# ENVELOPING ALGEBRAS. (THIS WAS THE SUBJECT OF LECTURE 9.)

## 1. Enveloping algebra

1.1. **Enveloping algebra of $\mathfrak{sl}_2(\mathbb{C})$.** Take $k := \mathbb{C}$. Fix the standard basis $h, e, f$ of $\mathfrak{sl}_2$. Recall that

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

**1.1.1.** Let $\rho : \mathfrak{sl}(2) \to \mathfrak{gl}(V)$ be a representation. Denote

$$E := \rho(e), \ F := \rho(f), H := \rho(h).$$

They satisfy the relations:

$$\begin{array}{rcl} HE - EH & = & 2E, \\ HF - FH & = & -2F, \\ EF - FE & = & H. \end{array} \tag{1}$$

We have shown that the linear endomorphism

$$Q := H^2 + 2FE + 2EF \in \mathrm{End}(V)$$

is an $\mathfrak{sl}_2$-endomorphism. By Schur's lemma, if $V$ is simple, $Q$ acts by a scalar (that is by a multiple of the identity map).

Let $V_1, V_2$ be two module and $Q_1 \in \mathrm{End}(V_1), Q_2 \in \mathrm{End}(V_2)$ be the endomorphism given by the above formula. If $\phi : V_1 \to V_2$ is a homomorphism then $Q_2\phi = \phi Q_1$ since

$$Q_2\phi = (\rho_2(h)^2 + \rho_2(e)\rho_2(f) + \rho_2(f)\rho_2(e))\phi = \phi(\rho_1(h)^2 + \rho_1(e)\rho_1(f) + \rho_1(f)\rho_1(e)) = \phi Q_1.$$

Moreover, if $V_1$ is submodule of $V_2$ then $Q_2(V_1) \subset V_1$ and the restriction of $Q_2$ to $V_1$ is $Q_1$.

**1.1.2.** We have a strange situation: there is an interesting element $Q$ which has the same formula in each module. However, $Q$ is not defined by an element of the Lie algebra, as $E$, $F$ or $H$. It turns out that it is possible to construct a bigger algebra which contains all these elements. This algebra, however, is not a Lie algebra. This is an associative algebra.

**Definition 1.1.3.** Let $U$ be an associative algebra. A module $V$ over $U$ is a vector space together with an algebra homomorphism $f : U \to \mathrm{End}(V)$.

As usual, if $x \in U$ and $v \in V$, we denote $xv \in V$ the image of $v$ under the endomorphism $f(x)$.

**Definition 1.1.4.** Let $A$ be an associative algebra. A Lie algebra $A^{\mathrm{Lie}}$ is defined as follows. As a vector space, this is the same algebra $A$. The bracket in $A^{\mathrm{Lie}}$ is defined by the formula $[a, b] := ab - ba$.

For example, $\mathrm{End}(V)^{\mathrm{Lie}} = \mathfrak{gl}(V)$.

Let now a homomorphism $f : L \to A^{\mathrm{Lie}}$ from a Lie algebra $L$ to the Lie algebra corresponding to $A$ is given. This means simply that $f$ is a linear map satisfying the property

$$f([x, y]) = f(x)f(y) - f(y)f(x). \tag{2}$$

Then, if $V$ is an $A$-module, $f$ defines on $V$ a structure of $L$-module by the formula

$$xv = f(x)v. \tag{3}$$

The property (2) assures that $[x, y]v = xyv - yxv$.

**1.1.5.** We are ready to present now an associative algebra $U$ having the same modules as $\mathfrak{sl}(2)$.

As a vector space $U$ is generated by expressions of type $F^aH^bE^c$ where $a, b, c \in \mathbb{N}$. We have to indicate how to multiply two expressions of this form. It is enough to know how to multiply $F^aH^bE^c$ from the right on $E, H, F$. A general idea is simple: to use identities (1) so that the result once more has the required form. Multiplication by $E$ is the easiest: the result is $F^aH^bE^{c+1}$. Let us show how to multiply by $H$. Since $EH = HE - 2E$, we can move recursively $H$ to the left through the $E's$. Similarly one proceeds with multiplication by $F$.

The mathematical proof of existence of multiplication on $U$ satisfying the properties (1) is nontrivial. A much more general statement will be proven at the next lecture.

Let us indicate that $Q = H^2 + 2EF + 2FE = H^2 + 4EF - 2H$ belongs to $U$.

**1.1.6.** One can easily see that in any associative algebra

$$[f, gh] = [f, g]h + g[f, h]$$

where $[f,g] = fg - gf$. Actually, this means that the map $ad_f : g \mapsto [f,g]$ is a derivation. Also in any associative algebra one has

$$[f,[g,h]] = [[f,g],h] + [g,[f,h]].$$

Recall that

$$[E,Q] = [F,Q] = [H,Q] = 0 \tag{4}$$

**Definition 1.1.7.** The centre of an algebra $A$ is the set

$$Z(A) := \{x \in A | \ \forall a \in A \ ax = xa\}.$$

Remark that for Lie algebras $[a,x] = [x,a]$ implies $[a,x] = 0$ and so the definition coincides with the previous definition of the centre of Lie algebra. The centre is a linear subspace and, moreover, a subalgebra, but it is not always an ideal. The relations (4) shows that $Q \in Z(U)$.

