LECTURES 25-26/5

1. Lecture 25/5

1.1. **Bilinear forms.** Consistent (even): $B(\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}) = 0$,

Supersymmetric $B(a,b) = (-1)^{d(a)d(b)} B(b,a)$,

Invariant: $B([a,b],c) = B(a,[b,c])$.

Example: if $\rho : \mathfrak{g} \to \mathrm{End}(V)$ is a fin. dim. representation of $\mathfrak{g}$, put $B(a,b) := \mathrm{str}\rho(a)\rho(b)$.

1.1.1. A non-zero consistent non-degenerate supersymmetric invariant form B on a finite dimensional Lie superalgebra gives rise to a Casimir element. Indeed, $B \in \mathrm{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes \mathfrak{g}, \mathrm{triv})$ induces a dual map $B^* \in \mathrm{Hom}_{\mathfrak{g}}(\mathrm{triv}^*, (\mathfrak{g} \otimes \mathfrak{g})^*)$. Since $\mathrm{triv}^* \cong \mathrm{triv}$ we obtain a $\mathfrak{g}$-invariant element in $(\mathfrak{g} \otimes \mathfrak{g})^* \cong \mathfrak{g}^* \otimes \mathfrak{g}^*$. On the other hand, B induces a $\mathfrak{g}$-homomorphism $\iota : \mathfrak{g} \to \mathfrak{g}^*$ by the formula $\iota(g)(g') := B(g, g')$. Since B is non-degenerate, ι is injective and now $\dim \mathfrak{g} < \infty$ implies that $\mathfrak{g} \cong \mathfrak{g}^*$. Hence we obtain a $\mathfrak{g}$-invariant element C' in $\mathfrak{g}^* \otimes \mathfrak{g}^* \cong \mathfrak{g} \otimes \mathfrak{g}$. Supersymmetricity and consistency of B means that $C' \in \mathcal{S}^2(\mathfrak{g}_{\bar{0}}) + \Lambda^2(\mathfrak{g}_{\bar{1}}) \subset \mathrm{gr}\mathcal{U}(\mathfrak{g})$. Since $\mathcal{U}(\mathfrak{g})$ and $\mathrm{gr}\mathcal{U}(\mathfrak{g})$ are isomorphic as $\mathfrak{g}$-modules, there exists $C \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}} = \mathcal{Z}(\mathfrak{g})$ such that $\mathrm{gr}C = C'$. If $\{v_i\}, \{v^i\}$ are dual bases of $\mathfrak{g}$ wrt B, the Casimir C can be choosen as $\sum v_i v^i$.

If $\dim \mathfrak{g} = \infty$, the Casimir can be constructed from B via the procedure described in Ch.2 of Kac's book.

1.2. **Classification.** The finite-dimensional simple Lie superalgebras were classified by V. Kac in [K1]. A finite-dimensional simple Lie superalgebra with a reductive even part is called *classical*; it is called *basic classical* if it admits a consistent non-degenerate invariant bilinear form. The basic classical Lie superalgebras can be described in terms of Cartan mtarices: they are finite-dimensional Kac-Moody superalgebras (Kac-Moody superalgebras are called also contragredient).

1.2.1. Example:$\mathfrak{osp}(m,n)$

Let F be a non-degenerate consistent supersymmetric bilinear form on V where $\dim V = (m|n)$; this means $\dim V_{\bar{0}} = m,\ \dim V_{\bar{1}} = n$. View $\mathrm{End}(V)$ as a superalgebra and put for $s \in \mathbb{Z}_2$

$$\mathfrak{osp}(m,n)_s := \{a \in \mathrm{End}(V)_s |\ F(a(v), v') = -(-1)^{sd(v)} F(v, a(v'))\}.$$

Observe that $\mathfrak{osp}(m,0)$ is the orthogonal Lie algebra and $\mathfrak{osp}(0,n)$ is the symplectic Lie algebra. The algebra $\mathfrak{osp}(m,n)$ is basic classical.

