BOUNDED HIGHEST WEIGHT MODULES OVER $\mathfrak{osp}(1, 2n)$

THOMAS FERGUSON, MARIA GORELIK, DIMITAR GRANTCHAROV

Abstract. We classify all simple bounded highest weight modules of the ortosymplectic superalgebras $\mathfrak{osp}(1, 2n)$. The classification is obtained in two independent ways: using equivalence of categories of $\mathfrak{osp}(1, 2n)$-modules and $\mathfrak{osp}(1, 2n)_{\bar{0}}$-modules, and by finding primitive vectors in tensor products of bounded and finite-dimensional $\mathfrak{osp}(1, 2n)$-modules. We also obtain character formulae for the simple bounded highest weight modules of $\mathfrak{osp}(1, 2n)$.

1. Introduction

Weight modules are modules that are semisimple as modules over a fixed Cartan subalgebra. Examples of weight modules include quotients of parabolically induced modules (in particular, highest weight modules) and some generalized Harish-Chandra modules. The classification of all simple weight $\mathfrak{g}$-modules with finite weight multiplicities over finite-dimensional simple Lie superalgebras $\mathfrak{g}$ is not completed yet. This classification for Lie algebras $\mathfrak{g}$ was completed in the breakthrough paper [M] by classifying all simple cuspidal $\mathfrak{g}$-modules, i.e. modules on which all root elements of $\mathfrak{g}$ act bijectively. In the Lie superalgebra case, the classification was obtained in [DMP] for all $\mathfrak{g}$ except for the Lie superalgebra series $\mathfrak{osp}(m; 2n)$, $m = 1, 3, 4, 5, 6$; $\mathfrak{psq}(n)$, $D(2, 1, \alpha)$, and the Cartan series of type S and H. It is interesting to note that the Lie superalgebras $\mathfrak{osp}(m, 2n)$, $m \geq 7$, are not in the list because their even parts do not have cuspidal modules. For the classical Lie superalgebras $\mathfrak{g}$, the classification of simple weight modules with finite weight multiplicities was reduced to the classification of the so-called bounded highest weight modules, [Gr]. The latter classification was obtained for $\mathfrak{g} = \mathfrak{psq}(n)$ and $\mathfrak{g} = D(2, 1, \alpha)$ in [GG] and [H], respectively, leaving the orthosymplectic series the only remaining classical Lie superalgebras to consider.

Fix a triangular decomposition of $\mathfrak{g}$ and denote by $L(\lambda)$ the simple highest weight $\mathfrak{g}$-module of highest weight λ. If the set of weight multiplicities of $L(\lambda)$ is uniformly bounded we call the module $L(\lambda)$ *bounded* and the weight λ $\mathfrak{g}$-*bounded*. In this paper we make the first step towards the classification of the bounded highest weight modules $L(\lambda)$ (and, hence, of the simple weight modules with finite weight multiplicities) of the

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orthosymplectic superalgebras - we solve the problem for $\mathfrak{osp}(1,2n)$. Our result can be written in short as follows.

Theorem. *An infinite-dimensional module $L(\lambda)$ is $\mathfrak{osp}(1,2n)$-bounded if and only if λ and $\lambda-\delta_n$ are $\mathfrak{sp}(2n)$-bounded.*

We believe that our classification will play crucial role in the classification of the simple bounded highest weight modules for the remaining four orthosymplectic superlagebras series. In addition to the classification of the bounded $\mathfrak{osp}(1,2n)$-modules $L(\lambda)$, we obtain character formulae of $L(\lambda)$ in terms of characters of simple finite-dimensional $\mathfrak{so}(2n)$-modules, see (2).

We present two alternative proofs of the classifcation of the bounded modules $L(\lambda)$. The first is based on the equivalence of categories of graded $\mathfrak{osp}(1,2n)$-modules and of $\mathfrak{sp}(2n)$-modules established in [G2]. The second relies on finding primitive vectors of tensor products of the Weyl module $L(-\frac{1}{2}\sum_{i=1}^{n}\delta_i)$ and finite-dimensional modules.

The paper is organized as follows. The main result together with character formulae for bounded $L(\lambda)$ are presented in Section 3. The alternative proof of the classification of bounded highest weight modules is included in Section 4. Some important facts on the equivalences of categories and character formulae are collected in the Appendix.

2. Notation and conventions

Except for the appendix, throughout the paper, $\mathfrak{g}=\mathfrak{osp}(1,2n)$. We fix a triangular decomposition of $\mathfrak{g}$, hence of $\mathfrak{g}_{\bar{0}}$, and by $\mathfrak{h}$ we denote the Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{g}_{\bar{0}}$.

