INTRODUCTION TO REPRESENTATION THEORY. 22/11/04. LECTURE 6

1. Algebras. Derivations. Definition of Lie algebra

1.1. **Algebras.** Let k be a field. An algebra over k (or k-algebra) is a vector space A endowed with a bilinear operation

$$a, b \in A \mapsto a \cdot b \in A.$$

Recall that bilinearity means that for each $a \in A$ left and right multiplications by a are linear transformations of vector spaces (i.e. preserve sum and multiplication by a scalar).

1.1.1. *Some extra properties.* An algebra A is called associative if $a \cdot (b \cdot c) = (a \cdot b) \cdot c$).

An algebra A is commutative if $a \cdot b = b \cdot a$.

Usually commutative algebras are supposed to be associative as well.

1.2. **Examples.** Example: if $K \subset K'$ are fields, K' is an algebra over K (for instance, $\mathbb{C}$ is an $\mathbb{R}$-algebra).

Example: $\mathbb{R}^3$ wrt the vector product is an algebra.

Example: the set of functions $X \to F$ where X is any set and F is a field is an algebra over F wrt the usual (point-wise) multipication of functions.

Example: group algebra.

Notice that to describe an algebra it is enough to determine the product of basis elements. To check that an algebra is associative (commutative) it is enough to check this on basis elements.

An element $e \in A$ is called the *unity* if $e \neq 0$ and $ae = ea = a$ for any $a \in A$. Such an element is unique (but sometimes does not exists).

If V is a vector space, $\mathrm{End}(V)$, the set of (linear) endomorphisms of V is an associative algebra with respect to composition. If $V = k^n$ $\mathrm{End}(V)$ is just the algebra of $n \times n$ matrices over k.

The ring of polynomials $k[x]$ over k is a commutative k-algebra. The same for $k[x_1, \ldots, x_n]$, the algebra of polynomials of n variables.

If V is a vector space, define an operation by the formula

$$a \cdot b = 0.$$

This is an algebra operation.

1.3. **Subalgebras, ideals, quotient algebras.** A linear map $f : A \to B$ of k-algebras is called *homomorphism* if $f(a \cdot b) = f(a) \cdot f(b)$ for each $a, b \in A$. A homomorphism is called *isomorphism* if it is bijective.

The image of a homomorphism is a *subalgebra*. Kernel of f defined as $\{a \in A | f(a) = 0\}$ is *an ideal* in A. Here are the appropriate definitions.

Definition. vector subspace $B \subseteq A$ is called a subalgebra if

$$a, b \in B \Longrightarrow a \cdot b \in B.$$

Definition. vector subspace $I \subseteq A$ is called an ideal if

$$a \in A \& x \in I \Longrightarrow a \cdot x \in I \& x \cdot a \in I.$$

Notice that any ideal is a subalgebra (in group theory a similar assertion is that any normal group is a subgroup).

Lemma. *Let $f : A \to B$ be a homomorphism of algebras. Then* $\mathrm{Ker}(f)$ *is an ideal in A.*

Proof. Exercise. □

An important property of ideals is that one can form a quotient algebra "modulo I". Here is the construction.

Let A be an algebra and I an ideal in A. We define the quotient algebra A/I as follows.

As a set this is the quotient of A modulo the equivalence relation

$$a \sim b \text{ iff } a - b \in I.$$

Thus, this is the set of equivalence classes having form $a + I$, where $a \in A$.

Structure of vector space on A/I is given by the formulas

$$(a + I) + (b + I) = (a + b) + I; \quad \lambda(a + I) = \lambda a + I.$$

Algebra structure on A/I is given by the formula

$$(a + I) \cdot (b + I) = a \cdot b + I.$$

One has a canonical homomorphism

$$\rho : A \to A/I$$

defined by the formula $\rho(a) = a + I$.

As usual, the following theorem (Theorem on homomorphism) is straighforward.

Theorem. *Let $f : A \to B$ be a homomorphism of algebras and let I be an ideal in A. Suppose that $I \subseteq \mathrm{Ker}(f)$. Then there exists a unique homomorphism $\overline{f} : A/I \to B$ such that $f = \overline{f} \circ \rho$ where $\rho : A \to A/I$ is the canonical homomorphism.*

Moreover, $\overline{f}$ is surjective iff f is surjective; $\overline{f}$ is injective iff $I = \mathrm{Ker}(f)$.

