INTRODUCTION TO REPRESENTATION THEORY. 29/11/04. LECTURE 7

1. Adjoint action

Let L be a Lie algebra, $x \in L$. Define a linear transformation

$$\operatorname{ad} x : L \to L$$

by the formula $\operatorname{ad}_x(y) = [x, y]$.

1.1. **Lemma.** ad_x *is a linear transformation.* In fact, this follows from the linearity of $[-,-]$ in the first argument.

1.2. **Lemma.** ad_x *is a derivation.* In fact,

$$\operatorname{ad}_x[y, z] = [\operatorname{ad}_x(y), z] + [y, \operatorname{ad}_x(z)]$$

— this follows from the Jacobi identity.

1.3. Assembling together ad_x for all $x \in L$ we get therefore a map

$$\operatorname{ad} : L \to \operatorname{Der}(L).$$

Lemma. *The map* $\operatorname{ad} : L \to \operatorname{Der}(L)$ *is a homomorphism of Lie algebras.* One has to check that

$$\operatorname{ad}_{[x,y]} = \operatorname{ad}_x \circ \operatorname{ad}_y - \operatorname{ad}_y \circ \operatorname{ad}_x .$$

This also follows from the Jacobi identity.

1.4. **Definition.** Center of a Lie algebra L is defined by the formula

$$Z(L) = \{x \in L | \forall y \in L \quad [x, y] = 0\}.$$

By definition of ad, one has $Z(L) = \operatorname{Ker}(\operatorname{Ad})$.

1.5. The following easy lemma is useful in calculations.

1.5.1. **Lemma.** *Let $f, g, h \in \mathrm{End}(V)$. Then*

$$[f, gh] = [f, g]h + g[f, h].$$

Here, as usual, the bracket is defined by the formula $[f, g] = fg - gf$.

Proof.

$$[f, g]h + g[f, h] = fgh - gfh + gfh - ghf = fgh - ghf = [f, gh].$$

□

The above lemma means that the operator $d_f : \mathrm{End}(V) \to \mathrm{End}(V)$ given by $d_f(g) := [f, g]$ is a derivation of the associative algebra $\mathrm{End}(V)$ (where the operation is the composition).

2. Representations

Definition. An L-module is a vector space M endowed with a Lie algebra homomorphism

$$\rho : L \to \mathfrak{gl}(M).$$

2.1. Examples.

2.1.1. $\dim L = 1$. If L is one-dimensional, say, $L = ke$, a module structure

$$\rho : L \to \mathfrak{gl}(M)$$

is given by an endomorphism of M (the image $\rho(e)$).

2.1.2. *L is commutative.* A representation of L is a Lie algebra homomorphism. If $L = \langle e_1, \ldots, e_n \rangle$, a homomorphism $r : L \to \mathfrak{gl}(M)$ is given by the images $r(e_i)$. Since r is a homomorphism, $r(e_i)$ commute. Vice versa, any collection of n commuting endomorphisms of M define on M a structure of L-module.

2.1.3. *Natural representation.* By definition, Lie algebra $\mathfrak{gl}_n$ admits an n-dimensional representation. It is given by the identity map

$$\mathrm{id} : \mathfrak{gl}_n \to \mathfrak{gl}(k^n).$$

It is called *the natural representation.* Similarly, if $\mathfrak{g} \subseteq \mathfrak{gl}_n$ is a Lie subalgebra, we have a natural n-dimensional representation of $\mathfrak{g}$.

Examples of subalgebras of $\mathfrak{gl}_n$:

$$\mathfrak{sl}_n := \{x \in \mathfrak{gl}_n \mid \mathrm{Tr}\, x = 0\},$$
$$\mathfrak{b}_n := \{(a_{ij}) \in \mathfrak{gl}_n | \forall i > j \ a_{ij} = 0\},$$
$$\mathfrak{n}_n := \{(a_{ij}) \in \mathfrak{gl}_n | \forall i \geq j \ a_{ij} = 0\}.$$

2.2. **Category of L-modules.** Fix a Lie algebra L.

A linear map $f : M \to N$ is an L-module homomorphism if

$$f(ax) = af(x)$$

for each $a \in L,\ x \in M$. Clearly, composition of homomorphisms is a homomorphism. The collection of L-homomorphism of modules is denoted $\mathrm{Hom}_L(M, N)$. It forms a vector space over k.

Thus,

$$\begin{aligned}\mathrm{Hom}_L(M, N) : \ &= \{\phi \in \mathrm{Hom}_k(M, N) |\ \forall v \in M, \forall x \in L\ \phi(xv) = x\phi(v)\} \\ &= \{\phi \in \mathrm{Hom}_k(M, N) |\ \forall x \in L\ \phi\rho(x) = \rho(x)\phi\} \\ &= \{\phi \in \mathrm{Hom}_k(M, N) |\ \forall x \in L\ [\phi, \rho(x)] = 0\}.\end{aligned}$$

If $a \in L$ is central then $\rho(a) \in \mathrm{End}_L(V)$.

