

# THE MINIMAL PRIMITIVE SPECTRUM OF THE ENVELOPING ALGEBRA OF THE LIE SUPERALGEBRA $\mathfrak{osp}(1, 2l)$

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ABSTRACT. A well known theorem of Duflo, the “annihilation theorem”, claims that the annihilator of a Verma module in the enveloping algebra of a complex semisimple Lie algebra is centrally generated. For the Lie superalgebra  $\mathfrak{osp}(1, 2l)$ , this result does not hold. In this article, we introduce a “correct” analogue of the centre for which the annihilation theorem does hold in the case  $\mathfrak{osp}(1, 2l)$ . This substitute of the centre is the centralizer of the even part of the enveloping algebra. This algebra shares some nice properties with the centre. As a consequence of the annihilation theorem we obtain the description of the minimal primitive spectrum of the enveloping algebra of the Lie superalgebra  $\mathfrak{osp}(1, 2l)$ . We also deduce a criterium for a  $\mathfrak{osp}(1, 2l)$ -Verma module to be a direct sum of  $\mathfrak{sp}(2l)$ -Verma modules.

## 1. INTRODUCTION

**1.1.** The goal of this paper is to give a description of the minimal primitive spectrum of the enveloping algebra of the Lie superalgebra  $\mathfrak{osp}(1, 2l)$ . Recall that for a complex semisimple Lie algebra  $\mathfrak{k}$ , the minimal primitive spectrum of the enveloping algebra  $\mathcal{U}(\mathfrak{k})$  is equal to the set of ideals of the form  $\mathcal{U}(\mathfrak{k})\mathfrak{m}$  where  $\mathfrak{m}$  runs through the set of maximal ideals of the centre  $\mathcal{Z}(\mathfrak{k})$ . This result follows from two theorems of Duflo: the first one claiming that any primitive ideal is the annihilator of a simple highest weight module and the second one, called annihilation theorem in what follows, claiming that the annihilator of any Verma module is generated by its intersection with the centre of the enveloping algebra.

In [Mu2], Musson generalized the first theorem to the classical simple Lie superalgebras  $\mathfrak{g} \neq Q(n)$ . However, the annihilation theorem seemed to fail to be true for the Lie superalgebras. Thus in [GL], we gave a necessary and sufficient condition on a Verma module over the Lie superalgebra  $\mathfrak{g} := \mathfrak{osp}(1, 2l)$  for its annihilator to be centrally generated.

In this paper we show that the annihilation theorem remains actually true — at least for the case  $\mathfrak{osp}(1, 2l)$  — if we use a “correct” analogue of the centre  $\mathcal{Z}(\mathfrak{k})$ .

The role of  $\mathcal{Z}(\mathfrak{g})$  for the superalgebra  $\mathfrak{g} = \mathfrak{osp}(1, 2l)$  is played by the centralizer  $\mathcal{A}$  of the even part  $\mathcal{U}(\mathfrak{g})_0$  of the enveloping algebra  $\mathcal{U}(\mathfrak{g})$  (we believe that this is a general

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phenomenon). The algebra  $\mathcal{A}$  comes along with a canonical involution  $\sigma$  so that  $\mathcal{A}^\sigma$  is the supercentre of  $\mathcal{U}(\mathfrak{g})$ . This reflects the fact that the category of the (super) representations of a Lie superalgebra admits a canonical parity change involution  $\Pi$ .

Now, in terms of the pair  $(\mathcal{A}, \sigma)$  the main results of the paper look as follows:

1. *The annihilator of any Verma module over  $\mathfrak{g}$  is generated by its intersection with  $\mathcal{A}$ — see Theorem 6.2.*
2. *The set of Verma module annihilators in  $\mathcal{A}$  coincides with the set of maximal  $\sigma$ -invariant ideals of  $\mathcal{A}$ — see Proposition 6.1.1 (i).*

Note that  $\mathcal{A}$  is a polynomial algebra and  $\sigma$  is a reflection; thus the maximal  $\sigma$ -invariant ideals of  $\mathcal{A}$  are explicitly known.

