WEYL DENOMINATOR IDENTITY FOR THE AFFINE LIE SUPERALGEBRA $\mathfrak{gl}(2|2)\hat{}$

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ABSTRACT. We prove the Weyl denominator identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)\hat{}$ conjectured by V. Kac and M. Wakimoto in [KW]. As it was pointed out in [KW], this gives a new proof of the Jacobi identity for the number of presentations of a given integer as a sum of 8 squares.

0. INTRODUCTION

The denominator identities for Lie superalgebras were formulated and partially proven in the paper of V. Kac and M. Wakimoto [KW]. In the same paper it was shown how various classical identities in number theory as the number of representation of a given integer as a sum of d squares can be obtained, for some d, by evaluation of certain denominator identities. The following cases are considered in the paper [KW]:

(a) basic Lie superalgebras, i.e. the finite-dimensional simple Lie superalgebras, which have a reductive even part and admit an even non-degenerate invariant bilinear form;

(b) the affinization of basic Lie superalgebras with non-zero dual Coxeter number;

(c) the (twisted) affinization of a strange Lie superalgebras $Q(n)$;

(d) the affinization of $\mathfrak{gl}(2|2)$ (this is the smallest basic Lie superalgebras with zero dual Coxeter number).

Some of the cases (a), (b) are proven in [KW]; the proof is based on combinatorics of root systems and a certain result from representation theory. The rest of (a) was proven in [G1] using only combinatorics of root systems. The rest of (b) was proven in [G2] using (a) and the existence of Casimir operator. The case (c) was proven in [Z] analytically. In the present paper we prove (d), i.e. the identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)\hat{}$ conjectured in [KW], 7.1. The proof uses the existence of Casimir operator and an idea of [Z].

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In order to write down the identity, we introduce the following infinite products after [DK]: for a parameter q and a formal variable x we set

$$(1+x)_q^\infty := \prod_{n=0}^{\infty}(1+q^n x), \quad \text{and} \quad (1-x)_q^\infty := \prod_{n=0}^{\infty}(1-q^n x).$$

These infinite products converge for any $x \in \mathbb{C}$ if the parameter q is a real number $0<q<1$. In particular, they are well-defined for $0<x=q<1$ and $(1\pm q)_q^\infty := \prod_{n=1}^{\infty}(1\pm q^n)$.

Take the formal variables x, y_1, y_2. The denominator identity for $\mathfrak{gl}(2|2)\hat{}$ can be written in the following form

(1)
$$\frac{(1-x)_q^\infty(1-qx^{-1})_q^\infty(1-xy_1y_2)_q^\infty(1-q(xy_1y_2)^{-1})_q^\infty\big((1-q)_q^\infty\big)^4}{\prod_{i=1}^{2}(1+y_i)_q^\infty(1+qy_i^{-1})_q^\infty(1+xy_i)_q^\infty(1+qx^{-1}y_i^{-1})_q^\infty}=$$
$$=\frac{((1-q)_q^\infty)^2}{(1-qy_1^{-1}y_2)_q^\infty(1-qy_1y_2^{-1})_q^\infty}\cdot\sum_{n=-\infty}^{\infty}\Big(\frac{q^n}{(1+q^ny_1)(1+q^ny_2)}-\frac{q^nx}{(1+q^nxy_1)(1+q^nxy_2)}\Big).$$

Expanding the factor $\frac{((1-q)_q^\infty)^2}{(1-qy_1^{-1}y_2)_q^\infty(1-qy_1y_2^{-1})_q^\infty}$ in the region $q<|\frac{y_1}{y_2}|<q^{-1}$ we obtain (see Lemma 1.3.1)

$$\frac{((1-q)_q^\infty)^2}{(1-qy_1^{-1}y_2)_q^\infty(1-qy_1y_2^{-1})_q^\infty}=1+\sum_{n=1}^{\infty}f_n(\tfrac{y_1}{y_2}),$$
$$\text{where } f_n(y):=\big(y^n+y^{-n}-y^{n-1}-y^{1-n}\big)\sum_{j=0}^{\infty}(-1)^jq^{(j+1)(j+2n)/2}$$

and this gives the identity conjectured by V. Kac and M. Wakimoto.

The left-hand side of the identity represents the Weyl denominator $\hat{R}$ for the affine Lie superalgebra $\mathfrak{gl}(2|2)\hat{}$; the second factor in the right-hand side is the analogue of the right-hand side of the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number. Note that the denominator identity for the affine Lie superalgebra $\mathfrak{sl}(2|2)\hat{}$ can be obtained from the denominator identity for $\mathfrak{gl}(2|2)\hat{}$ by taking $y_1=y_2$; as a result, the denominator identity for $\mathfrak{sl}(2|2)\hat{}$ is almost similar to the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number with one extra-factor $(1-q)_q^\infty$ in the left-hand side (since the dimension of Cartan subalgebra for $\mathfrak{sl}(2|2)$ is less by one than the dimension of Cartan subalgebra for $\mathfrak{gl}(2|2)$).

As it is shown in [KW], the evaluation of this identity gives the following Jacobi identity [J]:

$$\Box(q)^8=1+16\sum_{j,k=1}^{\infty}(-1)^{(j+1)k}k^3q^{jk}, \tag{2}$$

where $\Box(q)=\sum_{j\in\mathbb{Z}}q^{j^2}$ and thus the coefficient of q^m in the power series expansion of $\Box(q)^d$ is the number of representation of a given integer as a sum of d squares (taking into the account the order of summands).

