INTRODUCTION TO REPRESENTATION THEORY. 27/12/04. LECTURE 11

1. Semisimple Lie algebras

2. Killing form

2.1. Let $\mathfrak{g}$ be a Lie algebra and V be a $\mathfrak{g}$-module. A bilinear form $B : V \times V \to \mathbb{C}$ is called $\mathfrak{g}$-*invariant* if

$$B(xv, v') + B(v, xv') = 0$$

for all $x \in \mathfrak{g}, v, v' \in V$.

If B is a $\mathfrak{g}$-invariant symmetric bilinear form, its kernel $\operatorname{Ker} B := \{v \in V | \ B(v, v') = 0, \ \forall v' \in V\}$ is a submodule of V. More generally, for such a B the orthogonal compliment

$$W^{\perp} := \{v \in V | \ B(v, w) = 0, \ \forall w \in W\}$$

of a submodule W is also submodule.

A symmetric bilinear form is called *non-degenerate* if its kernel is equal to zero.

2.2. **Definition.** Let $\mathfrak{g}$ be a Lie algebra. The form $K(x, y) := \operatorname{Tr}(\operatorname{ad} x \operatorname{ad} y)$ $(x, y \in \mathfrak{g})$ is called the *Killing form.*

The Killing form is symmetric and bilinear. Moreover it is easy to check that K is Ad-invariant:

$$K(\operatorname{ad} x(y), z) + K(y, \operatorname{ad} x(z)) = 0$$

for all $x, y, z \in \mathfrak{g}$.

In particular, $\operatorname{Ker} K$ is an ideal.

2.3. **Claim.** *Let $\mathfrak{g}$ be a Lie algebra and let K be its Killing form. The kernel of K is a solvable ideal and it contains the centre of $\mathfrak{g}$.*

2.4. **Theorem.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. The following conditions are equivalent*

(i) $\mathfrak{g}$ is a direct product of simple Lie algebras,

(ii) $\mathfrak{g}$ has zero radical,

(iii) the Killing form of $\mathfrak{g}$ is non-degenerate.

Proof. To obtain $(i) \implies (ii)$ observe that a semisimple Lie algebra is not solvable since its commutant coincides with the algebra itself and that a non-zero ideal of a semisimple Lie algebra is again a semisimple Lie algebra.

The implication $(ii) \implies (iii)$ follows from Claim 2.3.

Let us prove the remaining implication $(iii) \implies (i)$ by the induction on $\dim \mathfrak{g}$.

Recall the following fact from Linear Algebra: if V is a finite dimensional vector space equipped with a non-degenerate symmetric bilinear form and W is a subspace of V such that the restriction of B to W is again non-degenerate, then $V = W \oplus W^{\perp}$ where $W^{\perp} := \{v \in V |\ B(v, w) = 0,\ \forall w \in W\}$. Assume that $\mathfrak{g}$ is not simple and I is a non-trivial ideal of $\mathfrak{g}$. Since the Killing form K is Ad-invariant, $I^{\perp} := \{v \in \mathfrak{g} |\ K(v, w) = 0,\ \forall w \in I\}$ is an ideal. Let us verify that the restriction of K to I is non-degenerate.

Fix $x \in I$ such that $K(x, y) = 0$ for all $y \in I$. For any $y \in I, g \in \mathfrak{g}$ one has

$$0 = K(x, [y, g]) = K([x, y], g).$$

Since K is non-degenerate, this implies $[x, y] = 0$ for all $y \in I$. Choose a basis $v_1, \dots, v_k$ of I and extend it to a basis $v_1, \dots, v_k, v_{k+1}, \dots, v_m$ of $\mathfrak{g}$. Take $g \in \mathfrak{g}$. For $j \le k$ one has $v_j \in I$ and so $(\operatorname{ad} x)(\operatorname{ad} g)v_j = [x, [g, v_j]] = 0$ because $[g, v_j] \in I$. For $j > k$ one has $v_j \notin I$ and $(\operatorname{ad} x)(\operatorname{ad} g)v_j = [x, [g, v_j]] \in I$. Therefore $K(x, g) = \operatorname{Tr}(\operatorname{ad} x)(\operatorname{ad} g) = 0$. Since g is an arbitary element of $\mathfrak{g}$, we obtain $x = 0$. Hence the restriction of K to I is non-degenerate.

