WEYL DENOMINATOR IDENTITY FOR THE AFFINE LIE SUPERALGEBRA $\mathfrak{gl}(2|2)$

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Abstract. We prove the Weyl denominator identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)$ conjectured by V. Kac and M. Wakimoto in [KW]. As it was pointed out in [KW], this gives a new proof of the Jacobi identity for the number of presentations of a given integer as a sum of 8 squares.

0. Introduction

The denominator identities for Lie superalgebras were formulated and partially proven in the paper of V. Kac and M. Wakimoto [KW]. In the same paper it was shown how various classical identities in number theory as the number of representations of a given integer as a sum of $d$ squares can be obtained, for some $d$, by evaluation of certain denominator identities. The following cases are considered in the paper [KW]:

(a) basic Lie superalgebras, i.e. the finite-dimensional simple Lie superalgebras, which have a reductive even part and admit an even non-degenerate invariant bilinear form;

(b) the affinization of basic Lie superalgebras with non-zero dual Coxeter number;

(c) the (twisted) affinization of a strange Lie superalgebras $Q(n)$;

(d) the affinization of $\mathfrak{gl}(2|2)$ (this is the smallest basic Lie superalgebras with zero dual Coxeter number).

Some of the cases (a), (b) are proven in [KW]; the proof is based on combinatorics of root systems and a certain result from representation theory. The rest of (a) was proven in [G1] using only combinatorics of root systems. The rest of (b) was proven in [G2] using (a) and the existence of Casimir operator. The case (c) was proven in [Z] analytically. In the present paper we prove (d), i.e. the identity for the affine Lie superalgebra $\mathfrak{gl}(2|2)$ conjectured in [KW], 7.1. The proof uses the existence of Casimir operator and an idea of [Z].

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In order to write down the identity, we introduce the following infinite products after [DK]: for a parameter $q$ and a formal variable $x$ we set

$$(1 + x)^\infty_q := \prod_{n=0}^{\infty}(1 + q^n x), \quad \text{and} \quad (1 - x)^\infty_q := \prod_{n=0}^{\infty}(1 - q^n x).$$

These infinite products converge for any $x \in \mathbb{C}$ if the parameter $q$ is a real number $0 < q < 1$. In particular, they are well-defined for $0 < x = q < 1$ and $(1 \pm q)^\infty_q := \prod_{n=1}^{\infty}(1 \pm q^n)$.

Take the formal variables $x, y_1, y_2$. The denominator identity for $\mathfrak{gl}(2|2)$ can be written in the following form

$$(1 - x)^\infty_q (1 - qx^{-1})^\infty_q (1 - xy_1y_2)^\infty_q (1 - q(xy_1y_2)^{-1})^\infty_q \left((1 - q)^\infty_q\right)^4 = \prod_{i=1}^{2} (1 + y_i)^\infty_q (1 + qy_i^{-1})^\infty_q (1 + xy_i)^\infty_q (1 + qx^{-1}y_i^{-1})^\infty_q$$

$$= \frac{((1 - q)^\infty_q)^2}{(1 - qy_1y_2)^\infty_q (1 - qy_1y_2)^\infty_q} \cdot \sum_{n=-\infty}^{\infty} \frac{q^n}{(1 + q^n y_1)(1 + q^n y_2)} - \frac{q^n x}{(1 + q^n xy_1)(1 + q^n xy_2)}.$$ 

Expanding the factor $\frac{((1 - q)^\infty_q)^2}{(1 - qy_1y_2)^\infty_q (1 - qy_1y_2)^\infty_q}$ in the region $q < |\frac{y_1}{y_2}| < q^{-1}$ we obtain (see Lemma [1.3.1])

$$\frac{((1 - q)^\infty_q)^2}{(1 - qy_1y_2)^\infty_q (1 - qy_1y_2)^\infty_q} = 1 + \sum_{n=1}^{\infty} f_n(y_1 y_2),$$

where $f_n(y) := \left(y^n + y^{-n} - y^{n-1} - y^{1-n}\right) \sum_{j=0}^{\infty} (-1)^j q^{(j+1)(j+2n)/2}$

and this gives the identity conjectured by V. Kac and M. Wakimoto.

The left-hand side of the identity represents the Weyl denominator $\hat{R}$ for the affine Lie superalgebra $\mathfrak{gl}(2|2)$; the second factor in the right-hand side is the analogue of the right-hand side of the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number. Note that the denominator identity for the affine Lie superalgebra $\mathfrak{sl}(2|2)$ can be obtained from the denominator identity for $\mathfrak{gl}(2|2)$ by taking $y_1 = y_2$; as a result, the denominator identity for $\mathfrak{sl}(2|2)$ is almost similar to the denominator identity for affine Lie superalgebras with non-zero dual Coxeter number with one extra-factor $(1 - q)^\infty_q$ in the left-hand side (since the dimension of Cartan subalgebra for $\mathfrak{sl}(2|2)$ is less by one than the dimension of Cartan subalgebra for $\mathfrak{gl}(2|2)$).

As it is shown in [KW], the evaluation of this identity gives the following Jacobi identity [J]:

$$\Box(q)^8 = 1 + 16 \sum_{j,k=1}^{\infty} (-1)^{j+1} k^3 q^{jk},$$

(2)
where $\Box(q) = \sum_{j \in \mathbb{Z}} q^{j^2}$ and thus the coefficient of $q^m$ in the power series expansion of $\Box(q)^d$ is the number of representation of a given integer as a sum of $d$ squares (taking into the account the order of summands).

In Section 1 we introduce notation. In Section 2 we prove the identity (1). In Section 3 we recall how to deduce the Jacobi identity from the identity (1).

