INTRODUCTION TO REPRESENTATION THEORY. 1/11/04. LECTURE 3

1. Comments to homeworks

1.1. To check that a non-empty subset H of a group G is a subgroup, there is no need in checking associativity. It is enough to verify

$$h, h' \in H \implies hh' \in H.$$

$$h \in H \implies h^{-1} \in H.$$

1.2. Why the natural representation of S_n has two irreducible submodules ($\operatorname{char} k = 0$)?

The natural representation V has a basis $v_1, \dots, v_n$ and the action is $\sigma v_i := v_{\sigma(i)}$. Clearly, $W_1 := span\{\sum v_i\}$, $W_{n-1} = \{\sum a_i v_i | \sum a_i = 0\}$ are submodules, $V = W_1 \oplus W_{n-1}$, $W_1 \cong$ triv. It remains to check that W_{n-1} is irreducible. Let W be a non-zero submodule of W_{n-1} and $v = \sum a_i v_i$ be a non-zero vector. Then there exist i, j $(i \neq j)$ such that $a_i = a_j$; one has $(ij)v - v = (a_j - a_i)(v_i - v_j)$ and so $v' := v_i - v_j \in W$. Then $(ir)v' = v_r - v_j \in W$; clearly, the set $\{v_r - v_j\}_{r \neq j}$ form a basis of W_{n-1}. Hence $W = W_{n-1}$ is irreducible.

1.3. In the Homework 2, problem 3, one needs to add the conditions $\dim V < \infty$ and k is algebraically closed.

In what follows we assume that all modules are finite dimensional.

2. Complete reducibility

2.1. **Theorem.** *Let G be a finite group and $|G|$ is not divisible by* $\operatorname{char} F$. *Let V be a G-module and $W \subset V$ be a submodule. Then there exists W' such that $V = W \oplus W'$ as G-modules.*

2.2. **Corollary.** *Complete reducibility (semisimplicity): any representation is a direct sum of irreducible ones.* For finite-dimensional ones there Cor. is immediate. For infinite dimensions this is a standard (non-trivial) fact.

2.3. **Corollary.** *Any repr. of a finite group takes form*

$$V = V_1^{\oplus a_1} \oplus V_2^{\oplus a_2} \oplus \ldots$$

where the V_i are distinct irred. repr. The decomposition is unique in a sense that V_i that occurs and multiplicities a_i are uniquely determined. Submodule $V_i^{\oplus a_i}$ is called an *isotypical component.*

Proof. The existence follows from the above Cor. Let us prove uniqueness. One has

$$\mathrm{Hom}_G(V_i, V) = \mathrm{Hom}_G(V_i, V_i^{\oplus a_i}) = \mathrm{Hom}_G(V_i, V_i)^{\oplus a_i}$$

and thus $\dim \mathrm{Hom}_G(V_i, V) = a_i \dim \mathrm{Hom}_G(V_i, V_i)$. Then

$$a_i = \frac{\dim \mathrm{Hom}_G(V_i, V)}{\dim \mathrm{Hom}_G(V_i, V_i)}$$

(the denominator is non-zero since $\mathrm{id} \in \mathrm{Hom}_G(V_i, V_i)$) and so the multiplicities a_i are uniquely determined. Each isotypical component is uniquely determined since $\sum_{\phi \in \mathrm{Hom}_G(V_i, V)} \phi(V_i) = V_i^{\oplus a_i}$. □

2.3.1. *Remark.* If the field is alg. closed then, by Schur's lemma, $\dim \mathrm{Hom}_G(V_i, V_i) = 1$ that is $a_i = \dim \mathrm{Hom}_G(V_i, V)$.

2.3.2. *Remark.* If we write $V = V_1 \oplus V_2 \oplus \ldots V_n$ and drop the condition that V_i are distinct, the decomposition is not unique. Example: $\mathrm{triv} \oplus \mathrm{triv}$.

2.3.3. *Remark.* Let G be a finite group, $\rho : G \to \mathfrak{GL}(V)$ be a repr.. Then all eigenvalues of $\rho(g)$ are roots of unity. Moreover, if our base field is $\mathbb{C}$,then $\rho(g)$ is always diagonalizable.

Indeed, g generates a cyclic subgroup H which is abelian and finite. This implies the first assertion. If $k = \mathbb{C}$, then V as an H-module is a direct sum of one-dim. submodules $V_i = \mathbb{C}v_i$. Then $\{v_i\}$ is a basis of V and $\rho(g)$ in this basis is diagonal.

3. An adjoint representation

An adjoint representation: take a vector space with a basis $v_g : g \in G$ (indexed by the elements of G). Define a G-module structure by setting $gv_{g'} := v_{gg'g^{-1}}$.