Hence $\mathfrak{g} = I \oplus I^{\perp}$ where both $I, I^{\perp}$ are ideals. Since $[I, I^{\perp}] \subset I \cap I^{\perp} = 0$, one has $\mathfrak{g} = I \times I^{\perp}$. To finish the proof, it remains to check that the Killing form of any non-zero ideal of $\mathfrak{g}$ is non-degenerate.

Let K_I be the Killing form of I. As we have shown, for any non-zero $x \in I$ there exists $y \in I$ such that $K(x, y) \neq 0$. The elements $(\operatorname{ad} x)(\operatorname{ad} y)v_j$ lie in I for any $j = 1, \dots, m$ and so

$$0 \neq K(x, y) = \operatorname{Tr}_{\mathfrak{g}}(\operatorname{ad} x)(\operatorname{ad} y) = \operatorname{Tr}_I(\operatorname{ad} x)(\operatorname{ad} y) = K_I(x, y).$$

Therefore K_I is non-degenerate.

Finally, we obtain $\mathfrak{g} = I \times I^{\perp}$ where both $I, I^{\perp}$ are ideals with non-degenerate Killing forms. By the induction hypothesis, both $I, I^{\perp}$ are direct products of simple Lie algebras. Therefore $\mathfrak{g}$ is also a direct product of simple Lie algebras. □

2.5. **Some linear algebra.** The linear algebra setting introduced earlier can be generalized in the following way.

Let V be a f.d. vector space and $f_1, \dots, f_k$ be diagonalizable endomorphisms. Then they pairwise commute (that is span a commutative Lie algebra) iff they have a common eigenbasis. Indeed, if such eigenbasis exists then the matrices of $f_1, \dots, f_k$ in this basis are diagonal; all diagonal matrices commute and so $f_1, \dots, f_k$ pairwise commute. Conversely, for any $\lambda \in \operatorname{Spec} f_k$ the equalities $f_k f_j = f_j f_k$ implies $f_j(V_\lambda) \subset V_\lambda$ for $j = 1, \dots, k-1$. Using the induction on k, we can assume that $f_1, \dots, f_{k-1}$ have a common eigenbasis in

V_λ; this basis is a common eigenbasis for $f_1, \dots, f_k$ because $f_1 v = \lambda v$ for all $v \in V_\lambda$. Since $V = \oplus_{\lambda \in \mathrm{Spec}\, f_k} V_\lambda$, this implies the existence of a common eigenbasis in V.

If v is an eigenvector for $\vec{f} = (f_1, \dots, f_k)$, the (generalized) eigenvalue is $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$ where λ_i is the eigenvalue of f_i. For a (generalized) eigenvalue $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$ define the (generalized) eigenspace by

$$V_{\vec{\lambda}} := \{v \in V |\ f_i v = \lambda_i v\}.$$

The above equivalence can be reformulated as follows: diagonalizable endomorphisms $f_1, \dots, f_k$ pairwise commute iff

$$V = \oplus V_{\vec{\lambda}}.$$

Lemma. *Let $f_1, \dots, f_k \in \mathrm{End}(V)$ be diagonalizable pairwise commuting endomorphisms and let W be a subspace of V such that $f_i(W) \subseteq W$ for $i = 1, \dots, k$. Then*

$$W = \oplus_{\vec{\lambda}} W_{\vec{\lambda}} \text{ where } W_{\vec{\lambda}} = W \cap V_{\vec{\lambda}}.$$

