INTRODUCTION TO REPRESENTATION THEORY. 17/1/05. LECTURE 14

1. Highest weight modules

Fix a semisimple Lie algebra $\mathfrak{g}$, its Cartan subalgebra $\mathfrak{h}$ and decomposition $\Delta = \Delta^+ \coprod \Delta^-$. Take the corresponding triangular decomposition

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

Recall that $\mathfrak{n}^+$ and $\mathfrak{n}^-$ are nilpotent subalgebras of $\mathfrak{g}$

$$\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+.$$

This is called a *triangular decomposition*. Set

$$\mathfrak{b}^+ := \mathfrak{n}^+ + \mathfrak{h}, \quad \mathfrak{b}^- := \mathfrak{n}^- + \mathfrak{h}.$$

The algebras $\mathfrak{b}^+, \mathfrak{b}^-$ are solvable ($[\mathfrak{b}^+, \mathfrak{b}^+] = \mathfrak{n}^+$). The algebra $\mathfrak{b}^+$ is called a *Borel subalgebra.*

1.1. **Verma modules.** Recall that Verma modules of $\mathfrak{g}$ are parametrized by the elements of $\mathfrak{h}^*$. Taking $\lambda \in \mathfrak{h}^*$ we can define $M(\lambda)$ as

$$M(\lambda) := U(\mathfrak{g})/\big(U(\mathfrak{g})\mathfrak{n}^+ + \sum_{h\in\mathfrak{h}} U(\mathfrak{g})(h - \lambda(h))\big).$$

Clearly, the image of 1 is a cyclic vector; denote it by v_λ. Moreover, as $\mathfrak{n}^-$-modules

$$U(\mathfrak{n}^-) \cong M(\lambda)$$

where $u \mapsto uv_\lambda$ for any $u \in U(\mathfrak{n}^-)$. Viewing $U(\mathfrak{n}^-)$ as $\mathfrak{h}$-module wrt the adjoint action one has

$$\operatorname{ch} U(\mathfrak{n}^-) = \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{-1}.$$

Therefore

$$\operatorname{ch} M(\lambda) = e^\lambda \prod_{\alpha\in\Delta^+} (1 - e^{-\alpha})^{-1}.$$

1.1.1. Set

$$Q^+ := \sum_{\alpha\in\Delta^+} \mathbb{Z}_{\geq 0}\alpha = \sum_{\alpha\in\pi} \mathbb{Z}_{\geq 0}\alpha.$$

Define a partial order on $\mathfrak{h}^*$: say $\lambda \geq \nu$ if $\lambda - \nu \in Q^+$.

1.1.2. Call a $\mathfrak{g}$-module V a *module of highest weight* λ if V admits a character and

$$\operatorname{ch} V = c_\lambda e^\lambda + \sum_{\mu<\lambda} c_\mu e^\mu$$

where $c_\lambda \neq 0$.

Any quotient of $M(\lambda)$ is a module of highest weight λ. Conversely, if V is a module of highest weight λ and V has a cyclic vector v of weight λ then V is isomorphic to a quotient of $M(\lambda)$. Indeed, since λ is a maximal weight, $\mathfrak{n}^+ v = 0$. Therefore $\operatorname{Ann}_{U(\mathfrak{g})} v \subset \operatorname{Ann}_{U(\mathfrak{g})} v_\lambda$ and so the map $\phi : M(\lambda) \to V$ given by $v_\lambda \mapsto v$ is a $U(\mathfrak{g})$-homomorphism. This homomorphism is surjective since v generates V. Hence $V \cong M(\lambda)/\operatorname{Ker}\phi$.

Lemma. *Each Verma module has a unique maximal submodule $\overline{M}(\lambda)$.*

Proof. Notice that for any family $V^{(i)} : i \in I$ of weight modules the sum $\sum_{i\in I} V^{(i)}$ is again a weight module that is

$$\sum_{i\in I} V^{(i)} = \sum_\mu (\sum_{i\in I} V^{(i)})_\mu$$

and, moreover,

$$(\sum_{i\in I} V^{(i)})_\mu = \sum_{i\in I} V^{(i)}_\mu.$$

Observe that $\mathfrak{h}$ acts diagonally to $M(\lambda)$ and so any submodule V of $M(\lambda)$ is weight module that is

$$V = \sum_\mu M(\lambda)_\mu \cap V.$$

Clearly, $V \neq M(\lambda)$ iff $V_\lambda = 0$. Let S be the sum of all proper submodules of $M(\lambda)$ (proper means $V \neq M(\lambda)$). By the above, S is again weight module and $S_\lambda = 0$. Thus S is proper. Hence S is the unique maximal submodule of $M(\lambda)$. ◻

Corollary. *$V(\lambda) := M(\lambda)/\overline{M}(\lambda)$ is a unique simple module of highest weight λ.*

1.1.3. **Corollary.** *If V is finite-dim. simple module then $V \cong V(\lambda)$ for some λ.*

Proof. The set of weight of V is finite and so it contains a maximal element λ. Let v be a non-zero element of V_λ. Since λ is a maximal weight, $\mathfrak{n}^+ v = 0$. Therefore $\operatorname{Ann}_{U(\mathfrak{g})} v \subset \operatorname{Ann}_{U(\mathfrak{g})} v_\lambda$ and so the map $M(\lambda) \to V$ given by $v_\lambda \mapsto v$ is a $U(\mathfrak{g})$-homomorphism. Since V is simple, this map is surjective; hence $V \cong V(\lambda)$. ◻

1.2. Finite-dimensional modules.

1.2.1. **Proposition.** *If V, V' are finite-dimensional modules and* $\operatorname{ch} V = \operatorname{ch} V'$ *then* $V \cong V'$.