1. Notation

1.1. Root system. Consider $V := \mathbb{R}^5$ endowed by a bilinear form and an orthogonal basis $\varepsilon_1, \varepsilon_2, \delta_1, \delta_2, \delta$ such that $(\varepsilon_i, \varepsilon_i) = 1 = -(\delta_i, \delta_i)$ for $i = 1, 2$ and $(\delta, \delta) = 0$. Set

$$\beta_1 := \delta_1 - \varepsilon_1, \ \alpha := \varepsilon_1 - \varepsilon_2, \ \beta_2 := \varepsilon_2 - \delta_2, \ \gamma := \beta_1 + \alpha + \beta_2 = \delta_1 - \delta_2.$$  

The root system of $\mathfrak{gl}(2|2)$ is $\Delta_0 = \{\pm \alpha, \pm \gamma\}$, $\Delta_1 = \{\pm \beta_i; \pm (\alpha + \beta_i)\}_{i=1,2}$. The affine root system is $\hat{\Delta}_i = \bigcup_{s \in \mathbb{Z}} (\Delta_i + s\delta)$, $i = 0, 1$.

We consider the following sets of simple roots for $\mathfrak{gl}(2|2)$ and $\mathfrak{gl}(2|2)^\tau$ respectively:

$$\Pi := \{\beta_1, \alpha, \beta_2\}, \ \text{and} \ \hat{\Pi} = \{\beta_1, \alpha, \beta_2, \delta - \gamma\}.$$  

One has

$$\Delta_+ = \{\alpha, \gamma; \beta_i, \alpha + \beta_i\}_{i=1,2}, \ \hat{\Delta}_+ = \Delta_+ \cup \bigcup_{s=1}^\infty (\Delta + s\delta), \ \hat{\rho} = \rho = -\frac{\beta_1 + \beta_2}{2}.$$  

Set

$$Q^+ := \sum_{\mu \in \Pi} \mathbb{Z}_{\geq 0} \mu, \ \hat{Q}^+ := \sum_{\mu \in \Pi} \mathbb{Z}_{\geq 0} \mu.$$  

1.1.1. For $\nu \in \Delta_0$ let $s_\nu \in \text{Aut}(V)$ be the reflection with respect to $\nu$, i.e. $s_\nu(\lambda) = \lambda - \frac{\langle \lambda, \nu \rangle}{\langle \nu, \nu \rangle}\nu$. The Weyl group $W$ of $\Delta_0$ takes form $W = W_\alpha \times W_\gamma$, where $W_\alpha$ (resp., $W_\gamma$) is generated by the reflection $s_\alpha$ (resp., $s_\gamma$).

For $\nu \in V$ introduce $t_\nu \in \text{Aut}(V)$ by the formula

$$t_\mu(\lambda) = \lambda - (\lambda, \mu)\delta.$$  

Then $t_\mu t_\nu = t_{\mu + \nu}$. For $\nu \in \Delta_0$ we denote by $T_\nu$ the infinite cyclic group generated by $t_\nu$ and by $\hat{T}_\nu$ the group generated by $s_\nu$ and $t_\nu$. The Weyl group of $\mathfrak{gl}(2|2)$ is $\hat{W} = W_\alpha \times \hat{W}_\gamma$. Notice that $\delta$ and $\beta_1 - \beta_2$ lie in the kernel of the bilinear form so these vectors are $\hat{W}$-stable.

For a subgroup $G$ of the Weyl group we introduce the following operator:

$$F_G := \sum_{w \in G} \text{sgn} \ w \cdot w.$$  

1.2. Algebra $\mathcal{R}$. We are going to use notation of [G2], 1.4, which we recall below.
1.2.1. Consider the space $\hat{\mathfrak{h}}^* = V \oplus \mathbb{R}\Lambda_0$ and extend our bilinear form by $(\Lambda_0, \delta) = 1$, $(\Lambda_0, \delta) = (\Lambda_0, \epsilon_i) = (\Lambda_0, \delta_i) = 0$ for $i = 1, 2$. The Weyl group $\hat{W}$ acts on $\hat{\mathfrak{h}}^*$ as follows: the reflections act by the same formulas and the action of $t_\mu$ extends by the standard formula

$$t_\mu(\lambda) = \lambda + (\lambda, \delta)\mu - ((\lambda, \mu) + (\mu, \mu)/2)(\lambda, \delta), \quad \mu \in V, \lambda \in \hat{\mathfrak{h}}^*$$

Call a $\hat{Q}^+$-cone a set of the form $(\lambda - \hat{Q}^+)$, where $\lambda \in \hat{\mathfrak{h}}^*$.

1.2.2. For a formal sum of the form $Y := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$, $b_\nu \in \mathbb{Q}$ define the support of $Y$ by $\text{supp}(Y) := \{\nu \in \hat{\mathfrak{h}}^*| b_\nu \neq 0\}$. Let $\mathcal{R}$ be a vector space over $\mathbb{Q}$, spanned by the sums of the form $\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{\lambda - \nu}$, where $\lambda \in \hat{\mathfrak{h}}^*$, $b_\nu \in \mathbb{Q}$. In other words, $\mathcal{R}$ consists of the formal sums $Y = \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu$ with the support lying in a finite union of $\hat{Q}^+$-cones.

Clearly, $\mathcal{R}$ has a structure of commutative algebra over $\mathbb{Q}$. If $Y \in \mathcal{R}$ is such that $YY' = 1$ for some $Y' \in \mathcal{R}$, we write $Y^{-1} := Y'$.

1.2.3. Action of the Weyl group. For $w \in \hat{W}$ set $w(\sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^\nu) := \sum_{\nu \in \hat{\mathfrak{h}}^*} b_\nu e^{w\nu}$. One has $wY \in \mathcal{R}$ iff $w(\text{supp } Y)$ is a subset of a finite union of $\hat{Q}^+$-cones.