For any simple highest weight module  $\tilde{V}$  there exists (see Proposition 6.1.1 (ii)) a simple Verma module  $\tilde{M}$  such that  $\text{Ann}_{\mathcal{A}} \tilde{V} \supseteq \text{Ann}_{\mathcal{A}} \tilde{M}$ . This implies that the minimal primitive spectrum of  $\mathcal{U}(\mathfrak{g})$  coincides with the set of Verma module annihilators. Therefore any minimal primitive ideal is generated by its intersection with  $\mathcal{A}$  (see Corollary 6.3).

Using Theorem 6.2 we give in Section 7 a necessary and sufficient condition on a Verma module to be a direct sum of  $\mathfrak{g}_0$ -Verma modules (where  $\mathfrak{g}_0$  is the even part of  $\mathfrak{g}$ ).

**1.2.** Let us point out the main steps of the proof of the main Theorem 6.2. As for the proof of the main result of [GL], the Parthasarathy–Ranga-Rao–Varadarajan (PRV) matrices play a crucial role.

**1.2.1.** In Section 4.1 we study the centralizer  $\mathcal{A}$  of  $\mathcal{U}(\mathfrak{g})_0$  in  $\mathcal{U}(\mathfrak{g})$ . We show that  $\mathcal{A} = \mathcal{Z}(\mathfrak{g}) \oplus \mathcal{Z}(\mathfrak{g})T$  for some element  $T$  satisfying  $T^2 \in \mathcal{Z}(\mathfrak{g})$  (this element appears in [ABF] and [Mu1]). Let  $t \in \mathcal{S}(\mathfrak{h})$  be the Harish-Chandra projection of  $T$ .

In [GL] we proved that the annihilator of a Verma module  $\tilde{M}(\mu)$  of the highest weight  $\mu$  is a centrally generated ideal iff  $t(\mu) \neq 0$ . In particular Theorem 6.2 obviously holds for  $\tilde{M}(\mu)$  such that  $t(\mu) \neq 0$ . Note that  $t$  is a product of different linear factors so the “degenerate” values of  $\mu$  form a union of hyperplanes in  $\mathfrak{h}^*$ .

**1.2.2.** The next important ingredient is the following variant of separation theorem Theorem 5.2.

*There exists an ad  $\mathfrak{g}_0$ -submodule  $\mathcal{K}$  of  $\mathcal{U}(\mathfrak{g})$  such that the multiplication map induces an isomorphism of  $\mathfrak{g}_0$ -modules  $\mathcal{K} \otimes \mathcal{A} \rightarrow \mathcal{U}(\mathfrak{g})$ .*

**1.2.3.** Since  $\mathcal{A}$  is a free rank two  $\mathcal{Z}(\mathfrak{g})$ -module the multiplicity of any simple  $\mathfrak{g}_0$ -module in the standard harmonics is twice as its multiplicity in  $\mathcal{K}$ . This fact, together with some results of [GL] about PRV matrices, proves the main theorem for  $\tilde{M}(\mu)$  such that  $t$  has at the point  $\mu$  a simple zero.

**1.2.4.** The above considerations allow us to reformulate the main theorem as the following claim (see Claim 6.6.6) about the ranks of PRV matrices.

*The corank of any PRV matrix is constant on the set*

$$\mathfrak{h}_{sd}^* := \{\mu \mid \widetilde{M}(\mu) \text{ is simple and } t(\mu) = 0\}.$$

**1.2.5.** To check the above claim we use our notion of generalized PRV matrices described in 3.2. The classical PRV matrices are constructed using the subspace of harmonics in the enveloping algebra. In generalized PRV matrices, we are allowed to substitute the harmonics by another linear subspace (see details in 3.2). The subspace we choose allows one to calculate the rank of the PRV matrices in the set  $\mathfrak{h}_{sd}^*$ .

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## 2. BACKGROUND

In this Section we fix the main notations we use throughout this paper. The notation  $\mathbb{N}^+$  will stand for the set of positive integers. The base field we are going to work with is  $\mathbb{C}$ .

