INTRODUCTION TO REPRESENTATION THEORY. 24/1/05. LECTURE 15

1. Finite-dimensional $\mathfrak{sl}(n)$-modules

Recall the

Claim. *$V(\lambda)$ is finite dimensional iff $\lambda \in P^+(\pi)$.*

The part "if $V(\lambda)$ is finite dimensional then $\lambda \in P^+(\pi)$" is easy. Indeed, $\lambda \in P^+(\pi)$ means $<\lambda, \alpha>= 2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z}_{\geq 0}$ for all $\alpha \in \pi$. Take $\alpha \in \pi$ and the corresponding $\mathfrak{sl}(2)$-triple $e_\alpha \in \mathfrak{g}_\alpha, f_\alpha \in \mathfrak{g}_{-\alpha}, h_\alpha = [e_\alpha, f_\alpha]$. If $V(\lambda)$ is finite dimensional a highest weight vector v_λ generates a finite-dimensional module over this copy of $\mathfrak{sl}(2)$; since $e_\alpha v_\lambda = 0$ this implies $2(\lambda,\alpha)/(\alpha,\alpha) \in \mathbb{Z}_{\geq 0}$ as required. Notice that the second part "$V(\lambda)$ is finite dimensional if $\lambda \in P^+(\pi)$" is equivalent to the fact that $V(\omega_i)$ are finite-dimensional for all fundamental roots ω_i. Indeed, let $V(\omega_i)$ be finite-dimensional. Take $\lambda \in P^+(\pi)$ that is $\lambda = \sum k_i\omega_i$ for some $k_i \in \mathbb{Z}_{\geq 0}$. The module $N := V(\omega_1)^{\otimes k_1} \otimes V(\omega_2)^{\otimes k_2} \otimes \ldots \otimes V(\omega_l)^{\otimes k_l}$ has highest weight λ and so there is a non-zero map $M(\lambda) \to N$. Thus $V(\lambda)$ is a subquotient of N and, in particular, $V(\lambda)$ is finite-dimensional.

Now we the claim for $\mathfrak{sl}(n)$ follows from the fact that $\Lambda^k V(\omega_1) \cong V(\omega_k)$ (for $k = 2$ it is shown in Example 1.4, Lect. 14; for other k the proof is similar) and so $V(\omega_k)$ is finite-dimensional.

1.1. Examples of simple finite dimensional modules.

Recall Example 1.3 from Lecture 13. Define an $\mathfrak{gl}(n)$-module structure on the polynomial algebra $\mathbb{C}[x_1, \ldots, x_n]$ by setting $E_{ij}(p) = x_i \frac{\partial}{\partial x_j} p$. For each $m \geq 0$, the space of homogeneous polynomials of total degree m is a submodule $V(m)$ and its character wrt $\mathfrak{sl}(n)$ is given by

$$\operatorname{ch} V(m) = \sum_{r_1, r_2, \ldots, r_n \geq 0,\ r_1 + \ldots r_n = m} e^{r_1\varepsilon_1 + r_2\varepsilon_2 + \ldots + r_n\varepsilon_n}.$$

We claim that $V(m)$ is simple. Indeed, all weight spaces are of $V(m)$ are one-dimensional and are spanned by monomials. Thus, if L is a non-trivial submodule of $V(m)$, L contains a monomial $p := x_1^{r_1} x_2^{r_2} \ldots x_n^{r_n}$ (where $\sum r_i = m$). Now acting by $E_{ij} = x_i \frac{\partial}{\partial x_j}$ on p we obtain all possible monomials of the total degree m; hence $L = V(m)$.

A highest weight of $V(m)$ is $m\varepsilon_1$ so $V(m) \cong V(m\varepsilon_1) = V(m\omega_1)$. This gives us an explicit realisation of $V(m\omega_1)$.

Observe that $\mathbb{C}[x_1, \dots, x_n]$ is the symmetric algebra of a vector space spanned by $x_1, \dots, x_n$. Moreover, as $\mathfrak{sl}(n)$-module $\mathbb{C}[x_1, \dots, x_n]$ is isomorphic to $SymV(\varepsilon_1)$. Hence

$$V(m) \cong V(m\omega_1) \cong Sym^m V(\omega_1).$$

2. Weyl character formula

Set

$$\rho := \sum_{\alpha \in \Delta^+} \alpha/2.$$

Define the twisted action of W on $\mathfrak{h}^*$ by

$$w.\lambda := w(\lambda + \rho) - \rho.$$

Recall that W is generated by the reflections $s_\alpha : \alpha \in \Delta^+$. It turns out that it is actually generated by the smaller set of "simple reflections" $s_\alpha : \alpha \in \pi$. Thus each w can be written as a product of simple reflections; let $l(w)$ be the minimal length (number of factors) of such a product. It is possible to show that if w is a product of k reflections then k has the same parity as $l(w)$. In particular, the map $w \mapsto (-1)^{l(w)}$ is a group homomorphism $W \to \mathbb{Z}_2$.