Lemma. *Let $f : M \to N$ be a bijective homomorphism of L-modules. Then $f^{-1} : N \to M$ is also a homomorphism.*

Proof. Straighforward. □

The notion of submodule and quotient-module are defined in a standard way.

2.2.1. Recall that ad_f is a derivation of the Lie algebra $\mathfrak{gl}(V)$ (see Lemma 1.2) that is

$$[f, [g, h]] = [[f, g,], h] + [g, [f, h]].$$

This formula has the following application. Suppose that V is an L-module and L is generated by the set $x_i, i \in I$ (this means that the only Lie subalgebra containing all elements $x_i, i \in I$ is L itself). Then in order to check that an operator $\phi \in \mathrm{End}(V)$ is an L-endomorphism, it is enough to verify $\phi\rho(x_i) = \rho(x_i)\phi$ for all $i \in I$ (since $[\phi, \rho(x_i)] = 0,\ [\phi, \rho(x_j)] = 0$ implies $[\phi, [\rho(x_i), \rho(x_j)]] = 0$).

2.2.2. *Direct sum.* Given two L-modules M and N, one defines an L-module structure on $M \oplus N$ by the formula

$$a(m, n) = (am, an).$$

One has $\mathrm{Hom}_L(M \oplus N, M' \oplus N') = \mathrm{Hom}_L(M, M') \oplus \mathrm{Hom}_L(M, N') \oplus \mathrm{Hom}_L(M', N) \oplus \mathrm{Hom}_L(M', N')$.

2.3. **Schur's lemma. Theorem.** *Suppose the base field k is algebraically closed. If V is a simple finite dimensional module over a Lie algebra L then $\mathrm{Hom}_L(V, V) = k \cdot \mathrm{id}$.*

Proof. Take $\phi \in \mathrm{Hom}_L(V, V)$. For any $c \in k$ the linear operator $(\phi - c \cdot \mathrm{id})$ is an L-homomorphism and so it is either an isomorphism or zero. Let c be an eigenvalue of ϕ;

then the operator $(\phi - c \cdot \mathrm{id})$ has a non-zero kernel and so it is not an isomorphism. Hence $\phi - c \cdot \mathrm{id} = 0$ as required. □

2.4. **Representation theory.** Representation theory of Lie algebras studies the category of modules over a Lie algebra. Here are the typical questions and the typical notions studied.

2.4.1. *Classification.* Description of all isomorphism classes of L-modules. Sometimes only modules satisfying special properties are considered (e.g., finite dimensional modules).

Today we will see that in the case L is one-dimensional we already know the answer from Linear Algebra.

2.4.2. *Simple modules.* A module is called simple if it does not admit non-trivial submodules. (A synonym: irreducible representation).

We say that vectors generate a submodule M' if M' is the minimal submodule containing these vectors. A simple module is generated by any non-zero vector.

2.4.3. *Semisimple modules.* A module is called semisimple if it is isomorphic to a direct sum of simple modules (there are other equivalent definitions). Synonym: a completely reducible representation.

We will study soon the following result.

Theorem. *All finite dimensional representations of $\mathfrak{sl}_2$ are completely reducible.*

2.5. **Representations of a one-dimensional Lie algebra.** .

2.5.1. *Isomorphism classes.* We are looking for isomorphism classes of n-dimensional representations. A map $f : M \to N$ is a homomorphism of representations if $f\alpha_M = \alpha_N f$. Since $M = N = k^n$ as vector spaces, we deduce that endomorphisms α_1 and α_2 define isomorphic representations iff there exists an automorphism f such that $\alpha_2 = f\alpha_1 f^{-1}$.

Thus the problem of classifications of n-dimensional representations is equivalent to that of classification of square matrices up to conjugation.

Theory of Jordan normal form answers this question in the case k is algebraically closed.

If k is algebraically closed, simple modules are one-dimensional. Finite dimensional representations are not completely reducible.

2.5.2. *Recollections from Linear Algebra.* Let $f : V \to V$ be an endomorphism of a finite dimensional vector space over an algebraically closed field k. Recall that $\lambda \in k$ is an eigenvalue of f if $f - \lambda I$ is not invertible. The collection of eigenvalues of f is therefore the set of roots of the characteristic polynomial of f defined as

$$P_f(t) = \det(f - tI).$$

In what follows $S(f)$ will denote the set of eigenvalues of f.

Let $\lambda \in S(f)$. A vector $v \in V$ is called an eigenvector corresponding to λ if $f(v) = \lambda v$. Each eigenvalue admits a non-zero eigenvector. Furthermore, $v \in V$ is called generalized eigenvector if there exists $n \in \mathbb{N}$ such that $(f - \lambda I)^n v = 0$.