In Section 1 we introduce notation. In Section 2 we prove the identity (1). In Section 3 we recall how to deduce the Jacobi identity from the identity (1).

1. Notation

1.1. **Root system.** Consider $V:=\mathbb{R}^5$ endowed by a bilinear form and an orthogonal basis $\varepsilon_1,\varepsilon_2,\delta_1,\delta_2,\delta$ such that $(\varepsilon_i,\varepsilon_i)=1=-(\delta_i,\delta_i)$ for $i=1,2$ and $(\delta,\delta)=0$. Set

$$\beta_1:=\delta_1-\varepsilon_1,\ \alpha:=\varepsilon_1-\varepsilon_2,\ \beta_2:=\varepsilon_2-\delta_2,\ \gamma:=\beta_1+\alpha+\beta_2=\delta_1-\delta_2.$$

The root system of $\mathfrak{gl}(2|2)$ is $\Delta_0=\{\pm\alpha,\pm\gamma\}$, $\Delta_1=\{\pm\beta_i;\pm(\alpha+\beta_i)\}_{i=1,2}$. The affine root system is $\hat{\Delta}_i=\cup_{s\in\mathbb{Z}}(\Delta_i+s\delta)$, $i=0,1$.

We consider the following sets of simple roots for $\mathfrak{gl}(2|2)$ and $\mathfrak{gl}(2|2)\hat{}$ respectively:

$$\Pi:=\{\beta_1,\alpha,\beta_2\},\quad\text{and}\quad\hat{\Pi}=\{\beta_1,\alpha,\beta_2,\delta-\gamma\}.$$

One has

$$\Delta_+=\{\alpha,\gamma;\beta_i,\alpha+\beta_i\}_{i=1,2},\quad\hat{\Delta}_+=\Delta_+\cup\cup_{s=1}^{\infty}(\Delta+s\delta),\quad\hat{\rho}=\rho=-\frac{\beta_1+\beta_2}{2}.$$

Set

$$Q^+:=\sum_{\mu\in\Pi}\mathbb{Z}_{\geq 0}\mu,\quad\hat{Q}^+=\sum_{\mu\in\hat{\Pi}}\mathbb{Z}_{\geq 0}\mu.$$

1.1.1. For $\nu\in\Delta_0$ let $s_\nu\in\operatorname{Aut}(V)$ be the reflection with respect to ν, i.e. $s_\nu(\lambda)=\lambda-\frac{(\lambda,\nu)}{(\nu,\nu)}\nu$. The Weyl group W of Δ_0 takes form $W=W_\alpha\times W_\gamma$, where W_α (resp., W_γ) is generated by the reflection s_α (resp., s_γ).

For $\nu\in V$ introduce $t_\nu\in\operatorname{Aut}(V)$ by the formula

$$t_\mu(\lambda)=\lambda-(\lambda,\mu)\delta.$$

Then $t_\mu t_\nu=t_{\mu+\nu}$. For $\nu\in\Delta_0$ we denote by T_ν the infinite cyclic group generated by t_ν and by $\hat{W}_\nu$ the group generated by s_ν and t_ν. The Weyl group of $\mathfrak{gl}(2|2)\hat{}$ is $\hat{W}=\hat{W}_\alpha\times\hat{W}_\gamma$. Notice that δ and $\beta_1-\beta_2$ lie in the kernel of the bilinear form so these vectors are $\hat{W}$-stable.

For a subgroup G of the Weyl group we introduce the following operator:

$$\mathcal{F}_G:=\sum_{w\in G}\operatorname{sgn}w\cdot w.$$

1.2. **Algebra $\mathcal{R}$.** We are going to use notation of [G2], 1.4, which we recall below.

1.2.1. Consider the space $\hat{\mathfrak{h}}^* = V \oplus \mathbb{R}\Lambda_0$ and extend our bilinear form by $(\Lambda_0, \delta) = 1, (\Lambda_0, \Lambda_0) = (\Lambda_0, \varepsilon_i) = (\Lambda_0, \delta_i) = 0$ for $i = 1, 2$. The Weyl group $\hat{W}$ acts on $\hat{\mathfrak{h}}^*$ as follows: the reflections act by the same formulas and the action of t_μ extends by the standard formula

$$t_\mu(\lambda) = \lambda + (\lambda, \delta)\mu - ((\lambda, \mu) + \frac{(\mu, \mu)}{2}(\lambda, \delta))\delta, \quad \mu \in V, \lambda \in \hat{\mathfrak{h}}^*$$

Call a $\hat{Q}^+$-*cone* a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

1.2.2. For a formal sum of the form $Y := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu,\ b_\nu \in \mathbb{Q}$ define the *support* of Y by $\mathrm{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^* |\ b_\nu \neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu \in \hat{Q}^+} b_\nu e^{\lambda - \nu}$, where $\lambda \in \hat{\mathfrak{h}}^*,\ b_\nu \in \mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y = \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of $\hat{Q}^+$-cones.

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

1.2.3. *Action of the Weyl group.* For $w \in \hat{W}$ set $w(\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu) := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. One has $wY \in \mathcal{R}$ iff $w(\mathrm{supp}\, Y)$ is a subset of a finite union of $\hat{Q}^+$-cones.

Let W' be a subgroup of $\hat{W}$. Let $\mathcal{R}_{W'} := \{Y \in \mathcal{R} |\ wY \in \mathcal{R}$ for each $w \in W'\}$. Clearly, $\mathcal{R}_{W'}$ is a subalgebra of $\mathcal{R}$.