2.1. **Theorem.** *For $\lambda \in P^+(\pi)$ one has*

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w.\lambda}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}.$$

For $\lambda = 0$ the Weyl character formula gives

$$1 = V(0) = \frac{\sum_{w \in W} (-1)^{l(w)} e^{w.0}}{\prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})}$$

so $\sum_{w \in W} (-1)^{l(w)} e^{w.0} = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$.

Now we can reformulate the theorem in the form

2.2. **Corollary.** *For $\lambda \in P^+(\pi)$ one has*

$$\operatorname{ch} V(\lambda) = \frac{\sum_{w \in W} (-1)^w e^{w.\lambda}}{\sum_{w \in W} (-1)^w e^{w.0}} = \frac{\sum_{w \in W} (-1)^w e^{w(\lambda+\rho)}}{\sum_{w \in W} (-1)^w e^{w\rho}}.$$

2.3. An outline of a proof of Theorem 2.1.

Since $\mathfrak{h}$ is commutative, $U(\mathfrak{h})$ is a polynomial algebra. Identify $U(\mathfrak{h})$ with $S(\mathfrak{h})$ and view elements of $U(\mathfrak{h})$ as polynomial functions on $\mathfrak{h}^*$. The projection $\mathrm{HC} : U(\mathfrak{g}) \to U(\mathfrak{h}) = S(\mathfrak{h})$ with the kernel $U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g})$ is called the *Harish-Chandra* projection.

Let $Z(\mathfrak{g})$ be the centre of $U(\mathfrak{g})$.

2.3.1. **Lemma.** *Any element $z \in Z(\mathfrak{g})$ acts on a Verma module $M(\lambda)$ by the scalar* $\mathrm{HC}(z)(\lambda)$.

Proof. Let v_λ be a highest weight vector of $M(\lambda)$. Since v_λ generates $M(\lambda)$, it is enough to show that $zv_\lambda = \mathrm{HC}(z)(\lambda)v_\lambda$. Write $z = \mathrm{HC}(z) + u$. Since $\mathrm{HC}(z) \in U(\mathfrak{h})$ one has $\mathrm{HC}(z)v_\lambda = \mathrm{HC}(z)(\lambda)v_\lambda$. It remains to show that $uv_\lambda = 0$.

Recall that $u \in U(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^- U(\mathfrak{g})$. Clearly, $U(\mathfrak{g})\mathfrak{n}^+ v_\lambda = 0$. Thus $uv_\lambda \in \mathfrak{n}^- U(\mathfrak{g})v_\lambda = \mathfrak{n}^- M(\lambda)$. Notice that $\mathfrak{n}^- M(\lambda) = \sum_{\mu<\lambda} M(\lambda)$. Since z is central, it has zero weight; $\mathrm{HC}(z) \in U(\mathfrak{h})$ has weight zero and so u has weight zero as well. Therefore $uv_\lambda \in M(\lambda)_\lambda$. Finally, $M(\lambda)_\lambda \cap \mathfrak{n}^- M(\lambda) = 0$ so $uv_\lambda = 0$ as required. ☐

If $V(\mu)$ is a subquotient of $M(\lambda)$, then $z \in Z(\mathfrak{g})$ acts on $V(\mu)$ by $\mathrm{HC}(z)(\lambda)$. Therefore $\mathrm{HC}(z)(\lambda) = \mathrm{HC}(z)(\mu)$ for any $z \in Z(\mathfrak{g})$.

2.3.2. Set

$$S(\mathfrak{h})^{W\cdot} = \{f \in S(\mathfrak{h}) |\ w.f = f \ \text{ for all } w \in W\}.$$

The following Harish-Chandra theorem describes the centre $Z(\mathfrak{g})$.

Theorem. *The retsriction of* HC *to* $Z(\mathfrak{g})$ *is the algebra isomorphism* $Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{h})^{W\cdot}$.