Proof. We prove the statement by induction on k. For $k = 1$ it is already proven. Write

$$V = \oplus_{\nu \in \mathrm{Spec}\, f_k} V_\nu.$$

We know that

$$W = \oplus_\nu W_\nu, \quad W_\nu := W \cap V_\nu.$$

As it is shown above, for each ν the eigenspace V_ν is $f_1, \dots, f_{k-1}$-invariant and so

$$V_\nu = \oplus_{\vec{\lambda}:\lambda_k = \nu} V_{\vec{\lambda}}.$$

Each summand $V_{\vec{\lambda}}$ in the above expression is the intersection of V_ν and an eigenspace of the family $f_1, \dots, f_{k-1}$. Using the statement for $n := k - 1$ we obtain

$$W_\nu = \oplus_{\vec{\lambda}:\lambda_k = \nu} W_{\vec{\lambda}}$$

and so

$$W = \oplus_{\vec{\lambda}} W_{\vec{\lambda}}$$

as required. □

2.6. Simplicity of $\mathfrak{sl}(n)$.

2.6.1. Let $\mathbb{F}$ be a field of zero characteristic. Set

$$\mathfrak{h} := \sum \mathbb{F} E_{ii}$$

this is the set of diagonal matrices. Notice that $\mathfrak{h}$ consists of ad-diagonilizable elements (i.e., for any $x \in \mathfrak{gl}(n)$, $\mathrm{ad}\, x$ is diagonal iwrt the natiral basis of $\mathfrak{gl}(n)$) and $[\mathrm{ad}\, x, \mathrm{ad}\, y] = \mathrm{ad}[x, y] = 0$ for any $x, y \in \mathfrak{h}$. Set $\phi_i := \mathrm{ad}(E_{ii}) \in \mathrm{End}(\mathfrak{gl}(n))$. Then the eigenspaces wrt

$\{\phi_i\}_{i=1}^n$ are $\mathfrak{h}$ (eigenvalue is $(0,\dots,0)$ and $\mathbb{F}E_{ij}$ $(i \neq j)$. Clearly, any element of $\mathfrak{h}$ acts on each eigenspace by a scalar. View eigenvalues as elements of $\mathfrak{h}^*$ and write

$$\mathfrak{gl}(n) = \oplus_{\alpha \in S} \mathfrak{g}_\alpha$$

where $S \subset \mathfrak{h}^*$. Choose a basis $\{\varepsilon_i\}$ in $\mathfrak{h}$ which is dual to E_{ii} $(\varepsilon_j(E_{ii}) = \delta_{i,j})$. Then

$$S := \{0; \varepsilon_i - \varepsilon_j : i \neq j\}$$

and

$$\mathfrak{g}_0 = \mathfrak{h}, \quad \mathfrak{g}_{\varepsilon_i - \varepsilon_j} = \mathbb{F}E_{ij}.$$

Notice that $\alpha \in S$ implies $-\alpha \in S$ and that $\mathfrak{g}_\alpha$ are one-dimensional for $\alpha \neq 0$.

2.6.2. Let us show that $\mathfrak{sl}(n)$ is simple. Indeed, $\mathfrak{gl}(n) = \mathfrak{sl}(n) \times \mathbb{F}\mathrm{id}$. It is enough to show that $\mathbb{F}\mathrm{id}$ and $\mathfrak{sl}(n)$ are the only non-trivial ideals in $\mathfrak{gl}(n)$. Let I be an ideal of $\mathfrak{gl}(n)$, $I \neq 0, \mathbb{F}\mathrm{id}$. Let us show that $\mathfrak{sl}(n) \subset I$. Any ideal of $\mathfrak{gl}(n)$ is $\mathfrak{h}$-invariant and so, by Lemma 2.5, I either contains E_{ij} for some $i \neq j$ or $I \cap \mathfrak{h} \neq 0, \mathbb{F}\mathrm{id}$. In the second case, take $x \in I \cap \mathfrak{h}, x \notin \mathbb{F}\mathrm{id}$. Then $x = \sum a_i E_{ii}$ where $a_i \neq a_j$ for some i, j and so I contains E_{ij} (because $[x, E_{ij}] = (a_i - a_j)E_{ij} \neq 0$). Hence in both cases, I contains E_{ij} for some $i \neq j$; then I contains E_{ik} for $k \neq i$ because $[E_{ij}, E_{jk}] = E_{ik}$ and then I contains $E_{ii} - E_{kk}$ for all $k \neq i$. The elements $E_{ii} - E_{kk}$ span $\mathfrak{h} \cap \mathfrak{sl}(n) = \{\sum b_r E_{rr} : \sum b_r = 0\}$. Finally, for any pair (r, s) with $r \neq s$ the formula $2E_{rs} = [E_{rr} - E_{ss}, E_{rs}]$ forces $E_{rs} \in I$. Hence $\mathfrak{sl}(n) \subset I$ as required.