1.2.4. *Infinite products.* An infinite product of the form $Y = \prod_{\nu \in X}(1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q},\ \ r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \hat{\Delta}_+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu \in X}(1 + a_\nu e^{-w\nu})^{r(\nu)},$$

is again an infinite product of the above form, since, as one easily sees ([G2], Lem. 1.2.8), the set $w\hat{\Delta}_+ \setminus \hat{\Delta}_+$ is finite. Hence $\mathcal{Y}$ is a $\hat{W}$-invariant multiplicative subset of $\mathcal{R}_{\hat{W}}$.

The elements of $\mathcal{Y}$ are invertible in $\mathcal{R}$: using the geometric series we can expand Y^{-1} (for example, $(1 - e^\alpha)^{-1} = -e^{-\alpha}(1 - e^{-\alpha})^{-1} = -\sum_{i=1}^\infty e^{-i\alpha}$).

1.2.5. *The subalgebra* $\mathcal{R}'$. Denote by $\mathcal{R}'$ the localization of $\mathcal{R}_{\hat{W}}$ by $\mathcal{Y}$. By above, $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$. Observe that $\mathcal{R}' \not\subset \mathcal{R}_{\hat{W}}$: for example, $(1 - e^{-\alpha})^{-1} \in \mathcal{R}'$, but $(1 - e^{-\alpha})^{-1} = \sum_{j=0}^\infty e^{-j\alpha} \notin \mathcal{R}_{\hat{W}}$. We extend the action of $\hat{W}$ from $\mathcal{R}_{\hat{W}}$ to $\mathcal{R}'$ by setting $w(Y^{-1}Y') := (wY)^{-1}(wY')$ for $y \in \mathcal{Y},\ Y' \in \mathcal{R}_{\hat{W}}$.

An infinite product of the form $Y=\prod_{\nu\in X}(1+a_\nu e^{-\nu})^{r(\nu)}$, where a_ν, X are as above and $r(\nu)\in\mathbb{Z}$ lies in $\mathcal{R}'$, and $wY=\prod_{\nu\in X}(1+a_\nu e^{-w\nu})^{r(\nu)}$. One has

$$\operatorname{supp}(Y)\subset\lambda'-\hat{Q}^+,\quad\text{where }\lambda':=-\sum_{\nu\in X\setminus\hat{\Delta}_+:a_\nu\neq 0} r_\nu\nu.$$

Remark. Set $q:=e^{-\delta}, x:=e^{-\alpha}, y_i:=e^{-\beta_i}$ and write elements of $\mathcal{R}'$ as power series in these variables. Since $\{e^{-\nu},\nu\in\hat{\Pi}\}=\{x,y_1,y_2,q(xy_1y_2)^{-1}\}$, the support of $Y\in\mathcal{R}'$ correspond to the expansion of Y in the region $|q|<|xy_1y_2|;|x|,|y_1|,|y_2|<1$.

1.2.6. Let W' be a subgroup of $\hat{W}$. For $Y\in\mathcal{R}'$ we say that Y *is W'-invariant (resp., W'-anti-invariant)* if $wY=Y$ (resp., $wY=\operatorname{sgn}(w)Y$) for each $w\in W'$.

Let $Y=\sum a_\mu e^\mu\in\mathcal{R}_{W'}$ be W'-anti-invariant. Then $a_{w\mu}=(-1)^{\operatorname{sgn}(w)}a_\mu$ for each μ and $w\in W'$. In particular, $W'\operatorname{supp}(Y)=\operatorname{supp}(Y)$, and, moreover, for each $\mu\in\operatorname{supp}(Y)$ one has $\operatorname{Stab}_{W'}\mu\subset\{w\in W'|\ \operatorname{sgn}(w)=1\}$. The condition $Y\in\mathcal{R}_{W'}$ is essential: for example, for $W'=\{\mathrm{id},s_\alpha\}$, the expressions $Y:=e^\alpha-e^{-\alpha}$, $Y^{-1}=e^{-\alpha}(1-e^{-2\alpha})^{-1}$ are W'-anti-invariant, but $\operatorname{supp}(Y^{-1})=-\alpha,-3\alpha,\dots$ is not s_α-invariant.

Take $Y=\sum a_\mu e^\mu\in\mathcal{R}_{W'}$. The sum $\mathcal{F}_{W'}(Y)=\sum_{w\in W'}\operatorname{sgn}(w)wY$ is an element of $\mathcal{R}$ if for each μ the sum $\sum_{w\in W'}\operatorname{sgn}(w)a_{w\mu}$ is finite (i.e., $W'\mu\cap\operatorname{supp}(Y)$ is finite). In this case $\mathcal{F}_{W'}(Y)\in\mathcal{R}$ and, writing $\mathcal{F}_{W'}(Y)=\sum b_\mu e^\mu$, we obtain $b_\mu=\sum_{w\in W'}\operatorname{sgn}(w)a_{w\mu}$ so $b_\mu=\operatorname{sgn}(w)b_{w\mu}$ for each $w\in W'$. We conclude that

$$Y\in\mathcal{R}_{W'}\ \&\ \mathcal{F}_{W'}(Y)\in\mathcal{R}\implies\begin{cases}\mathcal{F}_{W'}(Y)\in\mathcal{R}_{W'};\\ \mathcal{F}_{W'}(Y)\text{ is }W'\text{-anti-invariant};\\ \operatorname{supp}(\mathcal{F}_{W'}(Y))\text{ is }W'\text{-stable.}\end{cases}$$

