INTRODUCTION TO REPRESENTATION THEORY. 3/1/05. LECTURE 12

1. Root systems

Fix a Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ and denote by Δ the set of roots:

$$\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Delta} \mathfrak{g}_\alpha).$$

For any $\alpha \in \Delta$ the subspace $\mathfrak{g}_\alpha$ is one-dimensional and it consists of ad-nilpotent elements (this means that $\operatorname{ad} x$ is nilpotent for $x \in \mathfrak{g}_\alpha$. Moreover if α is a root, $-\alpha$ is also a root and there exists $h_\alpha \in \mathfrak{h}$ such that

$$\mathfrak{g}_{-\alpha} \oplus \mathbb{C}h_\alpha \oplus \mathfrak{g}_\alpha \cong \mathfrak{sl}(2).$$

The element h_α is defined up to an invertible scalar; we choose it such that

$$\alpha(h_\alpha) = 2.$$

One has

1) $0 \notin \Delta$;

2) Δ spans $\mathfrak{h}^*$;

3) $\alpha \in \Delta \implies -\alpha \in \Delta, \quad \mathbb{Z}\alpha \cap \Delta = \pm\alpha$;

4) $\forall \alpha, \beta \in \Delta \quad \beta(h_\alpha) \in \mathbb{Z}, \quad \beta - \beta(h_\alpha)\alpha \in \Delta$.

We set

$$< \alpha, \beta > := \alpha(h_\beta).$$

Note that $< -, - >$ is linear on the first argument.

Recall that the restriction of Killing form is a non-degenerate symmetric bilinear form on $\mathfrak{h}$. Since $\mathfrak{h}$ is finite dimensional, this induces a linear isomorphism $\beta : \mathfrak{h} \to \mathfrak{h}^*$ where $\beta(v)(v') := K(v, v')$. Now we can obtain a non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ by the formula

$$(\lambda, \nu) := K(\beta^{-1}(\lambda), \beta^{-1}(\nu)).$$

Let $\mathfrak{h}^*_{\mathbb{R}}$ be the $\mathbb{R}$-span of Δ in $\mathfrak{h}^*$. It turns out that $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ and that the restriction of $(-, -)$ is a scalar product on $\mathfrak{h}^*_{\mathbb{R}}$. It turns out that

$$< \alpha, \beta > = 2\frac{(\alpha, \beta)}{(\beta, \beta)}.$$

Notice that changing scalar product $(-,-)$ by a non-zero multiple, does not change the above formula. In particular, for a simple Lie algebra, we can use any non-degenerate invariant symmetric bilinear form for computations of numbers $<\alpha,\beta>$.

1.1. **Example:** $\mathfrak{sl}(n)$. Define a scalar product by $(x,y) := \operatorname{Tr} xy$ for $x,y \in \mathfrak{h} \subset \mathfrak{sl}(n)$. Then $\mathfrak{h}_{\mathbb{R}} = \{\sum a_i E_{ii} | \ a_i \in \mathbb{R}, \sum a_i = 0\}$ and

$$(\sum a_i E_{ii}, \sum b_i E_{ii}) = \sum a_i b_i.$$

Moreover

$$\beta(\sum a_i E_{ii}) \sum a_i \varepsilon_i$$

and so

$$(\sum a_i \varepsilon_i, \sum b_i \varepsilon_i) = \sum a_i b_i.$$

Recall that $\mathfrak{h}^*_{\mathbb{R}} = span\{\varepsilon_i\}/\sum \varepsilon_i$. The scalar product is induced by the natural scalar product on $span\{\varepsilon_i\}$: $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$.

1.2. The set Δ contains a basis π of $\mathfrak{h}$ such that any element of Δ^+ is an integer linear of basis elements and all coefficients are either non-negative or non-positive. The set π is called a *set of simple roots*. Fix π and let Δ^+ be the subset of Δ where all coefficients are non-negative.

