1. Preliminaries

1.1. We start from a complex Kac-Moody Lie algebra $\mathfrak{g}$ with a symmetrizable Cartan matrix. Recall that such an algebra $\mathfrak{g}$ is generated by a commutative subalgebra (Cartan subalgebra) $\mathfrak{h}$ and symbols e_i and f_i ($i = 1, \dots, n = \dim \mathfrak{h}$). The algebra $\mathfrak{h}$ contains linearly independent elements $\alpha_i^\vee$ and the dual space $\mathfrak{h}^*$ contains linearly independent elements α_i satisfying the relations

$$[e_i, f_j] = \delta_{ij}\alpha_i^\vee,\ [h, e_i] = \alpha_i(h)e_i,\ [h, f_i] = -\alpha_i(h)f_i.$$

On $\mathfrak{h}^*$, there is the standard bilinear form $(\cdot,\cdot)$. An element $\xi \in \mathfrak{h}^*$ is called *isotropic* if $(\xi, \xi) = 0$. There is a decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+,$$

where $\mathfrak{n}^+$ (resp., $\mathfrak{n}^-$) is the subalgebra of $\mathfrak{g}$ generated by the e_i (resp., f_i). Denote by Δ^+ the multiset of positive roots: the multiplicity of α is $\dim \mathfrak{g}_\alpha$.

Set

$$Q := \sum_{i=1}^n \mathbb{Z}\alpha_i, \quad Q^+ := \sum_{i=1}^n \mathbb{Z}_{\geq 0}\alpha_i,$$

where $\mathbb{Z}_{\geq 0}$ stands for the set of non-negative integers. Denote by $\mathbb{Z}_{>0}$ the set of positive integers.

For a given $\mathfrak{g}$-module N, we denote by N_ν ($\nu \in \mathfrak{h}^*$) the corresponding weight space:

$$N_\nu := \{v \in N \mid hv = \nu(h)v \text{ for any } h \in \mathfrak{h}\}.$$

1.1.1. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the antiautomorphism (a llinear bijection satisfying $\sigma([x, y]) = [\sigma(y), \sigma(x)]$) given by

$$\sigma|_{\mathfrak{h}} = \mathrm{id}, \quad \sigma(e_i) = f_i, \sigma(f_i) = e_i.$$

Notice that σ interchanges $\mathfrak{n}^+$ and $\mathfrak{n}^-$ and $\sigma^2 = \mathrm{id}$,.

1.1.2. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The linear map σ can be uniquely extended to a linear bijection $U(\mathfrak{g}) \to U(\mathfrak{g})$ satisfying the rule $\sigma(xy) = \sigma(y)\sigma(x)$.

We identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$. The projection onto the first summand in the decomposition $U(\mathfrak{g}) = U(\mathfrak{h}) \oplus (U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g}))$ is called the *Harish-Chandra projection*: $\mathrm{HC} : U(\mathfrak{g}) \to S(\mathfrak{h})$. Notice that $\mathrm{HC} \circ \sigma = \mathrm{HC}$.

1.2. **Shapovalov forms.** The bilinear map $S : U(\mathfrak{g}) \times U(\mathfrak{g}) \to S(\mathfrak{h})$ defined by the formula

$$(a, b) \mapsto \mathrm{HC}(\sigma(a)b)$$

is called the *Shapovalov map* (see [Sh],[KK]).

1.2.1. The Shapovalov map is symmetric:

$$S(a,b)=S(b,a).$$

It has the following property known as *contravariance*:

$$S(ca,b)=S(a,\sigma(c)b).$$

For any $\lambda\in\mathfrak{h}^*$, denote by $S(\lambda)$ the *evaluated* Shapovalov map $S(\lambda):U(\mathfrak{g})\times U(\mathfrak{g})\to\mathbb{C}$ given by

$$(a,b)\mapsto \mathrm{HC}(\sigma(a)b)(\lambda).$$

1.2.2. Let $\mathbb{C}(\lambda)$ be a one-dimensional $\mathfrak{h}$-module corresponding to $\lambda\in\mathfrak{h}^*$. Let $M(\lambda)$ be the Verma module with highest weight λ that is

$$M(\lambda):=U(\mathfrak{g})\otimes_{U(\mathfrak{b})}\mathbb{C}(\lambda)$$

where $\mathbb{C}(\lambda)$ is viewed as a $\mathfrak{b}$-module via the trivial $\mathfrak{n}^+$-action. In $M(\lambda)$, there is a unique maximal proper submodule which we denote by $\overline{M}(\lambda)$. As $U(\mathfrak{n}^-)$-modules, $M(\lambda)\cong U(\mathfrak{n}^-)$ (the isomorphism is defined up to a scalar factor). The restriction of $S(\lambda)$ onto $U(\mathfrak{n})^-\times U(\mathfrak{n})^-$ induces a bilinear form $M(\lambda)\otimes M(\lambda)\to\mathbb{C}$. The contravariance easily implies that the kernel of $S(\lambda)$ coincides with the maximal proper submodule $\overline{M}(\lambda)$ of $M(\lambda)$.

