INTRODUCTION TO REPRESENTATION THEORY. 20/12/04. LECTURE 10

1. Nilpotent, solvable and semisimple Lie algebras

Set
$$D_1\mathfrak{g} := D^1\mathfrak{g} := [\mathfrak{g},\mathfrak{g}], \quad D_k\mathfrak{g} := [\mathfrak{g}, D_{k-1}\mathfrak{g}], \quad D^k\mathfrak{g} := [D^{k-1}\mathfrak{g}, D^{k-1}\mathfrak{g}].$$

Definition. $\mathfrak{g}$ is nilpotent if $D_k(\mathfrak{g}) = 0$ for some k.

$\mathfrak{g}$ is solvable if $D^k(\mathfrak{g}) = 0$ for some k.

$\mathfrak{g}$ is semisimple if $\mathfrak{g}$ is finite dimensioanl and has no nonero solvable ideals.

We always assume that $\mathfrak{g}$ is finite dimensional.

The Lie algebra $\mathfrak{n}_n$ of strictly upper triangular matrices is nilpotent. Any nilpotent Lie algebra is solvable since $D^k \subset C^k$. The Lie algebra $\mathfrak{b}_n$ of upper triangular matrices is solvable. However $\mathfrak{b}$ is not nilpotent since $D_1(\mathfrak{b}) = D^1(\mathfrak{b}) = \mathfrak{n}$ and $D_1(\mathfrak{b}) = D_2(\mathfrak{b}) = \dots$. The quotients and subalgebras of a nilpotent (solvable) Lie algebra are nilpotent (solvable).

1.1. Nilpotent Lie algebras.

1.1.1. **Proposition.** *The following conditions are equivalent*

(i) $D_n(\mathfrak{g}) = 0$.

(ii) For any $x_0, \dots, x_n \in \mathfrak{g}$

$$[x_n, [..[x_2, [x_1, x_0]]...] = (\operatorname{ad} x_n)(\operatorname{ad} x_{n-1}) \dots (\operatorname{ad} x_1)(x_0) = 0.$$

(iii) There exists a chain of ideals
$$\mathfrak{g} = I_0 \supset I_1 \supset \dots \supset I_n = 0$$
such that $[\mathfrak{g}, I_k] \subset I_{k+1}$ for all k.

Proof. Obviously, (i)$\iff$ (ii). (i)$\implies$ (iii) since the chain $I_k := D_k(\mathfrak{g})$ satisfies the condition of (iii). Moreover, this is the minimal chain which satisfies the condition: if $I_0, \dots, I_n$ is such that $[\mathfrak{g}, I_k] \subset I_{k+1}$ for all k, then $I_k \supset D_k(\mathfrak{g})$. Hence (iii)$\implies$ (i). $\square$

1.1.2. **Definition.** An element $x \in \mathfrak{g}$ is called *ad-nilpotent* if $\operatorname{ad} x$ is nilpotent that is $(\operatorname{ad} x)^n = 0$ for some n.

1.1.3. The above proposition implies that any element of a nilpotent Lie algebra is ad-nilpotent (see (ii)). Suprisingly, for finite dimensional algebras the converse is also true.

Engel's Theorem.

If a finite dimensional Lie algebra consists of ad-nilpotent elements then it is nilpotent.

We prove this theorem later.

1.2. We need the following

Lemma. *If $g \in \mathfrak{gl}(V)$ is a nilpotent linear endomorphism of V then g is ad-nilpotent.*

Proof. Let $g \in \mathfrak{gl}(V)$ be a nilpotent linear endomorphism. This means that $g^n = 0$ for a suitable n. Denote by $L : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ the linear operator $\psi \mapsto g\psi$ and by $R : \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ the linear operator $\psi \mapsto \psi g$. For any k the linear operator L^k maps $\psi \in \mathfrak{gl}(V)$ to $g^k\psi$. Thus $L^n = 0$ and similarly $R^n = 0$. Observe that the operators L and R commute: $LR(\psi) = g\psi g = RL(\psi)$ for all $\psi \in \mathfrak{gl}(V)$. One has $\operatorname{ad} g = L - R$ and $(\operatorname{ad} g)^{2n} = (L-R)^{2n} = 0$ since $LR = RL$ and $L^n = R^n = 0$. $\square$

Warning: the converse does not hold: $\mathrm{id} \in \mathfrak{gl}(V)$ is ad-nilpotent.

