# INTRODUCTION TO REPRESENTATION THEORY. 25/10/04. LECTURE 2

## 1. One dimensional representations

Any one dimensional representation is irreducible. Any group has a so-called *trivial* representation: $\dim V = 1, \rho(g) = \mathrm{id}$.

### 1.1. Irreducible complex representations of $\mathbb{Z}_n$.

As we checked last time all irreducible complex representations of a finite abelian group are one-dimensional. Let $x$ be a generator of $\mathbb{Z}_n$ that is $\mathbb{Z}_n = \{e, x, x^2, \dots, x^{n-1}\}$. For a one dimensional representation $\rho$ one has $\rho(x) = \omega\,\mathrm{id}$ where $\omega^n = 1$ since $x^n = 1$. Let $\rho, \rho'$ be one dimensional representations and $\rho(x) = \omega\,\mathrm{id}$, $\rho'(x) = \omega'\,\mathrm{id}$. It is easy to see that $\rho \cong \rho'$ iff $\omega = \omega'$. Hence we obtain $n$ isomorphism classes of one-dimensional representations: $\rho_0, \rho_1, \dots, \rho_{n-1}$ where $\rho_r(x) = e^{2r\pi/n}\,\mathrm{id}$, $\rho_r(x^j) = e^{2rj\pi/n}\,\mathrm{id}$.

1.2. If $G, H$ are two groups one can define the product $G \times H$: as the set it is the Cartesian product and the group operation is given by

$$(g, h) * (g', h') = (gg', hh').$$

**Theorem.** *Any finite abelian group is isomorphic to* $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_r}$.

### 1.3. Irreducible complex representations of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$.

Let $x_1$ (reps., $x_2$) be a generator of $\mathbb{Z}_{n_1}$ (resp., of $\mathbb{Z}_{n_2}$). An irreducible complex representations of $\mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2}$ is one dimensional. There are $n_1 n_2$ isomorphism classes of one-dimensional representations $\rho_{r,s} : r = 0, n_1 - 1,\ s = 0, n_2 - 1$ where $\rho_{r,s}(x_1) = e^{2r\pi/n_1}\,\mathrm{id}$, $\rho_{r,s}(x_2) = e^{2s\pi/n_2}\,\mathrm{id}$.

## 2. Quotient representation

If $V$ is a module and $W \subset V$ is a submodule the quotient space $V/W$ inherits a $G$-module structure:

$$g(v + W) := gv + W.$$

2.1. **Example.** Let $V$ be a two-dim. $\mathbb{Z}$-module

$$\rho(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

Then $W := kv_1$ is a submodule; one has $W \cong V/W \cong$ triv.

## 3. Direct sum, tensor product and duality

We extend operations between vector spaces to representations.

3.1. $V \oplus W$**.** Let $\rho : G \to \mathrm{End}(V)$ and $\rho' : G \to \mathrm{End}(W)$ be representations. Define $\rho \oplus \rho' : G \to \mathrm{End}(V \oplus W)$ by the formula

$$(\rho \oplus \rho')(g)(v + w) := \rho(g)(v) + \rho'(g)(w).$$

This gives $V \oplus W$ the structure of $G$-module. We identify $V$ and $W$ with the corresponding submodules of $V \oplus W$.

3.1.1. Let $\rho : G \to \mathrm{End}(V)$ and $\rho' : G \to \mathrm{End}(W)$ be representations. Let $\{e_i\}_{i=1}^r$ be a basis of $V$ and let $\{e_i\}_{i=r+1}^n$ be a basis of $W$. Then $\{e_i\}_{i=1}^n$ is a basis of $V \oplus W$ and the matrix $(\rho \oplus \rho')(g)$ written in this bases is a block matrix:

$$(\rho \oplus \rho')(g) = \begin{pmatrix} \rho(g) & 0 \\ 0 & \rho'(g) \end{pmatrix}.$$

3.1.2. A representation is called *indecomposable* if it is not isomorphic to a direct sum of two representations.

3.1.3. *Example.* Consider a complex finite-dimensional representations of $\mathbb{Z}$. Since 1 generates $\mathbb{Z}$, any representation takes form

$$\rho(m) = A^m$$

where $A \in GL(V)$. Indecomposable representation is indecomposable if the Jordan form of $A$ is a block.