1.3. **Triangular decompositions.** A triangular decomposition of a Lie superalgebra $\mathfrak{g}$ can be constructed as follows (see [PS2]). A Cartan subalgebra is a nilpotent subalgebra which coincides with its normalizer. It is proven in [PS2] that any two Cartan subalgebras are conjugate by an inner automorphism. Fix a Cartan subalgebra $\mathfrak{h}$. Then $\mathfrak{g}$ has a generalized root decomposition

$$\mathfrak{g} := \mathfrak{h} + \oplus_{\alpha\in\Delta}\mathfrak{g}_\alpha$$

where Δ is a subset of $\mathfrak{h}^*$ and

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | (ad(h) - \alpha(h))^{dim\mathfrak{g}}(x) = 0\}.$$

If $\mathfrak{g}$ is basic classical, a root space $\mathfrak{g}_\alpha$ is either odd or even. That allows one to define the parity on the set of roots Δ. Denote by $\mathfrak{g}_{\overline{0}}$ (resp., $\mathfrak{g}_{\overline{1}}$) the even (resp., odd) component of $\mathfrak{g}$. Denote by Δ_0 (resp. Δ_1) the set of non-zero weights of $\mathfrak{g}_{\overline{0}}$ (resp., $\mathfrak{g}_{\overline{1}}$) with respect to $\mathfrak{h}$. Then Δ is a disjoint union of Δ_0 and Δ_1.

Now fix $h \in \mathfrak{h}_0$ satisfying $Re\alpha(h) \neq 0$ for all $\alpha \in \Delta$. Set

$$\begin{array}{l}\Delta^+ := \{\alpha \in \Delta |\ \alpha(h) > 0\},\\ \mathfrak{n}^+ := \sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha\end{array}$$

where $\mathfrak{g}_\alpha$ is the weight space corresponding to α.

Define Δ^- and $\mathfrak{n}^-$ similarly. Then $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$ is a triangular decomposition.

1.4. **Notation.** Denote by Δ_0^+ (resp., Δ_1^+) the set of even (resp., odd) positive roots. Let $Q \subset \mathfrak{h}^*$ be the root lattice that is the $\mathbb{Z}$-span of Δ^+ and let Q^+ be the $\mathbb{Z}_{\geq 0}$-span of Δ^+. Introduce the standard partial ordering on $\mathfrak{h}^*$ by setting $\mu \leq \nu$ if $\nu - \mu \in Q^+$.

The element ρ for a basic classical Lie superalgebra is given by

$$2\rho := \sum_{\alpha\in\Delta_{\overline{0}}^+}\alpha - \sum_{\alpha\in\Delta_{\overline{1}}^+}\alpha.$$

1.4.1. Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Fix a consistent non-degenerate invariant bilinear form (it is unique up to a scalar). The restriction of this form gives a form on $\mathfrak{h}$ which induces a form $(-,-)$ on $\mathfrak{h}^*$.

Definition A weight $\lambda \in \mathfrak{h}^*$ is called *typical* if $(\lambda+\rho,\beta) \neq 0$ for any isotropic $\beta \in \Delta_1$.

1.5. **Odd reflections.** The odd reflections were introduced by I. Penkov and V. Serganova— see [PS2], 3.1. If $\mathfrak{g}$ is a basic classical Lie superalgebra, the definition takes the following form. Two Borel subsuperalgebras $\mathfrak{b}, \mathfrak{b}' \subset \mathfrak{g}$ are *connected by an odd reflection along* β iff β is a simple odd isotropic root of $\mathfrak{b}'$ and

$$\Delta(\mathfrak{b}) = \{-\beta\} \cup \Delta(\mathfrak{b}') \setminus \{\beta\}.$$

Note that $\Delta_0(\mathfrak{b}) = \Delta_0(\mathfrak{b}')$ and so $\mathfrak{b}_0 = \mathfrak{b}'_0$. As it is shown in [PS2], 3.1, any Borel subsuperalgebras $\mathfrak{b}, \mathfrak{b}' \subset \mathfrak{g}$ satisfying $\mathfrak{b}_0 = \mathfrak{b}'_0$ are connected by a chain of odd reflections.