Fix $\lambda \in S(f)$. Let V^λ denote the set of eigenvectors and V_λ the set of generalized eigenvectors corresponding to λ. These are vector subspaces of V and

$$V_\lambda \supseteq V^\lambda \neq 0.$$

The following are the main results of this study.

— $V = \bigoplus_{\lambda \in S(f)} V_\lambda$.
— $\dim V_\lambda$ equals the multiplicity of λ in the characteristic polynomial of f.
— Each V_λ is isomorphic to a direct sum of Jordan blocks having eigenvalue λ (definition of J. b. see below).

Jordan block having eigenvalue λ is the matrix $A = (a_{ij})$ defined by the formulas

$$a_{ij} = \begin{cases} \lambda & \text{if} \quad i = j, \\ 1 & \text{if} \quad i = j - 1, \\ 0 & \quad\quad \text{otherwise.} \end{cases}$$

2.5.3. *Simple modules.* Direct consequence of the above: Any simple module has dimension 1; it is defined up to isomorphism by its (only) eigenvalue λ.

This module will be sometimes denoted by k_λ.

2.5.4. *Semisimple modules.* Semisimple module is a direct sum of simple modules. Thus (for k algebraically closed) (V, f) is semisimple iff f is diagonalizable.

2.5.5. *Example of non-semisimple modules.* It is given by the matrix

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

2.6. **Examples of representations.**

2.6.1. *Adjoint representation.* Let L be any Lie algebra. The map $\mathrm{Ad} : L \to \mathfrak{gl}(L)$ defines a repersentation of L called adjoint representation.

Note that submodules of the adjoint representation are the ideals. Therefore, the adjoint representation is irreducible iff L is simple or one-dimensional.

2.6.2. *... and its restrictions.* If $M \subseteq L$ is a Lie subalgebra, one can consider L as a M-module restricting the adjoint representation of L on M.

Consider, for example, $L = \mathfrak{sl}_2$ and $M = \langle h \rangle$. Algebra M is one-dimensional and L is an M-module. It is semisimple with eigenvalues $-2,\ 0,\ 2$.

If we take another M, say, $\langle e \rangle$, the picture will change. All eigenvalues are zero and the M-module L is not semisimple.

3. The Lie algebra $\mathfrak{sl}(2)$

3.1. It is convenient to choose a basis of $\mathfrak{sl}_2$ as follows.

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

Than the bracket in $\mathfrak{sl}_2$ is given by the formulas

$$[ef] = h,\ [he] = 2e,\ [hf] = -2f.$$

3.2. **Simplicity of $\mathfrak{sl}_2$. Definition.** A Lie algebra L is *simple* if it is not one-dimensional and if it has no non-trivial ideals.

Our aim is to prove the following

Theorem. *$\mathfrak{sl}_2$ is simple.*

3.2.1. *Some linear algebra.* Let V be a f.d. vector space and $f \in \mathrm{End}(V)$.

Endomorphism f is called *diagonalizable* if V has a basis of eigenvectors.

If f is diagonalizable then $V = \oplus_{\lambda \in S} V_\lambda$ where $V_\lambda = \{x \in V | f(x) = \lambda x\}$ is the eigenspace corresponding to the eigenvalue λ and S is the set of eigenvalues of f (*spectrum of f*).

Lemma. *Let $f \in \mathrm{End}(V)$ be diagonalizable and let W be an f-invariant subspace of V (i.e., $f(W) \subseteq W$). Then*

$$W = \oplus_{\lambda \in S} W_\lambda \text{ where } W_\lambda = W \cap V_\lambda.$$

Proof. We have to prove that if $x \in W$ and if $x = \sum x_\lambda$ with $x_\lambda \in V_\lambda$ then $x_\lambda \in W$.

In fact, $W \ni f^k(x) = \sum f^k(x_\lambda) = \sum \lambda^k x_\lambda$ for each k.

Let $T = \{\lambda \in S | x_\lambda \neq 0\}$ and let $t = |T|$. This is the number of non-zero summands in the decomposition of x. The vectors $x, f(x), \dots, f^{t-1}(x)$ can be expressed as linear combinations of t linearly independent vectors x_λ. The transition matrix has form

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_t \\ \dots & \dots & \dots & \dots \\ \lambda_1^{t-1} & \lambda_2^{t-1} & \dots & \lambda_t^{t-1} \end{pmatrix}.$$

This is a Vandermonde matrix. Its determinant is

$$\prod_{i<j} (\lambda_i - \lambda_j) \neq 0.$$

This proves that x_λ can be expressed through $f^k(x)$ and therefore belong to W. □

3.2.2. *Proof of Theorem 3.2.* Consider endomorphism ad_h of $\mathfrak{sl}_2$. It is diagonalizable with eigenvalues $-2,\ 0,\ 2$. Any ideal I is invariant with respect to ad_h. Therefore, I should be spanned by a subset of f, h, g. It is easy to check that this is impossible for any nonempty proper subset of generators.