If $X \subset G$ is a conjugate class then the elements $v_g : g \in X$ span a submodule; such a submodule is not simple since $\sum_{g \in X} v_g$ spans a trivial submodule.

3.1. **Representations of S_3.** Set $k := \mathbb{C}$. Recall that we have shown that there are 3 complex irreducible representations of S_3: two-dimensional W_2, sgn and triv where W_2 corresponds to

$$\rho((12)) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho((123)) = \begin{pmatrix} \omega & 0 \\ 0 & \omega^2 \end{pmatrix}.$$

In S_3 there are three conjugate classes.

$$\mathbb{C}e = \text{triv}$$

$$span\{(123), (132)\} = \text{triv} \oplus \text{sgn}\,.$$

$$span\{(12), (13), (23)\} = \text{triv} \oplus W_2.$$

4. A regular representation

4.1. **Definition.** A representation $\rho : G \to \mathrm{End}(V)$ is called *faithful* if $\mathrm{Ker}\,\rho = 0$.

A regular representation $\mathrm{Reg}(G)$ of a group G corresponds to the left action of G on itself: take a vector space as in 3 and define a G-action by $gv_{g'} = v_{gg'}$.

A regular representation is faithful.

4.2. **Proposition.**

$$\dim \mathrm{Hom}_G(\mathrm{Reg}(G), V) = \dim V.$$

Proof. Define a map $\mathrm{Hom}_G(\mathrm{Reg}(G), V) \to V$ by $\Psi : \phi \mapsto \phi(e)$. Then Ψ is injective: if $\phi(e) = 0$ then $\phi(g) = g\phi(e) = 0$ so $\phi = 0$. Moreover Ψ is surjective: if $\phi(e) = v$ set $\phi(g) = gv$; since the elements of G form a basis of $\mathrm{Reg}(G)$ the map ϕ is well-defined. □

4.3. **Corollary.** *If G is finite and $|G|$ is not divisible by $\mathrm{char}\, k$ then any irred. repr. V occurs in $\mathrm{Reg}(G)$ and its multiplicity is $\frac{\dim V}{\dim \mathrm{Hom}_G(V,V)}$.*

Proof. Since $\mathrm{Reg}(G)$ is completely reducible, the multiplicity a of an irreducible module V satisfies

$$a \dim \mathrm{Hom}_G(V, V) = \dim \mathrm{Hom}_G(V, \mathrm{Reg}(G)) = \dim \mathrm{Hom}_G(\mathrm{Reg}(G), V).$$

□

4.4. Let Irr be the set of isomorphism classes of irred. repr. of G.

Corollary. *Under the assumptions of Corollary 4.3 one has*
$$|G| = \sum_{V\in \mathrm{Irr}} \dim^2 V$$
if k is algebraically closed.

Proof. If k is algebraically closed, $\dim \mathrm{Hom}_G(V,V) = 1$ by Schur's lemma. Hence the multiplicity $a(V) = \dim V$ for any irreducible module V. Therefore $|G| = \dim \mathrm{Reg}(G) = \sum_{V\in\mathrm{Irr}} \dim^2 V$ as required. □

Remark. The group $G := \mathbb{Z}_3$ has a two-dimensional irred. real represenation W_2 so the above corollary does not hold for this case. Let us see what happens. One has $\dim \mathrm{Hom}_G(W_2, W_2) = 2$ and so W_2 has multiplicity one in $\mathrm{Reg}(G)$. Indeed,
$$\mathrm{Reg}(G) = W_2 \oplus \mathrm{triv} :, \ \mathrm{triv} = \mathbb{R}(e + x + x^2), \quad W_2 = \mathbb{R}(e - x, x - x^2).$$

5. Complex representations of S_4 and A_4

5.1. **Complex representations of S_4.** We have triv, sgn, W_3; let us find the remaining irred. repr.

Let us see what can be $\mathrm{Ker}\,\rho$: S_4 has the following normal non-trivial subgroups: A_4 and
$$H := \{(12)(34), (13)(24), (14)(23), e\}.$$

$S_4/A_4 = \mathbb{Z}_2$ gives us two one dim. representations: triv and sgn.

$S_4/H = S_3$ gives two one dim. representations: triv, sgn and two-dim. repr. W_2.

Notice that if ρ is one-dimensional ($\rho : G \to k$) then $G/Ker\,\rho$ is abelian. Hence triv, sgn are the only one-dim. repr.

