INTRODUCTION TO REPRESENTATION THEORY. 8/11/04. LECTURE 4

1. Character theory

We continue to study the character theory. Recall that we work over $\mathbb{C}$, our group G is finite and all representations are finite-dimensional.

1.1. **Recollection. Definition.**

$$\chi_V(g) := \mathrm{Tr}(\rho(g)).$$

1.2. Clearly, χ_V is constant on conj. class; such function is called a *class function.* They form a vector space denoted by $\mathbb{C}_{class}(G)$.

Define a Hermitian inner product on $\mathbb{C}_{class}(G)$ by

$$(\alpha, \beta) := \frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g).$$

Our goal is to show that the characters of irred. repr. form an orthonormal basis of $\mathbb{C}_{class}(G)$.

1.3. Define an average of all endomorphisms $\rho(g)$ that is

$$\phi := \frac{1}{|G|} \sum_{g \in G} \rho(g).$$

Lemma. *Let $\rho : G \to \mathrm{GL}(V)$ be a representation. The map ϕ is a projection of V to V^G.*

Proof. Clearly, $\phi(v) = v$ for any $v \in V^G$. It is easy to check that $\rho(h)\phi(v) = \phi(v)$ for any $v \in V, h \in G$. Hence $\mathrm{Im}\, \phi = V^G$. □

1.3.1. **Corollary.**

$$\dim V^G = \mathrm{Tr}\, \phi = \frac{1}{|G|} \sum_{g \in G} \chi_V(g).$$

1.3.2. One has $\mathrm{Hom}_G(V,W) = \mathrm{Hom}(V,W)^G$ and $\chi_{\mathrm{Hom}(V,W)} = \overline{\chi_V}\chi_W$. Hence

$$\dim \mathrm{Hom}_G(V,W) = \frac{1}{|G|}\sum_{g\in G} \overline{\chi_V}(g)\chi_W(g) = (\chi_V, \chi_W).$$

1.3.3. **Corollary.** *The characters of irred. modules form an orthonormal subset in $\mathbb{C}_{class}(G)$.*

1.3.4. **Corollary.** *The number of elements in* Irr *is at most the number of conjugacy classes of G.*

1.3.5. **Corollary.** *If V is irred. and W is any finite dimensional module, (χ_V, χ_W) is equal to the multiplicity of V in W.*

1.3.6. **Corollary.** *A finite dim. module is completely determined by its character: $\chi_V = \chi_W$ forces $V \cong W$.*

1.3.7. **Corollary.** *A module V is simple iff $(\chi_V, \chi_V) = 1$.*

1.3.8. Now we give another proof of the fact that the multiplicity of a simple module V in $\mathrm{Reg}(G)$ is $\dim V$. Indeed, $\chi_{\mathrm{Reg}(G)}(g) = 0$ if $g \neq e$, $\chi_{\mathrm{Reg}(G)}(e) = |G|$. One has $\chi_V(e) = \dim V$ and so

$$a = (\chi_V, \mathrm{Reg}_V) = \frac{1}{|G|}|G| \dim V = \dim V.$$

1.3.9. **Corollary.** *If $g \neq e$ one has*

$$\sum_{V\in\mathrm{Irr}} \dim V \chi_V(g) = 0.$$

Proof. One has

$$\mathrm{Reg}(G) = \sum_{V\in\mathrm{Irr}} V^{\oplus \dim V}.$$

Hence

$$0 = \chi_{\mathrm{Reg}(G)}(g) = \sum_{V\in\mathrm{Irr}} \dim V \chi_V(g)$$

for $g \neq e$. □

In particular, if we know characters of all but one irred. repr. we can calculate the remaining one via Corollary 1.3.9.

1.4. **Applications:: S_4.** : For S_4 we find the last line via Corollary 1.3.9

$$\begin{pmatrix} & e & (12) & (123) & (1234) & (12)(34) \\ \text{triv} & 1 & 1 & 1 & 1 & 1 \\ \text{sgn} & 1 & -1 & 1 & -1 & 1 \\ W_3 & 3 & 1 & 0 & -1 & -1 \\ W_3 \otimes \text{sgn} & 3 & -1 & 0 & 1 & -1 \\ W_2 & 2 & 0 & -1 & 0 & 2 \end{pmatrix}$$

1.5. To prove that the characters of irred. repr. form an othonormal basis in $\mathbb{C}_{class}$ it remains to verify that any class function orthogonal to the characters of irred. repr. is zero.