3.2. $V \otimes W$**.** Tensor product of vector spaces $V \otimes W$ can be defined as follows: let $U$ be a vector space with basis consisting of all pairs $(v, w)$: $v \in V, w \in W$; let $U' \subset U$ be $U' := span\{(v + v', w) - (v, w) - (v', w); (v, w + w') - (v, w) - (v, w'); (cv, w) - c(v, w); (v, cw) - c(v, w) : v \in V, w \in W\}$. Set

$$V \otimes W := U/U'.$$

The image of $(v, w)$ in $V \otimes W$ is denoted by $v \otimes w$. One has

$$(v + v') \otimes w = v \otimes w + v' \otimes w, \quad v \otimes (w + w') = v \otimes w + v \otimes w',$$

$$cv \otimes w = v \otimes cw = c(v \otimes w).$$

3.2.1. *Basis.* Let $\{e_i\}_{i\in I}$ be a basis of $V$ and let $\{f_j\}_{j\in J}$ be a basis of $W$. Then $\{e_i \otimes f_j\}_{i\in I, j\in J}$ is a basis of $V \otimes W$.

3.2.2. *G-module structure.* Define the $G$-module structure on $U := span\{(v,w)/v \in V, w \in W\}$ by setting

$$\rho''(g)(v,w) := (\rho(g)v, \rho'(g)w)$$

and then extending linearly to $U$ Then $\rho''(g) \in \mathrm{End}(U)$ since the pairs $(v,w)$ form a basis of $U$. Clearly, $\rho''(gh) = \rho''(g)\rho''(h)$. Hence $\rho''$ is a representation of $G$. It is easy to check that $U'$ is a submodule of $U$. The quotient $U/U'$ is the representation $\rho \otimes \rho' : G \to \mathrm{End}(V \otimes W)$. One has

$$g(v \otimes w) := gv \otimes gw.$$

3.2.3. *Example.* For any $G$-module $V$ one has

$$V \otimes \mathrm{triv} \cong V$$

3.2.4. *Example.* Let $V$ be a two-dim. $\mathbb{Z}$-module

$$\rho(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

and $W$ be one-dim. $\rho'(m) = (-1)^m$.

$$(\rho \otimes \rho')(m) = \begin{pmatrix} (-1)^m & m(-1)^m \\ 0 & (-1)^m \end{pmatrix}.$$

Recall that $V$ has a submod. $V'$ which is trivial; clearly, $V \otimes W$ has no trivial submodules.

3.2.5. Set

$$V^{\otimes 0} := \mathrm{triv}, V^{\otimes 1} := V, \quad V^{\otimes n+1} := V^{\otimes n} \otimes V.$$

One has

$$V \otimes (W \otimes U) \cong (V \otimes W) \otimes U,$$

$$V \otimes W \cong W \otimes V,$$

$$V \otimes (W \oplus U) \cong (V \otimes W) \oplus (U \otimes W).$$

3.3. **Dual representation.** Let $F$ be our base field. Recall that $V^* := \mathrm{Hom}(V, F)$. If $\rho : G \to \mathrm{End}(V)$ is a representation, set

$$\rho^*(g)(f)(v) := f(\rho(g^{-1})(v)).$$

.

In the matrix form

$$\rho^*(g) = \rho(g^{-1})^t.$$

3.4. $\mathrm{Hom}(V, W)$**.** By definition, $\mathrm{Hom}(V, W)$ is the set of linear homomorphisms $V \to W$. Let $W$ be finite dimensional. Then, as a vector space, $V^* \otimes W \cong \mathrm{Hom}(V, W)$ via the map $(f \otimes w) \mapsto \Psi_{f,w}$, $\Psi_{f,w}(v) = f(v)w$.

Define a $G$-module structure on $\mathrm{Hom}(V, W)$ via this isomorphism. Then

$$(g\psi)(v) := g\psi(g^{-1}v).$$

One has $V^* = \mathrm{Hom}(V, \mathrm{triv})$.

## 4. Complete reducibility

Our goal is to classify all representations of a finite group over field $\mathbb{C}$. Using direct sums we can construct representations from smaller ones. Let us see what are indecomposable representations of a finite group.

4.1. **Theorem.** *Let $G$ be a finite group and $|G|$ is not divisible by* $\mathrm{char}\, F$. *Let $V$ be a $G$-module and $W \subset V$ be a submodule. Then there exists $W'$ such that $V = W \oplus W'$ as $G$-modules.*

*Proof.* Take any vector space $U$ such that $V = W \oplus U$ and let $\psi_0 : V \to W$ be the corr. projection. Take

$$\psi := \sum_{g \in G} \rho(g)^{-1} \psi_0 \rho(g).$$

Then $\psi : V \to W$ is $G$-homomorphism since

$$\psi(g'v) = \sum_{g \in G} g^{-1} \psi_0(gg'v) = \sum_{g \in G} g'(gg')^{-1} \psi_0(gg'v) = g'\psi(v).$$

One has $\psi(w) = |G|w$ and so $\psi/|G|$ is projection. Then $\mathrm{Ker}\, \psi \oplus W = V$ and $\mathrm{Ker}\, \psi$ is a submodule. Putting $W' := \mathrm{Ker}\, \psi$ we complete the proof $\square$

*Remark.* For the case $F = \mathbb{C}$ the theorem can be proven as follows.