Let $\mathfrak{b}$ and $\mathfrak{b}'$ be Borel subsuperalgebras connected by the odd reflection along an odd isotropic root β. Then $\mathfrak{b}' = \mathfrak{b} \cap \mathfrak{b}' + \mathbb{C}x$ where x is a non-zero element of $\mathfrak{g}_\beta$. If v is a $\mathfrak{b}$-primitive vector such that $xv = 0$ then v is also a $\mathfrak{b}'$-primitive. This implies $V_{\mathfrak{b}}(\lambda) = V_{\mathfrak{b}'}(\lambda)$ for λ satisfying $(\lambda, \beta) = 0$. If $xv \neq 0$ then xv is a $\mathfrak{b}'$-primitive since $[x, x] = 0$. This implies $V_{\mathfrak{b}}(\lambda) = V_{\mathfrak{b}'}(\lambda + \beta)$ and $M_{\mathfrak{b}}(\lambda) = M_{\mathfrak{b}'}(\lambda + \beta)$ for $(\lambda, \beta) \neq 0$. Remark that $(\rho_{\mathfrak{b}}, \beta) = 0$, since β is a simple isotropic root, and so $(\lambda, \beta) = 0$ iff $(\lambda + \rho_{\mathfrak{b}}, \beta) = 0$. Taking into account that $\rho_{\mathfrak{b}} = \rho_{\mathfrak{b}'} + \beta$, one concludes that for Borel subsuperalgebras $\mathfrak{b}$, $\mathfrak{b}'$ satisfying $\mathfrak{b}_0 = \mathfrak{b}'_0$ and a $\mathfrak{b}$-typical weight λ one has $V_{\mathfrak{b}}(\lambda) = V_{\mathfrak{b}'}(\lambda')$ and $M_{\mathfrak{b}}(\lambda) = M_{\mathfrak{b}'}(\lambda')$ where $\lambda + \rho_{\mathfrak{b}} = \lambda' + \rho_{\mathfrak{b}'}$.

1.5.1. Example: $\mathfrak{sl}(3,2)$ The "natural" triangular decomposition $(\mathfrak{n}^+_{\bar 1} = \mathfrak{g}_1, \mathfrak{n}^-_{\bar 1} = \mathfrak{g}_{-1})$ gives

$$\Delta^+_0 = \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_1 - \varepsilon_3; \delta_1 - \delta_2\},\ \Delta^+_1 = \{\varepsilon_i - \delta_j : i = 1, 2, 3, j = 1, 2\}.$$

As in the Lie algebra case, $\alpha^\vee(\beta) = 2(\alpha, \beta)/(\alpha, \alpha)$. Up to a scalar, one has

$$(\varepsilon_i, \varepsilon_{i'}) = \delta_{ii'},\ (\delta_j, \delta_{j'}) = -\delta_{jj'},\ (\varepsilon_i, \delta_j) = 0.$$

The corresponding set of simple roots is

$$\pi := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3 - \delta_1, \delta_1 - \delta_2\}.$$

The odd reflection with respect to $\varepsilon_3 - \delta_1$ gives

$$(\Delta^+_0)' = \Delta^+_0,\ \ \Delta^+_1 = \{\varepsilon_i - \delta_2 : i = 1, 2, 3, \varepsilon_1 - \delta_1, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3\}.$$

The corresponding set of simple roots is

$$\pi' := \{\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \delta_1, \delta_1 - \varepsilon_3, \varepsilon_3 - \delta_2\}.$$

1.6. **Completely reducible Lie superalgebras.** A Lie superalgebra is called completely reducible if all its finite dimensional modules are completely reducible. According to the theorem of Djoković and Hochschild (see [Sch], p. 239), any finite dimensional completely reducible Lie superalgebra is a direct sum of semisimple Lie algebras and algebras $\mathfrak{osp}(1,2l)$ $(l \geq 1)$. The superalgebra $\mathfrak{g} = \mathfrak{osp}(1,2l)$ has many features of the semisimple Lie algebras; in particular, $\mathcal{U}(\mathfrak{g})$ is a domain and $\mathcal{Z}(\mathfrak{g})$ is a polynomial algebra.

Rank 1 *and their representations*

1.7. $\mathfrak{sl}(1|1)$. $\mathfrak{g}_0 = \mathbb{C}h,\ \mathfrak{g}_1 = \mathbb{C}E, \mathfrak{g}_{-1} = \mathbb{C}F.$

The relations

$$[h,E] = [h,F] = 0, \quad [E,F] = h.$$

This algebra is not simple.