Let $W'$ be a subgroup of $\hat{W}$. Let $\mathcal{R}_{W'} := \{Y \in \mathcal{R}| wY \in \mathcal{R} \text{ for each } w \in W'\}$. Clearly, $\mathcal{R}_{W'}$ is a subalgebra of $\mathcal{R}$.

1.2.4. Infinite products. An infinite product of the form $Y = \prod_{\nu \in X} (1 + a_\nu e^{-\nu})^{r(\nu)}$, where $a_\nu \in \mathbb{Q}$, $r(\nu) \in \mathbb{Z}_{\geq 0}$ and $X \subset \hat{\Delta}$ is such that the set $X \setminus \hat{\Delta}^+$ is finite, can be naturally viewed as an element of $\mathcal{R}$; clearly, this element does not depend on the order of factors. Let $\mathcal{Y}$ be the set of such infinite products. For any $w \in \hat{W}$ the infinite product

$$wY := \prod_{\nu \in X} (1 + a_\nu e^{-w\nu})^{r(\nu)},$$

is again an infinite product of the above form, since, as one easily sees ([G2], Lem. 1.2.8), the set $w\hat{\Delta}^+ \setminus \hat{\Delta}^+$ is finite. Hence $\mathcal{Y}$ is a $\hat{W}$-invariant multiplicative subset of $\mathcal{R}_{\hat{W}}$.

The elements of $\mathcal{Y}$ are invertible in $\mathcal{R}$: using the geometric series we can expand $Y^{-1}$ (for example, $(1 - e^\alpha)^{-1} = -e^{-\alpha}(1 - e^{-\alpha})^{-1} = -\sum_{i=1}^{\infty} e^{-i\alpha}$).

1.2.5. The subalgebra $\mathcal{R}'$. Denote by $\mathcal{R}'$ the localization of $\mathcal{R}_{\hat{W}}$ by $\mathcal{Y}$. By above, $\mathcal{R}'$ is a subalgebra of $\mathcal{R}$. Observe that $\mathcal{R}' \nsubseteq \mathcal{R}_{\hat{W}}$: for example, $(1 - e^{-\alpha})^{-1} \in \mathcal{R}'$, but $(1 - e^{-\alpha})^{-1} = \sum_{j=0}^{\infty} e^{-j\alpha} \notin \mathcal{R}_{\hat{W}}$. We extend the action of $\hat{W}$ from $\mathcal{R}_{\hat{W}}$ to $\mathcal{R}'$ by setting $w(Y^{-1}Y') := (wY)^{-1}(wY')$ for $y \in \mathcal{Y}$, $Y' \in \mathcal{R}_{\hat{W}}$. 
An infinite product of the form $Y = \prod_{\nu \in \Delta} (1 + a_{\nu}e^{-\nu})r_{\nu}$, where $a_{\nu}, X$ are as above and $r(\nu) \in \mathbb{Z}$ lies in $\mathcal{R}$, and $wY = \prod_{\nu \in \Delta} (1 + a_{\nu}e^{-w\nu})r_{\nu}$. One has

\[ \operatorname{supp}(Y) \subseteq \lambda' - \hat{Q}^{+}, \quad \text{where} \quad \lambda' := -\sum_{\nu \in \Delta \setminus \Delta_{+}, a_{\nu} \neq 0} r_{\nu} \nu. \]

**Remark.** Set $q := e^{-\delta}$, $x := e^{-\alpha}$, $y_{i} := e^{-\beta_{i}}$, and write elements of $\mathcal{R}$ as power series in these variables. Since $\{e^{-\nu}, \nu \in \Pi\} = \{x, y_{1}, y_{2}, q(xy_{1}y_{2})^{-1}\}$, the support of $Y \in \mathcal{R}$ correspond to the expansion of $Y$ in the region $|q| < |xy_{1}y_{2}||x||y_{1}||y_{2}| < 1$.

1.2.6. Let $W'$ be a subgroup of $\hat{W}$. For $Y \in \mathcal{R}$ we say that $Y$ is $W'$-invariant (resp., $W'$-anti-invariant) if $wY = Y$ (resp., $wY = \operatorname{sgn}(w)Y$) for each $w \in W'$.

Let $Y = \sum a_{\mu}e^{\mu} \in \mathcal{R}_{W}$ be $W'$-anti-invariant. Then $a_{w\mu} = (-1)^{\operatorname{sgn}(w)}a_{\mu}$ for each $\mu$ and $w \in W'$. In particular, $W'$-supp$(Y) = \operatorname{supp}(Y)$, and, moreover, for each $\mu \in \operatorname{supp}(Y)$ one has $\text{Stab}_{W'} \mu \subseteq \{w \in W' \mid \operatorname{sgn}(w) = 1\}$. The condition $Y \in \mathcal{R}_{W'}$ is essential: for example, for $W' = \{\text{id}, s_{\alpha}\}$, the expressions $Y := e^{\alpha} - e^{-\alpha}$, $Y^{-1} = e^{-\alpha}(1 - e^{-2\alpha})^{-1}$ are $W'$-anti-invariant, but supp$(Y^{-1}) = -\alpha, -3\alpha, \ldots$ is not $s_{\alpha}$-invariant.