The root system of $(\mathfrak{g},\mathfrak{h})$ is $\Delta=\Delta_{\bar{0}}\cup\Delta_{\bar{1}}$, where $\Delta_{\bar{1}}=\{\pm\delta_i \mid i=1,...,n\}$, $\Delta_{\bar{0}}=\{\pm\delta_i\pm\delta_j,\pm 2\delta_i \mid 1\leq i<j\leq n\}$. We will also use the notation $\Delta_C=\Delta_{\bar{0}}$ (root system of $C_n=\mathfrak{sp}(2n)$) and $\Delta_D=\{\pm\delta_i\pm\delta_j \mid 1\leq i<j\leq n\}$ (root system of $D_n=\mathfrak{so}(2n)$). By Δ^+ we will denote the set of all positive roots of $\mathfrak{g}$ and W will stand for the Weyl group. Fix $(-,-)$ to be the symmetric bilinear form on $\mathfrak{h}^*$ such that $(\delta_i,\delta_j)=\delta_{i,j}$.

In most of the paper we will fix $\Pi=\{\delta_1-\delta_2,...,\delta_{n-1}-\delta_n,\delta_n\}$ to be the base of Δ. We will also use $\widetilde{\Pi}=\{-\delta_1,\delta_1-\delta_2,...,\delta_{n-1}-\delta_n\}$. Denote by θ the automorphism of Δ defined by $\delta_i\mapsto-\delta_{n+1-i}$. This automorphism extends to an automorphism of $\mathfrak{g}$ that we will denote by the same letter. Note that $\widetilde{\Pi}=\theta(\Pi)$.

We will call a vector v in a $\mathfrak{g}$-module primitive (respectively, $\widetilde{\Pi}$-primitive), if the elements of the α-root spaces of $\mathfrak{g}$ for all $\alpha\in\Pi$ (respectively, $\alpha\in\widetilde{\Pi}$) annihilate v.

We denote by $W(C)$ and $W(D)$ the Weyl groups of C_n and D_n, respectively, and by ρ_C,ρ_D the corresponding half sums of positive roots. Also, by $\rho=\rho_0-\rho_1$ we denote the difference of the half sums of the positive even roots and the positive odd roots of $\mathfrak{g}$. We have:

$$\rho = \rho_C - \frac{1}{2}\sum_{i=1}^{n}\delta_i = \rho_D + \frac{1}{2}\sum_{i=1}^{n}\delta_i.$$

We denote by $L(\lambda)$ (resp., $M(\lambda)$) the irreducible (respectively, the Verma) $\mathfrak{g}$-module of highest weight λ. Also, by $L_{\widetilde{\Pi}}(\lambda)$ we denote the simple $\widetilde{\Pi}$-highest weight module with highest weight λ. We denote by $L_C(\lambda)$ (resp., $L_D(\lambda)$) an irreducible C_n (resp., D_n) module of the highest weight λ. We consider two shifted actions of $W(C)$ to $\mathfrak{h}^*$:

$$w\circ\lambda := w(\lambda+\rho)-\rho, \quad w\circ_C\lambda := w(\lambda+\rho_C)-\rho_C,$$

and the shifted action of $W(D)$: $w\circ_D\lambda := w(\lambda+\rho_D)-\rho_D$. For each $\lambda\in\mathfrak{h}^*$, $W(\lambda)$ stands for the corresponding integral Weyl group, i.e. the subgroup of W generated by the reflection r_α for the even roots α satisfying $2(\lambda,\alpha)\in\mathbb{Z}(\alpha,\alpha)$.

We say that a $\mathfrak{g}$-module M is a weight module if $M=\bigoplus_{\lambda\in\mathfrak{h}^*}M_\lambda$, where $M_\lambda=\{m\in M \mid hm=\lambda(h)m$ for every $h\in\mathfrak{h}\}$ is the λ-weight space. We call a weight module M bounded if there is C such that $\dim M_\lambda < C$ for all $\lambda\in\mathfrak{h}^*$. A weight λ is called $\mathfrak{g}$-bounded, or simply bounded, if $L(\lambda)$ is a bounded module. Analogously, we introduce the notions of $\mathfrak{sp}(2n)$-bounded modules and bounded weights.

For each root α we introduce $\alpha^\vee := 2\alpha/(\alpha,\alpha)$. Recall that for a simple Lie algebra an irreducible module of a highest weight module λ is finite-dimensional if and only if $(\lambda+\rho,\alpha^\vee)\in\mathbb{Z}_{>0}$ for each positive root α. A $\mathfrak{g}$-module $L(\lambda)$ is finite-dimensional if and only if $L_C(\lambda)$ is finite-dimensional, see [K1], Thm. 8. For each $\lambda\in\mathfrak{h}^*$ we set $\Delta(\lambda)=\{\alpha\in\Delta_C|\ (\lambda,\alpha^\vee)\in\mathbb{Z}\}$ and $\Delta(\lambda)^+=\Delta(\lambda)\cap\Delta^+$.

3. Bounded highest weight modules of $\mathfrak{osp}(1,2n)$

In this section we classify the bounded highest weight $\mathfrak{osp}(1,2n)$-modules and obtain their character formulae.

3.1. By [M], Lemma 9.1, $L_C(\lambda)$ is bounded if and only if

(i) $(\lambda,\delta_i-\delta_{i+1})\in\mathbb{Z}_{\geq 0}$ for $i=1,\dots,n-1$, and
(ii) either $(\lambda,\delta_n)\in\mathbb{Z}_{\geq 0}$ (then $L_C(\lambda)$ is finite-dimensional), or $(\lambda,\delta_n)\in\mathbb{Z}+1/2$ and $(\lambda,\delta_{n-1}+\delta_n)\in\mathbb{Z}_{\geq -2}$.