1.7.1. Finite-dimensional simple modules For each $c \in \mathbb{C}$ there is a simple module $V(c)$: $V(0)$ is one-dimensional and for $c \neq 0$, $V(c)$ is two-dimensional

$$h \mapsto \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}, E \mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad F \mapsto \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}.$$

Strictly speaking, we have two trivial modules and two modules $V(c)$. Namely, for superalgebras we have the parity functor Π: for each module V, the module $\Pi(V)$ is the same vector space with switched parity that is $d(\Pi(v)) = d(v) + 1$ where for $v \in V$ we denote by $\Pi(v)$ the corresponding vector in $\Pi(V)$. The action is given by

$$a(\Pi(v)) = (-1)^{d(a)}\Pi(av).$$

Lemma 1.7.2. *Let $\mathfrak{g}$ be a basic classical Lie superalgebra. One has*

$$\alpha \in \Delta_{\bar{1}}\ (\alpha,\alpha) = 0 \implies 2\alpha \notin \Delta.$$

Moreover, for any $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ the algebra generated by x, y is isomorphic to $\mathfrak{sl}(1|1)$.

Proof. Recall that $\mathfrak{g}_{\bar{0}}$ is reductive and so $(\beta,\beta) > 0$ for any even root β. This gives $\implies$. Assume that $2\alpha \notin \Delta$. Since $\mathfrak{g}$ admits a non-degenerate consistent invariant bilinear form B, $-2\alpha \notin \Delta$ as well. Take $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$. Since $\pm 2\alpha \notin \Delta$ one has $[x,x] = [y,y] = 0$. Then $[x,[x,y]] = [[x,x],y] - [x,[x,y]] = -[x,[x,y]]$ so for $h := [x,y]$ one has $[h,x] = 0$ and, similarly, $[h,y] = 0$. Hence the algebra generated by x, y is isomorphic to $\mathfrak{sl}(1|1)$. $\square$

Remark Assume that $2\alpha \notin \Delta$. Notice that $B(h,h) = B([x,y],[x,y]) = B(x,[y,[x,y]]) = B(x,0) = 0$. Recall that B induces an isomorphism $\mathfrak{h} \to \mathfrak{h}^*$. Up to a scalar, the image of h is α and this gives $(\alpha,\alpha) = 0$.

1.8. $\mathfrak{osp}(1|2)$. $\mathfrak{g}_{\overline{0}} = \mathbb{C}f + \mathbb{C}h + \mathbb{C}e \cong \mathfrak{sl}(2)$ and $\mathfrak{g}_{\overline{1}}$ is isomorphic to the natural (2-dimensional) representation $V'(1)$ of $\mathfrak{sl}(2)$. The bracket on $\mathfrak{g}_{\overline{1}}$ comes from an $\mathfrak{sl}(2)$-isomorphism $S^2(V'(1)) \cong \mathrm{Ad}$. Fix the generators in $\mathfrak{g}_{\overline{0}} \cong \mathfrak{sl}(2)$. Fix $E \in \mathfrak{g}_{\overline{1}}$ in such a way that $[E,E] = 2e$ and put $F := [f,E]$. Then $\mathfrak{g}_{\overline{1}} = \mathbb{C}E + \mathbb{C}F$ and the $\mathfrak{sl}(2)$-structure on $V(1) \cong \mathfrak{g}_{\overline{1}}$ gives $[h,E] = E,\ [h,F] = -F,\ [e,F] = E$.

The remaining non-trivial relation are $[E,F] =?,\ [F,F] =?$. One has

$$[E,F] = [E,[f,E]] = [[E,f],E]+[f,[E,E]] = [-F,E]+[f,2e] = -[E,F]+2h$$

that is $[E,F] = h$.

$$[F,F] = [F,[f,E]] = [f,[F,E]] = [f,h] = f$$

since $[f,F] = 0$. Finally,

$$[e,f] = h,\ [h,e] = 2e,\ [h,f] = -2f;$$

$$[h,E] = E,\ [h,F] = -F,\ [e,F] = E, [f,E] = F,\ [E,F] = h,\ [E,E] = 2e,\ [F,F] = f.$$

Triangular decomposition $\mathfrak{h} := \mathbb{C}h,\ \mathfrak{n}^+ := \mathbb{C}e+\mathbb{C}E,\ \mathfrak{n}^- := \mathbb{C}f+\mathbb{C}F$. Let α be the weight of E.