Let us call a vector $\lambda\in\hat{\mathfrak{h}}^*$ *W'-regular* if $\operatorname{Stab}_{W'}\lambda=\{\mathrm{id}\}$. Say that the orbit $W'\lambda$ is W'-regular if λ is W'-regular (so the orbit consists of W'-regular points). If W' is an affine Weyl group, then for any $\lambda\in\hat{\mathfrak{h}}^*$ the stabilizer $\operatorname{Stab}_{W'}\lambda$ is either trivial or contains a reflection. Thus for $W'=\hat{W}_\alpha,\ \hat{W}_\gamma$ one has

$$Y\in\mathcal{R}_{W'}\ \&\ \mathcal{F}_{W'}(Y)\in\mathcal{R}\implies\operatorname{supp}(\mathcal{F}_{W'}(Y))\ \text{ is a union of }W'\text{-regular orbits.}$$

1.2.7. *Remark.* For $Y\in\mathcal{R}'$ the sum $\mathcal{F}_{W'}(Y)$ is not always W'-anti-invariant: for example, for $W'=\{\mathrm{id},s_\alpha\}$ one has $\mathcal{F}_{W'}((1-e^{-\alpha})^{-1})=(1-e^{-\alpha})^{-1}-(1-e^{\alpha})^{-1}=1+2e^{-\alpha}+2e^{-2\alpha}+\dots$ which is not W'-anti-invariant.

1.3. **Another form of denominator identity.** Introduce the following elements of $\mathcal{R}$:

$$R_0:=\prod_{\nu\in\Delta_{0,+}}(1-e^{-\nu}),\quad R_1:=\prod_{\nu\in\Delta_{1,+}}(1+e^{-\nu}),\quad R:=\frac{R_0}{R_1},$$
$$\hat{R}_0:=\prod_{\nu\in\hat{\Delta}_{0,+}}(1-e^{-\nu}),\quad \hat{R}_1:=\prod_{\nu\in\hat{\Delta}_{1,+}}(1+e^{-\nu}),\quad \hat{R}:=\frac{\hat{R}_0}{\hat{R}_1}.$$

The products Re^ρ and $\hat{R}e^\rho$ are $\hat{W}$-anti-invariant elements of $\mathcal{R}'$ (see, for instance, [G2], Lem. 1.5.1).

1.3.1. **Lemma.** *In the region $q<|y|<q^{-1}$ one has*

$$\frac{((1-q)_q^\infty)^2}{(1-qy)_q^\infty(1-qy^{-1})_q^\infty}=\sum_{n=1}^\infty\big(y^n+y^{-n}-y^{n-1}-y^{1-n}\big)\sum_{j=0}^\infty(-1)^j q^{(j+1)(j+2n)/2}$$

and this expression lies in $\mathcal{R}$ for $y=e^{\beta_2-\beta_1}$.

Proof. Consider the root system $\mathfrak{sl}(2|1)$ with the odd simple roots β_1',β_2' and the even positive root $\alpha'=\beta_1'+\beta_2'$. Note that the corresponding element ρ' is equal to zero. Consider the corresponding affine root system, let δ' be the minimal imaginary root and $\hat{W}'$ be its Weyl group. The affine denominator identity for $\mathfrak{sl}(2|1)$ written for $z:=e^{-\delta}$ takes form

$$\frac{(1-e^{-\alpha'})_z^\infty(1-ze^{\alpha'})_z^\infty\big((1-z)_z^\infty\big)^2}{\prod_{i=1}^2(1+e^{-\beta_i})_z^\infty(1+ze^{\beta_i})_z^\infty}=\sum_{n=-\infty}^\infty z^{n^2}\big(\frac{e^{n\alpha}}{1+z^ne^{-\beta_1}}-\frac{e^{-n\alpha}}{1+z^ne^{\beta_2}}\big).$$

Both sides are well-defined for real $0<z<1$ and β_i' such that $e^{\beta_i'}\neq z^n$ for $n\in\mathbb{Z}$. Taking $e^{\alpha'}=-1$ and $e^{-\beta_1'}:=-\xi$ we obtain $e^{-\beta_2'}=e^{-\alpha}e^{\beta_1'}=\xi^{-1}$ and the evaluation gives

$$\frac{2\big((1+z)_z^\infty(1-z)_z^\infty\big)^2}{(1-\xi)_z^\infty(1+\xi^{-1})_z^\infty(1-z\xi^{-1})_z^\infty(1+z\xi)_z^\infty}=\sum_{n=-\infty}^\infty(-1)^n z^{n^2}\big(\frac{1}{1-z^n\xi}-\frac{1}{1+z^n\xi}\big).$$

For $z^2=q,\xi^2=y$ we get

$$\frac{\big((1-q)_q^\infty\big)^2}{(1-qy)_q^\infty(1-qy^{-1})_q^\infty(1-y)}=2\sum_{m=-\infty}^\infty(-1)^m q^{\frac{m^2+m}{2}}\frac{1}{1-q^my}.$$

For $m>0$ one has $\frac{1}{1-q^my}=\sum_{k=0}^\infty q^{mk}y^k$ and $\frac{1}{1-q^{-m}y}=-\sum_{k=1}^\infty q^{mk}y^{-k}$ so

$$\begin{aligned}\frac{\big((1-q)_q^\infty\big)^2}{(1-qy)_q^\infty(1-qy^{-1})_q^\infty}&=1+(1-y)\textstyle\sum_{m=1}^\infty(-1)^m\sum_{k=0}^\infty(q^{\frac{m^2+m+2mk}{2}}y^k-q^{\frac{m^2-m+2mk}{2}}y^{-k})\\&=1+\textstyle\sum_{m=0}^\infty(-1)^m\sum_{k=1}^\infty(q^{\frac{(m+1)(m+2k)}{2}}(y^k-y^{k-1}+y^{-k}-y^{1-k})\end{aligned}$$

as required. One readily sees that the right-hand side of the above expression lies in $\mathcal{R}$ for $y=e^{\beta_2-\beta_1}$. □

