INTRODUCTION TO REPRESENTATION THEORY. 15/11/04. LECTURE 5

1. Exterior and Symmetric powers

1.1. Both $\Lambda^r V$ and $\operatorname{Sym}^r V$ can be realized as subspaces of $V^{\otimes n}$ if $\operatorname{char} k = 0$. The embedding $\iota : \Lambda^r V \to V^{\otimes n}$ is given by

$$v_1 \wedge \ldots \wedge v_r \mapsto \sum_{\sigma \in S_r} \operatorname{sgn}(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(r)}.$$

Notice that $\operatorname{Im} \iota$ is the isotypical component of sgn into the representation $S_r \to \operatorname{End}(V^{\otimes r})$ given by

$$\sigma(v_1 \otimes \ldots \otimes v_r) := v_{\sigma^{-1}(1)} \otimes \ldots \otimes v_{\sigma^{-1}(r)}.$$

2. Real representations

If V_0 is a real vector space $V := \mathbb{C} \otimes_{\mathbb{R}} V_0$ is a complex vector space. As a real vector space $V = V_0 \oplus iV_0$.

If V_0 is a real vector space and G-module V inherits a G-module structure; V is called the *complexification of* V_0. We call such a complex repr. *realizable over* $\mathbb{R}$.

We always assume $\dim V < \infty$.

2.1. Let V be the complexification of V_0. Choose a basis $\{e_i\}$ in V_0; then $\{1 \otimes e_i\}$ is a basis of V. If $\rho : G \to \operatorname{End}(V)$ is a complexification of $\rho_0 : G \to \operatorname{End}(V_0)$ then the matrix $\rho(g)$ written wrt the basis $\{1 \otimes e_i\}$ is equal to the matrix $\rho_0(g)$ written wrt the basis $\{e_i\}$. In particular, any repr. realizable over $\mathbb{R}$ has a real-valued character. The converse does not hold: for instance, the quaternion group Q_8 has a two-dim. repr. $\rho : Q_8 \to \operatorname{End}(\mathbb{C}^2)$ where

$$i \mapsto \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$j \mapsto \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$k \mapsto \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$-1 \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus

$$\operatorname{Tr}(-1) = -2, \operatorname{Tr} i = \operatorname{Tr} j = \operatorname{Tr} k = 0$$

and so the character is real-valued. This representation is not realizable over $\mathbb{R}$: this can be checked directly (there is no matrix A such that the conjugation by A reduces the above matrices to the ones with real entries) or as follows. Suppose ρ is a complexification of ρ_0. Since Q_8 is a finite group, any real repr. of Q_8 has an invariant scalar product: we can pick up any scalar product and then construct an invariant one $(-,-)_n$ by setting

$$(v, v')_n := \sum_{g \in Q_8} (gv, gv').$$

Fix an invariant scalar product and notice that $\rho_0(g)$ is an isometry for any $g \in G$. Since ρ is a complexification of ρ_0, ρ_0 maps i, j, k to matrices with determinant 1. An isometry of $\mathbb{R}^2$ with determinant 1 is a rotation and all rotations commute with each other. However, $\rho_0(ij) = \rho_0(k) \neq \rho_0(ji) = -\rho_0(k)$. This gives a contradiction.

2.1.1. Suppose that χ_V is real. Then $\chi_V = \chi_{V^*}$ and so $V \cong V^*$. The last is equivalent to the existence of a G-invariant non-degenerate bilinear form on V: if an isomorphism $\iota : V \to V^*$ then $B(x, y) := \iota(x)(y)$ is a G-invariant non-degenerate bilinear form since

$$B(gx, gy) = \iota(gx)(gy) = (g\iota(x))(gy) = \iota(x)(g^{-1}gy) = \iota(x)(y) = B(x, y)$$

and if $B(x, y)$ a G-invariant non-degenerate bilinear form then $\iota : V \to V^*$ defined by $\iota(x)(y) := B(x, y)$ is an injective G-homomorphism and so is an isomorphism.

2.1.2. Now suppose that V is realizable over $\mathbb{R}$ that is $V = V_0 \oplus iV_0$ and V_0 is irreducible. Then V_0 has a non-degenerate invariant symmertic bilinear form B (scalar product) which induces a non-degenerate invariant symmertic bilinear form on V.

Let us show that if an irred. complex repr. V has a non-degenerate invariant symmertic bilinear form B then V is realizable over $\mathbb{R}$. Indeed, take an arbitrary non-degenerate Hermitian invariant form H on V. One has

$$V \to^{\phi_B} V^* \to^{\phi_H} V$$

and so $\phi := \phi_H \phi_B : V \to V$ is such that

$$B(x, y) = H(\phi(x), y)$$

$\phi(gv) = g\phi(v)$ and $\phi(\lambda v) = \overline{\lambda}\phi(v)$. Then $\phi^2 : V \to V$ is a G-homomorphism and so $\phi^2 = c\,\mathrm{id}$. Moreover, $B(x, y) = B(y, x)$ implies

$$H(\phi(x), y) = \overline{H(x, \phi(y))}$$

that is $H(\phi^2(x), y) = H(x, \phi^2(y))$ and so $c \in \mathbb{R}, c \neq 0$. Since $\phi^2 = c\,\mathrm{id}$, ϕ is diagonalizable: write $V = V_+ \oplus V_-$ where V_+, V_- correspond to $\pm\sqrt{c}$. Both V_+, V_- are G-invariant real subspace of V and $\phi(ix) = -i\phi(x)$ implies $iV_+ = V_-$. Hence $V = V_+ \otimes_{\mathbb{R}} \mathbb{C}$ as required.