**2.1.** For  $\mathbb{Z}_2$ -graded vector space  $M$  we denote by  $M_0$  its even part and by  $M_1$  its odd part.

Let  $\mathfrak{g}$  be the Lie superalgebra  $\mathfrak{osp}(1, 2l)$ ,  $l \geq 1$  (see Kac [K2] for a presentation of this Lie superalgebra by generators and relations). Denote by  $\mathfrak{g}_0$  the even part and by  $\mathfrak{g}_1$  the odd part of  $\mathfrak{g}$ . We recall that  $\mathfrak{g}_0 \simeq \mathfrak{sp}(2l)$ . Fix a Cartan subalgebra  $\mathfrak{h}$  in  $\mathfrak{g}_0$ . Denote by  $\Delta_0$  (resp.,  $\Delta_1$ ) the set of even (resp., odd) roots of  $\mathfrak{g}$ . Set  $\Delta = \Delta_0 \cup \Delta_1$ . Let  $\Delta_{irr}$  be the set of irreducible roots of  $\Delta$ . Then  $\Delta_{irr} = \overline{\Delta}_0 \cup \Delta_1$ , where  $\overline{\Delta}_0 := \Delta_0 \setminus 2\Delta_1$ .

Fix a basis of simple roots  $\pi$  of  $\Delta$ , and define correspondingly the sets  $\Delta^\pm, \Delta_0^\pm, \Delta_1^\pm, \overline{\Delta}_0^\pm, \Delta_{irr}^\pm$ . Denote by  $W$  the Weyl group of  $\Delta$ . Set

$$\rho_0 := \frac{1}{2} \sum_{\alpha \in \Delta_0^+} \alpha, \quad \rho_1 := \frac{1}{2} \sum_{\alpha \in \Delta_1^+} \alpha, \quad \rho := \rho_0 - \rho_1 = \frac{1}{2} \sum_{\alpha \in \Delta_{irr}^+} \alpha.$$

Introduce the standard partial order relation on  $\mathfrak{h}^*$ :  $\lambda \leq \mu \iff \mu - \lambda \in \mathbb{N}\pi$ . Denote by  $(-, -)$  the non-degenerate bilinear form on  $\mathfrak{h}^*$  coming from the restriction of the Killing form of  $\mathfrak{g}_0$  to  $\mathfrak{h}$ . Let  $\varphi : \mathfrak{h}^* \rightarrow \mathfrak{h}$  be the isomorphism given by  $\varphi(\lambda)(\mu) := (\lambda, \mu)$ . For any  $\lambda, \mu \in \mathfrak{h}^*$ ,  $(\mu, \mu) \neq 0$  one defines

$$\langle \lambda, \mu \rangle := 2 \frac{(\lambda, \mu)}{(\mu, \mu)}.$$

**2.2.** One has the following useful realization of  $\Delta$ . Identify  $\mathfrak{h}^*$  with  $\mathbb{C}^l$  and consider  $(-, -)$  as a scalar product on  $\mathbb{C}^l$ . Then there exists an orthonormal basis  $\{\beta_1, \dots, \beta_l\}$  such that

$$\begin{aligned} \pi &= \{\beta_1 - \beta_2, \dots, \beta_{l-1} - \beta_l, \beta_l\} & \pi_0 &= \{\beta_1 - \beta_2, \dots, \beta_{l-1} - \beta_l, 2\beta_l\} \\ \Delta_0^+ &= \{\beta_i \pm \beta_j, 1 \leq i < j \leq l, 2\beta_i, 1 \leq i \leq l\}, & \Delta_1^+ &= \{\beta_i, 1 \leq i \leq l\} \\ \Delta_{irr}^+ &= \{\beta_i \pm \beta_j, \beta_i, 1 \leq i < j \leq l\}, & \overline{\Delta}_0^+ &= \{\beta_i \pm \beta_j, 1 \leq i < j \leq l\} \\ \rho &= \sum_{i=1}^l (l - i + \frac{1}{2})\beta_i, & \rho_0 &= \sum_{i=1}^l (l - i + 1)\beta_i \end{aligned}$$

and the Weyl group  $W$  is just the group of the signed permutations of the  $\beta_i$ .

**2.3. Weyl group.** Define the translated action of  $W$  on  $\mathfrak{h}^*$  by the formula:

$$w.\lambda = w(\lambda + \rho) - \rho \quad \forall \lambda \in \mathfrak{h}^*, w \in W.$$

Define the left translated action of  $W$  on  $\mathcal{S}(\mathfrak{h})$  by setting  $w.f(\lambda) = f(w^{-1}.\lambda)$  for any  $\lambda \in \mathfrak{h}^*$ .