1.3. **Theorem.** *Let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$ where $\dim V = m < \infty$. If all elements of $\mathfrak{g}$ are nilpotent linear endomorphisms of V then there exists a basis of V relative to which $\mathfrak{g}$ is a subalgebra of $\mathfrak{n}(m)$.*

Proof. The statement is an easy consequence of the following assertion: under the assumptions of the theorem, if $\dim V \neq 0$ then V contains a vector $v \neq 0$ such that $\mathfrak{g}v = 0$.

Let us prove this assertion by induction on $\dim \mathfrak{g}$.

For $\dim \mathfrak{g} = 1$ one has $\mathfrak{g} = \mathbb{F}x$ where x is nilpotent. Any $v \in \ker x$ satisfies $\mathfrak{g}v = 0$.

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$ and $\mathfrak{h} \neq \mathfrak{g}$. The Lie algebra $\mathfrak{h}$ acts (via ad) on the vector space $\mathfrak{g}$ and, by Lemma 1.2, all elements of $\mathfrak{h}$ act nilpotently. Therefore all elements of $\mathfrak{h}$ act nilpotently also on the quotient module $\mathfrak{g}/\mathfrak{h}$. By the induction hypothesis, there exists $x \in \mathfrak{g}$ such that $x \notin \mathfrak{h}$ and $[x, \mathfrak{h}] \subset \mathfrak{h}$. Therefore $\mathfrak{h}' := \mathfrak{h} + \mathbb{F}x$ is a subalgebra of $\mathfrak{g}$. Thus any subalgebra of $\mathfrak{g}$ is an ideal of codimension 1 in another subalgebra of $\mathfrak{g}$. Hence $\mathfrak{g}$ has an ideal I of codimension 1: $\mathfrak{g} = I \oplus \mathbb{F}X$.

By the induction hypothesis, the space $V' := \{v \in V | \; Iv = 0\}$ is non-zero. The space V' is x-invariant, since for any $y \in I, v \in V'$ one has $yxv = xyv + [y,x]v = 0$. The nilpotence of X guarantees the existence of a non-zero v in V' killed by X. Then $\mathfrak{g}v = 0$ as required. $\square$

1.4. *Engel's Theorem.*

If a finite dimensional Lie algebra $\mathfrak{g}$ consists of ad-nilpotent elements then $\mathfrak{g}$ is nilpotent.

Proof. Consider the adjoint representation $\mathrm{Ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$. All images $\mathrm{Ad}(g)$ are nilpotent linear endomorphisms. Therefore the algebra $\mathrm{Ad}(\mathfrak{g})$ satisfies the assumption of Theorem 1.3. As a consequence, there exists a basis $x_1, , \dots , x_r$ of $\mathfrak{g}$ such that for any $g \in \mathfrak{g}$ one has $gx_1 = 0$ and $gx_i \in \sum_{j<i} \mathbb{F}x_j$ for $j = 2, \dots , n$. Then the chain of ideals $I_n := 0$, $I_k := \sum_{j\leq n-k} \mathbb{F}x_j$ for $k = 0, \dots , n-1$ satisfies $[g, I_k] \subset I_{k+1}$. By Proposition 1.1.1 (iii), $\mathfrak{g}$ is nilpotent. □

1.5. Solvable Lie algebras.

1.5.1. **Proposition.** *The following conditions are equivalent*

(i) $D^n(\mathfrak{g}) = 0$.

(ii) There exists a chain of ideals

$$\mathfrak{g} = I_0 \supset I_1 \supset \dots \supset I_n = 0$$

such that I_k/I_{k+1} *is a commutative Lie algebra that is* $[I_k, I_k] \subset I_{k+1}$ *for all* k.

Proof. (i)$\Longrightarrow$ (ii) since the chain $I_k := D^k(\mathfrak{g})$ satisfies the condition of (ii). Moreover, this is the minimal chain which satisfies the condition: if $I_0, \dots , I_n$ is such that $[I_k, I_k] \subset I_{k+1}$ for all k, then $I_k \supset D^k(\mathfrak{g})$. Hence (ii)$\Longrightarrow$ (i). □

1.5.2. **Theorem.** *(*Lie's theorem.*) Assume that the base field* $\mathbb{F}$ *is algebraically closed and has characteristic zero. Let* $\mathfrak{g}$ *be a solvable finite dimensional Lie algebra and let* $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ *be a finite dimensional representation of* $\mathfrak{g}$*. Then there exists a basis of* V *relative to which* $\rho(\mathfrak{g})$ *is a subalgebra of* $\mathfrak{b}(m)$*,* $m := \dim V$*.*

1.5.3. *Remark.* Theorem 1.5.2 and Theorem 1.3 have different nature. Theorem 1.5.2 provides us an information about the structure of a representation of a solvable Lie algebra. Theorem 1.3 does not provide us an information about the structure of a representation of a nilpotent Lie algebra. Instead, it deals with a representation whose image consists of nilpotent endomorphisms. To understand this difference better, consider a one-dimensional Lie algebra. Its representation is determined by a single operator A. This operator A need not be nilpotent. If the base field is algebraically closed, we can choose a basis relative to which the matrix of A is upper triangular (compare to Lie's theorem).