2.7. **Structure of semisimple complex Lie algebra.** Our base field is $\mathbb{C}$. Let $\mathfrak{g}$ be a semisimple complex Lie algebra. Recall that the Killing form

$$K(x, y) := \mathrm{Tr}(\mathrm{ad}\, x \,\mathrm{ad}\, y)$$

is a non-degenerate invariant bilinear form on $\mathfrak{g}$.

2.7.1. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called a *Cartan subalgebra* if it is nilpotent and coinicides with its centralizer: $[x, \mathfrak{h}] \subset \mathfrak{h} \implies x \in \mathfrak{h}$. It truns out that in a semisimple complex Lie algebra $\mathfrak{g}$, any Cartan subalgebra is commutative and consists of *ad*-semisimple elements. Moreover all Cartan subalgebras are conjugate: if $\mathfrak{h}, \mathfrak{h}'$ are Cartan subalgebras of $\mathfrak{g}$ then there exists an automorphism $\psi : \mathfrak{g} \to \mathfrak{g}$ such that $\psi(\mathfrak{h}) = \mathfrak{h}'$.

We can write

$$\mathfrak{g} = \oplus_{\alpha \in S} \mathfrak{g}_\alpha$$

where $S \subset \mathfrak{h}^*$. Remark that $\mathfrak{g}_0 = \mathfrak{h}$. Notice that

$$K(x, y) = 0$$

if $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ and $\alpha + \beta \neq 0$. In particular, the restriction of the Killing form K to $\mathfrak{h}$ is non-degenerate.

2.7.2. Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and denote by Δ the set of roots:

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha).$$

For any $\alpha \in \Delta$ the subspace $\mathfrak{g}_\alpha$ is one-dimensional and it consists of ad-nilpotent elements (this means that $\operatorname{ad} x$ is nilpotent for $x \in \mathfrak{g}_\alpha$. Moreover if α is a root, $-\alpha$ is also a root and there exists $h_\alpha \in \mathfrak{h}$ such that

$$\mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_\alpha \cong \mathfrak{sl}(2).$$

The element h_α is defined up to an invertible scalar; we choose it such that

$$\alpha(h_\alpha) = 2.$$

One has

1) $0 \notin \Delta$;

2) Δ spans $\mathfrak{h}^*$;

3) $\alpha \in \Delta \implies -\alpha \in \Delta, \quad \mathbb{Z}\alpha \cap \Delta = \pm\alpha$;

4) $\forall \alpha, \beta \in \Delta \quad \beta(h_\alpha) \in \mathbb{Z}, \quad \beta - \beta(h_\alpha)\alpha \in \Delta$.

We set

$$< \alpha, \beta > := \alpha(h_\beta).$$

Note that $< -, - >$ is linear on the first argument.

2.7.3. It turns out that the root system Δ completely determines $\mathfrak{g}$ (see the book of Serre for a proof). To illustrate this fact let me give two more properties of Δ.