For  $\alpha \in \Delta_{irr}^+$  we denote by  $s_\alpha$  the reflection with respect to  $\alpha$ . Let  $D$  be the subgroup of  $W$  generated by the reflections with respect to the roots  $\alpha \in \overline{\Delta}_0^+$ . Note that  $D$  is isomorphic to the Weyl group of the root system  $D_l := \{\pm\beta_i \pm \beta_j, 1 \leq i < j \leq l\}$ . Moreover  $D$  is a subgroup of index 2 in  $W$  so it is normal.

Denote by  $S$  the subgroup of  $W$  generated by the reflections with respect to the simple even roots. Clearly  $S$  is isomorphic to the symmetric group  $S_l$ .

**2.4. Enveloping algebra.** As usual, if  $\mathfrak{k}$  is a Lie superalgebra,  $\mathcal{U}(\mathfrak{k})$  denotes its enveloping algebra. If  $V$  is a vector superspace, we denote by  $\mathcal{S}(V)$  its symmetric superalgebra endowed with the natural  $\mathbb{Z} \times \mathbb{Z}_2$ -grading.

Set  $\mathcal{F}$  the canonical filtration of  $\mathcal{U}(\mathfrak{g})$  defined by  $\mathcal{F}^{(n)}(\mathcal{U}(\mathfrak{g})) = (\mathfrak{g}^n)_{n \in \mathbb{N}}$ . The graded algebra of  $\mathcal{U}(\mathfrak{g})$  associated to  $\mathcal{F}$  is the symmetric superalgebra  $\mathcal{S}(\mathfrak{g}) \simeq \mathcal{S}(\mathfrak{g}_0) \otimes \wedge \mathfrak{g}_1$  which is not a domain. Nevertheless, Aubry and Lemaire proved in [AL] that  $\mathcal{U}(\mathfrak{g})$  is a domain.

We define the supercentre to be the vector subspace of  $\mathcal{U}(\mathfrak{g})$  generated by the homogeneous elements  $a$  such that  $ax = (-1)^{|a||x|}xa$  for all homogeneous elements  $x$  in  $\mathcal{U}(\mathfrak{g})$ . For  $\mathfrak{g} = \mathfrak{osp}(1, 2l)$ , the supercentre coincides with the genuine centre.

The Lie superalgebra  $\mathfrak{g}$  acts on  $\mathcal{U}(\mathfrak{g})$  and  $\mathcal{S}(\mathfrak{g})$  by superderivation via the adjoint action. We denote these actions by  $\text{ad}$ . Throughout this paper, an action of any element of  $\mathfrak{g}$  on  $\mathcal{U}(\mathfrak{g})$  means always the adjoint action.

We identify  $\mathcal{U}(\mathfrak{h})$  with  $\mathcal{S}(\mathfrak{h})$ .

**2.5.** Let  $M$  be a  $\mathfrak{g}$ -module. For any  $\lambda \in \mathfrak{h}^*$ , set

$$M_\lambda = \{m \in M \mid hm = \lambda(h)m, \forall h \in \mathfrak{h}\}.$$

For  $\lambda = 0$ , we use notation  $M|_0$  in order to prevent confusion with the corresponding homogeneous component.

A non-zero vector  $v \in M$  has weight  $\lambda$  if  $v \in M_\lambda$ . For any subspace  $N$  of  $M$  we denote by  $\Omega(N)$  the set of weights  $\lambda \in \mathfrak{h}^*$  such that  $N \cap M_\lambda \neq \{0\}$ . The module  $M$  is said to be diagonalizable if  $M = \bigoplus_{\lambda \in \mathfrak{h}^*} M_\lambda$ . If  $M$  is a diagonalizable module and  $\dim M_\lambda < \infty$  for all  $\lambda \in \mathfrak{h}^*$ , we set  $\text{ch } M = \sum_{\lambda \in \mathfrak{h}^*} \dim M_\lambda e^\lambda$ .