1.3.2. Set $x:=e^{-\alpha},y_i:=e^{-\beta_i}$ for $i=1,2$ and $q:=e^{-\delta}$. Under this substitution, the left-hand side of (1) becomes $\hat{R}$ and, using Lemma 1.3.1, we rewrite (1) in the following form

$$\hat{R}e^\rho=(1+\sum f_n)e^{-\rho}\mathcal{F}_{\hat{W}_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big).\tag{3}$$

Denote by LHS (resp., RHS) the left-hand (resp., right-hand) side of the identity (3).

The denominator identity for $\mathfrak{gl}(2|2)$ takes the form

$$\mathcal{F}_{W_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)=Re^\rho=\mathcal{F}_{W_\gamma}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big). \tag{4}$$

so (3) can be rewritten as

$$\hat{R}e^\rho=(1+\sum f_n)\mathcal{F}_{T_\alpha}(Re^\rho).$$

In the sequel we need the following lemma.

1.3.3. **Lemma.** *If $\mathcal{F}_{T_\alpha}(Re^\rho)$ is well-defined (as an element of $\mathcal{R}$), then*

$$\mathcal{F}_{T_\alpha}(Re^\rho)=\mathcal{F}_{T_\gamma}(Re^\rho).$$

Proof. Note that $(\gamma-\alpha,\rho)=(\gamma-\alpha,\beta_i)=0$ for $i=1,2$ so $\frac{e^\rho}{(1-e^{-\beta_1})(1-e^{-\beta_2})}$ is invariant with respect to the action of $t_{\gamma-\alpha}$. Therefore

$$\mathcal{F}_{T_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)=\mathcal{F}_{T_\gamma}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big).$$

Using the formula (4), we obtain

$$\begin{array}{l}\mathcal{F}_{T_\alpha}(Re^\rho)=\mathcal{F}_{T_\alpha}\circ\mathcal{F}_{W_\gamma}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)==\mathcal{F}_{W_\gamma}\circ\mathcal{F}_{T_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)\\ =\mathcal{F}_{W_\gamma}\circ\mathcal{F}_{T_\gamma}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)==\mathcal{F}_{T_\gamma}\circ\mathcal{F}_{W_\gamma}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)=\mathcal{F}_{T_\gamma}(Re^\rho),\end{array}$$

as required. $\square$

As a corollary, (3) can be rewritten as $\hat{R}e^\rho=(1+\sum f_n)\mathcal{F}_{T_\gamma}(Re^\rho)$.

2. Proof of the denominator identity

2.1. By 1.2.4, LHS is an invertible element of $\mathcal{R}'$. In this subsection we show that RHS is a well-defined element of $\mathcal{R}$.

For $w\in\hat{W}_\alpha$ set

$$S_w:=\operatorname{supp}\big(w\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)\big).$$

One has

$$t_{n\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)=\frac{e^\rho q^n}{(1+q^ne^{-\beta_1})(1+q^ne^{-\beta_2})}.$$

Then for $n > 0$ one has

$$
(5)\qquad
\begin{array}{l}
S_{\mathrm{id}} = \{\rho - k_1\beta_1 - k_2\beta_2\}, \quad S_{s_\alpha} = \{\rho - (1+k_1+k_2)\alpha - k_1\beta_1 - k_2\beta_2\},\\
S_{t_{n\alpha}} = \{\rho - n(1+k_1+k_2)\delta - k_1\beta_1 - k_2\beta_2\},\\
S_{t_{-n\alpha}} = \{\rho - n(1+k_1+k_2)\delta + (k_1+1)\beta_1 + (k_2+1)\beta_2\},\\
S_{t_{n\alpha}s_\alpha} = \{\rho - n(1+k_1+k_2)\delta - (1+k_1+k_2)\alpha - k_1\beta_1 - k_2\beta_2\},\\
S_{t_{-n\alpha}s_\alpha} = \{\rho - n(1+k_1+k_2)\delta + (1+k_1+k_2)\alpha + k_1\beta_1 + k_2\beta_2\},
\end{array}
$$

where $k_1, k_2 \geq 0$. Observe that the above sets are pairwise disjoint so the sum $\mathcal{F}_{\hat{W}_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)$ is well-defined and its support lies in $\rho - \hat{Q}^+$. Clearly, the sum $1 + \sum f_n$ is well-defined.

One readily sees that

$$
(6)\qquad n\delta + k\beta_1 - k\beta_2 \in \hat{Q}^+ \iff |k| \leq n
$$

Thus the support of $1 + \sum f_n$ lies in $\{0\} \cup \{-m\delta + k\beta_1 - k\beta_2 |\ m > 0\} \cap -\hat{Q}^+$ (in particular, $(1 + \sum f_n) \in \mathcal{R}$). Hence RHS is a well-defined element of $\mathcal{R}$.