Suppose that $\alpha, \beta, \alpha + \beta \in \Delta$ then

1) $\mathfrak{g}_{\alpha+\beta} = [\mathfrak{g}_\alpha, \mathfrak{g}_\beta]$.

2) Let r, q be the maximal integers such that $\beta - r\alpha \in \Delta, \beta + q\alpha \in \Delta$. Then $\beta + k\alpha \in \Delta$ for all $-r \leq k \leq q$ and $\beta(h_\alpha) = r - q$.

In other words, the integers $< \alpha, \beta >= \alpha(h_\beta)$ can be read from the structure of Δ.

2.7.4. *Example:* $\mathfrak{sl}(n)$. To describe a root system of $\mathfrak{sl}(n)$, it is convinient to start from $\mathfrak{gl}(n)$. The latter is reductive: $\mathfrak{gl}(n) = \mathfrak{sl}(n) \times \mathbb{C}z$ and it is easy to check that a Cartan subalgebra is of the form $\mathfrak{h}' = \mathfrak{h} \times \mathbb{C}z$ where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{sl}(n)$.

The natural choice for $\mathfrak{h}'$ is the set of diagonal matrices

$$\mathfrak{h}' := \{\sum a_i E_{i,i}, \ a_i \in \mathbb{C}\}.$$

The natural choice for $\mathfrak{h}$ is the set of traceless diagonal matrices

$$\mathfrak{h} := \{\sum a_i E_{i,i} | \ \sum a_i = 0\}.$$

Let $\{\varepsilon_i\}_{i=1}^n$ be a basis of $(\mathfrak{h}')^*$ which is dual to the basis $E_{i,i}$. Since $\mathfrak{h}$ is a subspace of $\mathfrak{h}'$, the dual space $\mathfrak{h}^*$ may be naturally viewed as a factor space

$$\mathfrak{h}^* = span\{\varepsilon_i\}_{i=1}^n / \sum_{i=1}^n \varepsilon_i.$$

In this notation, one has

$$\Delta := \{\varepsilon_i - \varepsilon_j\}_{i \neq j}.$$

The $\mathfrak{sl}(2)$ triple corresponing to $(\varepsilon_i - \varepsilon_j)$ is

$$E_{i,j},\ h_{\varepsilon_i - \varepsilon_j} := E_{i,i} - E_{j,j},\ E_{j,i}.$$

The $(\varepsilon_i - \varepsilon_j)$-strings take form

$$\varepsilon_k - \varepsilon_i; \varepsilon_k - \varepsilon_j.$$

Therefore the integers $\beta(h_\alpha)$ are

$$(\varepsilon_k - \varepsilon_i)(h_{\varepsilon_i - \varepsilon_j}) = -1$$
$$(\varepsilon_i - \varepsilon_j)(h_{\varepsilon_i - \varepsilon_j}) = 2$$
$$(\varepsilon_k - \varepsilon_j)(h_{\varepsilon_i - \varepsilon_j}) = 1$$

where $k \neq i, j$ and zero for all remaining cases.

2.7.5. *Example:* $\mathfrak{sp}(n)$ *$(n = 2l)$.* This is a Lie subalgebra of $\mathfrak{gl}(n)$ which consists of all matrices T satisfying $TA + AT^t = 0$ where

$$A = \begin{pmatrix} 0 & | & I_l \\ -- & - & -- \\ -I_l & | & 0 \end{pmatrix}$$

and I_l stands for the identity $l \times l$ matrix.

The matrices in $\mathfrak{sp}(n)$ are of the form

$$T_{x,y,z} := \begin{pmatrix} x & | & y \\ -- & - & -- \\ z & | & -x^t \end{pmatrix}$$

where x, y, x are $l \times l$ matrices and y, z are symmetric: $y^t = y, z^t = z$. We have a natural embedding $\mathfrak{gl}(l) \subset \mathfrak{sp}(2l)$ $(x \mapsto T_{x,0,0})$.