If  $M$  is a completely reducible  $\mathfrak{g}$  (resp.,  $\mathfrak{g}_0$ ) module and  $V$  is a simple  $\mathfrak{g}$  (resp.,  $\mathfrak{g}_0$ ) module, we shall denote by  $[M : V]$  the multiplicity of  $V$  in  $M$ .

We say that element  $a \in \mathcal{U}(\mathfrak{g})$  acts by a scalar on the subspace  $N$  of a  $\mathcal{U}(\mathfrak{g})$ -module if there exists  $c \in \mathbb{C}$  such that  $av = cv$  for any  $v \in N$ .

**2.6. Harish-Chandra projection.** For any  $\alpha \in \Delta$ , let  $\mathfrak{g}_\alpha$  be the subspace of weight  $\alpha$  of  $\mathfrak{g}$  (which is always one-dimensional). Set  $\mathfrak{n}^\pm = \bigoplus_{\alpha \in \Delta^\pm} \mathfrak{g}_\alpha$ ,  $\mathfrak{b}^\pm = \mathfrak{h} \oplus \mathfrak{n}^\pm$ .

The Harish-Chandra projection  $\Upsilon : \mathcal{U}(\mathfrak{g}) \rightarrow \mathcal{U}(\mathfrak{h})$  is the projection with respect to the following triangular decomposition  $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{h}) \oplus (\mathcal{U}(\mathfrak{g})\mathfrak{n}^+ + \mathfrak{n}^-\mathcal{U}(\mathfrak{g}))$ .

Clearly  $\Upsilon(\mathcal{U}(\mathfrak{g})_\mu) = 0$  for any  $\mu \neq 0$ . An element  $a$  of  $\mathcal{U}(\mathfrak{g})|_0$  acts on a primitive vector of weight  $\mu$  ( $\mu \in \mathfrak{h}^*$ ) by multiplication by the scalar  $\Upsilon(a)(\mu)$ . Thus the restriction of  $\Upsilon$  on  $\mathcal{U}(\mathfrak{g})|_0 = \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$  is an algebra homomorphism from  $\mathcal{U}(\mathfrak{g})|_0$  to  $\mathcal{U}(\mathfrak{h})$ .

The restriction of  $\Upsilon$  on  $\mathcal{Z}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})^{\mathfrak{g}}$  is an algebra isomorphism from  $\mathcal{Z}(\mathfrak{g})$  onto  $\mathcal{U}(\mathfrak{h})^W$ . Moreover, Musson proved in [Mu1], 5.3 that the restriction of  $\Upsilon$  on  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is an algebra isomorphism from  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  onto  $\mathcal{S}(\mathfrak{h})^S$ .

**2.7. Graded Verma modules.** For a fixed  $\lambda \in \mathfrak{h}^*$ , let  $\tilde{\mathcal{C}}_\lambda(\bar{i})$  ( $\bar{i} \in \mathbb{Z}_2$ ) be the one dimensional space of degree  $\bar{i}$  endowed with the structure of  $\mathfrak{b}^+$ -module through  $\mathfrak{n}^+v = 0$  and  $hv = \lambda(h)v$  for all  $h \in \mathfrak{h}$  and  $v \in \tilde{\mathcal{C}}_\lambda(\bar{i})$ .

Set

$$\tilde{M}(\lambda, \bar{i}) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{b}^+)} \tilde{\mathcal{C}}_\lambda(\bar{i}).$$

The Verma module  $\tilde{M}(\lambda, \bar{i})$  has a unique simple quotient denoted by  $\tilde{V}(\lambda, \bar{i})$ .

Denote by  $\Pi$  the parity change functor. Then  $\Pi(\tilde{M}(\lambda, \bar{i})) = \tilde{M}(\lambda, \bar{i} + 1)$ .