1.8.1. Let $\mathfrak{g}$ be a basic classical Lie superalgebra. Let α be an odd root such that 2α is a root. Then for any $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ the algebra generated by x,y is isomorphic to $\mathfrak{osp}(1|2)$. As a consequence, if $(\alpha,\alpha) \neq 0$ then $x \in \mathfrak{g}_\alpha, y \in \mathfrak{g}_{-\alpha}$ generate a copy of $\mathfrak{osp}(1|2)$.

1.8.2. Finite-dimensional simple modules Remark that a finite-dimensional $\mathfrak{g}$-module is completely reducible over $\mathfrak{g}_{\overline{0}} \cong \mathfrak{sl}(2)$.

Denote by $M(c)$ the Verma $\mathfrak{g}$-module with an even highest weight vector v_c. Then

$$M(c)_{\overline{0}} = \mathrm{span}\{v_c, fv_c, f^2v_c, \dots\},\ M(c)_{\overline{1}} = \mathrm{span}\{Fv_c, fFv_c, f^2Fv_c, \dots\}.$$

Shapovalov matrices are 1 by 1.

$$HC(e^kf^k) = k!h(h-1)\dots(h-(k-1)).$$

$$HC(Ee^kf^kF) = k!h(h-1)\dots(h-k).$$

Thus $M(c)$ is simple iff $c \notin \mathbb{N}$. If $m \in \mathbb{N}$ the module $V(m)$ is finite-dimensional; for $m \geq 1$ one has $V(m)_{\bar{0}} = V'(m), V(m)_{\bar{1}} = V'(m-1)$ as $\mathfrak{sl}(2)$-modules.

1.8.3. *Warning*: if $\mathfrak{p}$ is a classical Lie superalgebra the even (odd) part of Verma $\mathfrak{p}$-module is not a Verma module over $\mathfrak{p}_{\bar{0}}$. It has a filtration with subquotients isomorphic to Verma modules over $\mathfrak{p}_{\bar{0}}$ with the number of terms equal to $2^{\dim \mathfrak{p}_{\bar{1}}-1}$.

Lemma 1.8.4. *Let $\mathfrak{g}$ be a classical Lie superalgebra. If α is simple then $2(\rho, \alpha) = (\alpha, \alpha)$.*

Proof. If α is even or 2α is an even root then both $\Delta_{\bar{0}}$ $\Delta_{\bar{1}}$ are s_α-invariant since $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ (resp., $\mathfrak{g}_{2\alpha} + \mathfrak{g}_{-2\alpha} + [\mathfrak{g}_{2\alpha}, \mathfrak{g}_{-2\alpha}]$) contain a copy of $\mathfrak{sl}(2)$ (actually, it is isomorphic to $\mathfrak{sl}(2)$ since $\dim \mathfrak{g}_\beta = 1$ for any even root β). If β is a positive root which is not proportional to α then $s_\alpha \beta$ is also positive: otherwise $-\gamma := \beta - m\alpha \in \Delta^-$ so $m\alpha = \beta + \gamma$ contradicts to the simplicity of α. If α is even we conclude that $\Delta_{\bar{0}} \setminus \{\alpha\}$ and $\Delta_{\bar{1}}$ are s_α-invariant and so

$$s_\alpha(2\rho) = s_\alpha \sum_{\beta \in \Delta_{\bar{0}}} \beta - s_\alpha \sum_{\beta \in \Delta_{\bar{1}}} \beta = s_\alpha \alpha = -\alpha$$

and thus $2(\rho, \alpha) = (\alpha, \alpha)$ as required. If α is odd we conclude that $\Delta_{\bar{0}} \setminus \{2\alpha\}$ and $\Delta_{\bar{1}} \setminus \{\alpha\}$ are s_α-invariant and so

$$s_\alpha(2\rho) = s_\alpha \sum_{\beta \in \Delta_{\bar{0}}} \beta - s_\alpha \sum_{\beta \in \Delta_{\bar{1}}} \beta = s_\alpha 2\alpha - s_\alpha \alpha = -\alpha$$

and thus again $2(\rho, \alpha) = (\alpha, \alpha)$.

