1. Gradings

1.1. **Definition.** Let A be a k-algebra and I be an abelian group. A decomposition $A = \oplus_{i \in I} A_i$ is called a *grading* if $A_i A_j \subset A_{i+j}$.

1.2. **Definition.** Let A be a graded algebra and N be an A-module. A decomposition $N = \oplus_{i \in I} N_i$ is called a *grading* if $A_i N_j \subset A_{i+j}$.

2. Filtrations

2.1. **Definition.** Let A be a k-algebra. An increasing sequence of subspaces $\{F^i(A)\}_{i \in \mathbb{Z}}$ is called an *increasing filtration of* A if $F^i(A)F^j(A) \subseteq F^{i+j}(A)$.

Set

$$\operatorname{gr}_F A := \oplus_{i \in \mathbb{Z}} F^i(A)/F^{i-1}(A).$$

An increasing filtration is called *exhausting* if $\cup_{i \in \mathbb{Z}} F^i(A) = A$.

Let A be an algebra with an exhausting increasing filtration $F^i(A)$. We say that $x \in A$ has degree n if $x \in F^n(A) \setminus F^{n-1}(A)$; we denote by $\operatorname{gr} x$ the element of $\operatorname{gr}_F A$ which is the image of x under the canonical map $F^n(A) \to F^n(A)/F^{n-1}(A) \subset \operatorname{gr}_F A$.

2.2. Let A be a filtered algebra. Let ε be a formal variable and $k[\varepsilon]$ be the polynomial algebra.

The *Rees algebra* $\tilde{A} := \oplus_{i \in \mathbb{Z}} \varepsilon^i F^i(A)$ is an algebra over $k[\varepsilon]$ with the operation induced by the operation in A. Viewed as an algebra over k, the Rees algebra is a graded algebra with jth homogeneous component $\varepsilon^j F^j(\mathfrak{g})$.

The *evaluation* of a $k[\varepsilon]$-algebra B at a point $c \in k$ is the quotient algebra $B/(\varepsilon - c)B$. Notice that for $c = 0$ the ideal εB is graded and thus the quotient $B/\varepsilon B$ inherits a grading.

2.3. *Exercise.* The evaluation of the Rees algebra $\tilde{A}$ at $\varepsilon = 0$ is canonically isomorphic (as a graded vector space) to $\operatorname{gr}_F A$.

Define an algebra structure on $\operatorname{gr}_F A$ via the above isomorphism. The algebra $\operatorname{gr}_F A$ is called the *assocated graded* of A. The algebra $\operatorname{gr}_F A$ is naturally graded:

$$\operatorname{gr}_F A = \oplus_{i \in \mathbb{Z}} (\operatorname{gr}_F A)^i, \quad \text{where } (\operatorname{gr}_F A)^i := F^i(A)/F^{i-1}(A).$$

Exercise. Check that for a of degree m and b of degree n $\operatorname{gr}_F a \operatorname{gr}_F b$ is the image of ab under the map $F^{m+n}(A) \to F^{m+n}(A)/F^{m+n-1}(A) \subset \operatorname{gr}_F A$.

Exercise. The evaluation of the Rees algebra $\tilde{A}$ at $\varepsilon = 1$ is canonically isomorphic to A.

2.4. **Definition.** Let A be a filtered algebra and N be an A-module. An increasing sequence of subspaces $\{F^i(N)\}_{i\in\mathbb{Z}}$ is called a *filtration of* N if $F^i(A)F^j(N) \subseteq F^{i+j}(A)$. Let A be a filtered algebra and N be a filtered A-module. One can define the notion of Rees module over Rees algebra and define a structure of $\operatorname{gr}_F A$-module $\operatorname{gr}_F N := \oplus_{i\in\mathbb{Z}} F^i(N)/F^{i-1}(N)$.

3. The canonical filtration

Let $\mathfrak{g}$ be a Lie algebra over k. The canonical filtration on $U(\mathfrak{g})$ is given by $\mathcal{F}^i(U(\mathfrak{g})) = 0$ for $i < 0$, $\mathcal{F}^0(U(\mathfrak{g})) = k$, $\mathcal{F}^1(U(\mathfrak{g})) = k + \mathfrak{g}$ and $\mathcal{F}^r(U(\mathfrak{g})) = \operatorname{span}\{x_1 \dots x_r |\ x_i \in k + \mathfrak{g}\}$. The canonical filtration is an increasing exhausted filtration.