2.2. **Lemma.** *The expansion of $\frac{RHS}{LHS}$ in the region $|q| < |xy_1y_2|, |x|, |y_1|, |y_2| < 1$ is of the form $1 + \sum_{n=1}^{\infty}\sum_{j=-n}^{n} a_{n,j}q^n(\frac{y_1}{y_2})^j$, where $a_{n,j} \in \mathbb{Z}$.*

Proof. Recall that $LHS = \hat{R}e^\rho$ and that $\hat{R} \in \mathcal{Y}$ (see 1.2.4 for notation). By 2.1, $RHS \in \mathcal{R}$. Therefore the fraction

$$
Y := \frac{RHS}{LHS} = \hat{R}^{-1}e^{-\rho} \cdot RHS
$$

lies in $\mathcal{R}$.

Clearly, $-\rho \in \mathrm{supp}(\hat{R}^{-1}e^{-\rho}) \subset (-\rho - \hat{Q}^+)$. Since $\mathrm{supp}(RHS) \subset \rho - \hat{Q}^+$, we conclude that $\mathrm{supp}(Y) \subset -\hat{Q}^+$. By (5) the coefficient of e^ρ in RHS is 1; clearly, the coefficient of $e^{-\rho}$ in $\hat{R}^{-1}e^{-\rho}$ is also 1, so the coefficient of $e^0 = 1$ in Y is 1. In the light of Remark 1.2.5, the required assertion is equivalent to the inclusion

$$
(7)\qquad \mathrm{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2) | n \geq 0, |j| \leq n\}.
$$

Retain notation of 1.2.1. The element $\hat{\rho}_\alpha := 2\Lambda_0 + \frac{\alpha}{2}$ is the standard element for the corresponding copy of $\hat{\mathfrak{sl}_2} \subset \hat{\mathfrak{gl}(2|2)}$. Recall that $\hat{R} = \frac{\hat{R}_0}{\hat{R}_1}$ (see 1.3 for notation) so $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho} = \hat{R}_0 e^{\hat{\rho}_\alpha} \cdot (\hat{R}e^\rho)^{-1}$. By 1.2.4, $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ belongs to $\mathcal{R}_{\hat{W}}$. It is a standard fact that $\hat{R}_0 e^{\hat{\rho}_\alpha}$ is $\hat{W}_\alpha$-anti-invariant. Recall that $\hat{R}e^\rho$ is $\hat{W}$-anti-invariant. Thus $\hat{R}_1 e^{\hat{\rho}_\alpha - \rho}$ is a $\hat{W}_\alpha$-invariant element of $\mathcal{R}_{\hat{W}}$. One has

$$
\hat{R}_0 e^{\hat{\rho}_\alpha} Y = \hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot RHS = (1 + \sum f_n) \cdot \hat{R}_1 e^{\hat{\rho}_\alpha - \rho} \cdot \mathcal{F}_{\hat{W}_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big).
$$

The $\hat{W}_\alpha$-invariance of $\hat{R}_1 e^{\hat{\rho}_\alpha-\rho}$ gives

$$\hat{R}_1 e^{\hat{\rho}_\alpha-\rho}\cdot\mathcal{F}_{\hat{W}_\alpha}\Big(\frac{e^{\rho}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\Big)=\mathcal{F}_{\hat{W}_\alpha}\Big(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\Big)$$

so

$$\hat{R}_0 e^{\hat{\rho}_\alpha} Y=(1+\sum f_n)\cdot\mathcal{F}_{\hat{W}_\alpha}\Big(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\Big).$$

By 2.1, $\mathcal{F}_{\hat{W}_\alpha}\big(\frac{e^{\rho}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)$ lies in $\mathcal{R}$ so $\mathcal{F}_{\hat{W}_\alpha}\big(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)$ lies in $\mathcal{R}$. By 1.2.4, the term

$$\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}=e^{\hat{\rho}_\alpha}\prod_{\beta\in\hat{\Delta}_{1,+}\setminus\{\beta_1,\beta_2\}}(1+e^{-\beta})$$

lies in $\mathcal{R}_{\hat{W}}$. Therefore, in the light of 1.2.6, $\mathcal{F}_{\hat{W}_\alpha}\big(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Observe that $\frac{y_1}{y_2}=e^{\beta_2-\beta_1}$ is $\hat{W}$-invariant so f_n is $\hat{W}$-invariant. Thus $(1+\sum f_n)\mathcal{F}_{\hat{W}_\alpha}\big(\frac{\hat{R}_1 e^{\hat{\rho}_\alpha}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big)$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. As a result, $\hat{R}_0 e^{\hat{\rho}_\alpha} Y$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Write $Y=Y_1+Y_2$, where $Y_1=\sum_{n=0}^{\infty}\sum_{j=-\infty}^{\infty}a_{n,j}q^n(\frac{y_1}{y_2})^j$ and Y_2 does not have monomials of the form $q^n(\frac{y_1}{y_2})^j$, i.e $\mathrm{supp}(Y)=\mathrm{supp}(Y_1)\coprod\mathrm{supp}(Y_2)$. One has $Y_i\in\mathcal{R}$ because $\mathrm{supp}(Y_i)\subset\mathrm{supp}(Y)\subset-\hat{Q}^+$ $(i=1,2)$.