Take $Y = \sum a_{\mu}e^{\mu} \in \mathcal{R}_{W'}$. The sum $\mathcal{F}_{W'}(Y) = \sum_{w \in W'} \operatorname{sgn}(w)wY$ is an element of $\mathcal{R}$ if for each $\mu$ the sum $\sum_{w \in W'} \operatorname{sgn}(w)a_{w\mu}$ is finite (i.e., $W' \mu \cap \operatorname{supp}(Y)$ is finite). In this case $\mathcal{F}_{W'}(Y) \in \mathcal{R}$ and, writing $\mathcal{F}_{W'}(Y) = \sum b_{\mu}e^{\mu}$, we obtain $b_{\mu} = \sum_{w \in W'} \operatorname{sgn}(w)a_{w\mu}$ so $b_{\mu} = \operatorname{sgn}(w)b_{w\mu}$ for each $w \in W'$. We conclude that

\[ Y \in \mathcal{R}_{W'} \land \mathcal{F}_{W'}(Y) \in \mathcal{R} \implies \begin{cases} \mathcal{F}_{W'}(Y) \in \mathcal{R}_{W'}; \\ \mathcal{F}_{W'}(Y) \text{ is } W'-\text{anti-invariant}; \\ \operatorname{supp}(\mathcal{F}_{W'}(Y)) \text{ is } W'-\text{stable}. \end{cases} \]

Let us call a vector $\lambda \in \hat{h}^{*}$ $W'$-regular if $\text{Stab}_{W'} \lambda = \{\text{id}\}$. Say that the orbit $W'\lambda$ is $W'$-regular if $\lambda$ is $W'$-regular (so the orbit consists of $W'$-regular points). If $W'$ is an affine Weyl group, then for any $\lambda \in \hat{h}^{*}$ the stabilizer $\text{Stab}_{W'} \lambda$ is either trivial or contains a reflection. Thus for $W' = \hat{W}_{\alpha}$, $\hat{W}_{\gamma}$ one has

\[ Y \in \mathcal{R}_{W'} \land \mathcal{F}_{W'}(Y) \in \mathcal{R} \implies \operatorname{supp}(\mathcal{F}_{W'}(Y)) \text{ is a union of } W'-\text{regular orbits}. \]

1.2.7. **Remark.** For $Y \in \mathcal{R}'$ the sum $\mathcal{F}_{W'}(Y)$ is not always $W'$-anti-invariant: for example, for $W' = \{\text{id}, s_{\alpha}\}$ one has $\mathcal{F}_{W'}((1 - e^{-\alpha})^{-1}) = (1 - e^{-\alpha})^{-1} - (1 - e^{\alpha})^{-1} = 1 + 2e^{-\alpha} + 2e^{-2\alpha} + \ldots$ which is not $W'$-anti-invariant.

1.3. **Another form of denominator identity.** Introduce the following elements of $\mathcal{R}$:

\[ R_{0} := \prod_{\nu \in \Delta_{0,+}} (1 - e^{-\nu}), \quad R_{1} := \prod_{\nu \in \Delta_{1,+}} (1 + e^{-\nu}), \quad R := \frac{R_{0}}{R_{1}}, \]

\[ \hat{R}_{0} := \prod_{\nu \in \Delta_{0,+}} (1 - e^{-\nu}), \quad \hat{R}_{1} := \prod_{\nu \in \Delta_{1,+}} (1 + e^{-\nu}), \quad \hat{R} := \frac{\hat{R}_{0}}{\hat{R}_{1}}. \]

The products $Re^{\theta}$ and $\hat{R}e^{\theta}$ are $\hat{W}$-anti-invariant elements of $\mathcal{R}'$ (see, for instance, [G2], Lem. 1.5.1).
1.3.1. Lemma. In the region \( q < |y| < q^{-1} \) one has

\[
\frac{(1 - q)^\infty}{(1 - qy)^\infty(1 - qy^{-1})_q^\infty} = \sum_{n=1}^\infty (y^n + y^{-n} - y^{n-1} - y^{1-n}) \sum_{j=0}^\infty (-1)^j q^{(j+1)(j+2n)/2}
\]

and this expression lies in \( \mathcal{R} \) for \( y = e^{\beta_2 - \beta_1} \).

Proof. Consider the root system \( \mathfrak{sl}(2|1) \) with the odd simple roots \( \beta'_1, \beta'_2 \) and the even positive root \( \alpha' = \beta'_1 + \beta'_2 \). Note that the corresponding element \( \rho' \) is equal to zero. Consider the corresponding affine root system, let \( \delta' \) be the minimal imaginary root and \( \hat{W'} \) be its Weyl group. The affine denominator identity for \( \mathfrak{sl}(2|1) \) written for \( z := e^{-\delta} \) takes form

\[
\frac{2((1 + z)_z^\infty(1 - z)^\infty)^2}{(1 - \xi)_z^\infty(1 + \xi^{-1})_z^\infty(1 - z\xi^{-1})_z^\infty(1 + z\xi)_z^\infty} = \sum_{n=-\infty}^\infty (-1)^n z^n \left( \frac{1}{1 - z^n \xi} - \frac{1}{1 + z^n \xi} \right).
\]

Both sides are well-defined for real \( 0 < z < 1 \) and \( \beta'_1 \) such that \( e^{\beta'_1} \neq z^n \) for \( n \in \mathbb{Z} \). Taking \( e^{\alpha'} = -1 \) and \( e^{-\beta'_1} := -\xi \) we obtain \( e^{-\beta_1} = e^{-\alpha} e^{\beta_1} = \xi^{-1} \) and the evaluation gives

\[
\frac{2((1 + z)_z^\infty(1 - z)^\infty)^2}{(1 - \xi)_z^\infty(1 + \xi^{-1})_z^\infty(1 - z\xi^{-1})_z^\infty(1 + z\xi)_z^\infty} = \sum_{n=-\infty}^\infty (-1)^n z^n \left( \frac{1}{1 - z^n \xi} - \frac{1}{1 + z^n \xi} \right).
\]