The above conditions can be rewritten as follows: $L_C(\lambda)$ is bounded if and only if $(\lambda+\rho_C,\alpha^\vee)\in\mathbb{Z}_{>0}$ for each $\alpha\in\Delta(\lambda)^+$ and $\Delta(\lambda)=\Delta_C$ (then $\dim L_C(\lambda)<\infty$) or $\Delta(\lambda)=\Delta_D$.

3.2. Lemma. *If $(\lambda+\rho_C,\alpha^\vee)\in\mathbb{Z}_{>0}$ and $\Delta(\lambda)=\Delta_D$ (equivalently, if $L_C(\lambda)$ is infinite-dimensional $\mathfrak{sp}(2n)$-bounded module), then*

$$e^{\rho_C}\operatorname{ch} L_C(\lambda)=\prod_{i=1}^n(1-e^{-2\delta_i})^{-1}e^{\rho_D}\operatorname{ch} L_D(\lambda+\rho_C-\rho_D),\quad \dim L_D(\lambda+\rho_C-\rho_D)<\infty.$$

Proof. Since $(\lambda+\rho_C,\alpha^\vee)\in\mathbb{Z}_{>0}$ for each $\alpha\in\Delta(\lambda)^+$, we have

$$\operatorname{ch} L_C(\lambda)=\sum_{w\in W(\Delta(\lambda))}(-1)^{l(w)}\operatorname{ch} M_C(y\circ_C\lambda). \tag{1}$$

If $\Delta(\lambda)=\Delta_D$, then $W(\Delta(\lambda))=W(D)$. The condition $(\lambda+\rho,\alpha^\vee)\in\mathbb{Z}_{>0}$ for each $\alpha\in\Delta(\lambda)^+$ is equivalent to $\dim L_D(\lambda+\rho_C-\rho_D)<\infty$. Combining (1), the Weyl character formula for $L_D(\lambda+\rho_C-\rho_D)$ and the formula $\operatorname{ch} M_C(\nu)=\prod_{i=1}^n(1-e^{-2\delta_i})^{-1}\operatorname{ch} M_D(\nu)$ we obtain the desired identity. □

3.3. Lemma 3.2 implies that for each $\mu\in\mathfrak{h}^*$

$$\dim L_C(\lambda)_\mu\leq\dim L_D(\lambda+\rho_C-\rho_D).$$

3.4. Lemma. *If $L(\lambda)$ is an infinite-dimensional bounded module, then $L_C(\lambda)$ and $L_C(\lambda-\delta_n)$ are bounded $\mathfrak{sp}(2n)$-modules.*

Proof. Since $L_C(\lambda)$ is an $\mathfrak{sp}(2n)$-subquotient of $L(\lambda)$, $L_C(\lambda)$ is $\mathfrak{sp}(2n)$-bounded. It remains to prove that $L_C(\lambda-\delta_n)$ is $\mathfrak{sp}(2n)$-bounded. Since $L(\lambda)$ is an infinite-dimensional bounded module, we have that $\Delta(\lambda)=\Delta_D$, in particular $(\lambda,\delta_n)\neq 0$. But then one easily checks that $X_{-\delta_n}v$ is a nonzero $\mathfrak{g}_{\bar 0}$-primitive vector in $L(\lambda)$, where v is a highest weight vector of $L(\lambda)$ and $X_{-\delta_n}$ is in $\mathfrak{g}^{-\delta_n}$. Hence $L(\lambda)$ has a $\mathfrak{g}_{\bar 0}$-subquotient isomorphic to $L_C(\lambda-\delta_n)$. □

3.5. Proposition.

(i) *The module $L(\lambda)$ is bounded if and only if $\dim L(\lambda)<\infty$ or $\Delta(\lambda)=\Delta_D$ and $(\lambda+\rho,\alpha^\vee)\in\mathbb{Z}_{>0}$ for each $\alpha\in\Delta(\lambda)^+$. In other words, $L(\lambda)$ is bounded if and only if $L_C(\lambda)$ is bounded and $(\lambda,\delta_{n-1}+\delta_n)\neq -2$.*

(ii) *If $L(\lambda)$ is bounded and infinite-dimensional, then $\dim L_D(\lambda+\rho-\rho_D)<\infty$ and*

$$e^{\rho}\operatorname{ch} L(\lambda)=\prod_{i=1}^n(1-e^{-\delta_i})^{-1}e^{\rho_D}\operatorname{ch} L_D(\lambda+\rho-\rho_D). \tag{2}$$

In particular, for each $\mu\in\mathfrak{h}^*$ one has

$$\dim L(\lambda)_\mu\leq\dim L_D(\lambda+\rho-\rho_D).$$

Proof. Assume that $L(\lambda)$ is bounded.

Since $L_C(\lambda)$ is a submodule of $L(\lambda)$, we conclude that $\Delta(\lambda)$ is Δ_C or Δ_D. If $\Delta(\lambda) = \Delta_C$, then $\dim L_C(\lambda) < \infty$, which is equivalent to $\dim L(\lambda) < \infty$.