1.2. **Enveloping algebra: general construction.**

**Definition 1.2.1.** Let $L$ be a Lie algebra. Enveloping algebra of $L$ is an associative algebra $U$ endowed with a homomorphism

$$f : L \to U^{\mathrm{Lie}}$$

such that for any homomorphism $g : L \to A^{\mathrm{Lie}}$ there exists a unique associative algebra homomorphism $h : U \to A$ satisfying $g = h \circ f$.

The existence of enveloping algebra of any Lie algebra and its uniqueness is proven below. Let us first understand why the enveloping algebra has the same modules as the original Lie algebra.

In fact, as it have been already explained, any $U$-module $V$ admits a structure of $L$-module by the formula $xv = f(x)v$. In the other direction, let $V$ be an $L$-module. This means that a homomorphism

$$L \to \mathfrak{gl}(V) = \mathrm{End}(V)^{\mathrm{Lie}}$$

is given. By the universal property of $U$, this homomorphism can be uniquely presented as a composition of $f : L \to U^{\mathrm{Lie}}$ and a homomorphism $U \to \mathrm{End}(V)$. Thus we have got a structure of $U$-module on $V$. The two described operations being mutually inverse, we get the claim.

**1.2.2. Uniqueness.**

In what follows we will write $f : L \to A$ instead of $f \to A^{\mathrm{Lie}}$.

Suppose $f : L \to U$ and $f' : L \to U'$ are two enveloping algebras. Since $U$ is an enveloping algebra, there exists a unique associative algebra homomorphism $h : U \to U'$ such that $f' = hf$. Similarly, since $U'$ is an enveloping algebra, there exists a unique associative algebra homomorphism $h' : U' \to U$ such that $f = h'f'$.

Then one deduces $f = h'hf$ which implies that $h'h = \mathrm{id}_U$ (Why?). Similarly $hh' = \mathrm{id}_{U'}$ and we have proven the following

**Theorem 1.2.3.** *The enveloping algebra of a Lie algebra (if it exists) is unique up to a unique isomorphism.*

The proof of existence of enveloping algebra is not difficult, but requires some algebraic technique (mambo-jumbo?) which is useful in any case.

1.3. **Free associative algebra.** As a first step in construction of enveloping algebra, we will define and construct a free associative algebra.

**Definition 1.3.1.** Let $V$ be a vector space. A free associative algebra of $V$ is an associative algebra $T$ together with a linear map $f : V \to T$ such that any linear map $g : V \to A$ from $V$ to an associative algebra $A$ can be uniquely presented as a composition of $f$ with an algebra map $h : T \to A$.

Uniqueness of the free algebra of $V$ is proven along the same lines as the uniqueness of the enveloping algebra. Let us prove the existence. Choose a basis in $V$, $V = \langle v_1, \dots , v_n\rangle$. Define $T$ as the algebra of non-commutative polynomials in variables $v_1, \dots , v_n$. As a vector space, it is generated by monomials in these variables which are finite sequences of $v_i$ in arbitrary order (repetitions are allowed). The product is defined by concatenation of the monomials. The map $f : V \to T$ is the natural embedding ($v_i \mapsto v_i$).

The property of free algebra generated by $V$ says that to define an algebra homomorphism from $T$ to $A$ it is necessary and sufficient to define the images of $v_i$. This is obvious. Hence $T$ is a free associative algebra.

We denote the free associative algebra of $V$ by $T(V)$.

**1.3.2. Construction of enveloping algebra.**

Let now $L$ be a Lie algebra and let $f : L \to T(L)$ be the free associative algebra of $L$. For each pair $x, y \in L$ consider the element $f([x, y]) - f(x)f(y) + f(y)f(x) \in T(L)$. Let $I$ be the (two-sided) ideal, see Lecture 1, generated by all these elements. Finally, put $U = T(L)/I$.

Let us check all necessary properties. The composition $L \to^f T(L) \to^\rho U$ of $f$ with the natural projection $\rho$ gives a map (also denoted by $f$) from $L$ to $U$. This is a map $L \to U^{\mathrm{Lie}}$ of Lie algebras (since we have intentionally factored out by the ideal $I$). If a homomorphism $g : L \to A$ is given, one can uniquely present $g$ as a composition of $f : L \to T(L)$

with an algebra homomorphism $h : T(L) \to A$. Since $g$ is a homomorphism, one has $g([x, y]) - g(x)g(y) + g(y)g(x) = 0$ for all $x, y \in L$. Therefore the elements $f([x, y]) - f(x)f(y) + f(y)f(x) \in T(L)$ belong to the kernel of $h$. By a standard theorem on homomorphism, this gives rise to a unique homomorphism $U \to A$. This proves the universality property of the algebra $U$ constructed.

Enveloping algebra of $L$ will be denoted by $U(L)$.

**1.3.3. Example** Let $L$ be a commutative Lie algebra. Then $U(L)$ is the quotient of $T(L)$ by the ideal generated by the elements $f(x)f(y) - f(y)f(x)$. If we choose a basis $x_1, \dots, x_n$ of $L$, $T(L)$ identifies with the algebra of non-commutative polynomials of $x_1, \dots, x_n$ and the enveloping algebra is the algebra of commutative polynomials $\mathbb{C}[X_1, \dots, X_n]$.

### 1.4. Poincaré-Birkhoff-Witt theorem.

**Theorem 1.4.1.** *Let $x_1, x_2, \dots$ be any ordered basis of $\mathfrak{g}$. The elements $\pi(x_{i_1} x_{i_2} \dots x_{i_m})$ with $0 < m$ and $i_1 \leq i_2 \leq \dots \leq i_m$, along with $1$, form a basis of $\mathfrak{U}(\mathfrak{g})$.*