1.2.3. *An interpretation of the Shapovalov maps.* For every left $\mathfrak{g}$-module $M(\lambda)$, the dual module $M(\lambda)^*:=\mathrm{Hom}_{\mathbb{C}}(M(\lambda),\mathbb{C})$ has a natural structure of a *right* $\mathfrak{g}$-module. To convert $M(\lambda)^*$ into a left $\mathfrak{g}$-module, for any $g\in\mathfrak{g}$, $f\in M(\lambda)^*$ and $v\in M(\lambda)$, we set

$$(gf)(v)=f(\sigma(g)v).$$

The module $M(\lambda)^*$ has a submodule

$$M(\lambda)^{\#}:=\oplus_{\nu\in Q^+}\mathrm{Hom}_{\mathbb{C}}(M(\lambda)_{\lambda-\nu},\mathbb{C})$$

which, as $M(\lambda)$, is of highest weight λ thanks to the condition $\sigma|_{\mathfrak{h}}=\mathrm{id}$.

The Shapovalov map $S(\lambda)$ determines a $\mathfrak{g}$-homomorphism $M(\lambda)\to M(\lambda)^{\#}$ which is identical on the highest weight spaces identified with $\mathbb{C}(\lambda)$.

1.3. Obviously $S(a,b)=0$ if $a,b\in U(\mathfrak{g})$ are weight elements of distinct weights. For each $\nu\in Q^+$, denote by S_ν the restriction of S to $U(\mathfrak{n})^-_{-\nu}\times U(\mathfrak{n})^-_{-\nu}\to S(\mathfrak{h})$. Let $\{u_1,\dots,u_r$ be a basis of $U(\mathfrak{n})^-_{-\nu}$. Written with respect to this basis, the matrix of S_ν takes form

$$S_\nu:=\big(\mathrm{HC}(\sigma(u_i)u_j)\big)$$

This symmetric square matrix with entries in $S(\mathfrak{h})$ is called the *Shapovalov matrix.* For each $\lambda\in\mathfrak{h}^*$, the evaluated matrix $S_\nu(\lambda)$ is a complex matrix and

$$\mathrm{corank}\, S_\nu(\lambda)=\dim\overline{M}(\lambda)_{\lambda-\nu}.$$

In particular,

(1) $$M(\lambda) \text{ is simple } \iff \det S_\nu(\lambda) \neq 0 \text{ for any } \nu \in Q^+.$$

2. The Shapovalov determinants

The formulas for the Shapovalov determinants for a Kac-Moody Lie algebra with a symmetrizable Cartan matrix were obtained in [K2],[KK]. For $\mathfrak{g} = \mathfrak{osp}(1, 2l)$, a detailed proof is written in [M]; it is similar to that for a semisimple Lie algebra. For a Kac-Moody Lie algebra with a symmetrizable Cartan matrix the Shapovalov determinant is given by

$$\det S_\nu = \prod_{\alpha\in\Delta^+}\prod_{k=1}^{\infty} T_{k\alpha}^{\tau(\nu-k\alpha)}$$

where Δ^+ stands for the multiset of roots.

We will prove this formula for finite-dimensional and affine Kac-Moody algebras. A proof for the general case varies by some technical details, see [J].

2.1. A root $\alpha \in \Delta^+$ is called *real* if $\alpha = w\alpha_i$ for some simple root α_i and some $w \in W$. A root is called *imaginary* if it is not real.

If α is real then $(\alpha, \alpha) = (\alpha_i, \alpha_i) > 0$ since $(-,-)$ is W-invariant. Moreover, $\dim \mathfrak{g}_\alpha = 1$ and $\mathbb{Z}\alpha \cap |Delta = \{\pm\alpha\}$ since $\mathrm{ch}\, \mathfrak{g}_\alpha$ is W-invariant.

Finite-dimensional Kac-Moody algebras are semisimple. In this case all roots are real.

For affine Kac-Moody algebras all imaginary roots are isotropic and proportional in a sense that there exists δ such that any isotropic root α is of the form $k\delta$ for $k \in \mathbb{Z}$.

2.2. **Existence of linear factorization.** Every Kac-Moody Lie algebra with a symmetrizable Cartan matrix has a quadratic even Casimir element. This is a central element $z \in \mathcal{Z}(\mathfrak{g})$ satisfying

$$\mathrm{HC}(z)(\lambda) = (\lambda + \rho, \lambda + \rho),$$

where $\rho \in \mathfrak{h}^*$ is such that $2(\rho, \alpha_i) = (\alpha_i, \alpha_i)$ for all simple roots α_i (ρ is uniquely defined iff $\det A \neq 0$). For finite-dimensional $\mathfrak{g}$ one has

$$2\rho := \sum_{\alpha\in\Delta_0^+} \alpha - \sum_{\alpha\in\Delta_1^+} \alpha.$$

2.2.1. The existence of z implies that if $M(\lambda)$ has a primitive vector of weight $\lambda - \mu$, then

$$(\lambda + \rho, \lambda + \rho) = (\lambda + \rho - \mu, \lambda + \rho - \mu)$$

that is

(2) $$2(\lambda + \rho, \mu) = (\mu, \mu).$$

Now if $\det S_\nu(\lambda)=0$, then $M(\lambda)$ has a primitive vector of weight $\lambda-\mu$ for some $\mu\in Q^+$ such that $0<\mu\leq\nu$. In other words, $\det S_\nu(\lambda)=0$ forces (2) for some $\mu\in Q^+\setminus\{0\}$. For each $\mu\in Q^+\setminus\{0\}$, the solutions of (2) for λ form a hyperplane γ_μ in $\mathfrak{h}^*$. Hence all zeroes of $\det S_\nu$ belong to $\underset{\mu\in Q^+\setminus\{0\}}{\cup}\gamma_\mu$. Set

$$\mathrm{Irr}:=\{\eta\in Q^+\mid \eta=k\xi \ \text{ for } k\in\mathbb{Z}_{\geq 0},\ \xi\in Q^+ \implies k=1\}.$$