1.5.1. **Lemma.** *Let $f : G \to \mathbb{C}$ be a class function and $\rho : G \to \mathrm{GL}(V)$ be a repr. Let $\rho_f \in \mathrm{End}(V)$ be defined by*

$$\rho_f := \sum_{g \in G} f(g)\rho(g).$$

Then ρ_f is a G-homomorphism and if ρ is irreducible then

$$\rho_f = \frac{|G|}{\dim V}\overline{(f, \chi_{V^*})}\,\mathrm{id}\,.$$

Proof. One has

$$\rho_f\rho(g') = \sum_{g \in G} f(g)\rho(gg') = \sum_{g \in G} f(g{g'}^{-1})\rho(g)$$

and

$$\rho(g')\rho_f = \sum_{g \in G} f(g)\rho(gg') = \sum_{g \in G} f({g'}^{-1}g)\rho(g)$$

Since f is a class function $f({g'}^{-1}g) = f(g{g'}^{-1})$ and thus $\rho_f\rho(g') = \rho(g')\rho_f$ for any $g' \in G$. Hence ρ_f is a G-homomorphism. By Schur's lemma, if ρ is irreducible then $\rho_f = c\,\mathrm{id}$. To find c notice that $\mathrm{Tr}\,\rho_f = c \dim V$. One has

$$\mathrm{Tr}\,\rho_f = \mathrm{Tr}\sum_{g \in G} f(g)\rho(g) = \sum_{g \in G} f(g)\,\mathrm{Tr}\,\rho(g) = \sum_{g \in G} f(g)\chi_V(g) = |G|(\overline{f}, \chi_V) = |G|\overline{(f, \chi_{V^*})}.$$

$\square$

1.5.2. **Proposition.** *If $f \in \mathbb{C}_{class}$ is such that $(f, \chi_V) = 0$ for all $V \in \mathrm{Irr}$ then $f = 0$.*

Proof. If $(f, \chi_V) = 0$ for all $V \in \mathrm{Irr}$ then $\rho_f = 0$ for any irred. repr. ρ. Then ρ_f for any repr. ρ due to complete reducibility: if $\rho' : G \to \mathrm{GL}(V')$ is a subrepr. of $\rho : G \to \mathrm{GL}(V)$ then the restriction of ρ_f to V' is equal to ρ'_f. Let $\rho := \mathrm{Reg}(G)$. Then

$$0 = \rho_f = \sum_{g \in G} f(g)\rho(g)$$

and, in particular,

$$\rho_f(v_e) = \sum_{g \in G} f(g)\rho(g)(v_e) = \sum_{g \in G} f(g)v_g.$$

Since $\{v_g\}_{g \in G}$ form a basis of $\mathrm{Reg}(G)$ we obtain $f(g) = 0$ for all $g \in G$ as required. □

1.5.3. **Corollary.** *The cardinality of* Irr *is equal to the number of conjugasy classes in G.*

1.6. Let us formulate the main results.

1.6.1. **Corollary.** *For complex representations of a finite group G one has*

(i) *The number of elements in* Irr *is equal to the number of conjugacy classes of G.*
(ii) *A finite dim. module is completely determined by its character: $\chi_V = \chi_W$ forces $V \cong W$.*
(iii) *If V is irred. and W is any finite dimensional module, (χ_V, χ_W) is equal to the multiplicity of V in W.*
(iv) *A module V is simple iff $(\chi_V, \chi_V) = 1$.*

1.6.2. The last questions is how to describe an isotypical component of a given irreducible module W in a finite dimensional module V. Substitute $f := \mathrm{ch}\, W$ as a class function to Lemma 1.5.1; we obtain

$$\rho_f := \sum_{g \in G} f(g)g$$

is a G-homomorphism and if $\rho_i : G \to \mathrm{GL}(V_i)$ are irreducible subrepresentations then

$$\rho_f = \frac{|G|}{\dim V_i}\overline{(\chi_W, \chi_{V_i^*})}\,\mathrm{id}\,.$$

Write $V = \oplus V_i$ where V_i are irreducible; one has

1) $\rho_f(V_i) = V_i$ that is $\rho_f(V) = \sum \rho_{f,i}$ where $\rho_{f,i}$ is the restriction of ρ_f to V_i.

2) $\rho_{f,i} = 0$ if $V_i^* \ncong W$ that is $V_i \ncong W^*$.