2.1.3. Suppose again that V is an irred. and χ_V is real. Then $V \cong V^*$ i.e., there exists a G-invariant non-degenerate bilinear form on V.

Notice that a G-invariant non-degenerate bilinear form on V is a non-zero G-invariant element of $(V \otimes V)^* = V^* \otimes V^*$. One has

$$V^* \otimes V^* = \mathrm{Sym}^2(V^*) \oplus \Lambda^2(V^*).$$

If $V \cong V^*$ then $V^* \otimes V^*$ has a G-invariant element B which is unique up to a scalar. If $B \in \mathrm{Sym}^2(V^*)$ then B is symmetric; if $B \in \Lambda^2(V^*)$ then B is skew-symmetric.

2.1.4. **Corollary.** *If V is an irred. complex repr. then exactly one of three posibilities holds*

1) $\chi_V(g) \notin \mathbb{R}$ for some $g \in G$ and V does not have a non-degenerate invariant bilinear form

2) V has a non-degenerate invariant symmertic bilinear form and V is realizable over $\mathbb{R}$

3) V has a non-degenerate invariant skew-symmertic bilinear form and V is not realizable over $\mathbb{R}$.

2.1.5. Let V_0 be an irred. repr. over $\mathbb{R}$. Notice that V has an operation $\bar{\ } : \overline{x + iy} = x - iy,\ x, y \in V_0$ which satisfies $\overline{gv} = g\bar{v}$.

Suppose V is not irred. and $W \subset V$ be a simple submodule. Then $\overline{W}$ is also a submodule. We claim that $W + \overline{W} = V$. Indeed, if $x + iy \in W$ then $x - iy \in \overline{W}$ and so $x, y \in W + \overline{W}$; hence $(W + \overline{W}) \cap V_0 \neq 0$ and so $V_0 \subset W + \overline{W}$ because V_0 is simple and $(W + \overline{W}) \cap V_0$ is G-invariant. Hence $W + \overline{W} = V$. If $W \cap \overline{W} \neq 0$ then $W = \overline{W}$ since W is simple and then $V = W + W = W$ is simple which contradicts to our assumption. Therefore $W \cap \overline{W} = 0$ and so $V = W \oplus \overline{W}$ where W is irreducible and $\chi_{\overline{W}} = \overline{\chi_W}$ that is $\overline{W} \cong W^*$. Clearly, the pair $W, \overline{W}$ is uniquely determined by V_0.

Hence for any irred. repr. V_0 over $\mathbb{R}$ its complexification is either irreducible or the direct sum of two irreducible representations which are dual one to another. Let $\mathrm{Irr}_{\mathbb{R}}$ be the set of isomorphism classes of real irred. repr. and Irr be the set of isomorphism classes of complex irred. repr. Define an equivalence relation on the set Irr by putting $W \approx W^*$. Let $\mathrm{Irr}/\approx$ be the corresponding quotient set. We obtain a map $\Psi : \mathrm{Irr}_{\mathbb{R}} \to \mathrm{Irr}/\approx$. Let us show that Ψ is bijective.

To see that Ψ is injective take $V_0, W_0 \in \mathrm{Irr}_{\mathbb{R}}$ where $V_0 \ncong W_0$. Then $\mathrm{Hom}_{\mathbb{C}G}(V, W) = \mathrm{Hom}_{\mathbb{R}G}(V_0, W) = \mathrm{Hom}_{\mathbb{R}G}(V_0, W_0) \oplus i\,\mathrm{Hom}_{\mathbb{R}G}(V_0, W_0) = 0$ where $\mathrm{Hom}_{\mathbb{C}G}$ stands for G-homomorphisms of complex representations and $\mathrm{Hom}_{\mathbb{R}G}$ stands for G-homomorphisms of real representations (we can view W as a vector space over $\mathbb{R}$ and so as a real representation of G). Hence $\Psi(V_0) \neq \Psi(W_0)$.

Let us show that Ψ is surjective. Indeed, take an irred. complex repr. W of dimension n. Let us show that either $W = \mathbb{C} \otimes_{\mathbb{R}} W_0$ for an irred. real repr. W_0 or $V = W \oplus W^*$ is the complexification of a real irreducible representation (of dimension $2n$). Indeed, view W as a real vector space (of dimenion $2n$) and denote the corresponding real $2n$-dim. repr. by V_0. Let V be the complexification of V_0. It is easy to see that $\chi_V = 2Re\chi_W = \chi_W + \chi_{W^*}$. Therefore $V = W \oplus W^*$. If V_0 is simple then $V = W \oplus W^*$ is the complexification of a real irreducible representation of dimension $2n$. Assume that V_0 is not simple and write $V_0 = U_0 \oplus U_0'$ where U_0, U_0' are submodules. Then we obtain $U \oplus U' = V = W \oplus W^*$ that is $\{U, U'\} = \{W, W^*\}$. Hence $W = \mathbb{C} \otimes_{\mathbb{R}} W_0$ for $W_0 = U_0$ or $W_0 = U_0'$; since W is simple, W_0 is also simple.

3. Algebras

An algebra is a vector space endowed by a bilinear operation. An alegbra is associative if $(xy)z = x(yz)$.

If A is an algebra $I \subset A$ is called a *left ideal* (resp., right ideal) if I is a vector subspace of A and for any $a \in A, x \in I$ one has $ax \in I$ (resp., $xa \in I$). If I is a left ideal and a right ideal, it is called a *two-sided* ideal.

Let A, A' be algebras. A linear map $\phi : A \to A'$ is called an *algebra homomorphism* if $\phi(xy) = \phi(x)\phi(y)$.