It is clear that any $\mu\in Q^+\setminus\{0\}$ has a unique representation in the form $\mu=k\eta$ with $k\in\mathbb{Z}_{>0}$, $\eta\in\mathrm{Irr}$; the equation (2) takes the form

$$2(\lambda+\rho,\eta)-k(\eta,\eta)=0.$$

This gives the one-to-one correspondence between the hyperplanes γ_μ and the elements of Y, where

$$Y:=\{(\eta,k)|\ k\in\mathbb{Z}_{>0},\eta\in\mathrm{Irr}:(\eta,\eta)\neq 0\}\cup\{(\eta,1)|\ \eta\in\mathrm{Irr}:(\eta,\eta)=0\}.$$

2.2.2. As a consequence, if $\mathfrak{g}$ possesses a quadratic Casimir element, each Shapovalov determinant $\det S_\nu$ admits a linear factorization: up to a constant factor, we have

$$\det S_\nu=\prod T_{k\eta}^{d_{k\eta}(\nu)},$$

where $\eta\in\mathrm{Irr}$, $k\in\mathbb{Z}_{>0}$ and $k=1$ if ν is isotropic and

$$T_{k\eta}(\lambda)=2(\lambda+\rho,\eta)-k(\eta,\eta) \tag{3}$$

and $d_{k\eta}(\nu)\in\mathbb{Z}_{\geq 0}$.

3. The Jantzen filtration

The notion of *Jantzen filtration* was introduced for semisimple Lie algebras in [Ja]. In this subsection we recall the definition.

Observe that that $(\rho,\mu)\neq 0$ for all $\mu\in Q^+\setminus\{0\}$.

3.0.3. Let t be an even indeterminate and let $A:=\mathbb{C}[t]_{(t)}$ be the localization of $\mathbb{C}[t]$ by the principal ideal (t). The algebra A is a local ring and $F:=\mathbb{C}(t)$ is its field of fractions.

We shall extend the field of scalars from $\mathbb{C}$ to A, and sometimes to F and accordingly replace all tensorings over $\mathbb{C}$ by tensorings over A or over F. For any $\mathbb{C}$-vector superspace V, we denote by V_A the A-module $A\otimes V$. Since tensorings are over A, we can (and will) identify $U(\mathfrak{g}_A)$ with $U(\mathfrak{g})_A$.

For any $\xi\in\mathfrak{h}_A^*$, we can (and will) define a $U(\mathfrak{g}_A)$-module — a Verma module — $M(\xi)_A$ in the usual way. For each Verma module $M(\xi)_A$, let

$$\langle\cdot,\cdot\rangle_\xi:\ M(\xi)_A\times M(\xi)_A\to A$$

be the contravariant form defined in the same way as in 1.2.2. Consider the Shapovalov determinant $\det S_\nu\in S(\mathfrak{h})$ as an element of the ring $S(\mathfrak{h}_A)$. Then $\det S_\nu$ thus defined coincides with the Shapovalov determinant $\det S_\nu$ constructed for $U(\mathfrak{g}_A)$ (notice that the $\det S_\nu$ is defined up to an invertible element of A).

Similarly, for any A-module V, we denote by V_F the vector superspace $V\otimes_A F$. As above, we consider tensoring in $U(\mathfrak{g}_F)$ over F, and hence can identify $U(\mathfrak{g}_F)\ =U(\mathfrak{g})_F$. For any $\xi\in\mathfrak{h}_A^*$, the module $M(\xi)_F$ admits the natural structure of $U(\mathfrak{g}_F)$-Verma module of highest weight ξ. Let us show that $M(\lambda+t\rho)_F$ is simple for any $\lambda\in\mathfrak{h}^*$. Indeed, for any $\mu\in Q^+\setminus\{0\}$, we have

$$(\lambda+t\rho+\rho,\mu)=(\lambda+\rho,\mu)+t(\rho,\mu)\not\in\mathbb{C}$$

since $(\rho,\mu)\neq 0$. In particular, $(\lambda+t\rho+\rho,\mu)\not\in\mathbb{Z}_{\geq 0}(\mu,\mu)$. By 2.2.1, this implies the simplicity of $M(\lambda+t\rho)_F$ (since the reasoning of 2.2.1 can be applied to a Kac-Moody Lie algebra over any field).

3.0.4. For any $\lambda\in\mathfrak{h}^*$, define the decreasing filtration on the module $M(\lambda+t\rho)_A$ by setting

$$M(\lambda+t\rho)_A^i:=\begin{cases}M(\lambda+t\rho)_A & \text{for } i\in-\mathbb{Z}_{\geq 0}\\ \{m\in M(\lambda+t\rho)_A\mid\langle m,M(\lambda+t\rho)_A\rangle_{\lambda+t\rho}\subseteq t^iA\} & \text{for } i\in\mathbb{Z}_{>0}.\end{cases}$$

The "decreasing filtration" means that $M(\lambda+t\rho)_A^i\subset M(\lambda+t\rho)_A^{i-1}$ and that all terms $M(\lambda+t\rho)_A^i$ are $\mathfrak{g}_A$-submodules.