For \( z^2 = q, \xi^2 = y \) we get

\[
\frac{(1 - q)^\infty}{(1 - qy)^\infty(1 - qy^{-1})_q^\infty(1 - y)} = 2 \sum_{m=-\infty}^\infty (-1)^m q^{\frac{m^2 + m}{2}} \frac{1}{1 - q^m y}.
\]

For \( m > 0 \) one has \( \frac{1}{1 - q^{m+1}} = \sum_{k=0}^\infty q^{mk} y^k \) and \( \frac{1}{1 - q^{-m-1}} = -\sum_{k=0}^\infty q^{mk} y^{-k} \) so

\[
\frac{(1 - q)^\infty}{(1 - qy)^\infty(1 - qy^{-1})_q^\infty(1 - y)} = 1 + (1 - y) \sum_{m=1}^\infty (-1)^m \sum_{k=0}^\infty (q^{\frac{m^2 + m + 2mk}{2}} y^k - q^{m^2 - m + 2mk} y^{-k})
\]

as required. One readily sees that the right-hand side of the above expression lies in \( \mathcal{R} \) for \( y = e^{\beta_2 - \beta_1} \).

1.3.2. Set \( x := e^{-\alpha}, y_i := e^{-\beta_i} \) for \( i = 1, 2 \) and \( q := e^{-\delta} \). Under this substitution, the left-hand side of \( (1) \) becomes \( \hat{R} \) and, using Lemma 1.3.1, we rewrite \( (1) \) in the following form

\[
\hat{R} e^\rho = (1 + \sum_{n} f_n) e^{-\rho} \mathcal{F}_W (e^\rho \frac{1}{(1 + e^{\beta_1})(1 + e^{-\beta_2})})
\]

Denote by \( LHS \) (resp., \( RHS \)) the left-hand (resp., right-hand) side of the identity \( (3) \).
The denominator identity for $\mathfrak{gl}(2\mid 2)$ takes the form

$$
\mathcal{F}_{W_\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \text{Re}^\rho = \mathcal{F}_{W_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).
$$

so (3) can be rewritten as

$$
\hat{\text{Re}}^\rho = (1 + \sum f_n) \mathcal{F}_{T_\alpha}(\text{Re}^\rho).
$$

In the sequel we need the following lemma.

1.3.3. Lemma. If $\mathcal{F}_{T_\alpha}(\text{Re}^\rho)$ is well-defined (as an element of $\mathcal{R}$), then

$$
\mathcal{F}_{T_\alpha}(\text{Re}^\rho) = \mathcal{F}_{T_\gamma}(\text{Re}^\rho).
$$

Proof. Note that $(\gamma - \alpha, \rho) = (\gamma - \alpha, \beta_i) = 0$ for $i = 1, 2$ so $\frac{e^\rho}{(1 - e^{-\beta_1})(1 - e^{-\beta_2})}$ is invariant with respect to the action of $t_{\gamma - \alpha}$. Therefore

$$
\mathcal{F}_{T_\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{T_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).
$$

Using the formula (4), we obtain

$$
\mathcal{F}_{T_\alpha}(\text{Re}^\rho) = \mathcal{F}_{T_\alpha} \circ \mathcal{F}_{W_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = = \mathcal{F}_{W_\gamma} \circ \mathcal{F}_{T_\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{T_\gamma} \circ \mathcal{F}_{W_\gamma}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{T_\gamma}(\text{Re}^\rho),
$$

as required. \qed

As a corollary, (3) can be rewritten as $\hat{\text{Re}}^\rho = (1 + \sum f_n) \mathcal{F}_{T_\gamma}(\text{Re}^\rho)$.

2. Proof of the denominator identity

2.1. By LHS is an invertible element of $\mathcal{R}'$. In this subsection we show that RHS is a well-defined element of $\mathcal{R}$.

For $w \in \hat{W}_\alpha$ set

$$
S_w := \text{supp}(w(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})).
$$

One has

$$
\text{t}_{\alpha}(\frac{e^\rho}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \frac{e^\rho q^n}{(1 + q^n e^{-\beta_1})(1 + q^n e^{-\beta_2})}.
$$
Then for \( n > 0 \) one has
\[
S_{id} = \{ \rho - k_1\beta_1 - k_2\beta_2 \}, \quad S_{s_{\alpha}} = \{ \rho - (1 + k_1 + k_2)\alpha - k_1\beta_1 - k_2\beta_2 \}, \\
S_{p_{\alpha}} = \{ \rho - n(1 + k_1 + k_2)\delta - k_1\beta_1 - k_2\beta_2 \}, \\
S_{l_{\alpha}s_{\alpha}} = \{ \rho - n(1 + k_1 + k_2)\delta + (1 + k_1 + k_2)\alpha - k_1\beta_1 - k_2\beta_2 \}, \\
S_{l_{\alpha}s_{\alpha}} = \{ \rho - n(1 + k_1 + k_2)\delta + (1 + k_1 + k_2)\alpha + k_1\beta_1 + k_2\beta_2 \}.
\]
where \( k_1, k_2 \geq 0 \). Observe that the above sets are pairwise disjoint so the sum \( F_W(\frac{e^{\rho}}{(1+e^{-\rho_1})(1+e^{-\rho_2})}) \) is well-defined and its support lies in \( \rho - \hat{Q}^+ \). Clearly, the sum \( 1 + \sum f_n \) is well-defined.

One readily sees that
\[
n\delta + k\beta_1 - k\beta_2 \in \hat{Q}^+ \iff |k| \leq n
\]

Thus the support of \( 1 + \sum f_n \) lies in \( \{0\} \cup \{-m\delta + k\beta_1 - k\beta_2 | m > 0\} \cap -\hat{Q}^+ \) (in particular, \( (1 + \sum f_n) \in \mathcal{R} \). Hence \( RHS \) is a well-defined element of \( \mathcal{R} \).