Consider the case $\Delta(\lambda) = \Delta_D$, in particular, $L(\lambda)$ is infinite-dimensional. Since $L_C(\lambda)$ is bounded, for $i = 1, \dots, n-1$ we have

$$(\lambda + \rho, \delta_i - \delta_{i+1}) = (\lambda + \rho_C, \delta_i - \delta_{i+1}) \in \mathbb{Z}_{>0}.$$

Since $L(\lambda)$ is infinite dimensional, by Lemma 3.4, $L_C(\lambda - \delta_n)$ is bounded. This gives

$$(\lambda + \rho, \delta_{n-1} + \delta_n) = (\lambda - \delta_n + \rho_C, \delta_{n-1} + \delta_n) \in \mathbb{Z}_{>0}.$$

Since $\delta_i - \delta_{i+1}, i = 1, \dots, n-1$ and $\delta_{n-1} + \delta_n$ are simple roots of $\Delta(\lambda) = \Delta_D$, we have $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}$ for each $\alpha \in \Delta(\lambda)^+$.

Now let λ be such that $\Delta(\lambda) = \Delta_D$ and $(\lambda + \rho, \alpha^\vee) \in \mathbb{Z}_{>0}$ for each $\alpha \in \Delta(\lambda)^+$. This means that $W(\lambda) = W(D)$, λ is maximal in its $W(\lambda)$-orbit $W(\lambda) \circ \lambda$, and the stabilizer of λ is trivial. In particular, $\dim L_D(\lambda + \rho - \rho_D) < \infty$ and the Weyl character formula gives

$$e^{\rho_D} \operatorname{ch} L_D(\lambda + \rho - \rho_D) = \sum_{w \in W(D)} \operatorname{sgn} w \operatorname{ch} M_D(w(\lambda + \rho)).$$

On the other hand, by Corollary 5.5 we obtain

$$e^{\rho} \operatorname{ch} L(\lambda) = \sum_{w \in W(D)} \operatorname{sgn} w \operatorname{ch} M(w(\lambda + \rho)).$$

Combining the last two identities with the formula $\operatorname{ch} M(\nu) = \prod_{i=1}^n (1 - e^{-\delta_i})^{-1} \operatorname{ch} M_D(\nu)$ leads to (2). This implies $\dim L(\lambda)_\mu \leq \dim L_D(\lambda + \rho - \rho_D)$ as required. □

4. Bounded highest weight modules of $\mathfrak{osp}(1,2n)$: an alternative approach

In this section we establish the classification of bounded weights of $\mathfrak{g}$ in an alternative way. Namely, we will present every bounded $L(\lambda)$ as a subquotient of a tensor product of a bounded module and a finite-dimensional module.

By $\mathcal{D}(n)$ we will denote the Weyl algebra $\mathbb{C}[x_1, ..., x_n; \partial_1, ..., \partial_n]$ generated by x_i, ∂_j subject to the relations $x_i x_j - x_j x_i = \partial_i \partial_j - \partial_j \partial_i = 0$; $x_i \partial_j - \partial_j x_i = \delta_{ij}$. We consider $\mathcal{D}(n)$ as an associative superalgebra letting x_i and ∂_i to be odd. There are several ways to define a homomorphism $U(\mathfrak{g}) \to \mathcal{D}(n)$. In this paper we will use a presentation for which the δ_i- and $(-\delta_i)$-root vectors of $\mathfrak{g}$ act as $\frac{1}{\sqrt{2}} x_i$ and $\frac{1}{\sqrt{2}} \partial_i$, respectively.

Fix for convenience elements X_α in the α-root space of $\mathfrak{g}$ so that $[X_{\delta_i}, X_{\pm\delta_j}] = X_{\delta_i \pm \delta_j}$, $[X_{-\delta_i}, X_{-\delta_j}] = -X_{-\delta_i - \delta_j}$, $i \neq j$, $[X_{\pm\delta_i}, X_{\pm\delta_i}] = \pm 2 X_{\pm 2\delta_i}$. The complete list of relations $[X_\alpha, X_\beta] = c_{\alpha,\beta} X_{\alpha+\beta}$ can be found in §4, [F]. We also fix elements $h_{\delta_i - \delta_j} = [X_{\delta_i}, X_{-\delta_j}]$ and $h_{2\delta_i} = [X_{\delta_i}, X_{-\delta_i}] = [X_{2\delta_i}, X_{-2\delta_i}]$ in $\mathfrak{h}$.

The following proposition can be verified with a direct computation, [F].