The simplicity of $M(\lambda+t\rho)_F$ implies that the contravariant form $\langle\cdot,\cdot\rangle_{\lambda+t\rho}$ is non-degenerate. Thus $\bigcap\limits_{i\in\mathbb{Z}}M(\lambda+t\rho)_A^i=0$. We identify $M(\lambda)$ and $M(\lambda+t\rho)_A/tM(\lambda+t\rho)_A$. For each $i\in\mathbb{Z}_{\geq 0}$, define the $U(\mathfrak{g})$-submodule $M(\lambda)^i$ of $M(\lambda)$ by setting

$$M(\lambda)^i:=M(\lambda+t\rho)_A^i/tM(\lambda+t\rho)_A^{i-1}.$$

Since $M(\lambda+t\rho)_A^{i+1}\cap tM(\lambda+t\rho)_A=tM(\lambda+t\rho)_A^i$, it follows that $\{M(\lambda)^i\}_{i\in\mathbb{Z}_{\geq 0}}$ forms a decreasing filtration of $M(\lambda)$ and $\bigcap\limits_{i\in\mathbb{Z}_{\geq 0}}M(\lambda)^i=0$. This filtration is called the *Jantzen filtration.*

3.0.5. *Remark.* Observe that $\mathrm{HC}(\sigma(a)b)(\lambda+t\rho)$ is a polynomial in one variable t for any $a,b\in U(\mathfrak{n}^-)$ and $\lambda\in\mathfrak{h}^*$. Thus $\mathrm{HC}(\sigma(a)b)(\lambda+t\rho)\in(t)$ if and only if $\mathrm{HC}(\sigma(a)b)(\lambda)=0$. Hence $M(\lambda)^1$ coincides with the kernel of $\langle\cdot,\cdot\rangle_\lambda$ and $M(\lambda)/M(\lambda)^1=V(\lambda)$ is the simple module of highest weight λ.

3.0.6. We are going to use the following fact proved in [Ja]. Recall that $A=\mathbb{C}[t]_{(t)}$. Let N be a free A-module of finite rank r endowed by a bilinear form $N\otimes N\to A$, and let $D\in\mathrm{Mat}_{r\times r}(A)$ be the matrix of this bilinear form. Define a decreasing filtration

$$F^j(N):=\{v\in N\mid D(v,v')\in(t^j)\ \text{ for any } v'\in N\}.$$

Assume that $\det D \neq 0$. We claim that *the order of zero of* $\det D$ *at the origin is equal to*

$$\sum_{j=1}^{\infty} \dim\bigl(F^j(N)/(F^j(N) \cap tN)\bigr).$$

Indeed, it is easy to see that N has two systems of generators v_i and v'_i (for $i = 1, \dots, r$) such that $D(v_i, v'_j) = \delta_{ij} t^{s_i}$ (for $s_i \in \mathbb{Z}_{\geq 0}$). The order of zero of $\det D$ at the origin is $\sum_{i=1}^{r} s_i$ and

$$\dim F^j(N)/(F^j(N) \cap tN) = |\{i \mid s_i \geq j\}|.$$

The equality $\sum_i s_i = \sum_{j=1}^{\infty} |\{i \mid s_i \geq j\}|$ implies the claim. $\square$

3.0.7. Taking $D = S_\nu$, we conclude that, for any $\lambda \in \mathfrak{h}^*$ and any $\nu \in Q^+$, the sum $\sum_{r=1}^{\infty} \dim F^r(M(\lambda)_{\lambda-\nu})$ is equal to the order of zero of $\det S_\nu$ at λ.

3.0.8. *Remark.* The order m of zero of a polynomial q at λ is defined to be $m = 0$ if $q(\lambda) \neq 0$; it is $m = 1$ if $q(\lambda) = 0$ and there exists a non-zero partial derivative $\frac{dq}{dx}(\lambda) \neq 0$, and so on.

View $\mathfrak{h}^*$ as an affine space. Consider the set of zeros of q as an algebraic variety $X \subset \mathfrak{h}^*$, $X := \{\lambda | q(\lambda) = 0\}$. Write $X = \cup_{i=1}^{s} X_i$ where X_i are irreducible components of X. Let $\lambda \in X_i$ be *generic* in a sense that $\lambda \notin X_j$ for $i \neq j$. Let $\mu \in \mathfrak{h}^*$ be a vector which does not belong to the tangent space of X_i at point λ. Consider $q' := q(\lambda + t\mu)$ as a polynomial in one variable t. Then the order of zero of q' at the origin is equal to the order of q at λ.

3.0.9. It is clear that

$$\operatorname{corank} S_\nu(\lambda) \leq \text{ the order of zero of } \det S_\nu \text{ at } \lambda.$$

3.0.10. In the notation of sec. 2.2.2 we obtain

$$\sum_{k,\eta: T_{k\eta}(\lambda)=0} d_{k\eta}(\nu) = \sum_{r=1}^{\infty} \dim M(\lambda)^r_{\lambda-\nu} \text{ for any } \lambda \in \mathfrak{h}^* \text{ and any } \nu \in Q^+. \tag{4}$$

3.0.11. Fix a non-isotropic $\mu \in \mathrm{Irr}$ and $k \in \mathbb{Z}_{>0}$. Set

$$R := \{\lambda \in \mathfrak{h}^* \mid 2(\lambda + \rho, \mu) = k(\mu, \mu) \ \&\ 2(\lambda + \rho, \eta) \notin \mathbb{Z}_{\geq 0}(\eta, \eta) \text{ for any } \eta \in \mathrm{Irr},\ \eta \neq \mu\}.$$