Proof. To satisfy the relation $f = \overline{f} \circ \rho$ one needs $\overline{f}(a+I) = f(a)$ for all $a \in A$. This map is well-defined since $a + I = b + I$ implies $(a - b) \in I \subset \mathrm{Ker}\, f$ that is $f(a) = f(b)$. Now

$$\overline{f}((a+I)(b+I)) = f(ab) = f(a)f(b) = \overline{f}(a+I)\overline{f}(b+I)$$

and similarly

$$\overline{f}((a+I) + (b+I)) = \overline{f}(a+I) + \overline{f}(b+I), \quad \overline{f}(\lambda a + I) = \lambda \overline{f}(a+I).$$

Hence $\overline{f}$ is a homomorphism. Clearly, the image of $\overline{f}$ coincides with the image of f and $\mathrm{Ker}\, \overline{f} = \mathrm{Ker}\, f/I$. This implies the second assertion. □

1.4. **Representations of an associative algebra with unity.** We require that any homomorphism between two associative algebras with unity maps unity to unity.

Representation of an associative algebra with unity A is a homomorphism $\rho : A \to \mathrm{End}(V)$ where V is a vector space (over the same field as A) and $\rho(1) = \mathrm{id}$.

1.5. **Density theorem.** Let A be an associative algebra with unity and $\rho : A \to \mathrm{End}(V)$ be an irred. repr. Then ρ is surjective.

1.6. **Direct product of algebras.** As a set it is the Cartesian product; operation is coordinate-wise. If $\phi : A \to B, \phi : A \to C$ are algebra homomorphisms they induce an algebra homomorphism $\phi \times \phi' : A \to B \times C$:

$$\phi \times \phi'(a) := (\phi(a), \phi'(a)).$$

Notice that A, B are two-sided ideals in $A \times B$. Moreover, we have the natural embeddings $A, B \to A \times B$ and the images commute.

1.7.

$$\mathbb{C}[G \times G'] = \mathbb{C}[G] \times \mathbb{C}[G]$$

Theorem. *If G is a finite group then the irreducible representations $\mathbb{C}[G] \to \mathrm{End}(W)$ induce the algebra isomorphism*

$$\mathbb{C}[G] \cong \prod_{W \in \mathrm{Irr}} \mathrm{End}(W).$$

Proof. Clearly, $\iota : \mathbb{C}[G] \to \oplus_{W\in \mathrm{Irr}} \mathrm{End}(W)$ is an algebra homomorphism. We know that both sides have the same dimension. Let us show that ι is injective. Indeed, assume that $\iota(u) = 0$ for some $u \in \mathbb{C}[G]$. Then $uW = 0$ for all $W \in \mathrm{Irr}$. Since any finite dimensional repr. is a direct sum of irred. ones, $uV = 0$ for any finite dimensional module V. In particular, $u\,\mathrm{Reg} = 0$ that is $ua = 0$ for any $a \in \mathbb{C}[G]$. Since $1 \in \mathbb{C}[G]$ one has $0 = u1 = u$. Hence ι is injective. □

1.8. **Derivations.** A linear endomorphism $d : A \to A$ is called *derivation* if the following *Leibniz rule* holds.

$$d(a \cdot b) = d(a) \cdot b + a \cdot d(b).$$

The set of all derivations of A is denoted $\mathrm{Der}(A)$. This is clearly a vector subspace of $\mathrm{End}(A)$.

1.8.1. *Composition.* Let $d, d' \in \mathrm{Der}(A)$ let us check that the composition dd' is not a derivation.

$$\begin{aligned} (1)\quad dd'(a \cdot b) = d(d'(a) \cdot b + a \cdot d'(b)) = d(d'(a) \cdot b) + d(a \cdot d'(b)) = \\ dd'(a) \cdot b + d'(a) \cdot d(b) + d(a) \cdot d'(b) + a \cdot dd'(b) \end{aligned}$$

which is not exactly what we need.

1.8.2. *Bracket.* Thus, we suggest another operation. Given $d, d' \in \mathrm{Der}(A)$, define $[d, d'] = dd' - d'd$.

Theorem. *f $d, d' \in \mathrm{Der}(A)$ then $[d, d'] \in \mathrm{Der}(A)$.*

Proof. Direct calculation. □

1.8.3. *Properties of this bracket.* 1. $[x, x] = 0$.

2. (Jacobi identity) $[[xy]z] + [[zx]y] + [[yz]x] = 0$

Exercise: check this.

1.9. **Definition of Lie algebra. First examples.** A Lie algebra is an algebra with an operation satisfying the properties 1.8.3.

The operation in a Lie algebra is usually denoted $[-, -]$ and called (Lie) bracket.

Notice that Lie algebra does not have a unity.

1.9.1. *Anticommutativity.* The first property of a Lie algebra saying $[xx] = 0$ is called anticommutativity. In fact, it implies that $[xy] = -[yx]$ for all x, y.

Proof: $0 = [x+y, x+y] = [xx] + [xy] + [yx] + [yy]$. This implies $[xy] = -[yx]$. The converse is true if $\operatorname{char} k \neq 2$. Indeed, $[xx] = -[xx]$ implies that $2[xx] = 0$ and, if the charactersitic of k is not 2, this implies $[xx] = 0$.