3) If $V_i \cong W^*$ then $\dim V_i = \dim W$ and one has

$$\rho_{f,i} = \frac{|G|}{\dim W}\,\mathrm{id}\,.$$

Hence $\frac{\dim W}{|G|}\rho_f$ is the projection of V onto the subpace consisting of the sum of all copies of W^* appearing in V; this subspace is the isotypical component of W^*.

Changing W and W^* and using $\chi_{W^*} = \overline{\chi_W}$ we obtain

Corollary. *If W is a simple G-module and V is a finite dimensional G-module then*

$$\pi_W := \frac{\dim W}{|G|}\sum_{g\in G}\overline{\chi_W(g)}g$$

is the projection of V onto the isotypical component of W. Remark that Lemma 1.3 is a particular case of the above Corollary.

1.7. **Applications. Theorem.** *Let V be a faithful module. Then any irreducible G-module is a submodule of $V^{\otimes n}$ for infinitely many values of n.*

Proof. Let W be an irreducible G-module. We need to show that $(\chi_W, \chi_V^n) \neq 0$ for infinitely many values of n. Consider

$$\sum_{n=0}^{\infty}(\chi_W, \chi_V^n)t^n = \sum_{n=0}^{\infty}\frac{1}{|G|}\sum_{g\in G}\overline{\chi_W}(g)\chi_V^n(g) = \frac{1}{|G|}\sum_{g\in G}\frac{\overline{\chi_W}(g)}{1-\chi_V(g)t}$$

Since V is faithful, $\chi_V(g) \neq \chi_V(e)$ for all $g \neq e$. Indeed, all eigenvalues of g on V has length 1 (they are roots of 1) and some eigenvalues of g are not equal to 1; hence $\mathrm{Tr}_V(g) \neq \dim V = \mathrm{Tr}_V(e)$. Hence in the last sum there is exactly one summand whose denominator is divisible by $1 - \dim V t$; this summand is non-zero since $\overline{\chi_W}(e) = \dim W$. Thus the sum is a rational function which is not a polynomial. Hence $(\chi_W, \chi_V^n) \neq 0$ for infinitely many values of n. $\square$

2. Exterior and Symmetric powers

2.1. **Exterior powers.** The exterior power $\Lambda^k V$ $(k > 0)$ can be constructed as the quotient space of $V^{\otimes k}$ by the subspace spanned by all $v_1 \otimes \ldots \otimes v_n$ with two of the vectors are equal.

The image of $v_1 \otimes \ldots \otimes v_n$ is denoted by $v_1 \wedge \ldots \wedge v_n$. One has

$$v_{\sigma(1)} \wedge \ldots \wedge v_{\sigma(n)} = \mathrm{sgn}(\sigma)(v_1 \wedge \ldots \wedge v_n). \tag{1}$$

Set $\Lambda^0 V := k$.

2.1.1. A linear endomorphism $A : V \to V$ induces linear endomorphisms $\Lambda^r A : \Lambda^r V \to \Lambda^r V$:

$$\Lambda^r A(v_1 \wedge \ldots \wedge v_r) := Av_1 \wedge \ldots \wedge Av_r.$$

Clearly, $\Lambda^r(A \circ B) = \Lambda^r A \circ \Lambda^r B$ for any $A, B \in \mathrm{End}(V)$. In particular, if V is a G-module, $\Lambda^r(V)$ inherits the structure of G-module.

2.1.2. If $\{e_i\}$ is a basis of V then

$$\{e_{i_1} \wedge e_{i_2} \wedge \ldots \wedge e_{i_r} : i_1 < i_2 \ldots < i_r\}$$

is a basis for $\Lambda^r V$. In particular, $\Lambda^k V = 0$ for $k > \dim V$. One has

$$\Lambda^m(V \oplus W) = \oplus_{i=0}^m \Lambda^i(V) \otimes \Lambda^{m-i}(W)$$

where the identification comes from the maps

$$v_1 \wedge \ldots \wedge v_i \otimes w_1 \wedge \ldots \wedge w_{m-i} \mapsto v_1 \wedge \ldots \wedge v_i \wedge w_1 \wedge \ldots \wedge w_{m-i}.$$

2.1.3. If $\dim V = n$ then $\Lambda^n V$ is one-dimensional and $\Lambda^n A = \det A \,\mathrm{id}$. Indeed, let $\{e_i\}$ be a basis of V; set $a_i := Ae_i$. Write $\Lambda^n A = f(a_1, \ldots, a_n)\,\mathrm{id}$. We need to check that

(i) $f(e_1, \ldots, e_n) = 1$

(ii) $f(a_1, \ldots, a_n + cb_n) = f(a_1, \ldots, a_n) + cf(a_1, \ldots, b_n)$

(iii) $f(a_1, \ldots, a_n) = -f(b_1, \ldots, b_n)$ if $a_r = b_s, a_s = b_r$ for some r, s and $a_i = b_i$ for $i \neq r, s$.