The set R is dense in the hyperplane $T_{k\mu} = 0$. For any $\lambda \in R$, the only element $\xi \in Q^+$ satisfying

$$(\lambda + \rho, \lambda + \rho) = (\lambda + \rho - \xi, \lambda + \rho - \xi)$$

is $\xi = k\mu$. Therefore the module $M(\lambda - k\mu)$ is simple. Moreover, $M(\lambda)$ is either simple or its socle is the sum of several copies of $M(\lambda - k\mu)$. In any case, for all $r \geq 1$, there exists $j_r \geq 0$ such that

$$\operatorname{ch} M(\lambda)^r = j_r \operatorname{ch} M(\lambda - k\mu)$$

(if $M(\lambda)$ is simple, then $j_r = 0$ for all $r \geq 1$). Taking (4) into account, we obtain

$$d_{k\mu}(\nu) = j \dim M(\lambda - k\mu)_{\lambda-\nu} = j\tau(\nu - k\mu),$$

where $j := \sum_{r=1}^{\infty} j_r$. Finally, for any non-isotropic $\mu \in \operatorname{Irr}$, and any $k \in \mathbb{Z}_{>0}$, there exists $j \in \mathbb{Z}_{\geq 0}$ such that

$$d_{k\mu}(\nu) = j\tau(\nu - k\mu) \ \text{ for all } \nu \in Q^+. \tag{5}$$

4. The leading term of $\det S_\nu$

For each root $\alpha \in \Delta^+$, denote by $\alpha^\vee$ the corresponding coroot, i.e. $\alpha^\vee \in \mathfrak{h}$ is such that $(\alpha, \xi) = \alpha^\vee(\xi)$ for all $\xi \in \mathfrak{h}^*$. Notice that $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is spanned by $\alpha^\vee$.

In this section, we will prove the

Proposition. *The leading term of the polynomial* $\det S_\nu$ *is equal, up to a non-zero scalar factor, to*

$$\prod_{\alpha \in \Delta^+} (\alpha^\vee)^{\sum_{n=1}^{\infty} \tau(\nu - n\alpha)}.$$

4.1. **The choice of a PBW basis in $U(\mathfrak{n}^-)$.**

4.1.1. Define the *Kostant partition function* $\tau : Q \to \mathbb{Z}_{\geq 0}$ by setting $\tau(Q \setminus Q^+) = 0$ and

$$\operatorname{ch} U(\mathfrak{n}^-) = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})^{-1} =: \sum_{\eta \in Q^+} \tau(\eta) e^{-\eta}.$$

For any $\nu \in Q^+$, the vector $\mathbf{k} = \{k_\alpha\}_{\alpha \in \Delta^+}$ is called *a partition* of ν if $\nu = \sum_{\alpha \in \Delta^+} k_\alpha \alpha$, where $k_\alpha \in \mathbb{Z}_{\geq 0}$.

Set $|\mathbf{k}| = \sum_{\alpha \in \Delta^+} k_\alpha$. Denote by $\mathcal{P}(\nu)$ the set of all partitions of ν. According to PBW-theorem, $\mathcal{P}(\nu)$ has $\tau(\nu)$ elements.

4.1.2. The standard partial ordering on $\mathfrak{h}^*$ is given by the formula

$$\xi \geq \xi' \iff \xi - \xi' \in Q^+.$$

Fix a total ordering on Δ^+ compatible with the above partial ordering. For each $\alpha \in \Delta^+$, let $\{f_\alpha^{(i)}\}$ be a basis of $\mathfrak{g}_\alpha$ satisfying $(\sigma(f_\alpha^{(i)})|f_\alpha^{(i)}) = \delta_{ij}$. Then $[\sigma(f_\alpha^{(i)}), f_\alpha^{(i)}] = c_i\delta_{ij}\alpha^\vee$ for some non-zero scalar c_i.

The union $\cup_{\alpha\in\Delta^+}\{f_\alpha^{(i)}\}$ forms a basis of $\mathfrak{n}^-$. Take the corresponding PBW basis, i.e. for every $\mathbf{k} \in \mathcal{P}(\nu)$, define the monomial

$$\mathbf{f}^{\mathbf{k}} := \prod f_\alpha^{k_\alpha},$$

where the order of factors is given by the total ordering fixed above: the factors corresponding to smaller roots come first. The set $\{\mathbf{f}^{\mathbf{k}} \mid \mathbf{k} \in \mathcal{P}(\nu)\}$ forms a PBW basis of $U(\mathfrak{n}^-)_{-\nu}$.

4.2. For $u \in U(\mathfrak{g})$ denote by $\deg u$ the degree with respect to the canonical filtration. For $u \in U(\mathfrak{h})$ this is the total degree of u viewed as a polynomial via the identification $U(\mathfrak{h}) = S(\mathfrak{h})$.

4.2.1. **Lemma.**

(i) *If $u \in U(\mathfrak{g})$ is the product of n_- elements of $\mathfrak{n}^-$, n_+ elements of $\mathfrak{n}^+$ and n_0 elements of $\mathfrak{h}$ then* $\deg \mathrm{HC}(u) \leq \min(n_-, n_+) + n_0$.

(ii) *For $i = 1, \dots, k$ let x_i be an element of weight α_i and y_i be an element of weight $-\beta_i$, where $\alpha_i, \beta_i \in \Delta^+$. If $\mathrm{HC}(\prod_{i=1}^k x_i \prod_{i=1}^k y_i)$ has degree k then the multisets $\{\alpha_i\}_{i=1}^k$ and $\{\beta_i\}_{i=1}^k$ are equal.*

Proof. The proof of (i) is an easy induction on $n_- + n_+ + n_0$. Indeed, write $u = u'x$ or $u = u'xy_1 \dots y_r$ with $x \in (\mathfrak{h} + \mathfrak{n}^+), y_1 \dots y_r \in \mathfrak{n}^-, u' \in U(\mathfrak{g})$. If $u = u'x$ then $\mathrm{HC}(u) = \mathrm{HC}(u')\,\mathrm{HC}(x)$ and the assertion follows from by induction. In the case $u = u'xy_1 \dots y_r$, write

$$u = \pm u'y_1 \dots y_r x + \sum_{i=1}^r \pm u'y_1 \dots y_{i-1}((\operatorname{ad} x)y_i)y_{i+1} \dots y_r.$$

