INTRODUCTION TO REPRESENTATION THEORY. 10/1/05. LECTURE 13

1. Characters of finite dimensional modules

Let $\mathfrak{g}$ be a semisimple Lie algebra and $\mathfrak{h}$ be its Cartan subalgebra. Suppose that $\mathfrak{h}$ acts diagonally on V and all weight spaces V_λ are finite-dimensional. Set

$$\operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} \dim V_\lambda e^\lambda$$

where e^λ belongs to a multiplicative group isomorphic to $\mathfrak{h}^*$ that is $e^\lambda e^\mu = e^{\lambda+\mu}$ for all $\lambda, \mu \in \mathfrak{h}^*$. One has

$$\operatorname{ch}(V \oplus V') = \operatorname{ch} V + \operatorname{ch} V'.$$

1.1. Recall that $\mathfrak{h}$ acts diagonally on $\mathfrak{g}$ (via adjoint action). As a consequence, $\mathfrak{h}$ acts diagonally on $U(\mathfrak{g})$. Indeed, take an eigenbasis in $\mathfrak{g}$ wrt $\mathfrak{h}$: $a_1, \dots, a_r$; the corresponding PBW basis of $\mathfrak{g}$ is an eigenbasis wrt $\mathfrak{h}$ (if a_i has weight ν_i then $a_1^{r_1} \dots a_k^{r_k}$ has weight $\sum_i r_i \nu_i$).

Notice that $U(\mathfrak{g})_\nu V_\mu \subset V_{\nu+\mu}$ where $U(\mathfrak{g})_\nu V_\mu := span\{uv \mid u \in U(\mathfrak{g})_\nu, v \in V_\mu\}$.

1.1.1. **Proposition.** *If V is a simple finite-dimensional module, $\mathfrak{h}$ acts diagonally on V.*

Proof. Since V is finite-dimensional, it contains a non-zero vector v which is an eigenvector wrt $\mathfrak{h}$. Since V is simple, v generates V that is $V = U(\mathfrak{g})v$. Taking into account that $\mathfrak{h}$ acts diagonally on $U(\mathfrak{g})$, we obtain the assertion. □

Recall that any finite-dimensional module is completely reducible.

Corollary. *If V is a finite-dimensional module, $\mathfrak{h}$ acts diagonally on V.*

If V is a finite-dimensional module, $\operatorname{ch} V$ is a finite sum.

1.2. Define the action of W on $\operatorname{ch} V$ by

$$w \operatorname{ch} V := \sum_{\lambda \in \mathfrak{h}^*} \dim V_\lambda e^{w\lambda}.$$

1.2.1. Recall that for any $\alpha \in \Delta$ $s_\alpha(\mu) = \mu - \frac{2(\mu,\alpha)}{(\alpha,\alpha)}\alpha$ and for all $\beta \in \Delta$ $<\beta, \alpha> = \beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$ that is $2(\beta,\alpha) = \beta(h_\alpha)(\alpha,\alpha)$. Since Δ generate $\mathfrak{h}^*$, we conclude that for any $\mu \in \mathfrak{h}^*$ one has $2(\mu,\alpha) = \mu(h_\alpha)(\alpha,\alpha)$ that is

$$s_\alpha(\mu) = \mu - \mu(h_\alpha)\alpha.$$

1.2.2. **Theorem.** *If V is a finite-dimensional module, $\operatorname{ch} V$ is W invariant: $w \operatorname{ch} V = \operatorname{ch} V$.*

Proof. Let us check the case $\mathfrak{g} = \mathfrak{sl}(2)$. In this case, W contains two elements id, s_α and $\mathfrak{h}^* = \mathbb{C}\alpha$. Since a finite-dimensional module is completely reducible, it is enough to verify this for the simple V. One has $h_\alpha(\alpha) = 2$ and so $V(n)$ has a highest weight $n\alpha/2$. Recall that

$$\operatorname{ch} V(n) = e^{n\alpha/2} + e^{(n-2)\alpha/2} + \dots e^{-n\alpha/2}.$$

Since $s_\alpha \alpha = -\alpha$, we have $s_\alpha e^{x\alpha} = e^{-x\alpha}$ and so $s_\alpha \operatorname{ch} V(n) = \operatorname{ch} V(n)$ as required.