(i) is obvious. For (ii) let $B \in \mathrm{End}(V)$ be such that $B(e_i) = 0$ for $i < n$, $B(e_n) = cb_n$; one has

$$\Lambda^n(A + B)(e_1 \wedge \ldots \wedge e_n) = f(a_1, \ldots, a_n + cb_n)(e_1 \wedge \ldots \wedge e_n)$$

and

$$\begin{aligned}&\Lambda^n(A + B)(e_1 \wedge \ldots \wedge e_n) = a_1 \wedge \ldots \wedge (a_n + cb_n) = a_1 \wedge \ldots \wedge a_n + c(a_1 \wedge \ldots \wedge b_n)\\ &= f(a_1, \ldots, a_n)(e_1 \wedge \ldots \wedge e_n) + cf(a_1, \ldots, b_n)(e_1 \wedge \ldots \wedge e_n).\end{aligned}$$

Hence $f(a_1, \ldots, a_n + cb_n) = f(a_1, \ldots, a_n) + f(a_1, \ldots, cb_n)$.

Finally, for (iii) take $A = (a_1, \ldots, a_n)$ and $B = (b_1, \ldots, b_n)$. By (1) one has

$$\Lambda^n A(e_1 \wedge \ldots \wedge e_n) = a_1 \wedge \ldots \wedge a_n = -b_1 \wedge \ldots \wedge b_n = -\Lambda^n B(e_1 \wedge \ldots \wedge e_n)$$

as required.

2.1.4. One has

$$\chi_{\Lambda^2(V)}(g) = \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2)).$$

Indeed, let $\{e_i\}$ be an eigenbasis of g and $ge_i = \lambda_i e_i$. Then $\{e_i \wedge e_j : i < j\}$ is a basis of $\Lambda^2(V)$ and

$$g(e_i \wedge e_j) = \lambda_i e_i \wedge \lambda_j e_j = \lambda_i \lambda_j e_i \wedge e_j.$$

Thus

$$\chi_{\Lambda^2(V)}(g) = \sum_{i<j} \lambda_i \lambda_j = \frac{(\sum \lambda_i)^2 - \sum \lambda_i^2}{2} = \frac{\mathrm{Tr}_V(g)^2 - \mathrm{Tr}_V(g^2)}{2}$$

as required.

2.2. **Symmetric powers.** The symmetric power $\mathrm{Sym}^k V$ can be constructed as the quotient space of $V^{\otimes k}$ by the subspace spanned by all $v_1 \otimes \ldots \otimes v_n - v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}$ where $v_1, \ldots, v_n \in V$ and $\sigma \in S_n$.

The image of $v_1 \otimes \ldots \otimes v_n$ is denoted by $v_1 v_2 \ldots v_n$. One has

$$v_{\sigma(1)} \ldots v_{\sigma(n)} = v_1 \ldots v_n.$$

2.2.1. If $\{e_i\}$ is a basis of V then

$$\{e_{i_1} e_{i_2} \ldots e_{i_r} : i_1 \leq i_2 \ldots <\leq i_r\}$$

is a basis for $\mathrm{Sym}^r V$. Set $\mathrm{Sym}^0 V = k$. Then $\mathrm{Sym}^n(V)$ can be viewed as the space of homogeneous polynomials of degree n in variables e_i.

One has

$$\mathrm{Sym}^m(V \oplus W) = \oplus_{i=0}^m \mathrm{Sym}^i(V) \otimes \mathrm{Sym}^{m-i}(W).$$

2.3. Both $\Lambda^r V$ and $\mathrm{Sym}^r V$ can be realized as subspaces of $V^{\otimes n}$ if $\mathrm{char}\, k = 0$. The embedding $\iota : \Lambda^r V \to V^{\otimes n}$ is given by

$$v_1 \wedge \ldots \wedge v_r \mapsto \sum_{\sigma \in S_r} \mathrm{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)}.$$

Notice that $\mathrm{Im}\, \iota$ is the isotypical component of sgn into the representation $S_r \to \mathrm{End}(V^{\otimes r})$ given by

$$\sigma\big(v_{(1)} \otimes \ldots \otimes v_{(r)}\big) := v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)}.$$