2.2. **Lemma.** The expansion of \( \frac{RHS}{LHS} \) in the region \( |q| < |xy_1y_2|, |x|, |y_1|, |y_2| < 1 \) is of the form \( 1 + \sum_{n=1}^{\infty} \sum_{j=-n}^{n} a_{n,j} q^n (\frac{y_1}{y_2})^j \), where \( a_{n,j} \in \mathbb{Z} \).

**Proof.** Recall that \( LHS = \hat{R}e^\rho \) and that \( \hat{R} \in \mathcal{Y} \) (see [1.2.4] for notation). By [2.1] \( RHS \in \mathcal{R} \). Therefore the fraction
\[
Y := \frac{RHS}{LHS} = \hat{R}^{-1}e^-\rho \cdot RHS
\]
lies in \( \mathcal{R} \).

Clearly, \(-\rho \in \text{supp}(\hat{R}^{-1}e^-\rho) \subset (-\rho - \hat{Q}^+) \). Since \( \text{supp}(RHS) \subset \rho - \hat{Q}^+ \), we conclude that \( \text{supp}(Y) \subset -\hat{Q}^+ \). By [5] the coefficient of \( e^\rho \) in \( RHS \) is 1; clearly, the coefficient of \( e^-\rho \) in \( \hat{R}^{-1}e^-\rho \) is also 1, so the coefficient of \( e^0 = 1 \) in \( Y \) is 1. In the light of Remark [1.2.3] the required assertion is equivalent to the inclusion
\[
\text{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2) | n \geq 0, |j| \leq n \}.
\]

Retain notation of [1.2.1]. The element \( \hat{\rho}_\alpha := 2\Lambda_0 + \frac{\alpha}{2} \) is the standard element for the corresponding copy of \( \text{st}_2 \subset \mathfrak{gl}(2|2) \). Recall that \( \hat{R} = \frac{\hat{R}_0}{\hat{R}_1} \) (see [1.3] for notation) so \( \hat{R}_1e^{\hat{\rho}_\alpha - \rho} = \hat{R}_0e^{\hat{\rho}_\alpha} \cdot (\hat{R}e^\rho)^{-1} \). By [1.2.4] \( \hat{R}_1e^{\hat{\rho}_\alpha - \rho} \) belongs to \( \mathcal{R}_W \). It is a standard fact that \( \hat{R}_0e^{\hat{\rho}_\alpha} \) is \( \hat{W}_\alpha \)-anti-invariant. Recall that \( \hat{R}e^\rho \) is \( \hat{W} \)-anti-invariant. Thus \( \hat{R}_1e^{\hat{\rho}_\alpha - \rho} \) is a \( \hat{W}_\alpha \)-invariant element of \( \mathcal{R}_W \). One has
\[
\hat{R}_0e^{\hat{\rho}_\alpha} Y = \hat{R}_1e^{\hat{\rho}_\alpha - \rho} \cdot RHS = (1 + \sum f_n) \cdot \hat{R}_1e^{\hat{\rho}_\alpha - \rho} \cdot F_{\hat{W}_\alpha}(\frac{e^\rho}{(1+e^{-\rho_1})(1+e^{-\rho_2})}).
\]
The $\hat{W}_\alpha$-invariance of $\hat{R}_1 e^{\hat{\beta}_n - \rho}$ gives

$$\hat{R}_1 e^{\hat{\beta}_n - \rho} \cdot \mathcal{F}_{\hat{W}_\alpha}(\frac{e^{\rho}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}) = \mathcal{F}_{\hat{W}_\alpha}(\frac{\hat{R}_1 e^{\hat{\beta}_n}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})$$

so

$$\hat{R}_0 e^{\hat{\beta}_n} Y = (1 + \sum f_n) \cdot \mathcal{F}_{\hat{W}_\alpha}(\frac{\hat{R}_1 e^{\hat{\beta}_n}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).$$

By 2.1, $\mathcal{F}_{\hat{W}_\alpha}(\frac{e^{\rho}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})$ lies in $\mathcal{R}$ so $\mathcal{F}_{\hat{W}_\alpha}(\frac{\hat{R}_1 e^{\hat{\beta}_n}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})$ lies in $\mathcal{R}$. By 1.2.3, the term

$$\frac{\hat{R}_1 e^{\hat{\beta}_n}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})} = e^{\hat{\beta}_n} \prod_{\beta \in \Delta_{1,+} \setminus \{\beta_1, \beta_2\}} (1 + e^{-\beta})$$

lies in $\mathcal{R}_{\hat{W}}$. Therefore, in the light of 1.2.6, $\mathcal{F}_{\hat{W}_\alpha}(\frac{\hat{R}_1 e^{\hat{\beta}_n}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. Observe that $\frac{\mu}{y_2} = e^{\beta_2 - \beta_1}$ is $\hat{W}$-invariant so $f_n$ is $\hat{W}$-invariant. Thus $(1 + \sum f_n) \mathcal{F}_{\hat{W}_\alpha}(\frac{\hat{R}_1 e^{\hat{\beta}_n}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})})$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. As a result, $\hat{R}_0 e^{\hat{\beta}_n} Y$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Write $Y = Y_1 + Y_2$, where $Y_1 = \sum_{n=0}^{\infty} \sum_{j=-\infty}^{\infty} a_{n,j} q^n (\frac{\mu}{y_2})^j$ and $Y_2$ does not have monomials of the form $q^n (\frac{\mu}{y_2})^j$, i.e supp$(Y) = \text{supp}(Y_1) \bigcap \text{supp}(Y_2)$. One has $Y_i \in \mathcal{R}$ because supp$(Y_i) \subset \text{supp}(Y) \subset -\hat{Q}^+$ ($i = 1, 2$).