1.5.4. **Corollary.** *Assume that the base field* $\mathbb{F}$ *is algebraically closed and has characteristic zero. Let* $\mathfrak{g}$ *be a solvable finite dimensional Lie algebra. Then there exists a chain of ideals*

$$0 = I_0 \subset I_1 \subset \dots \subset I_n = \mathfrak{g}$$

such that $\dim I_k = k$*.*

Proof. Apply Theorem 1.5.2 to the adjoint representation of $\mathfrak{g}$. □

1.5.5. **Corollary.** *Assume that the base field $\mathbb{F}$ is algebraically closed and has characteristic zero. If $\mathfrak{g}$ is a solvable finite dimensional Lie algebra then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.*

Proof. Consider $\mathrm{Ad} : \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$. By Lie's theorem $\mathrm{Im}\,\mathrm{Ad} \subset \mathfrak{b}(\dim \mathfrak{g})$ for a suitable basis. Then $\mathrm{Ad}([\mathfrak{g}, \mathfrak{g}]) \subset \mathfrak{n}(\dim \mathfrak{g})$ and so $\mathrm{ad}\, x$ is nilpotent for any $x \in [\mathfrak{g}, \mathfrak{g}]$. By Engel's theorem $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. □

1.6. **Radical.** Observe that if $\mathfrak{h}$ is an ideal in $\mathfrak{g}$ then $\mathfrak{g}$ is solvable iff $\mathfrak{h}$ and $\mathfrak{g}/\mathfrak{h}$ are solvable. In particular, the sum of two solvable ideals is again a solvable ideal (because $(I_1 + I_2)/I_2 \cong I_1/(I_1 \cap I_2)$). It follows that the sum of all solvable ideals is the maximal solvable ideal called the *radical* and denoted $Rad(\mathfrak{g})$. The quotient $\mathfrak{g}/Rad(\mathfrak{g})$ is semisimple.

1.7. **Semisimple Lie algebras.** All simple modules of a complex solvable Lie algebra are one-dimensional. In constrast, semisimple Lie algebras have a lot of simple modules.

1.7.1. A semisimple Lie algebra has a trivial centre (since the centre is a solvable ideal). Thus the adjoint representation of a semisimple Lie algebra is faithful.

1.7.2. **Theorem.** *Any complex finite dimensional representation of a semisimple Lie algebra is completely reducible.*

1.7.3. Assume that Ad is completely reducible: $\mathrm{Ad} = I_1 \oplus I_2 \oplus \dots I_r$ where I_j are simple $\mathfrak{g}$-modules. Then I_j are ideals and so $\mathfrak{g} = I_1 \times I_2 \dots \times I_r$. The fact that I_j are simple $\mathfrak{g}$-modules imply that I_j are minimal ideals so I_j as a subalgebra does not have non-trival ideals. Hence I_j is either simple Lie algebra or one-dimensional Lie algebra. Hence $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2 \dots \times \mathfrak{g}_r \times Z$ where $\mathfrak{g}_j$ are simple Lie algebras and Z is the centre. Such algebras are called reductive (for instance, $\mathfrak{gl}(n)$). Hence any semisimple Lie algebra is a product of simple ones. Notice that these simple factors are unique: they coincide with the minimal ideals.

The fact that a product of simple Lie algebras is semisimple follows from Problem 2, Homework 10. Hence semisimple Lie algebras are products of finite dimensional simple Lie algebras. Using Problem 2 of Homework 10 one can easily sees that a quotient of semisimple Lie algebra is again semisimple.