As we have already checked $\deg \mathrm{HC}(u'y_1 \dots y_r x) \leq \min(n_-, n_+) + n_0$. The remaining summands are the products of n'_- elements of $\mathfrak{n}^-$, n'_+ elements of $\mathfrak{n}^+$ and n'_0 elements of $\mathfrak{h}$ where $n'_- + n'_+ + n'_0 = n_- + n_+ + n_0 - 1$. In all cases $\min(n'_-, n'_+) + n'_0 \leq \min(n_-, n_+) + n_0$ which implies (i); (ii) easily follows from (i). ∎

4.3. **Lemma.** *For any $\mathbf{m}, \mathbf{s} \in \mathcal{P}(\nu)$, we have*

(i) $\deg \mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}) \leq \min(|\mathbf{m}|, |\mathbf{s}|)$;

(ii) *If $|\mathbf{m}| = |\mathbf{s}|$, we have*

$$\deg \mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{s}}) = |\mathbf{m}\,| \iff \mathbf{m} = \mathbf{s};$$

(iii) *Up to a non-zero scalar factor*

$$\operatorname{gr}\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{m}}) = \prod_{\alpha\in\Delta^+}(\alpha^\vee)^{m_\alpha}.$$

Proof. (i) follows from Lemma 4.2.1 (i) and (ii) from Lemma 4.2.1 (ii). For (iii) write $\mathbf{f}^{\mathbf{m}} = f_\alpha^r\mathbf{f}^{\mathbf{k}}$ where $\mathbf{k}_\alpha = 0$ and recall that α is the minimal root among $\beta : m_\beta \neq 0$. One has

$$\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{m}}) = \mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^r f_\alpha^r\mathbf{f}^{\mathbf{k}}) = \mathrm{HC}\big(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}(\operatorname{ad}e_\alpha)\big(f_\alpha^r\mathbf{f}^{\mathbf{k}}\big)\big).$$

One has

$$\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}(\operatorname{ad}e_\alpha)\big(f_\alpha^r\mathbf{f}^{\mathbf{k}}\big) = \sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}(\operatorname{ad}e_\alpha)\big(f_\alpha^r\big)\mathbf{f}^{\mathbf{k}} + \sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}f_\alpha^r(\operatorname{ad}e_\alpha)\big(\mathbf{f}^{\mathbf{k}}\big)$$

since $\operatorname{ad}g$ is a derivation of $U(\mathfrak{g})$.

Due to the minimality of α the term $(\operatorname{ad}e_\alpha)(\mathbf{f}^{\mathbf{k}})$ lies in $\mathcal{U}(\mathfrak{n}^-)$ and thus, by Lemma 4.2.1 (i)

$$\deg\mathrm{HC}\big(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}f_\alpha^r(\operatorname{ad}e_\alpha)\big(\mathbf{f}^{\mathbf{k}}\big)\big) \leq |\mathbf{k}| + r - 1 < |\mathbf{m}|$$

since $|\mathbf{m}| = |\mathbf{k}| + r$.

Hence

$$\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{m}}) \equiv \mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}(\operatorname{ad}e_\alpha)\big(f_\alpha^r\big)\mathbf{f}^{\mathbf{k}}) \mod \mathcal{F}^{|\mathbf{m}|-1}(U(\mathfrak{g})).$$

Since $(\operatorname{ad}e_\alpha)f_\alpha = \alpha^\vee$

$$(\operatorname{ad}e_\alpha)\big(f_\alpha^r\big) = \sum_{i=0}^{r-1} f_\alpha^i\alpha^\vee f_\alpha^{r-1-i} = rf_\alpha^{r-1}\alpha^\vee + cf_\alpha^{r-1}$$

for some scalar c. Similarly,

$$f_\alpha^{r-1}\alpha^\vee\mathbf{f}^{\mathbf{k}} = f_\alpha^{r-1}\mathbf{f}^{\mathbf{k}}\alpha^\vee + c'f_\alpha^{r-1}\mathbf{f}^{\mathbf{k}}$$

for some scalar c'. Therefore

$$\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}(\operatorname{ad}e_\alpha)\big(f_\alpha^r\big)\mathbf{f}^{\mathbf{k}}) = r\,\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}f_\alpha^{r-1}\mathbf{f}^{\mathbf{k}}\alpha^\vee) + (c+c')\,\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}f_\alpha^{r-1}\mathbf{f}^{\mathbf{k}}).$$

By Lemma 4.2.1 (i), the degree of the last term is at most $|\mathbf{k}| + r - 1 = |\mathbf{m}| - 1$. Hence

$$\mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{m}})\mathbf{f}^{\mathbf{m}}) \equiv \mathrm{HC}(\sigma(\mathbf{f}^{\mathbf{k}})e_\alpha^{r-1}f_\alpha^{r-1}\mathbf{f}^{\mathbf{k}})\alpha^\vee \mod \mathcal{F}^{|\mathbf{m}|-1}(U(\mathfrak{g})).$$