Since $\frac{y_1}{y_2}=e^{\beta_2-\beta_1}$ is $\hat{W}$-invariant, Y_1 is a $\hat{W}$-invariant element of $\mathcal{R}_{\hat{W}}$. Since $\hat{R}_0 e^{\hat{\rho}_\alpha}$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$, the product $\hat{R}_0 e^{\hat{\rho}_\alpha}Y_1$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. By above, $\hat{R}_0 e^{\hat{\rho}_\alpha}Y$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Hence $\hat{R}_0 e^{\hat{\rho}_\alpha}Y_2$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Assume that $Y_2\neq 0$. Recall that $\mathrm{supp}(Y_2)\subset-\hat{Q}^+$. Let μ be a maximal element in $\mathrm{supp}(Y_2)$ with respect to the standard partial order $\mu\leq\nu$ if $(\nu-\mu)\in\hat{Q}^+$. Then $\hat{\rho}_\alpha+\mu$ is a maximal element in the support of $\hat{R}_0 e^{\hat{\rho}_\alpha}Y_2$. By 1.2.6, this support is the union of $\hat{W}_\alpha$-regular orbits, so $\hat{\rho}_\alpha+\mu$ is a maximal element in a regular $\hat{W}_\alpha$-orbit (regularity means that each element has the trivial stabilizer in $\hat{W}_\alpha$). Since $\mu\in-\hat{Q}^+$ one has $\frac{2(\mu,\alpha)}{(\alpha,\alpha)}\in\mathbb{Z}$. Therefore $\frac{2(\hat{\rho}_\alpha+\mu,\alpha)}{(\alpha,\alpha)}=1+(\mu,\alpha)$, $\frac{2(\hat{\rho}_\alpha+\mu,\delta-\alpha)}{(\delta-\alpha,\delta-\alpha)}=1+(\mu,\delta-\alpha)$ are positive integers so (μ,α), $(\mu,\delta-\alpha)\geq 0$. Since $\mu\in-\hat{Q}^+$ one has $(\mu,\delta)=0$ and thus $(\mu,\alpha)=0$.

The element $\hat{\rho}_\gamma:=-2\Lambda_0+\frac{\gamma}{2}$ is the standard element for the corresponding copy of $\hat{\mathfrak{sl}_2}$. Using Lemma 1.3.3 we obtain

$$\hat{R}_0 e^{\hat{\rho}_\gamma}Y=(1+\sum f_n)\mathcal{F}_{\hat{W}_\gamma}\Big(\frac{\hat{R}_1 e^{\hat{\rho}_\gamma}}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\Big).$$

Repeating the above reasoning for $\hat{W}_\gamma$ we obtain $(\mu,\gamma)=0$. Hence $(\mu,\alpha)=(\mu,\gamma)=0$ and $\mu\in-\hat{Q}^+$. This implies $\mu=-m\delta+k(\beta_1-\beta_2)$, which contradicts to the construction of Y_2. Hence $Y_2 = 0$ so $Y = Y_1$ that is $\mathrm{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2)\}$. Combining the condition $\mathrm{supp}(Y) \subset -\hat{Q}^+$ and (6), we obtain the required inclusion (7). $\square$

2.3. **Evaluation.** By Lemma 2.2, $\frac{RHS}{LHS}$ is a function of one variable $y := \frac{y_1}{y_2}$. In order to establish the identity $LHS = RHS$, it is enough to verify that $\frac{RHS}{LHS}(y) = 1$ for a fixed x and some y_2, y_1 satisfying $y_1 = yy_2$. We will check this for $x = -1, y_2 = y, y_1 = y^2$, (i.e. $e^{-\alpha} = -1, e^{-\beta_1} = y^2, e^{-\beta_2} = y$).

One has $\frac{RHS}{LHS} = \hat{R}^{-1}(RHS \cdot e^{-\rho})$. We write $RHS \cdot e^{-\rho} = AB$, where

$$A := \frac{((1-q)_q^\infty)^2}{(1-qy)_q^\infty(1-qy^{-1})_q^\infty}, \quad B := e^{-\rho} \cdot \mathcal{F}_{\hat{W}_\alpha}\big(\frac{e^\rho}{(1+e^{-\beta_1})(1+e^{-\beta_2})}\big), \quad y = \frac{y_1}{y_2},$$

2.3.1. Recall that an infinite product $\prod_{i=1}^\infty (1+g_i(z))$, where $g_i(z)$ are holomorphic functions in $U \subset \mathbb{C}$ is called *normally convergent in* U if $\sum g_i(z)$ normally converges in U. By [R], a normally convergent infinite product converges to a function $g(z)$, which is holomorphic in U; moreover, the set of zeros of $g(z)$ is the union of the sets of zeros of $1 + f_i(z)$ and the order of each zero is the sum of the orders of the corresponding zeros of $1 + g_i(z)$.

The denominator of $A(y)$ normally converges in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Thus $A(y)$ is a meromorphic function in the region $0 < |y|$ with simple poles at the points $y = q^n$, $n \in \mathbb{Z} \setminus \{0\}$.

2.3.2. The evaluation of $\hat{R}$ takes the form

$$\hat{R}(y) = \frac{2\prod_{n=1}^\infty (1-q^n)^4(1+q^n)^2(1+q^{n-1}y^3)(1+q^ny^{-3})}{\prod_{n=0}^\infty \prod_{s=1}^2 (1+q^ny^s)(1+q^{n+1}y^{-s})(1-q^ny^s)(1-q^{n+1}y^{-s})}.$$