Recall that a Hermitian inner product is a map $(\ ,\ ) : V \times V \to \mathbb{C}$ satisfying

$$(v, v' + v'') = (v, v') + (v, v''), \quad (cv, v') = c(v, v'), \quad (v, v') = \overline{(v', v)}.$$

The last condition gives $(v, v) \in \mathbb{R}$. Consider a Hermitian inner product satisfying $(v, v) > 0$ for $v \neq 0$. Define a new inner product by setting

$$(v_1, v_2)' := \sum_{g \in G} (gv_1, gv_2).$$

Clearly, $(v, v)' > 0$ for $v \neq 0$. Our new inner product is $G$-invariant:

$$(v_1, v_2)' = (gv_1, gv_2)'$$

for any $g \in G$. It is easy to see that the $G$-invariance implies that $W^{\perp} := \{v \in V |\ \forall w \in W\ (v, w) = 0\}$ is a submodule. Since $(v, v)' > 0$ for $v \neq 0$, one has $V = W \oplus W^{\perp}$. Putting $W' := W^{\perp}$ we conclude the assertion.

4.2. **Corollary.** *Complete reducibility (semisimplicity): any finite dimensional representation is a direct sum of irreducible ones.*

Remark: Complete reducibility holds for continuous representations of "compact groups" (integration substitutes the averaging); $\mathbb{Z}$ does not have this property:

$$\rho(m) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}.$$

4.3. **Corollary.** *Any representationof a finite group takes form*

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus ...$$

*where the $V_i$ are non-isomorphic irreducible modules. The decomposition is unique in a sense that $V_i$ that occurs and multiplicities $a_i$ are uniquely determined.* We will prove the uniqueness next time.

## 5. Complex representations of $S_3$.

The group $S_3$ has a subgroup isomorphic to $\mathbb{Z}_3$: $\{(123), (132), e\}$.

Let $V$ be a finite dimensional $S_3$-module. Write $V = V_1 \oplus V_{\omega} \oplus V_{\omega^2}$ where $\omega := e^{2\pi i/3}$ and $V_c = \{v \in V |\ (123)v = cv\}$. Notice that $V_c = \{v \in V |\ (132)v = c^2 v\}$.

Remark that $S_3$ is generated by $(123)$ and $(12)$. Hence if $W \subset V$ is s.t. $(123)W = W$, $(12)W = W$ then $W$ is a submodule.

One has $(123)(12) = (12)(132)$. This implies

$$(12)V_{\omega} = V_{\omega^2},\ (12)V_{\omega^2} = V_{\omega},$$

$$(12)V_1 = V_1.$$

Thus $V = V_1 \oplus (V_{\omega} \oplus V_{\omega^2})$ where both $V_1$ and $V_{\omega} \oplus V_{\omega^2}$ are submodules.

Choose any basis in $V_\omega$: $\{e_1, \dots, e_r\}$; put $e_i' := (12)e_i$. Then $\{e_i, e_i'\}$ spans 2-dim. submodule:

$$\rho((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho((123)) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

Hence

$$V_\omega \oplus V_{\omega^2} \cong W_2^{\oplus \dim V_\omega}$$

where $W_2$ is the 2-dimensional module corresponding to $\rho$.

Let us go back to $V_1$. Any subspace of $V_1$ is (123)-stable. The element (12) generates a subgroup isomorphic to $\mathbb{Z}_2$. Thus $V_1$ is a direct sum of one-dim. representations

$$V_1 = \mathrm{triv}^{\oplus a} \oplus \mathrm{sgn}^{\oplus b}$$

where sgn is one-dim. representation $\mathrm{sgn}((123)) = \mathrm{id}, \mathrm{sgn}((12)) = -\mathrm{id}$. Hence

$$V = \mathrm{triv}^{\oplus a} \oplus \mathrm{sgn}^{\oplus b} \oplus W_2^{\oplus c}$$

where

$$c := \dim V_\omega, \;\; a + c = \dim\{v \in V | \; (12)v = v\}, \;\; b + c = \dim\{v \in V | \; (12)v = -v\}.$$

In particular, there are three isomorphicm classes of irreducible $S_3$-represenations: two-dimensional module $W_2$ and one–dimensional modules $\mathrm{sgn}, \mathrm{triv}$.