Now (iii) follows by induction. □

4.3.1. **Corollary.** *Up to a non-zero scalar factor, the leading term of* $\det S_\nu$ *is equal to*

$$\prod_{\alpha\in\Delta^+}(\alpha^\vee)^{r_\alpha(\nu)},$$

where

$$r_\alpha(\nu) := \sum_{\mathbf{m}\in\mathcal{P}(\nu)} m_\alpha.$$

4.3.2. We need the following technical lemma.

4.3.3. **Lemma.** *For all $\alpha \in \Delta^+$, we have*

$$\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha = \sum_{n=1}^{\infty} \tau(\nu - n\alpha).$$

Proof. Let us use an induction on the partial order on Q^+. For any $\nu \not\geq \alpha$, the assertion obviously hold since both sides of both equations vanish. Fix $\nu \geq \alpha$ and assume that the assertions hold for all $\mu < \nu$. The map $\mathbf{k} \mapsto (\mathbf{k} - \alpha)$ gives a bijection

$$\{\mathbf{k} \in \mathcal{P}(\nu) \mid k_\alpha \neq 0\} \xrightarrow{\sim} \mathcal{P}(\nu - \alpha).$$

Hence

$$\begin{aligned}\sum_{\mathbf{k}\in\mathcal{P}(\nu)} k_\alpha \quad &= \quad \sum_{\mathbf{k}\in\mathcal{P}(\nu-\alpha)} (k_\alpha + 1) = \sum_{n=1}^{\infty} \tau(\nu - (n+1)\alpha) + \mid \mathcal{P}(\nu - \alpha) \mid \\ &\overset{\text{by the assumption}}{=} \sum_{n=2}^{\infty} \tau(\nu - n\alpha) + \tau(\nu - \alpha).\end{aligned}$$

The assertion follows. □

4.3.4. Substituting the formulas (4.3.3) in Corollary 4.3.1 we conclude that the leading term of the polynomial $\det S_\nu$ is equal, up to a non-zero scalar factor, to $\prod_{\alpha\in\Delta^+} (\alpha^\vee)^{\sum_{n=1}^{\infty} \tau(\nu - n\alpha)}$. This proves Proposition 4.

5. Computation of multiplicities of Shapovalov determinants

Combining Proposition 4 with 2.2.2 we conclude that $d_{k\eta} = 0$ if $\eta \notin \Delta^+$. Thus, up to a non-zero scalar factor,

$$\det S_\nu = \prod_{\alpha\in\Delta^+} \prod_{k=1}^{\infty} T_{k\alpha}^{d_{k\alpha}(\nu)}$$

where $T_{k\alpha}(\lambda) = (\alpha, \lambda + \rho) - k(\alpha, \alpha)/2$.

5.1. Consider the hyperplanes corresponding to the factor of Shapovalov determinants:

$$\gamma_{r,\alpha} := \{\lambda \in \mathfrak{h}^* \mid (\alpha, \lambda + \rho) - r(\alpha, \alpha)/2\}$$

where $\alpha \in \Delta^+, r \in \mathbb{Z}_{>0}$. Denote by Γ the set of these hyperplanes (some of these hyperplanes coincide, for instance, $\gamma_{r,\alpha} = \gamma_{s,\alpha}$ if α is isotropic). Let $\mathfrak{g}$ be finite-dimensional or affine Kac-Moody algebra. From 2.1, it follows that $\gamma_{r,\alpha} = \gamma_{s,\beta}$ for $(r, \alpha) \neq (s, \beta)$ iff α, β are isotropic that is proportional to the minimal isotropic root δ.

By Proposition 4 for real α one has

$$\sum_{k=1}^{\infty} d_{k\alpha}(\nu)=\sum_{n=1}^{\infty}\tau(\nu-n\alpha) \tag{6}$$

and

$$\sum_{k=1}^{\infty}\sum_{\alpha:(\alpha,\alpha)=0} d_{k\alpha}(\nu)=\sum_{n=1}^{\infty}\dim\mathfrak{g}_{n\delta}\sum_{m=1}^{\infty}\tau(\nu-mn\delta). \tag{7}$$

5.2. Let α be an isotropic root. Then $\gamma_{k,\alpha}=\gamma_\delta$ for all k and the total degree of T_δ is the right-hand side of (7). Thus, up to anon-zero scalar,

$$\prod_{\alpha\in\Delta^+:(\alpha,\alpha)=0}\prod_{k=1}^{\infty} T_{k\alpha}^{d_{k\alpha}(\nu)}=\prod_{\alpha\in\Delta^+:(\alpha,\alpha)=0}\prod_{m=1}^{\infty} T_{m\alpha}^{\tau(\nu-m\alpha)}.$$

5.2.1. Let α be a real positive root. By (5), for each $k=1,2,\dots$, there exists a $j_k\in\mathbb{Z}_{\geq 0}$ such that

$$d_{k\alpha}(\nu)=j_k\tau(\nu-k\alpha)\text{ for all }\nu\in Q^+.$$

Substituting this $d_{k\alpha}(\nu)$ in (6) we get

$$\sum_{k=1}^{\infty} j_k\tau(\nu-k\alpha)=\sum_{n=1}^{\infty}\tau(\nu-n\alpha)\text{ for }(\alpha,\alpha)\neq 0.$$

Consider $\tau_i:=\tau(\nu-i\alpha)$ as a function on $\nu\in Q^+$. The functions τ_i for distinct i's are linearly independent since $\tau_i(i\alpha)=1$ and $\tau_j(i\alpha)=0$ for $j>i$. Thus, we have $j_n=1$ for all $n\geq 1$.

5.2.2. **Corollary.** *Up to a non-zero scalar factor,*

$$\det S_\nu=\prod_{\alpha\in\Delta^+}\prod_{k=1}^{\infty} T_{k\alpha}^{\tau(\nu-k\alpha)}$$

where $T_{k\alpha}(\lambda)=(\alpha,\lambda+\rho)-k(\alpha,\alpha)/2$.