4.1. Proposition. *The following correspondences define a homomorphism $\phi : U(\mathfrak{g}) \longmapsto \mathcal{D}(n)$ of associative superalgebras:*

$$\begin{aligned}
X_{\delta_i-\delta_j} &\longmapsto x_i\partial_j, i \neq j;\\
X_{2\delta_i} &\longmapsto \frac{1}{2}x_i^2;\\
X_{-2\delta_i} &\longmapsto -\frac{1}{2}\partial_i^2;\\
X_{\delta_i+\delta_j} &\longmapsto x_ix_j, i \neq j;\\
X_{-\delta_i-\delta_j} &\longmapsto -\partial_i\partial_j, i \neq j;\\
h_{\delta_i-\delta_j} &\longmapsto x_i\partial_i - x_j\partial_j, i \neq j;\\
h_{2\delta_i} &\longmapsto x_i\partial_i + \frac{1}{2};\\
X_{\delta_i} &\longmapsto \frac{1}{\sqrt{2}}x_i;\\
X_{-\delta_i} &\longmapsto \frac{1}{\sqrt{2}}\partial_i.
\end{aligned}$$

From the above proposition we easily find that the $\mathcal{D}(n)$-module $\mathbb{C}[x_1, ..., x_n]$, when considered as a $\mathfrak{g}$-module though the homomorphism ϕ, is isomorphic to $L_{\widetilde{\Pi}}(\frac{1}{2}(\delta_1 + \cdots + \delta_n))$.

4.2. Lemma. *Let N be positive integer and v be a highest weight vector of $L_{\widetilde{\Pi}}(-N(\delta_1 + \cdots + \delta_n))$. Then the vector*

$$u = x_1^{2N} \otimes v + \sum_{k=1}^{2N} c_{2N-k}x_1^{2N-k} \otimes X_{\delta_1}^k(v)$$

is a primitive vector of $L_{\widetilde{\Pi}}(\frac{1}{2}(\delta_1 + \cdots + \delta_n)) \otimes L_{\widetilde{\Pi}}(-N(\delta_1 + \cdots + \delta_n))$, where the scalars c_i are defined as follows:

$$\begin{aligned}
c_{2N} &= 1\\
c_{2N-2j} &= \frac{(2N-1)(2N-3)...(2N-(2j-1))}{j!}, j > 0\\
c_{2N-(2j+1)} &= -\sqrt{2}c_{2N-2j}, j \geq 0
\end{aligned}$$

Proof. The identity $X_{\delta_i-\delta_{i+1}}u = 0$ is straightforward. It remains to show that $X_{-\delta_1}u = 0$. We easily check that

$$X_{-\delta_1}(x_1^{2N-k} \otimes X_{\delta_1}^k(v)) = \begin{cases} \frac{2N-k}{\sqrt{2}}x_1^{2N-k-1} \otimes X_{\delta_1}^k(v)) + \frac{k}{2}x_1^{2N-k} \otimes X_{\delta_1}^{k-1}(v)) & \text{if } k \text{ is even}\\ \frac{2N-k}{\sqrt{2}}x_1^{2N-k-1} \otimes X_{\delta_1}^k(v)) - \left(\frac{k-1}{2} - N\right) x_1^{2N-k} \otimes X_{\delta_1}^{k-1}(v)) & \text{if } k \text{ is odd}\end{cases}$$

Using the above identities we easily complete the proof. □

4.3. Corollary. *If λ is such that $(\lambda,\delta_n)\in\frac{1}{2}+\mathbb{Z}$ and $(\lambda,\delta_{n-1}+\delta_n)\in\mathbb{Z}_{\geq -1}$, then λ is bounded.*

Proof. We first prove that for every nonnegative integer N, the weight

$$\lambda_N=\left(N-\frac{1}{2}\right)\delta_1+\cdots+\left(N-\frac{1}{2}\right)\delta_{n-1}+\left(-N-\frac{1}{2}\right)\delta_n$$

is bounded. For this we note that the $\mathfrak{g}$-module $L(\lambda_N)^\theta$ obtained by twisting $L(\lambda_N)$ by θ is isomorphic to $L_{\widetilde{\Pi}}\left(\left(N+\frac{1}{2}\right)\delta_1+\left(\frac{1}{2}-N\right)\delta_2+\cdots+\left(\frac{1}{2}-N\right)\delta_n\right)$. Then, the latter by Lemma 4.2 is a subquotient of the tensor product of the bounded $\mathfrak{g}$-module $L_{\widetilde{\Pi}}(\frac{1}{2}(\delta_1+\cdots+\delta_n))$ and the finite-dimensional $\mathfrak{g}$-module $L_{\widetilde{\Pi}}(-N(\delta_1+\cdots+\delta_n))$. Therefore $L(\lambda_N)^\theta$ is bounded and hence λ_N is bounded.

Now let λ be a weight for which $(\lambda,\delta_n)\in\frac{1}{2}+\mathbb{Z}$ and $(\lambda,\delta_{n-1}+\delta_n)\in\mathbb{Z}_{\geq -1}$ and let $\mu=\lambda+\frac{1}{2}\sum_{i=1}^n\delta_i$. Also, set for simplicity $\mu_i=(\mu,\delta_i)$. Then we have that $\mu_i\in\mathbb{Z}$, $\mu_1\geq\mu_2\geq\cdots\geq\mu_n$ and $\mu_{n-1}+\mu_n\geq 0$. In particular $\mu_1\geq 0$ and hence $\lambda'=\left(\mu_1-\frac{1}{2}\right)\sum_{i=1}^{n-1}\delta_i-\left(\mu_1+\frac{1}{2}\right)\delta_n$ is a bounded weight. But then $\lambda=\lambda'+\lambda''$ for the bounded weight λ' and for the dominant integral weight $\lambda''=\sum_{i=2}^n(\mu_i-\mu_1)\delta_i$. Thus λ is bounded. □

The above corollary together with Lemma 3.4 leads to an alternative proof of the classification of bounded weights in Proposition 3.5(i).