Each term $\mathcal{F}^r(U(\mathfrak{g}))$ is $\operatorname{ad}\mathfrak{g}$-invariant and so $\operatorname{gr} U(\mathfrak{g}) := \operatorname{gr}_{\mathcal{F}} U(\mathfrak{g})$ inherits a structure of $\operatorname{ad}\mathfrak{g}$-module given by

$$(\operatorname{ad} g)\operatorname{gr} y := \operatorname{gr}(\operatorname{ad} g)(y), \quad g \in \mathfrak{g}, y \in U(\mathfrak{g}).$$

Observe that $U(\mathfrak{g})$ is generated by $\mathfrak{g}$ and thus $\operatorname{gr}_F U(\mathfrak{g})$ is generated by $(\operatorname{gr} U(\mathfrak{g}))^1$. The elements of $(\operatorname{gr} U(\mathfrak{g}))^1$ commute: for $x, y \in \mathfrak{g}$ one has $\operatorname{gr} x \operatorname{gr} y, \operatorname{gr} y \operatorname{gr} x \in (\operatorname{gr} U(\mathfrak{g}))^2$ and $\operatorname{gr} x \operatorname{gr} y - \operatorname{gr} y \operatorname{gr} x = \operatorname{gr} xy - \operatorname{gr} yx$, but $xy - yx = [x, y] \in \mathfrak{g} \subset \mathcal{F}^1(U(\mathfrak{g}))$ so $\operatorname{gr} xy = \operatorname{gr} yx$. Therefore $\operatorname{gr}_F U(\mathfrak{g})$ is a commutative algebra generated by $(\operatorname{gr} U(\mathfrak{g}))^1$ which is naturally isomorphic to $\mathfrak{g}$. This gives a natural algebra map $S(\mathfrak{g}) \to \operatorname{gr}_F U(\mathfrak{g})$. The PBW-theorem is equivalent to the statement that this map is bijection: $\operatorname{gr}_F U(\mathfrak{g}) \cong S(\mathfrak{g})$.

4. Homework

The exersice above (see 2.3) and below.

4.1. *Exercise.* Let e, f, h be the standard basis of $\mathfrak{sl}(2)$. Prove that

$$\operatorname{HC}(e^k f^k) = k! h(h-1)\dots(h-(k-1)).$$

4.2. *Exercise.* Show that the Harish-Chandra projection $\operatorname{HC} : U(\mathfrak{g}) \to U(\mathfrak{h})$ is not an algebra map, but its restriction to the zero weight space $\operatorname{HC} : U(\mathfrak{g})^{\mathfrak{h}} \to U(\mathfrak{h})$ is an algebra homomorphism.

4.3. *Exercise.* Show that $V(\lambda)$ has a unique (up to a scalar) primitive vector: $V(\lambda)^{\mathfrak{n}^+} = \mathbb{C}v_\lambda$.

4.4. *Exercise.* Let $\mathfrak{g}$ be a Kac-Moody algebra. Prove that $\dim V(\lambda) = 1$ iff $\lambda(h) = 0$ for all $h \in \mathfrak{h}'$.

4.5. *Exercise.* Show that $V(\lambda)$ is integrable iff $\lambda \in P^+(\pi)$ where

$$P^+(\pi) := \{\lambda \in \mathfrak{h}^* | \ <\lambda, \alpha_i^\vee> \in \mathbb{Z}_{\geq 0} \ \text{ for } i = 1, \dots, n\}.$$

4.6. *Exercise.* Let $\mathfrak{g}$ be a Kac-Moody algebra and M be a $\mathfrak{g}$-module in the category $\mathcal{O}$. Verify that

$$M^\# := \{f \in M^* | \ \dim U(\mathfrak{h}) f < \infty\}$$

is a $\mathfrak{g}$-module belonging to $\mathcal{O}$ ($\mathfrak{g}$-submodule of M^*).

Check that $M \mapsto M^\#$ is a contravariant functor that is

$$\mathrm{Hom}_{\mathfrak{g}}(M^\#, N^\#) = \mathrm{Hom}_{\mathfrak{g}}(N, M).$$

4.7. *Exercise.* Show that $V(\lambda)^\# \cong V(\lambda)$.