6. Applications to Verma modules

Retain notation of 5.1. Call $\lambda\in\mathfrak{h}^*$ *regular* if λ does not lie on a hyperplane from γ. Call $\lambda\in\mathfrak{h}^*$ *subregular* if λ belongs to a unique hyperplane from γ.

6.1. From (1) we get

Corollary. *The module $M(\lambda)$ is simple iff λ is regular.*

6.2. **Theorem.** *If $\lambda \in \gamma_{r,\alpha}$ is subregular and α is real then the Jantzen filtration of $M(\lambda)$ has length two, $M(\lambda)$ has length two and $\overline{M}(\lambda) = M(\lambda - r\alpha)$.*

"The Jantzen filtration has length two" means $M(\lambda)^2 = 0$ and "M has length two" means that its Jordan-Hölder filtration has length two that is M has a simple submodule M' such that M/M' is also simple. For $M = M(\lambda)$ this means that $\overline{M}(\lambda)$ is simple.

Proof. If α is real then $\mathbb{Z}\alpha \cap \Delta^+ = \{\alpha\}$ and so $\gamma_{r,\alpha} = \gamma_{s,\beta}$ forces $\alpha = \beta$, $r = s$. Then the order of zero of $\det S_\nu$ at λ is equal to $\tau(\nu - r\alpha)$. The sum formula gives

$$\sum_{m=1}^{\infty} \dim M(\lambda)^m_{\lambda-\nu} = \tau(\nu - r\alpha) = \dim M(\lambda - r\alpha)_{\lambda-\nu}$$

that is

$$\sum_{m=1}^{\infty} \operatorname{ch} M(\lambda)^m_{\lambda-\nu} = \operatorname{ch} M(\lambda - r\alpha)_{\lambda-\nu}. \tag{8}$$

Recall that $M(\lambda)^1 = \overline{M}(\lambda)$. Then $\overline{M}(\lambda)_\nu = 0$ if $\nu \not\geq r\alpha$ and $\overline{M}(\lambda)_{r\alpha}$ is one-dimensional. Therefore $\overline{M}(\lambda)$ has a primitive vector v' of weight $\lambda - r\alpha$. Since $U(\mathfrak{g})$ is a domain, v' generates a Verma module so $\overline{M}(\lambda)$ contains a submodule isomorphic to $M(\lambda - r\alpha)$. Now (8) gives $\overline{M}(\lambda) = M(\lambda - r\alpha)$ and $M(\lambda)^m = 0$ for $m > 1$.

Let us show that $\overline{M}(\lambda) = M(\lambda - r\alpha)$ is simple. This is equivalent to

$$2(\lambda + \rho - r\alpha, \beta) \not\in \mathbb{Z}_{>0}(\beta,\beta) \quad \text{for all } \beta \in \Delta^+. \tag{9}$$

Observe that $\lambda + \rho - r\alpha = s_\alpha(\lambda + \rho)$. Therefore

$$2(\lambda + \rho - r\alpha, \beta) = 2(s_\alpha(\lambda+\rho), \beta) = 2(\lambda + \rho, s_\alpha\beta).$$

Recall that $2(\lambda + \rho, \beta') \in \mathbb{Z}_{>0}(\beta', \beta')$ forces $\beta' = \alpha$. This implies (9) for isotropic β and for β such that $s_\alpha\beta \in \Delta^+$, because $(\beta,\beta) = (s_\alpha\beta, s_\alpha\beta)$. For $\beta = \alpha$ (9) holds since $2(\lambda + \rho, s_\alpha\alpha) = -2(\lambda+\rho,\alpha) = -r(\alpha,\alpha)$ and $-r \not\in \mathbb{Z}_{>0}$. Finally, assume that $s_\alpha\beta \not\in \Delta^+$ and β is non-isotropic, $\beta \neq \alpha$. Since Δ is W-invariant, $s_\alpha\beta \in \Delta^-$ and so $s_\alpha\beta = \beta - m\alpha$ where $m := \frac{2(\alpha,\beta)}{(\alpha,\alpha)} > 0$ that is $(\alpha,\beta) > 0$. We have

$$\begin{aligned} \tfrac{2(\lambda+\rho,s_\alpha\beta)}{(\beta,\beta)} &= \tfrac{2(\lambda+\rho,\beta)}{(\beta,\beta)} - \tfrac{2(\alpha,\beta)}{(\alpha,\alpha)}\tfrac{2(\lambda+\rho,\alpha)}{(\beta,\beta)} \\ &= \tfrac{2(\lambda+\rho,\beta)}{(\beta,\beta)} - \tfrac{2(\lambda+\rho,\alpha)}{(\alpha,\alpha)}\tfrac{2(\alpha,\beta)}{(\beta,\beta)} = \tfrac{2(\lambda+\rho,\beta)}{(\beta,\beta)} - r\tfrac{2(\alpha,\beta)}{(\beta,\beta)}. \end{aligned}$$

Recall that $\frac{2(\alpha,\beta)}{(\beta,\beta)}$ is an integer and it is positive because $(\alpha,\beta) > 0$. Since $\frac{2(\lambda+\rho,\beta)}{(\beta,\beta)} \not\in \mathbb{Z}_{>0}$, we conclude $\frac{2(\lambda+\rho,s_\alpha\beta)}{(\beta,\beta)} \not\in \mathbb{Z}_{>0}$ as required. ☐

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