5. Appendix: characters of some highest weight modules

5.1. Conventions. In this appendix $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ will be a basic classical Lie superalgebra with a fixed triangular decomposition $\mathfrak{g}=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}$. By $\mathfrak{h}$ we denote the Cartan subalgebra of $\mathfrak{g}$, W will be the Weyl group, and $\Delta=\Delta_0\cup\Delta_1$, Δ^+ will be the set of all roots and all positive roots, respectively. We fix an even non-degenerate bilinear invariant form $(-,-)$ on $\mathfrak{h}^*$. We write $\nu\geq\mu$ for weights ν,μ in $\mathfrak{h}^*$ if $\nu-\mu\in\mathbb{Z}_{\geq 0}\Delta^+$. Like in the case $\mathfrak{g}=\mathfrak{osp}(1,2n)$, for the shifted action of W on $\mathfrak{h}^*$ we write

$$w\circ\lambda:=w(\lambda+\rho)-\rho,$$

where $\rho=\rho_0-\rho_1$ is the difference of the half sums of the positive even roots and the positive odd roots.

By sgn : $W\to\{\pm 1\}$ we denote the sign homomorphism. For each $\lambda\in\mathfrak{h}^*$, like one the case $\mathfrak{g}=\mathfrak{osp}(1,2n)$, $W(\lambda)$ denotes the corresponding integral Weyl group.

As before, by $M(\nu)$ (resp., $L(\nu)$) we denote the Verma (resp., irreducible) module of highest weight ν. We denote by $\dot{M}(\nu)$ (resp., $\dot{L}(\nu)$) the corresponding Verma (resp., irreducible) $\mathfrak{g}_0$-modules. We introduce another shifted action of the Weyl group W on $\mathfrak{h}^*$ by

$$w \circ_{\mathfrak{g}_0} \lambda = w(\lambda + \rho_0) - \rho_0.$$

5.2. Character formulae for typical highest weight modules. Let λ be a maximal element in its $W(\lambda)$ orbit $W(\lambda) \circ \lambda$ (i.e., $\lambda \geq w \circ \lambda$ for each $w \in W$). Recall that if $\mathfrak{g}$ is a semisimple Lie algebra, then for each $w \in W(\lambda)$ the character of an irreducible highest weight module $L(w \circ \lambda)$ is given by the Kazhdan-Lusztig character formula:

$$\operatorname{ch} L(w \circ \lambda) = \sum_{y \in W(\lambda)} a_y^w \operatorname{ch} M(y \circ \lambda), \tag{3}$$

and the coefficients a_y^w are given by in terms of the inverse Kazhdan-Lusztig polynomial for the Weyl group $W(\lambda)$ and the stabilizer

$$\operatorname{Stab}_W(\lambda + \rho) = \{w \in W |\ w(\lambda + \rho) = \lambda + \rho\} = \{w \in W(\lambda) |\ w \circ \lambda = \lambda\}.$$

Note that any weight $\nu \in \mathfrak{h}^*$ is of the form $w \circ \lambda$, where λ is the maximal element in $W(\nu) \circ \nu$ (in this case λ is maximal in $W \circ \lambda$) and $w \in W(\nu) = W(\lambda)$. Hence, (3) gives the character of any irreducible highest weight module.

The character formula (3) also holds for a basic classical Lie superalgebra $\mathfrak{g}$ in the case when λ is strongly typical (i.e., $(\lambda + \rho, \beta) \neq 0$ for each $\beta \in \Delta_1$) or some weakly atypical weights, see §5.5 below; this gives a character formula for all strongly typical highest weight modules. The coefficients a_y^w are determined by the same formulae as for the Lie algebras case (they depend on $W(\lambda)$ and $\operatorname{Stab}_W(\lambda + \rho)$). This result easily follows from the equivalence of categories established in [PS1], [PS2] and [G2], see details below in §5.4 and §5.5.

If λ is strongly typical, maximal in $W(\lambda) \circ \lambda$, and has the trivial stabilizer, then $a_y^e = \operatorname{sgn} y$. Hence (3) takes the form

$$\operatorname{ch} L(\lambda) = \sum_{y \in W(\lambda)} \operatorname{sgn}(y) \operatorname{ch} M(y \circ \lambda). \tag{4}$$

This holds, in particular, if $L(\lambda)$ is typical and finite-dimensional. In the latter case $W(\lambda) = W$ and (3) becomes the Weyl-Kac character formula established in [K2].

5.3. Typicality and strong typicality. Recall that $\lambda \in \mathfrak{h}^*$ is *typical* if $(\lambda + \rho, \beta) \neq 0$ for each isotropic root β. We call $\lambda \in \mathfrak{h}^*$ *strongly typical* if $(\lambda + \rho, \beta) \neq 0$ for each odd root β.