All infinite product in the above expression normally converge in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Therefore $\hat{R}(y)$ is a meromorphic function in the region $0 < |y|$, and

$$\frac{A}{\hat{R}}(y) = \frac{(1-y)\prod_{n=0}^\infty (1-q^ny^2)(1-q^{n+1}y^{-2})\prod_{s=1}^2 (1+q^ny^s)(1+q^{n+1}y^{-s})}{2\prod_{n=0}^\infty (1-q^{2n})^2(1+q^ny^3)(1+q^{n+1}y^{-3})} \tag{8}$$

is a meromorphic function in the region $0 < |y|$ with simple poles, the zero of order two at $y = 1$ and all other zeros of order one; the set of poles (resp., zeros) is P (resp., Z):

$$P := \{y|\ y^3 = -q^m \ \&\ y \neq -q^k\}_{k,m\in\mathbb{Z}}, \qquad Z := \{y|\ y^2 = \pm q^m\}_{m\in\mathbb{Z}}.$$

One readily sees from (8) that

$$\lim_{y\to 1}(y-1)^{-2}\frac{A}{\hat{R}}(y) = 2, \qquad \frac{A}{\hat{R}}(qy) = \frac{A}{\hat{R}}(y) \cdot \frac{q(1-qy)}{1-y}. \tag{9}$$

2.3.3. Recall that

$$B=\sum_{n=-\infty}^{\infty}\Big(\frac{q^n}{(1+q^ne^{-\beta_1})(1+q^ne^{-\beta_2})}-\frac{q^ne^{-\alpha}}{(1+q^ne^{-\beta_1-\alpha})(1+q^ne^{-\beta_2-\alpha})}\Big)$$

so the evaluation takes the form

$$\begin{aligned} B(y) &= \textstyle\sum_{n=-\infty}^{\infty}\big(\frac{q^n}{(1+q^ny)(1+q^ny^2)}+\frac{q^n}{(1-q^ny)(1-q^ny^2)}\big)\\ &= \textstyle\frac{1}{1-y}\sum_{n=-\infty}^{\infty}\big(\frac{q^n}{1+q^ny}-\frac{q^ny}{1+q^ny^2}+\frac{q^n}{1-q^ny}-\frac{q^ny}{1-q^ny^2}\big).\end{aligned} \tag{10}$$

Each point $y\in\mathbb{C}$ such that $y^2\neq\pm q^n$ for $n\in\mathbb{Z}$ has a neighborhood U such that the above sums converge absolutely and uniformly. Thus $B(y)$ is a meromorphic function in the region $0<|y|$ with poles at the points $\{y|\ y^2=\pm q^n\}_{n\in\mathbb{Z}}$, where all poles are simple except the pole of order two at $y=1$. Let us verify that $B(y)=0$ for each $y\in P$. For $y^3=-q^k$, $y\not\in\{-q^m\}$ one has

$$\frac{y}{1\pm q^ny^2}=\frac{y}{1\mp q^{n+k}y^{-1}}=\mp\frac{1}{1\mp q^{-n-k}y}$$

so $B(y)=0$. Hence $\frac{AB}{\hat{R}}(y)$ is a holomorphic function in the region $0<|y|$.

From the second formula of (10) one sees that $B(qy)=q^{-1}\frac{1-y}{1-qy}$; combining with (9) we get $\frac{AB}{\hat{R}}(qy)=\frac{AB}{\hat{R}}(y)$. Since $\frac{AB}{\hat{R}}(y)$ is a holomorphic function in the region $0<|y|$, this function is constant. One has

$$\lim_{y\to 1}(1-y)^2\cdot B(y)=\lim_{y\to 1}(1-y)^2\frac{1}{(1-y)(1-y^2)}=\frac{1}{2}.$$

Using (9) we obtain $\frac{AB}{\hat{R}}(1)=1$ so $\frac{AB}{\hat{R}}(y)\equiv 1$ (for $0<|y|$). This completes the proof of denominator identity.

3. Application to Jacobi identity (2)

Recall the Gauss' identity (which follows easily from the Jacobi triple product)

$$\Box(-q)=\frac{(1-q)_q^{\infty}}{(1+q)_q^{\infty}}.$$

The evaluation of the identity (1) at $y_1=y_2=1$ gives

$$\frac{\big((1-x)_q^{\infty}(1-qx^{-1})_q^{\infty}\big)^2\big((1-q)_q^{\infty}\big)^4}{4((1+q)_q^{\infty})^4\big((1+x)_q^{\infty}(1+qx^{-1})_q^{\infty}\big)^2}=\sum_{n=-\infty}^{\infty}a_n,$$
$$\text{where } a_n:=\frac{q^n}{(1+q^n)(1+q^n)}-\frac{q^nx}{(1+q^nx)(1+q^nx)}.$$

We divide both sides of the above identity by $\frac{(1-x)^2}{16}$ and take the limit $x\mapsto 1$; we get

$$\Big(\frac{(1-q)_q^{\infty}}{(1+q)_q^{\infty}}\Big)^8=1-16\sum_{n=1}^{\infty}\frac{q^n(q^{2n}-4q^n+1)}{(1+q^n)^4},$$

since

$$\lim_{x\to 1}\frac{a_0}{(x-1)^2}=\frac{1}{16},\quad \lim_{x\to 1}\frac{a_n+a_{-n}}{(x-1)^2}=-\frac{q^n(q^{2n}-4q^n+1)}{(1+q^n)^4}.$$

Using the expansion $(a+1)^{-4}=\sum_{j=0}^{\infty}(-1)^j\frac{(j+1)(j+2)(j+3)}{6}a^j$, we obtain

$$\Box(-q)^8=\big(\tfrac{(1-q)_q^{\infty}}{(1+q)_q^{\infty}}\big)^8\ \ =1+16\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(-1)^j j^3 q^{nj},$$

which implies the required identity

$$\Box(q)^8=1+16\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}(-1)^{j+nj}j^3q^{nj}.$$

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