3.3. Hence we have two real simple Lie algebras of dimension 3: $\mathfrak{sl}_2$ and the algebra L described in Lecture VI, 1.9.7. They are non-isomorphic since for any $a, b \in L$ the bracket $[a, b] \notin \mathbb{R}^* a$ and $[f, h] = 2f$.

By contrast, there is only one (up to isomorphism) complex simple Lie algebras of dimension 3, namely, $\mathfrak{sl}_2$.

4. Complex representations of $\mathfrak{sl}(2)$.

4.1. **Applications of Schur's lemma to $\mathfrak{sl}(2)$.** Take $k := \mathbb{C}$. Fix the standard basis h, e, f of $\mathfrak{sl}_2$. Recall that

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h.$$

Let $\rho : \mathfrak{sl}(2) \to \mathfrak{gl}(V)$ be a representation. Denote

$$E := \rho(e),\ F := \rho(f), H := \rho(h).$$

Then the above relations imply

$$\begin{aligned} HE - EH &= 2E, \\ HF - FH &= -2F, \\ EF - FE &= H. \end{aligned} \tag{2}$$

Consider the endomorphism

$$Q := H^2 + 2FE + 2EF.$$

This is a linear endomorphism of V. We will check now that Q is an $\mathfrak{sl}_2$-endomorphism. To check this, it is enough to prove

$$QE = EQ,\ QF = FQ,\ QH = HQ.$$

4.1.1. **Lemma.** *The operator Q commutes with E, F, H.*

Proof. Recall that all calculations are done in $\mathrm{End}(V)$.

One has

$$[E,Q] = [E, H^2 + 2EF + 2FE] = [E,H]H + H[E,H] + \\ 2E[E,F] + 2[E,F]E = -2EH - 2HE + 2EH + 2HE = 0.$$

Similarly,

$$[F,Q] = [F, H^2 + 2EF + 2FE] = [F,H]H + H[F,H] + \\ 2[F,E]F + 2F[F,E] = 2FH + 2HF - 2HF - 2FH = 0$$

and therefore

$$[H,Q] = [[E,F],Q] = 0. \tag{3}$$

□

4.1.2. **Corollary.** *Let V be finite dimensional and simple $\mathfrak{sl}_2$-module. Then $Q = c \cdot \mathrm{id}$ for some $c \in \mathbb{C}$.*

4.2. **Finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.** Our next goal is to describe all finite dimensional representations of $\mathfrak{sl}_2(\mathbb{C})$.

As a first step, we will describe a collection of irreducible representations which will turn out to be the collection of all irreducible representations.

We denote by $\mathbb{N}$ the set of non-negative integers.

4.2.1. Recall that $\mathfrak{sl}_2 \subseteq \mathfrak{gl}_2 = \langle E_{11}, E_{12}, E_{21}, E_{22}\rangle$ where E_{ij} denotes the matrix whose only non-zero entry is 1 in the (ij) position.

Note that in this notation $E = E_{12},\ F = E_{21},\ H = E_{11} - E_{22}$.

Consider the polynomial algebra $\mathbb{C}[x,y]$ and define the action of $\mathfrak{gl}_2$ on it by the formulas

$$E_{11}(p) = xp'_x,\quad E_{22}(p) = yp'_y,\quad E_{12}(p) = xp'_y,\quad E_{21}(p) = yp'_x. \tag{4}$$

Lemma. *The formulas (4) define a $\mathfrak{gl}_2$-module structure on $\mathbb{C}[x,y]$.*

Proof. One can check this claim directly.

Here is another way which allows to almost avoid calculations. Note that the formulas (4) assign to E_{ij} *derivations* of $\mathbb{C}[x, y]$ (compare to Homework,6, # 1).

Any derivation of $\mathbb{C}[x, y]$ is uniquely defined by its value on the degree one polynomials x and y: if $d(x) = p,\ d(y) = q$ then $d(f) = pf'_x + qf'_y$ (once more, compare to Homework,6, # 1).

Then, in order to prove the formulas (4) are compatible with the brackets it is enough to check them on x and on y. One can see that the formulas (4) restricted on $\langle x, y\rangle$ give just the natural representation of $\mathfrak{gl}_2$. $\square$

The set of homogeneous polynomials of a degree n is, obviously, a $\mathfrak{gl}_2$-submodule and an $\mathfrak{sl}_2$-submodule. Denote this $\mathfrak{sl}_2$-submodule by $V(n)$. We will show that $V(n)$ is a simple $\mathfrak{sl}_2$-module and that any simple module of dimension $n + 1$ is isomorphic to $V(n)$.