1.8. **Appendix: Proof of Lie Theorem.**

1.8.1. **Theorem.** *(*Lie's theorem.*) Assume that the base field $\mathbb{F}$ is algebraically closed and has characteristic zero. Let $\mathfrak{g}$ be a solvable finite dimensional Lie algebra and let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite dimensional representation of $\mathfrak{g}$. Then there exists a basis of V relative to which $\rho(\mathfrak{g})$ is a subalgebra of $\mathfrak{b}(m)$, $m := \dim V$.*

Proof. The steps of the proof are similar to ones of Theorem 1.5 of the previous lecture. Step I is an observation that the statement is an easy consequence of the following assertion:

under the assumptions of the theorem, if $\dim V \neq 0$ then all endomorphisms $\rho(g), g \in \mathfrak{g}$ have a common eigenvector. (*)

We prove this assertion by induction on $\dim \mathfrak{g}$. In order to do this, we find in $\mathfrak{g}$ an ideal of codimension 1. By the induction hypothesis, all endomorphisms $\rho(g), g \in I$ have common eigenvectors. We show (and this is the most difficult point) that $\rho(\mathfrak{g})$ stabilizes a space consisting of such common eigenvector. Finally, we find in this space an eigenvector for a single x satisfying $\mathfrak{g} = I + \mathbb{F}x$. This proves (*) and completes the proof of the theorem.

Let us prove the assertion (*) by induction on $\dim \mathfrak{g}$.

For $\dim \mathfrak{g} = 1$ one has $\mathfrak{g} = \mathbb{F}x$. Since $\mathbb{F}$ is algebraically closed and $\dim V < \infty$, $\rho(x)$ has an eigenvector v. Obviously v is an eigenvector of all endomorphisms $\rho(g), g \in \mathfrak{g}$.

Since $\mathfrak{g}$ is solvable, $[\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$. Observe that any subspace of $\mathfrak{g}$ containing the commutant $[\mathfrak{g}, \mathfrak{g}]$ is an ideal of $\mathfrak{g}$. Hence $\mathfrak{g}$ has an ideal I of codimension 1: $\mathfrak{g} = I \oplus \mathbb{F}X$.

By the induction hypothesis, all endomorphisms $\rho(g), g \in I$ have a common eigenvector $v_0 \neq 0$. For any $g \in I$ denote by $\chi(g)$ such a scalar that $\rho(g)v_0 = \chi(g)v_0$. The space $V' := \{v \in V \mid \rho(g)v = \chi(g)v, \ \forall g \in I\}$ contains v_0. Any vector of V' is a common eigenvector for all endomorphisms $\rho(g), g \in I$. To finish the proof, it is enough to verify that $\rho(x)V' \subset V'$ since this inclusion implies the existence an eigenvector of $\rho(x)$ in V'.

Hence we need to show that $\rho(x)V' \subset V'$ or, equivalently, that $\rho(g)\rho(x)v = \chi(g)\rho(x)v$ for all $v \in V', g \in I$. Fix $v \in V'$ and for each $k \in \mathbb{N}$ denote by V_k the span of $v = \rho(x)^0 v, \rho(x)v, \dots, \rho(x)^k v$.

Let us show that for any $g \in I, k \in \mathbb{N}$ $\rho(g)V_k \subset V_k$ and, moreover, $\rho(g)\rho(x)^k v = \chi(g)\rho(x)^k v$ modulo V_{k-1} (set $V_{-1} = 0$). We show this by induction on k. For $k = 0$ the assertion follows from the definition of V'. For the induction step, observe that

$$\rho(g)\rho(x)^{k+1}v = \rho(x)\rho(g)\rho(x)^k v + \rho([g,x])\rho(x)^k v.$$

Since I is an ideal, $[g,x] \in I$. Then, by the induction hypothesis, $\rho([g,x])\rho(x)^k v \in V_k$ and $\rho(g)\rho(x)^k v = \chi(g)\rho(x)^k v$ modulo V_{k-1}. Therefore $\rho(g)\rho(x)^{k+1}v = \chi(g)\rho(x)^{k+1}v$ modulo V_k as required.

Let $k \in \mathbb{N}$ be maximal such that the vectors $v, \rho(x)v, \dots, \rho(x)^k v$ are linearly independent. Then $V_m = V_k$ for all $m \geq k$. As we showed, V_k is $\rho(I)$-stable and relative to the basis $v, \rho(x)v, \dots, \rho(x)^k v$ each matrix $\rho(g)$ $(g \in I)$ is upper triangular with the diagonal entries equal to $\chi(g)$. Obviously $\rho(x)V_k = V_k$ and so for any $g \in I$ $\rho([x,g]) = \rho(x)\rho(g) - \rho(g)\rho(x)$ is a traceless endomorphism. Thus $\chi([x,g]) = 0$ and so

$$\rho(g)\rho(x)v = \rho(x)\rho(g)v - \rho([x,g])v = \rho(x)\chi(g)v - \chi([x,g])v = \chi(g)\rho(x)v$$

as required. The theorem is proven. □