A central character $\chi : \mathcal{Z}(\mathcal{U}(\mathfrak{g})) \to \mathbb{C}$ is called typical (resp., strongly typical) if it is a central character of $L(\lambda)$ with typical (resp., strongly typical) λ. Note that this definition does not depend on the triangular decomposition in the following sense. If $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n} = \mathfrak{n}'_- \oplus \mathfrak{h} \oplus \mathfrak{n}'$ are two triangular decompositions such that $\mathfrak{n} \cap \mathfrak{g}_0 = \mathfrak{n}' \cap \mathfrak{g}_0$, and if λ is typical, then the Verma module $M(\lambda, \mathfrak{n})$ relative to the first triangular decomposition is isomorphic to a Verma module $M(\lambda', \mathfrak{n}')$ relative to the second triangular decomposition; in this case $\lambda+\rho=\lambda'+\rho'$. In particular, if λ is such that $(\lambda+\rho,\beta)\neq 0$ for all isotropic roots β, then $L(\lambda,\mathfrak{n})=L(\lambda',\mathfrak{n}')$, where $\lambda+\rho=\lambda'+\rho'$, so $(\lambda'+\rho',\beta)\neq 0$ for all isotropic roots β.

We call a $\mathfrak{g}$-module strongly typical if it has a strongly typical central character. The set of highest weights of irreducible highest weight modules with a fixed central character forms a single W-orbit if and only if this central character is typical: if $\chi:\mathcal{Z}(\mathcal{U}(\mathfrak{g}))\to\mathbb{C}$ is a central character, then $(\operatorname{Ker}\chi)L(\lambda)=(\operatorname{Ker}\chi)L(\lambda')=0$ implies $\lambda'+\rho\in W(\lambda+\rho)$ if and only if χ is typical.

5.4. Strongly typical case. Take any strongly typical central character χ. By [G2] Theorem 3.3.1, there exists a $\mathfrak{g}_0$-central character $\dot\chi:\mathcal{Z}(\mathcal{U}(\mathfrak{g}_0))\to\mathbb{C}$ such that the map $\Psi: N\mapsto N_{\dot\chi}:=\{v\in N|\ (\operatorname{Ker}\dot\chi)v=0\}$ provides an equivalence between the category of $\mathfrak{g}$-modules with the central character χ and the category of $\mathfrak{g}_0$-modules with the central character $\dot\chi$. The map Ψ maps a Verma $\mathfrak{g}$ module to a Verma $\mathfrak{g}_0$-module. Recall that $M(\nu), M(\lambda)$ (resp., $\dot M(\nu),\dot M(\lambda)$) have the same typical central character (resp., the same central character) if and only if $\nu\in W\circ\lambda$ (resp., $\nu\in W\circ_{\mathfrak{g}_0}\lambda$).

5.4.1. Lemma. *If $\Psi(M(\nu))=\dot M(\nu')$, then for each $w\in W$ one has*

$$\Psi(M(w\circ\nu))=\dot M(w\circ_{\mathfrak{g}_0}\nu').$$

Proof. Recall that $M(\lambda)$ has a filtration of $\mathfrak{g}_0$-modules with the factors $\{\dot M(\lambda-\gamma)\}_{\gamma\in\Gamma}$, where

$$\Gamma:=\left\{\sum_{\beta\in X}\beta\mid X\subset\Delta_1^+\right\}.$$

By [G1], Lemma 8.3.4(i), for each $M(\lambda)$ with central character χ, there exists a unique $\gamma\in\Gamma$ such that $\dot M(\lambda-\gamma)$ has the central character $\dot\chi$, and such that $\Psi(M(\lambda))=M(\lambda-\gamma)$. Thus it is enough to verify that for each $w\in W$ one has $w\circ\nu-w\circ_{\mathfrak{g}_0}\nu'\in\Gamma$ if $\nu-\nu'\in\Gamma$. Observe that

$$w\circ\nu-w\circ_{\mathfrak{g}_0}\nu'=w(\nu-\nu'-\rho_1)+\rho_1.$$

One readily sees that

$$\Gamma-\rho_1=\{\gamma-\rho_1|\ \gamma\in\Gamma\}=\left\{\sum_{\beta\in Y}\beta/2\mid Y\subset\Delta_1,\Delta_1=Y\sqcup(-Y)\right\}.$$

Since Δ_1 is W-invariant, $\Gamma-\rho_1$ is also W-invariant and thus $\nu-\nu'\in\Gamma$ implies $w\circ\nu-w\circ_{\mathfrak{g}_0}\nu'\in\Gamma$ as required. ∎

5.4.2. Proposition. *Let μ be strongly typical and a minimal element in its orbit $W(\mu)\circ\mu$. Let w_0 be the longest element in $W(\mu)$ and $\nu := w_0\circ\mu$. Then the character formula (3) holds for $\lambda=\nu$, i.e.*

$$\operatorname{ch} L(w\circ\nu)=\sum_{y\in W(\nu)} a_y^w \operatorname{ch} M(y\circ\nu).$$