Since $\frac{\mu}{y_2} = e^{\beta_2 - \beta_1}$ is $\hat{W}$-invariant, $Y_1$ is a $\hat{W}$-invariant element of $\mathcal{R}_{\hat{W}}$. Since $\hat{R}_0 e^{\hat{\beta}_n}$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$, the product $\hat{R}_0 e^{\hat{\beta}_n} Y_1$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$. By above, $\hat{R}_0 e^{\hat{\beta}_n} Y$ is a $\hat{W}_\alpha$-anti-invariant element of $\mathcal{R}_{\hat{W}_\alpha}$.

Assume that $Y_2 \neq 0$. Recall that supp$(Y_2) \subset -\hat{Q}^+$. Let $\mu$ be a maximal element in supp$(Y_2)$ with respect to the standard partial order $\mu \preceq \nu$ if $\nu - \mu \in \hat{Q}^+$. Then $\hat{\rho}_\alpha + \mu$ is a maximal element in the support of $\hat{R}_0 e^{\hat{\beta}_n} Y_2$. By 1.2.6, this support is the union of $\hat{W}_\alpha$-regular orbits, so $\hat{\rho}_\alpha + \mu$ is a maximal element in a regular $\hat{W}_\alpha$-orbit (regularity means that each element has the trivial stabilizer in $\hat{W}_\alpha$). Since $\mu \in -\hat{Q}^+$ one has $\frac{2(\mu, \alpha)}{(\alpha, \alpha)} \in \mathbb{Z}$.

Therefore $\frac{2(\hat{\rho}_\alpha + \mu, \alpha)}{(\alpha, \alpha)} = 1 + (\mu, \alpha), \frac{2(\hat{\rho}_\alpha + \mu, \delta - \alpha)}{(\delta - \alpha, \delta - \alpha)} = 1 + (\mu, \delta - \alpha)$ are positive integers so $(\mu, \alpha), (\mu, \delta - \alpha) \geq 0$. Since $\mu \in -\hat{Q}^+$ one has $(\mu, \delta) = 0$ and thus $(\mu, \alpha) = 0$.

The element $\hat{\rho}_\gamma := -2A_0 + \frac{\gamma}{2}$ is the standard element for the corresponding copy of $\mathfrak{sl}_2$. Using Lemma 1.3.3 we obtain

$$\hat{R}_0 e^{\hat{\rho}_\gamma} Y = (1 + \sum f_n) \mathcal{F}_{\hat{W}_\gamma}(\frac{\hat{R}_1 e^{\hat{\rho}_\gamma}}{(1 + e^{-\beta_1})(1 + e^{-\beta_2})}).$$

Repeating the above reasoning for $\hat{W}_\gamma$ we obtain $(\mu, \gamma) = 0$. Hence $(\mu, \alpha) = (\mu, \gamma) = 0$ and $\mu \in -\hat{Q}^+$. This implies $\mu = -m\delta + k(\beta_1 - \beta_2)$, which contradicts to the construction.
of $Y_2$. Hence $Y_2 = 0$ so $Y = Y_1$ that is $\text{supp}(Y) \subset \{-n\delta + j(\beta_1 - \beta_2)\}$. Combining the condition $\text{supp}(Y) \subset -\hat{Q}^+$ and \eqref{e:Y}, we obtain the required inclusion \eqref{e:Y1}. \hfill \Box

2.3. Evaluation. By Lemma \cite{2.2} $\frac{\text{RHS}}{\text{LHS}}$ is a function of one variable $y := \frac{y_1}{y_2}$. In order to establish the identity $\text{LHS} = \text{RHS}$, it is enough to verify that $\frac{\text{RHS}}{\text{LHS}}(y) = 1$ for a fixed $x$ and some $y_2, y_1$ satisfying $y_1 = y y_2$. We will check this for $x = -1, y_2 = y, y_1 = y^2$, (i.e. $e^{-\alpha} = -1, e^{-\beta_1} = y^2, e^{-\beta_2} = y$).

One has $\frac{\text{RHS}}{\text{LHS}} = \hat{R}^{-1}(\text{RHS} \cdot e^{-\rho})$. We write $\text{RHS} \cdot e^{-\rho} = AB$, where

$$A := \frac{((1 - q)^{\infty})^2}{(1 - qy)^{\infty}(1 - qy^{-1})^{\infty}}, \quad B := e^{-\rho} \cdot \mathcal{F}_W \left( \frac{e^\rho}{(1 + e^{-\beta})(1 + e^{-\beta_2})} \right), \quad y = \frac{y_1}{y_2},$$

2.3.1. Recall that an infinite product $\prod_{i=1}^{\infty} (1 + g_i(z))$, where $g_i(z)$ are holomorphic functions in $U \subset \mathbb{C}$ is called normally convergent in $U$ if $\sum g_i(z)$ normally converges in $U$. By \cite{R}, a normally convergent infinite product converges to a function $g(z)$, which is holomorphic in $U$; moreover, the set of zeros of $g(z)$ is the union of the sets of zeros of $1 + f_i(z)$ and the order of each zero is the sum of the orders of the corresponding zeros of $1 + g_i(z)$.

The denominator of $A(y)$ normally converges in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Thus $A(y)$ is a meromorphic function in the region $0 < |y|$ with simple poles at the points $y = q^n, n \in \mathbb{Z} \setminus \{0\}$.