Thus there is one missing irred. repr. of dimension 3. Let us show that it is $W_3 \otimes \mathrm{sgn}$. Indeed, if $(W_3 \otimes \mathrm{sgn}) \otimes \mathrm{sgn} = W_3$ and so $W_3 \otimes \mathrm{sgn}$ is irred. It remains to show that $W_3 \otimes \mathrm{sgn} \ncong W_3$. Consider W_3 as a submodule of the natural representation: $W_3 = span\{e_1 - e_2, e_2 - e_3, e_3 - e_4\}$. One has $\mathrm{sgn}((12)) = -\mathrm{id}$ and in W_3
$$\rho((12)) = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$
Thus $\rho((12))$ has eigenvalues $1, 1, -1$ and $(\rho\otimes \mathrm{sgn})((12))$ has eigenvalues $-1, -1, 1$. Hence $W_3 \otimes \mathrm{sgn} \ncong W_3$ as required.

Finally, $\mathrm{Irr} = \{\mathrm{triv}, \mathrm{sgn}, W_2, W_3, W_3 \otimes \mathrm{sgn}\}$.

5.2. **Complex repr. of A_4.** The only normal non-trivial subgroup of A_4 is H (see 5.1).

Since $A_4/H = \mathbb{Z}_3$ we obtain exactly 3 one-dim. modules.

One has $12 = 1+1+1+9$. Let us show that W_3 is irred. repr. of A_4. Indeed, if W_3 is not irred. it has a submodule V of dim. 1. Notice that H acts by id on any one-dim. repr. of A_4. If $v = \sum_1^4 a_i e_i \in V$ ($\sum a_i = 0$) then $(12)(34)(v) = v$ gives $a_1 = a_2$ and $a_3 = a_4$, $(13)(24)(v) = v$ gives $a_1 = a_3$ and $a_2 = a_4$. Hence $v = 0$. Thus W_3 is an irred. repr. of A_4.

Hence A_4 has 4 irred. repr.: 3 one-dimensional and three-dimensional W_3.

5.2.1. How it is possible $W_3 \otimes V \cong W_3$ for a one-dim. repr. V? For $g \notin H$ the eigenvalues of g in W_3 are $\omega, \omega^2, 1$ where $\omega^3 = 1, \omega \neq 1$. In a one-dim. repr. one has $\rho(g) = c\,\mathrm{id}$ where $c^3 = 1$. Thus in $W_3 \otimes V$ the eigenvalues of g are $\{c\omega, c\omega^2, c\} = \{\omega, \omega^2, 1\}$.

6. Character theory

We have shown that any repr. is a direct sum of irred. ones. Our next step is to describe somehow these irred. and find techniques for giving direct sum decomposition and, in particular, for determining multiplicities. Now we will develop a wonderful tool for this: character theory.

We work over $\mathbb{C}$ and our group G is finite.

6.1. **Definition.**

$$\chi_V(g) := \mathrm{Tr}(\rho(g)).$$

6.2. **Proposition.**

$$(i) \quad \chi_{V\oplus W} = \chi_V + \chi_W$$

$$(ii) \quad \chi_{V\otimes W} = \chi_V \chi_W$$

$$(iii) \quad \chi_{V^*} = \overline{\chi}_V, \qquad (iv) \quad \chi_{\mathrm{Hom}(V,W)} = \overline{\chi}_V \chi_W$$

Proof. For the second and third formulas use Remark 2.3.3: (ii) is immediate; for (iii) recall that $\rho^*(g) = \rho(g^{-1})^t$. If $\rho(g)$ is diagonal, $\rho(g^{-1}) = \rho(g)^{-1} = \overline{\rho(g)}$; hence (iii). Finally, (iv) follows from (iii). □

6.3. Let us show that S_n ($n \neq 3$) has at least two non-isomorphic irred. repr. of dimension $n-1$.

Let W be any repr. such that $W \otimes \mathrm{sgn} \cong W$. Then, by (ii), $\chi_W(g) = 0$ if $g \notin A_n$.

Let us check that $W_{n-1} \otimes \mathrm{sgn} \ncong W_{n-1}$. Recall that $W_{n-1} \oplus \mathrm{triv} = V$ where V corresponds to the natural repr. Then

$$\chi_V((12)) = n-2, \quad \chi_{\mathrm{triv}}((12)) = 1.$$

Thus $\chi_{W_{n-1}}((12)) = n-3 \neq 0$ as required.

6.4. Clearly, χ_V is constant on any conjugacy class; such function is called a *class function.* They form a vector space denoted by $\mathbb{C}_{class}(G)$.

Define a Hermitian inner product on $\mathbb{C}_{class}(G)$ by

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

Our goal is to show that the characters of irred. repr. form an orthonormal basis of $\mathbb{C}_{class}(G)$.

6.5. **Character tables of S_3 and S_4.** Using the natural repr. we obtain for S_3

$$\begin{pmatrix} & e & (12) & (123) \\ \mathrm{triv} & 1 & 1 & 1 \\ \mathrm{sgn} & 1 & -1 & 1 \\ W_2 & 2 & 0 & -1. \end{pmatrix}$$