The Weyl group W is generated by the reflections s_β, $\beta \in \Delta$. Thus it is enough to verify that $s_\beta \operatorname{ch} V = \operatorname{ch} V$ for all $\beta \in \Delta$ that is $\dim V_\mu = \dim V_{s_\beta \mu}$ for all $\mu \in \mathfrak{h}^*$. Fix $\alpha \in \Delta, \mu \in \mathfrak{h}^*$ and let $L := \sum_{r\in\mathbb{Z}} V_{\mu + r\beta}$. Recall that $\mathfrak{g}$ contains an $\mathfrak{sl}(2)$ triple $\mathfrak{s}$ spanned by $e_\alpha, f_\alpha, h_\alpha$. Clearly, L is an $\mathfrak{s}$-submodule. By the above, for any $c \in \mathbb{C}$

$$\dim\{v \in L |\ h_\alpha v = cv\} = \dim\{v \in L |\ h_\alpha v = -cv\}.$$

The element h_α acts on $v \in V_{\mu + r\alpha}$ by the scalar $h_\alpha(\mu) + 2r$. Therefore

$$\begin{aligned} \dim V_\mu &= \dim\{v \in L |\ h_\alpha v = h_\alpha(\mu)v\} = \dim\{v \in L |\ h_\alpha v = -h_\alpha(\mu)v\} \\ &= \dim V_{\mu - h_\alpha(\mu)\alpha} = \dim V_{s_\alpha \mu} \end{aligned}$$

as required. $\square$

1.3. **Example.** Define an $\mathfrak{gl}(n)$-module structure on the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$ by setting $E_{ij}(p) = x_i \frac{\partial}{\partial x_j} p$. For each $m \geq 0$, the space of homogeneous polynomials of total degree m is a submodule $V(m)$ and its character wrt $\mathfrak{sl}(n)$ is given by

$$\operatorname{ch} V(m) = \sum_{r_1, r_2, \dots, r_n \geq 0,\ r_1 + \dots r_n = m} e^{r_1\varepsilon_1 + r_2\varepsilon_2 + \dots + r_n\varepsilon_n}.$$

Recall that W acts on $\mathfrak{h}^*$ by permutations of ε_i. We see that $\operatorname{ch} V(m)$ is W-invariant.

2. Verma modules

2.1. **Triangular decomposition.** Fix a semisimple Lie algebra $\mathfrak{g}$, its Cartan subalgebra $\mathfrak{h}$ and decomposition $\Delta = \Delta^+ \coprod \Delta^-$. Set

$$\mathfrak{n}^+ := \sum_{\alpha \in \Delta^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}^- := \sum_{\alpha \in \Delta^-} \mathfrak{g}_\alpha.$$

Observe that $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent subalgebras of $\mathfrak{g}$: $\mathfrak{n}^+$ is a subalgebra since $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ with $\alpha, \beta \in \Delta^+$ implies $[x,y] \in \mathfrak{g}_{\alpha+\beta}$ so either $[x,y] = 0$ or $\alpha + \beta \in \Delta^+$ that is $[x,y] \in \mathfrak{n}^+$ as required. To check nilpotence of $\mathfrak{n}^+$ recall that Δ has a subset π such that any $\alpha \in \Delta^+$ takes form $\alpha = \sum_{\beta\in\pi} r_\beta \beta$ for some $r_\beta \in \mathbb{Z}, r_\beta \geq 0$; call $ht(\alpha) := \sum r_\beta$ a *height* of α. If $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_\beta$ then $[x,y] \in \mathfrak{g}_{\alpha+\beta}$ and $ht(\alpha+\beta) = ht(\alpha) + ht(\beta)$. Set $\mathfrak{n}_1^+ := [\mathfrak{n}^+, \mathfrak{n}^+]$, $\mathfrak{n}_{k+1}^+ := [\mathfrak{n}_k^+, \mathfrak{n}^+]$. Then $\mathfrak{n}_k^+ \subset \sum_{\alpha\in\Delta^+ : ht(\alpha)>k} \mathfrak{g}_\alpha$. Since Δ^+ is finite, there exists m such that $ht(\alpha) < m$ for all $\alpha \in \Delta^+$. Then $\mathfrak{n}_m^+ = 0$ as required.

One has

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

This is called a *triangular decomposition.* Set

$$\mathfrak{b}^+ := \mathfrak{n}^+ + \mathfrak{h}, \quad \mathfrak{b}^- := \mathfrak{n}^- + \mathfrak{h}.$$

The algebras $\mathfrak{b}^+, \mathfrak{b}^-$ are solvable because $[\mathfrak{b}^+, \mathfrak{b}^+] = \mathfrak{n}^+$. The algebra $\mathfrak{b}^+$ is called a *Borel subalgebra.*

2.2. **Verma modules.** Verma modules of $\mathfrak{g}$ are parametrized by the elements of $\mathfrak{h}^*$. For $\lambda \in \mathfrak{h}^*$ let $I(\lambda)$ be the left ideal generated by the elements of $\mathfrak{n}^+$ and the elements of the form $h - \lambda(h)$ for $h \in \mathfrak{h}$. The quotient

$$M(\lambda) := U(\mathfrak{g})/I(\lambda)$$

is a $\mathfrak{g}$-module called a *Verma module of hightest weight* λ. The image v_λ of 1 generates $M(\lambda)$ and, by PBW theorem, the map $U(\mathfrak{n}^-) \to M(\lambda)$ given by $u \mapsto uv_\lambda$ is bijective; clearly, this is a bijection of $\mathfrak{n}^-$-modules. The vector v_λ has weight λ since $(h - \lambda(h))v_\lambda = 0$.