Then

$$\Delta = \Delta^+ \coprod \Delta^-$$

where $\Delta^- = -\Delta^+$ and Δ^+ are closed under addition in the following sense:

$$\alpha, \beta \in \Delta^+, \ \alpha + \beta \in \Delta \quad \Longrightarrow \quad \alpha + \beta \in \Delta^+.$$

For instance, for $\mathfrak{sl}(n)$ one can choose

$$\pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n\}$$

and then

$$\Delta^+ := \{\varepsilon_i - \varepsilon_j : 1 \leq i < j \leq n\}.$$

Another choice for $\mathfrak{sl}(3)$ is

$$\pi := \{\varepsilon_2 - \varepsilon_1, \varepsilon_1 - \varepsilon_3\}$$

and then

$$\Delta^+ := \{\varepsilon_2 - \varepsilon_1; \varepsilon_1 - \varepsilon_3; \varepsilon_2 - \varepsilon_3\}.$$

For $\mathfrak{sp}(2l)$ one can choose

$$\pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{l-1} - \varepsilon_l, 2\varepsilon_l\}$$

and then

$$\Delta^+ := \{\varepsilon_i \pm \varepsilon_j : 1 \leq i < j \leq l; 2\varepsilon_i\}.$$

2. Weyl group

Let W be the subgroup of $\mathrm{Aut}\,\mathfrak{h}^*_{\mathbb{R}}$ generated by the reflections s_β ($\beta \in \Delta$) given by the formula

$$s_\beta(\lambda) := \lambda - 2\frac{(\lambda, \beta)}{(\beta, \beta)}\beta.$$

Notice that $s_\beta(\beta) = -\beta, s_\beta(\lambda) = \lambda$ if $(\lambda, \beta) = 0$. Thus s_β is the reflection wrt the hyperplane which contains the origin and is orthogonal to β. Therefore

$$(s_\beta(\lambda), s_\beta(\nu)) = (\lambda, \nu).$$

The property (4) of Δ implies that $s_\beta(\alpha) \in \Delta$ for any $\alpha \in \Delta$. Hence W acts on Δ; since Δ contains a basis of $\mathfrak{h}^*_{\mathbb{R}}$, this action is faithful. In particular, W is a finite group.

Notice that a reflection s_β does not change if we multiply the scalar products by a scalar. Thus for a simple $\mathfrak{g}$ we can use a scalar product induced by any non-degenerate symmetric bilinear form instead on induced by the Killing form.

2.1. **Example: $\mathfrak{sl}(n)$.**

In this case the Weyl group is isomorphic to the symmetric group S_n. The action of S_n to the vector space E spanned by $\{\varepsilon_i\}_{i=1}^n$ is given by

$$w(\varepsilon_i) = \varepsilon_{w(i)}.$$

Clearly, $w \sum \varepsilon_i = \sum \varepsilon_i$ and so W acts on $\mathfrak{h}^* = E/(\mathbb{C} \sum \varepsilon_i)$. To check that $W = S_n$ with the above action, notice that the reflection s_β for $\beta = \varepsilon_i - \varepsilon_j$ acts as the transposition (ij). Since the transpositions generate W we obtain $W = S_n$.

2.2. **Example: $\mathfrak{sp}(2l)$.**

Let $\{\varepsilon_i\}_{i=1}^l$ be the standard basis of $\mathfrak{h}^*$. Recall that

$$\Delta = \{\pm\varepsilon_i \pm \varepsilon_j; 2\varepsilon_i : 1 \leq i < j \leq l\}.$$

Clearly, $\{\varepsilon_i\}_{i=1}^l$ is also a basis of $\mathfrak{h}^*_{\mathbb{R}}$. Let us describe $\mathfrak{h}^*_{\mathbb{R}}$ as an Eucledian space. We can describe the scalar product computing the Killing form. Instead, we will describe a scalar product proportional to one induced by the Killing form which can be done much easier (see below).