1.9.2. *Example.* Let $k = \mathbb{R}$, $L = \mathbb{R}$. We are looking for possible Lie brackets on L. Bilinearity and anticommutativity require

$$[a, b] = [a \cdot 1, b \cdot 1] = ab[1, 1] = 0.$$

Thus, there is only one Lie bracket on $L = \mathbb{R}$.

Definition. Lie algebra L having a zero bracket is called *a commutative Lie algebra.*

Any vector subspace of a commutative Lie algebra is an ideal. All commutative Lie algebras of a given dimension are isomorphic. A quotient of a commutative Lie algebra is a commutative Lie algebra.

1.9.3. *Observation.* Fix a field k of characteristic $\neq 2$ and let $L = \langle e_1, \ldots, e_n \rangle$ be n-dimensional vector space over k. In order to define a bilinear operation, it is enough to define it on e_i:

$$[e_i, e_j] = \sum_{k=1}^{n} c_{ij}^k e_k.$$

(this is true for any type of algebra). Elements c_{ij}^k are called *structure constants* of L.

Since we want the bracket to be anti-commutative, one has to have

$$[e_i, e_j] = -[e_j, e_i].$$

Bilinearity and this condition imply anti-commutativity of the bracket (check this formally!).

Suppose now we have checked already anticommutativity. To check Jacobi identity let us denote

$$J(x, y, z) = [[xy]z] + [[zx]y] + [[yz]x].$$

One observes that J is trilinear (linear on each one of its three arguments) and antisymmetric (it changes sign if one interchanges any two arguments).

Thus, in order to check $J(x, y, z)$ is identically zero, it is enough to check

$$J(e_i, e_j, e_k) = 0 \text{ for } 1 \leq i < j < k \leq n.$$

1.9.4. *Example.* Suppose $\dim L = 2$. Then the Jacobi identity holds automatically. Suppose L is not commutative. Choose a basis $L = \langle e_1, e_2 \rangle$. One has

$$[e_1, e_1] = [e_2, e_2] = 0 \text{ and } [e_1, e_2] = -[e_2, e_1].$$

Let $[e_1, e_2] = y$. Then $y \neq 0$ and any bracket in L is proportional to y (by bilinearity).

Thus, it is convenient to take y as one of generators on L. Choose another one, say x. We have $L = \langle x, y \rangle$ and $[x, y] = \lambda y$. Since L is not commutative, $\lambda \neq 0$. Thus change variables once more setting $x := x/\lambda$.

We finally get

$$L = \langle x, y \rangle \text{ and } [x, y] = y. \tag{2}$$

We have therefore proven that there are only two two-dimensional Lie algebras over k up to isomorphism: a commutative Lie algebra and the one described in (2).

1.9.5. *Example.* If A is an associative algebra then the bracket defined by $[a, b] = a \cdot b - b \cdot a$ satisfy the axioms of Lie bracket. This gives A the structure of a Lie algebra.

The set of $n \times n$ matrices over k is an associative algebra with respect to the matrix multiplication. It becomes a Lie algebra if we define a bracket by the formula

$$[x, y] = xy - yx.$$

This Lie algebra is denoted $\mathfrak{gl}_n(k)$ (sometimes we do not mention the field k). Its dimension is, of course, n^2. It has the natural basis E_{ij}: the matrix E_{ij} has zero entries except 1 on the (i, j) place.

The Lie algebra $\mathfrak{gl}_n$ admits a remarkable Lie subalgebra.

Define $\mathfrak{sl}_n = \{a \in \mathfrak{gl}_n | \operatorname{Tr}(a) = 0\}$.

Here $\operatorname{Tr}(a) = \sum a_{ii}$ is the trace of a, the sum of the diagonal elements of a.

We claim this is a Lie subalgebra.

1.9.6. *Proof.* Recall that for each pair of matrices a, b one has

$$\operatorname{Tr}(ab) = \operatorname{Tr}(ba).$$

(Proof is just a direct calculation: both sides are equal to $\sum_{ij} a_{ij} b_{ji}$.)

Then $\operatorname{Tr}([a, b]) = \operatorname{Tr}(ab) - \operatorname{Tr}(ba) = 0$. This proves that $\mathfrak{sl}_n$ is closed under the bracket operation.

1.9.7. Let $k = \mathbb{R}$, $L = \mathbb{R}^3$. Define $[x, y] = x \times y$ — the cross-product. Recall that the latter is defined by the formulas

$$e_1 \times e_2 = e_3, \ e_2 \times e_3 = e_1, \ e_3 \times e_1 = e_2.$$

This algebra has no non-trivial ideals. Indeed, let I be a non-zero ideal and $a \in I$ be a non-zero element. Let a, b, c be a basis consisting of mutually orthogonal elements; then $[a, b]$ is a non-zero multiple of c and $[a, c]$ is a non-zero multiple of b. Thus $a \in I$ forces $a, b, c \in I$ that is $I = L$.