Finally, let α be odd and 2α is not a root. By Lme. 1.7.2 $\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha} + [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ contains a copy of $\mathfrak{sl}(1|1)$. Let h be a Cartan generator in this copy. View $\mathfrak{g}$ as a module over $\mathfrak{sl}(1|1) + \mathfrak{h}$ and consider its Jordan-Hölder filtartion. Each subquotient is a simple $\mathfrak{sl}(1|1) + \mathfrak{h}$ module; it can be one-dimensional of weight β where $\beta(h) = 0$ or two-dimensional of weights $\beta, \alpha + \beta$. Notice that β is positive iff $\alpha = \beta$ is positive. Since β and $\alpha + \beta$ have different parity, we conclude that $\rho(h) = 0$. Since α is proportional to the image of h under the isomorphism $\mathfrak{h} \to \mathfrak{h}^*$, we conclude $(\alpha, \rho) = 0$ as required. □

1.9. **Kac-Moody structure.** Kac-Moody superalgebras are defined via their Cartan matrices by the same procedure as in Lie algebra case.

1.9.1. Example: $\mathfrak{sl}(2|1)$ To write down the Cartan matrix for a given triangular decomposition we use the relation $[h_i, e_j] = a_{ij} e_j$.

For $\mathfrak{sl}(2|1)$ a Cartan matrix is

$$\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

The second root is odd.

1.9.2. Example: $\mathfrak{sl}(3|2)$ For $\mathfrak{sl}(3|2)$ for the first triangular decomposition

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

The third root is odd: $\pi_1 = \{\alpha_3\}$.
For $\mathfrak{sl}(3|2)$ for the second triangular decomposition

$$A' = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$\pi_1 = \{\alpha_2, \alpha_3, \alpha_4\}$.

1.9.3. Restrictions

1. $a_{ii} = 2$ or $a_{ii} = 0$.
2. if α_i is even then $a_{ii} = 2$, $a_{ij} \in -\mathbb{N}$.
3. if α_i is odd and $a_{ii} = 2$ then $a_{ij} \in -2\mathbb{N}$.
4. $a_{ij} = 0 \implies a_{ji} = 0$.

For $\mathfrak{osp}(1,2)$ the condition $A = (2)$ and so $\alpha^\vee \in \mathfrak{h}$ takes form $\alpha^{vee} = 2h$ in notaion of 1.8. Observe that if μ is the lowest weight of a finite dimensional $\mathfrak{osp}(1,2)$-module then $\alpha^{vee}(\mu) \in -2\mathbb{N}$. This gives the condition (iii) (it reflects the fact that $[f_i, e_j] = 0$ and $\mathrm{ad} e_i$ acts locally nilpotent on e_j so e_j is the lowest weight vector of a finite dimensional $\mathfrak{osp}(1,2)$-module associated with $e_i, f_i, \alpha_i^\vee$).

The conditions (ii), (iii) reflect the fact that $\mathrm{ad} e_i$ acts locally nilpotent on e_j (the condition (iii) follows from 1.8).

1.10. **Verma modules.**

1.10.1. Casimir operator If a Cartan matrix for $\mathfrak{g}$ is symmetrizable then $\mathfrak{g}$ admits a Casimir operator C where $HC(C) = (\lambda + \rho, \lambda + \rho)$.

1.10.2. Case $(\lambda + \rho, \alpha) = 0$ for an isotropic simple root α.

Fix $F_\alpha \in \mathfrak{g}_{-\alpha}$. Let v_λ be a highest weight vector of $M(\lambda)$. Then $F_\alpha v_\lambda$ is primitive, but it does not generate a Verma module since $F_\alpha(F_\alpha v_\lambda) = 0$. It is easy to see (using $\mathrm{gr}\mathfrak{U}(\mathfrak{n}^-)$ that $uF_\alpha \neq 0$ for $u \in \mathfrak{U}(\mathfrak{n}^-) \setminus \mathfrak{U}(\mathfrak{n}^-)F_\alpha$.

For λ such that $(\lambda+\rho,\alpha) = 0$ denote by $M'(\lambda)$ the submodule generated by $F_\alpha v_\lambda$. We obtain an embedding of $M'(\lambda-\alpha)$ to $M(\lambda)$. Thus $M(\lambda)$ contains a subquotient of $M(\lambda-\alpha)$.

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