Proof. Since $M(\mu)$ is an irreducible Verma module, the $\mathfrak{g}_0$-Verma module $\Psi(M(\mu))=\dot{M}(\mu')$ is also irreducible, that is μ' is minimal in its orbit $W(\mu')\circ_{\mathfrak{g}_0}\mu'$. Note that for each $\alpha\in\Delta_0$ one has $2(\alpha,\beta)/(\alpha,\alpha)\in\mathbb{Z}$ for all $\beta\in\Delta$ and thus for all $\beta\in\Gamma$. Hence $W(\mu')=W(\mu)$. Then $\nu=w_0\circ\mu$ (resp., $\nu':=w_0\circ_{\mathfrak{g}_0}\mu'$) is a (unique) maximal element in the orbit $W(\mu)\circ\mu$ (resp., $W(\mu)\circ_{\mathfrak{g}_0}\mu$). We have $W(\nu)=W(\mu)=W(\mu')=W(\nu')$. By (3), for each $w\in W(\nu)$ we have

$$\operatorname{ch}\dot{L}(w\circ_{\mathfrak{g}_0}\nu')=\sum_{y\in W(\nu)} a_y^w \operatorname{ch}\dot{M}(y\circ_{\mathfrak{g}_0}\nu').$$

Since Ψ is an equivalence of categories, $\Psi(M(y\circ\nu))=\dot{M}(y\circ_{\mathfrak{g}_0}\nu')$ for each $y\in W$ (by Lemma 5.4.1) and so $\Psi(L(y\circ\nu))=\dot{L}(y\circ_{\mathfrak{g}_0}\nu')$. A standard reasoning (see [BGG]) leads to

$$\operatorname{ch} L(w\circ\nu)=\sum_{y\in W(\nu)} a_y^w \operatorname{ch} M(y\circ\nu)$$

as needed. □

5.5. The special case $\mathfrak{g}=\mathfrak{osp}(1,2n)$. Let now $\mathfrak{g}:=\mathfrak{osp}(1,2n)$. Let $\nu\in\mathfrak{h}^*$ be such that $(\nu+\rho,\beta)=0$ for a unique odd positive root β and ν be the maximal element in its $W(\nu)$-orbit $W(\nu)\circ\nu$ (note that if $\lambda\in\mathfrak{h}^*$ is such that $(\lambda+\rho,\beta)=0$ for a unique odd positive root β, then $\lambda=w\circ\nu$, where ν is as above and $w\in W(\nu)$). We show that (3) holds for $\lambda=\nu$.

Let $\chi:\mathcal{Z}(\mathcal{U}(\mathfrak{g}))\to\mathbb{C}$ be the central character of $M(\nu)$. By [G2], Theorem 4.3 there exists a $\mathfrak{g}_0$-central character $\dot{\chi}:\mathcal{Z}(\mathcal{U}(\mathfrak{g}_0))\to\mathbb{C}$ such that the map $\Psi_0: N\mapsto N_{\bar{0},\dot{\chi}}:=\{v\in N_{\bar{0}}|\ (\operatorname{Ker}\dot{\chi})v=0\}$ provides an equivalence between the category of graded $\mathfrak{g}$-modules ($N=N_{\bar{0}}\oplus N_{\bar{1}}$) with the central character χ and the category of $\mathfrak{g}_0$-modules with the central character $\dot{\chi}$. The map Ψ_0 maps a graded Verma $\mathfrak{g}$-module to a Verma $\mathfrak{g}_0$-module.

Let μ be the minimal element in $W(\nu)\circ\nu$ (i.e., $\mu=w_0\circ\nu$, where w_0 is the longest element in $W(\nu)$). View $M(\mu)$ as a graded Verma module (with one of two possible gradings); note that $M(\mu)$ is irreducible, so $\Psi(M(\mu))=\dot{M}(\mu')$ is irreducible. Thus μ' is minimal in its $W(\mu')$-orbit $W(\mu')\circ_{\mathfrak{g}_0}\mu'$-orbit and $W(\nu)=W(\mu)=W(\mu')$. Arguing as in Lemma 5.4.1, we obtain that $\Psi^{-1}(\dot{M}(w\circ_{\mathfrak{g}_0}\mu'))$ is the Verma module $M(w\circ\mu)$ with one of two possible gradings (such that $M(w\circ\mu)_{w\circ_{\mathfrak{g}_0}\mu'}\subset M(w\circ\mu)_{\bar{0}}$). Now the argument in the proof of Proposition 5.4.2 shows that (3) holds for $\lambda=\nu$.

Corollary. *Let $\mathfrak{g} = \mathfrak{osp}(1, 2n)$, λ be maximal in its $W(\lambda)$-orbit $W(\lambda) \circ \lambda$, and the stabilizer $\mathrm{Stab}_W(\lambda + \rho)$ be trivial. Then the character formula (4) holds for λ.*

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Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019

E-mail address: thomas.ferguson@mavs.uta.edu

Department of Mathematics, Weizmann Institute of Science, Rehovot 76100, Israel

E-mail address: maria.gorelik@weizmann.ac.il

Department of Mathematics, University of Texas at Arlington, Arlington, TX 76019

E-mail address: grandim@uta.edu