Normalize the scalar product in such a way that $(\varepsilon_1, \varepsilon_1) = 1$. Notice that $2\varepsilon_i - k(2\varepsilon_j) \in \Delta$ iff $k = 0$. Hence $(2\varepsilon_i, 2\varepsilon_j) = 0$ by the property (4). To compute $(\varepsilon_i, \varepsilon_i)$ observe that $(\varepsilon_{i-1} - \varepsilon_i, \varepsilon_i - \varepsilon_{i+1}) \neq 0$ and so $< \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i - \varepsilon_{i+1} > \neq 0$. Then $< \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i - \varepsilon_{i+1} > = -1$ because $(\varepsilon_{i-1} - \varepsilon_i) - k(\varepsilon_i - \varepsilon_{i+1}) \in \Delta$ forces $k = 0$ or $k = -1$. Therefore

$$-1 = < \varepsilon_{i-1} - \varepsilon_i, \varepsilon_i - \varepsilon_{i+1} > = \frac{-2(\varepsilon_i, \varepsilon_i)}{(\varepsilon_i, \varepsilon_i) + (\varepsilon_{i+1}, \varepsilon_{i+1})}$$

and so $(\varepsilon_i, \varepsilon_i) = (\varepsilon_{i+1}, \varepsilon_{i+1})$. Hence $\{\varepsilon_i\}_{i=1}^l$ is an orthonormal basis.

2.2.1. Let us compute the Weyl group W. The reflection wrt $2\varepsilon_i$ takes form

$$s_{2\varepsilon_i}(\varepsilon_i) = -\varepsilon_i, \quad s_{2\varepsilon_i}(\varepsilon_j) = \varepsilon_j,\ i \neq j.$$

The reflection wrt $\varepsilon_i - \varepsilon_j$ interchanges ε_i with ε_j and preserves ε_r for $r \neq i, j$.

The reflection wrt $\varepsilon_i + \varepsilon_j$ interchanges ε_i with $-\varepsilon_j$ and preserves ε_r for $r \neq i, j$.

Hence the Weyl group W is isomorphic to the group of signed permutations C_l: these are permutations of the set $\{-l, -(l-1), \dots, -1, 1, 2, \dots, l\}$ satisfying $w(-i) = -w(i)$. The group C_l is isomorphic to a semidirect product of S_l and the abelian group $(\mathbb{Z}_2)^l$: this means that W contains both S_l and $(\mathbb{Z}_2)^l$ as subgroups, the subgroup $(\mathbb{Z}_2)^l$ is normal, any element of C_l can be uniquely written as gs where $g \in (\mathbb{Z}_2)^l$ and $s \in S_l$ ($C_l \not\cong (\mathbb{Z}_2)^l \times S_l$ since the elements of $(\mathbb{Z}_2)^l$ do not commute with the elements of S_l; in other words, S_l is not a normal subgroup). The isomorphism $W \xrightarrow{\sim} C_l$ is given by the following map:

$s_{2\varepsilon_i}$ maps to the permutation $w : w(i) = -i, w(j) = j$ for $j \neq i$;

$s_{\varepsilon_i - \varepsilon_j}$ maps to the permutation (ij)

$s_{\varepsilon_i + \varepsilon_j}$ maps to the permutation $w : w(i) = -j,\ w(j) = -i, w(r) = r$ for $r \neq i, j$.

3. Characters of finite dimensional modules

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. Suppose that $\mathfrak{h}$ acts diagonally on V and all weight spaces V_λ are finite-dimensional. Set

$$\operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} \dim V_\lambda e^\lambda$$

where e^λ belongs to a multiplicative group isomorphic to $\mathfrak{h}^*$ that is $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in \mathfrak{h}^*$. One has

$$\operatorname{ch}(V \oplus V') = \operatorname{ch} V + \operatorname{ch} V'.$$

If V is a finite-dimensional module, $\mathfrak{h}$ acts diagonally on V. In this case, $\operatorname{ch} V$ is a finite sum.

If V, V' are finite dimensional

$$\operatorname{ch}(V \otimes V') = \operatorname{ch} V \operatorname{ch} V'.$$

Define the action of W on $\operatorname{ch} V$ by

$$w \operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} \dim V_\lambda e^{w\lambda}.$$

Next time, we'll prove the

3.0.2. **Theorem.** *If V is a finite-dimensional module, $\operatorname{ch} V$ is W invariant: $w \operatorname{ch} V = \operatorname{ch} V$.*