Let $\mathfrak{h}$ be the set of diagonal matrices

$$\mathfrak{h} := \{\sum_{i=1}^l a_i(E_{i,i} - E_{l+i,l+i})\}$$

(it corresponds to $\mathfrak{h}'$ in the previous example). Retain notation for the dual basis.

Obviously this is a commutative Lie subalgebra. To check that $\mathfrak{h}$ is a maximal toral subalgebra, let us show that it consists of ad-semisimple elements and coincides with the own centralizer (so it is a maximal commutative subalgebra).

Indeed, if $x = E_{i,j}, y = z = 0$ then $T_{x,0,0}$ has weight $\varepsilon_i - \varepsilon_j$ (in this example i, j are assumed to be distinct integeres from 1 to l).

If $x = 0, y = E_{i,i}$ then $T_{0,y,0} = E_{i,l+i}$ and has weight $2\varepsilon_i$.

Similarly, if $x = 0, z = E_{i,i}$ then $T_{0,0,z} = E_{l+i,i}$ and has weight $-2\varepsilon_i$.

If $x = 0, y = E_{i,j} + E_{j,i}$ then $T_{0,y,0}$ has weight $\varepsilon_i + \varepsilon_j$.

If $x = 0, z = E_{i,j} + E_{j,i}$ then $T_{0,0,z}$ has weight $-(\varepsilon_i + \varepsilon_j)$.

Thus $\mathfrak{h}$ is a maximal toral subalgebra and

$$\Delta := \{\varepsilon_i - \varepsilon_j; \pm(\varepsilon_i + \varepsilon_j); \pm 2\varepsilon_i\}.$$

We have the following $\mathfrak{sl}(2)$-triples. The $\mathfrak{sl}(2)$ triple corresponing to $(\varepsilon_i - \varepsilon_j)$ comes from $\mathfrak{gl}(l) \subset \mathfrak{sp}(2l)$ and takes form

$$E_{i,j} - E_{l+j,l+i},\ h_{\varepsilon_i - \varepsilon_j} := (E_{i,i} - E_{j,j}) - (E_{l+i,l+i} - E_{l+j,l+j}),\ E_{j,i} - E_{l+i,l+j}.$$

The $\mathfrak{sl}(2)$ triple corresponing to $2\varepsilon_i$ is

$$E_{i,l+i}, h_{\varepsilon_i + \varepsilon_j} := E_{i,i} - E_{l+i,l+i}, E_{l+i,i}$$

Finally, the $\mathfrak{sl}(2)$ triple corresponing to $(\varepsilon_i + \varepsilon_j)$ is

$$E_{i,l+j} + E_{j,l+i},\ h_{\varepsilon_i + \varepsilon_j} := (E_{i,i} + E_{j,j}) - (E_{l+i,l+i} + E_{l+j,l+j}),\ E_{l+j,i} + E_{l+i,j}.$$

Examples of strings:

$$\begin{aligned} 2\varepsilon_2 - \text{string}: &\quad \varepsilon_1 - \varepsilon_2, \varepsilon_1 + \varepsilon_2; \\ (\varepsilon_1 - \varepsilon_2) - \text{string}: &\quad 2\varepsilon_2, \varepsilon_1 + \varepsilon_2, 2\varepsilon_1; \\ (\varepsilon_1 + \varepsilon_2) - \text{string}: &\quad -2\varepsilon_2, \varepsilon_1 - \varepsilon_2, 2\varepsilon_1. \end{aligned}$$

The numbers $< \alpha, \beta >= \alpha(h_\beta)$:

$$\begin{aligned} &< \varepsilon_1 - \varepsilon_2, 2\varepsilon_2 >= -1, \quad < \varepsilon_1 + \varepsilon_2, 2\varepsilon_2 >= 1 \\ &< 2\varepsilon_2, \varepsilon_1 - \varepsilon_2 >= -2, \quad < 2\varepsilon_2, \varepsilon_1 + \varepsilon_2 >= 2 \end{aligned}$$