2.3.2. The evaluation of $\hat{R}$ takes the form

$$\hat{R}(y) = \frac{2 \prod_{n=0}^{\infty} (1 - q^n)^4(1 + q^n)^2(1 + q^{n-1}y^3)(1 + q^n y^{-3})}{\prod_{n=0}^{\infty} \prod_{s=1}^{2} (1 + q^n y^s)(1 + q^{n+1} y^{-s})(1 - q^n y^s)(1 - q^{n+1} y^{-s})}.$$  

All infinite product in the above expression normally converge in any $U \subset X$, where $X \subset \mathbb{C}$ is a compact not containing 0. Therefore $\hat{R}(y)$ is a meromorphic function in the region $0 < |y|$, and

$$\frac{A}{\hat{R}}(y) = \frac{(1 - y) \prod_{n=0}^{\infty} (1 - q^n y^2)(1 - q^{n+1} y^{-2}) \prod_{s=1}^{2} (1 + q^n y^s)(1 + q^{n+1} y^{-s})}{2 \prod_{n=0}^{\infty} (1 - q^{2n})^2(1 + q^n y^3)(1 + q^{n+1} y^{-3})}$$

is a meromorphic function in the region $0 < |y|$ with simple poles, the zero of order two at $y = 1$ and all other zeros of order one; the set of poles (resp., zeros) is $P$ (resp., $Z$):

$$P := \{y | y^3 = -q^m \& y \neq -q^k \}_{k,m \in \mathbb{Z}}, \quad Z := \{y | y^2 = \pm q^m \}_{m \in \mathbb{Z}}.$$

One readily sees from \eqref{e:R} that

$$\lim_{y \to 1} (y - 1)^{-2} \frac{A}{\hat{R}}(y) = 2, \quad \frac{A(q y)}{\hat{R}(y)} = \frac{A(y)}{\hat{R}(y)} \cdot \frac{q(1 - q y)}{1 - y}.$$
2.3.3. Recall that

\[ B = \sum_{n=-\infty}^{\infty} \left( \frac{q^n}{(1 + q^n e^{-\beta})} - \frac{q^n e^{-\alpha}}{(1 + q^n e^{-\beta - \alpha})} \right) \]

so the evaluation takes the form

\[ B(y) = \sum_{n=-\infty}^{\infty} \left( \frac{q^n y}{1-q^n y} + \frac{q^n y}{1-q^n y^2} \right) \]

Each point \( y \in \mathbb{C} \) such that \( y^2 \neq \pm q^n \) for \( n \in \mathbb{Z} \) has a neighborhood \( U \) such that the above sums converge absolutely and uniformly. Thus \( B(y) \) is a meromorphic function in the region \( 0 < |y| \) with poles at the points \( \{ y \mid y^2 = \pm q^n \} \), where all poles are simple except the pole of order two at \( y = 1 \). Let us verify that \( B(y) = 0 \) for each \( y \in P \). For \( y^3 = -q^k \), \( y \not\in \{-q^m\} \) one has

\[ \frac{y}{1 \pm q^n y^2} = \frac{1}{1 \mp q^{n+k} y^{-1}} = \mp \frac{1}{1 \mp q^{n-k} y} \]

so \( B(y) = 0 \). Hence \( \frac{AB}{R}(y) \) is a holomorphic function in the region \( 0 < |y| \).

From the second formula of (10) one sees that \( B(qy) = q^{-1} \frac{1-y}{1-qy} \); combining with (9) we get \( \frac{AB}{R}(qy) = \frac{AB}{R}(y) \). Since \( \frac{AB}{R}(y) \) is a holomorphic function in the region \( 0 < |y| \), this function is constant. One has

\[ \lim_{y \to 1} (1-y)^2 \cdot B(y) = \lim_{y \to 1} (1-y)^2 \frac{1}{(1-y)(1-y^2)} = \frac{1}{2}. \]

Using (9) we obtain \( \frac{AB}{R}(1) = 1 \) so \( \frac{AB}{R}(y) \equiv 1 \) (for \( 0 < |y| \)). This completes the proof of denominator identity.

3. Application to Jacobi identity (2)

Recall the Gauss’ identity (which follows easily from the Jacobi triple product)

\[ \square(-q) = \frac{(1-q)_q}{(1+q)_q^2}. \]

The evaluation of the identity (11) at \( y_1 = y_2 = 1 \) gives

\[ \frac{(1-x)^{\infty}_q}{4((1+q)^{\infty}_q)^4}((1-x)^{\infty}_q)^2((1-q)^{\infty}_q)^4 = \sum_{n=0}^{\infty} a_n, \]

where \( a_n := \frac{q^n (q^n - 4q^n + 1)}{(1+q^n)(1+q^n)}. \)

We divide both sides of the above identity by \( \frac{1-x^2}{16} \) and take the limit \( x \to 1 \); we get

\[ \frac{(1-q)_q^{\infty}}{(1+q)_q^{\infty}}^8 = 1 - 16 \sum_{n=1}^{\infty} \frac{q^n (q^{2n} - 4q^n + 1)}{(1+q^n)^4}, \]
since

\[
\lim_{x \to 1} \frac{a_0}{(x-1)^2} = \frac{1}{16}, \quad \lim_{x \to 1} \frac{a_n + a_{-n}}{(x-1)^2} = -\frac{q^n(q^{2n} - 4q^n + 1)}{(1 + q^n)^4}.
\]

Using the expansion \((a + 1)^{-4} = \sum_{j=0}^{\infty} (-1)^j \frac{(j+1)(j+2)(j+3)}{6} a^j\), we obtain

\[
\Box(-q)^8 = \left(\frac{(1-q)^\infty}{(1+q)^\infty}\right)^8 = 1 + 16 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^j j^3 q^{nj},
\]

which implies the required identity

\[
\Box(q)^8 = 1 + 16 \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} (-1)^j j^3 q^{nj}.
\]

References


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