As a corollary one obtains

2.3.3. **Corollary.** *If $M(\lambda)$ has a subquotient isomorphic to $V(\mu)$ then $\mu \in W.\lambda$ and $\mu < \lambda$.*

It is not hard to show that $M(\lambda)$ admits a finite Jordan-Hölder series. Any simple subquotient of $M(\lambda)$ has a highest weight so is of the form $V(\mu)$. Therefore

$$\mathrm{ch}\, M(\lambda) = \sum_{\mu\in W.\lambda;\mu<\lambda} a_\mu^\lambda \,\mathrm{ch}\, V(\mu)$$

where a_μ is the multiplicity of $V(\mu)$ in the set of simple subquotients of $M(\lambda)$. Notice that $a_\lambda^\lambda = 1$. Fix $\lambda \in \mathfrak{h}^*$ and set

$$\Lambda := \{\nu \in \mathfrak{h}^* |\ \nu \in W.\lambda; \nu - \lambda \in Q(\pi)\}.$$

The partial order $\geq$ on $\mathfrak{h}^*$ induces a total order on Λ. Consider the matrix (a_μ^ν). This matrix is triangular (we order indexes wrt the total order on Λ) and all diagonal elements are equal to 1. Hence A is invertible, $B := A^{-1}$ is again triangular and all diagonal elements are equal to 1. One obtains

$$\mathrm{ch}\, V(\lambda) = \sum_{\mu\in W.\lambda;\mu<\lambda} b_\mu^\lambda \,\mathrm{ch}\, M(\mu) \tag{1}$$

where $b_\lambda^\lambda = 1$.

2.3.4. Now we are going to use that $V(\lambda)$ is finite-dimensional: we will deduce that $b^{\lambda}_{\mu}=(-1)^{l(w)}$ for $\mu=w.\lambda$ from the fact that $\operatorname{ch} V(\lambda)$ is W-invariant.

Rewrite (1) as

$$\operatorname{ch} V(\lambda)=\sum_{w\in W} b(w)\operatorname{ch} M(w.\lambda)=\frac{\sum_{w\in W} b(w)e^{w(\lambda+\rho)}}{e^{\rho}\prod_{\alpha\in\Delta^+}(1-e^{-\alpha})}$$

where for each $\mu\in W.\lambda$, $\mu<\lambda$ we choose $w\in W$ such that $\mu=w.\lambda$; for $\mu=\lambda$ we chooose $w=e$ and so $b(e)=1$.

Rewrite the denominator as

$$D:=e^{\rho}\prod_{\alpha\in\Delta^+}(1-e^{\alpha})=\prod_{\alpha\in\Delta^+}(e^{\alpha/2}-e^{-\alpha/2})$$

and observe that, by Lemma 2.3.5), $s_{\alpha}(D)=-D$ for any simple α. Since $s_{\alpha}(\operatorname{ch} V)=\operatorname{ch} V$, one obtains $s_{\alpha}(\sum_{w\in W} b(w)e^{w(\lambda+\rho)})=-\sum_{w\in W} b(w)e^{w(\lambda+\rho)}$. Now the condition $b(e)=1$ gives $b(w)=(-1)^{l(w)}$. This proves the Weyl character formula.

2.3.5. **Lemma.** *For any $\alpha\in\pi$ the set $\Delta\setminus\{\alpha\}$ is s_{α}-invariant.*

Proof. Take $\beta\in\Delta\setminus\{\alpha\}$ and write $\beta=\sum_{\gamma\in\pi}c_{\gamma}\gamma$; observe that $c_{\gamma_0}>0$ for some $\gamma_0\neq\alpha$ (because $\mathbb{Z}\alpha\cap\Delta^+=\{\alpha\}$). Then

$$s_{\alpha}(\beta)=\sum_{\gamma\in\pi,\gamma\neq\alpha}c_{\gamma}\gamma+c'_{\alpha}\alpha$$

is a root and the condition $c_{\gamma_0}>0$ implies that this root is positive. □

Corollary.

$$\rho=\sum\omega_i.$$

Proof. Since the fundamental weights $\{\omega_i\}$ is dual to the set of simple roots, they form a basis of $\mathfrak{h}^*$. Thus $\rho=\sum k_i\omega_i$. Let α_i be the ith simple root; then $s_{\alpha_i}(\rho)=\rho-k_i\alpha_i$. One has

$$s_{\alpha_i}(\rho)=s_{\alpha_i}(\sum_{\alpha\in\Delta^+}\alpha/2)=s_{\alpha_i}(\sum_{\alpha\in\Delta^+\setminus\{\alpha_i\}}\alpha/2+s_{\alpha_i}(\alpha_i/2)=\sum_{\alpha\in\Delta^+\setminus\{\alpha_i\}}\alpha/2-\alpha_i/2=\rho-\alpha_i.$$

Hence $k_i=1$ as required. □

2.4. The Weyl character formula gives the following dimension formula

$$\dim V(\lambda) = \prod_{\alpha \in \Delta^+} \frac{< \lambda + \rho, \alpha >}{< \alpha, \alpha >} = \prod_{\alpha \in \Delta^+} \frac{(\lambda + \rho, \alpha)}{(\alpha, \alpha)},$$

see W. Fulton, J. Harris "Representation Theory" p.402 for the proof.