Proof. Remark that
$$\operatorname{ch} V(\lambda) = e^{\lambda} + \sum_{\mu<\lambda} c_\mu e^{\mu}.$$
Write
$$V = \oplus V(\lambda_i)^{\oplus m_i}.$$
Let λ_1 be maximal that is $\lambda_1 \not< \lambda_i$ for all i. Then $\dim V_{\lambda_1} = m_1$ and λ_1 is a maximal weight of V. Since $\operatorname{ch} V = \operatorname{ch} V'$, λ_1 is a maximal weight of V' and so V' contains a submodule $V(\lambda_1)^{\oplus m_1}$. Hence
$$V = L \oplus V(\lambda_1)^{\oplus m_1}, \quad V' = L' \oplus V(\lambda_1)^{\oplus m_1}$$
and thus $\operatorname{ch} L = \operatorname{ch} L'$. Since $\dim L < \dim V$, the statement follows by induction on $\dim V$. □

1.2.2. If $V(\lambda)$ is finite dimensional then $<\lambda, \alpha>= 2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \Delta^+$ and, in particular, for all $\alpha \in \pi$. A weight lattice:
$$P(\pi) := \{\lambda \in \mathfrak{h}^* | \ \alpha \in \Delta^+ \quad <\lambda, \alpha>\in \mathbb{Z}\} = \{\lambda \in \mathfrak{h}^* | \ \alpha \in \pi \quad <\lambda, \alpha>\in \mathbb{Z}\}$$
(the equality is not obvious).

$$P^+(\pi) := \{\lambda \in \mathfrak{h}^* | \ \alpha \in \pi \quad <\lambda, \alpha>\in \mathbb{Z}\}.$$

One has the

Claim. *$V(\lambda)$ is finite dimensional iff $\lambda \in P^+(\pi)$.*

1.3. **Fundamental weights.** Let $\pi := \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots; define the *fundamental weights* $\omega_1, \dots, \omega_l$ ($l = \dim \mathfrak{h}$) by the condition $<\omega_i, \alpha_j>= \delta_{ij}$.

Observe that $P^+(\pi)$ is generated by $\omega_1, \dots, \omega_l$ ($l = \dim \mathfrak{h}$) that is $P^+(\pi) = \sum_i \mathbb{Z}_{\geq 0}\omega_i$.

1.4. **Example:** $\mathfrak{g} = \mathfrak{sl}(n)$. Recall that $\pi = \{\varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n\}$ where, as before, by ε_i we mean the image of ε_i in $\sum_{j=1}^n \mathbb{C}\varepsilon_j / \mathbb{C}(\sum_{j=1}^n \varepsilon_j)$.

In this case, $\omega_1 = \varepsilon_1$, $\omega_2 = \varepsilon_1 + \varepsilon_2$ and $\omega_r = \sum_{j=1}^r \varepsilon_j$ for $1 \leq r \leq n-1$.

The natural representation is isomorphic to $V(\varepsilon_1) = V(\omega_1)$. Let us show that $\Lambda^2 V(\varepsilon_1) \cong V(\omega_2)$. Recall that $\operatorname{ch} V(\varepsilon_1) = \sum_{i=1}^n e^{\varepsilon_i}$ and so
$$\operatorname{ch} \Lambda^2 V(\varepsilon_1) = \sum_{1\leq i<j\leq n} e^{\varepsilon_i + \varepsilon_j}.$$

We claim that $V := \Lambda^2 V(\varepsilon_1)$ is simple. Indeed, let L be a submodule of V; then $\dim L_{\varepsilon_i+\varepsilon_j} = 1$ for some i, j. Since the character of finite-dimensional module is W-invariant, this implies $\dim L_{\varepsilon_r+\varepsilon_s} = 1$ for all r, s. Finally, the maximal weight of $\Lambda^2 V(\varepsilon_1)$ is $\varepsilon_1 + \varepsilon_2$ so $\Lambda^2 V(\varepsilon_1) \cong V(\omega_2) = V(\varepsilon_1 + \varepsilon_2)$ as required.

2. Exercises

Put $\mathfrak{g} := \mathfrak{sl}(n)$.

1. Find $\operatorname{ch} V(\omega_i)$.

2. Is $Sym^k V(\omega_1) \cong V(k\omega_1)$?

Hint: use Example 1.3 of Lecture 13.