, *Modular invariance of characters of vertex operator algebras*, J. Amer. Math. Soc. **9** (1996), 237–302.

DEPT. OF MATHEMATICS, THE WEIZMANN INSTITUTE OF SCIENCE, REHOVOT 76100, ISRAEL

E-mail address: `maria.gorelik@weizmann.ac.il`

DEPT. OF MATHEMATICS, 2-178, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE, MA 02139-4307, USA

E-mail address: `kac@math.mit.edu`