**2.7.1.** Clearly as  $\mathfrak{g}$ -modules  $\tilde{M}(\lambda, \bar{i})$  are canonically isomorphic for  $\bar{i} \in \mathbb{Z}_2$ . We denote this  $\mathfrak{g}$ -module by  $\tilde{M}(\lambda)$  and by  $\tilde{V}(\lambda)$  its unique simple quotient. Set

$$\mathfrak{h}_s^* := \{\mu \in \mathfrak{h}^* \mid \tilde{M}(\mu) = \tilde{V}(\mu)\}.$$

**2.7.2.** By [Mu3] 2.4,  $\mu \in \mathfrak{h}^* \setminus \mathfrak{h}_s^*$  iff there exists an irreducible positive root  $\alpha$  such that  $n := \langle \mu + \rho, \alpha \rangle \in \mathbb{N}^+$  and  $n$  is odd if  $\alpha$  is odd. Moreover for  $\alpha \in \Delta_{irr}^+$  one has  $\widetilde{M}(\mu) \supset \widetilde{M}(s_\alpha \cdot \mu)$  iff  $\langle \mu + \rho, \alpha \rangle = n \in \mathbb{N}^+$  and  $n$  is odd if  $\alpha$  is odd.

Therefore for the graded Verma modules we have

$$\begin{aligned} \widetilde{M}(\mu, \bar{i}) \supset \widetilde{M}(s_\alpha \cdot \mu, \bar{i}) &\iff \langle \mu + \rho, \alpha \rangle \in \mathbb{N}^+, & \alpha \in \overline{\Delta}_0^+ \\ \widetilde{M}(\mu, \bar{i}) \supset \Pi(\widetilde{M}(s_\alpha \cdot \mu, \bar{i})) &\iff \langle \mu + \rho, \alpha \rangle \in (2\mathbb{N} + 1), & \alpha \in \Delta_1^+. \end{aligned} \quad (1)$$

As in the classical case, any (graded) Verma module contains a simple (graded) Verma submodule.

**2.8. Finite dimensional representations.** Define for  $r \in \{1, \dots, l\}$  the fundamental weight  $\omega_r = \sum_{i=1}^r \beta_i$ , and introduce the set

$$P^+(\pi) := \sum_{r=1}^l \mathbb{N}\omega_r = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \beta_i \rangle \in 2\mathbb{N}, \langle \lambda, \beta_i - \beta_{i+1} \rangle \in \mathbb{N}, \forall i = 1, \dots, l-1\}.$$

Kac (see [K1]) showed that  $\widetilde{V}(\lambda)$  is finite dimensional iff  $\lambda \in P^+(\pi)$ . For any  $\lambda \in \mathfrak{h}^*$ , let  $V(\lambda)$  be the simple  $\mathfrak{g}_0$ -module of highest weight  $\lambda$ . Remark that  $\{\beta_1 - \beta_2, \dots, \beta_{l-1} - \beta_l, 2\beta_l\}$  is a basis of simple roots of  $\Delta_0$  and that  $\langle \mu, 2\beta_l \rangle = \frac{1}{2}\langle \mu, \beta_l \rangle$  for all  $\mu \in \mathfrak{h}^*$ . Thus  $V(\lambda)$  is finite dimensional iff  $\lambda \in P^+(\pi)$ .

**2.9. The separation theorem.** Recall the separation theorem established by Musson in [Mu1], 1.4:

**Theorem.** *There exists an ad-invariant subspace  $\mathcal{H}$  in  $\mathcal{U}(\mathfrak{g})$  such that the multiplication map induces an ad  $\mathfrak{g}$ -isomorphism  $\mathcal{U}(\mathfrak{g}) \simeq \mathcal{Z}(\mathfrak{g}) \otimes \mathcal{H}$ . Moreover, for every simple finite dimensional module  $\widetilde{V}$ ,  $[\mathcal{H} : \widetilde{V}] = \dim \widetilde{V}|_0$ .*

**2.10.** We use the following fact: if  $B$  is a Zariski dense subset of  $\mathfrak{h}^*$  then

$$\bigcap_{\mu \in B} \text{Ann } \widetilde{M}(\mu) = 0$$

see [J1], 7.1

### 3. PRV MATRICES

In the first part of this Section we recall the standard definition of PRV matrices and the main results of [GL]. In the second part of the Section we define generalized PRV matrices. They provide a powerful tool for the calculation of the rank of the standard PRV matrices which we use in the proof of our main Theorem 6.2.

**3.1.** The separation theorem 2.9 leads to the following construction. Fix  $\nu \in P^+(\pi)$  and set  $n := \dim \tilde{V}(\nu)|_0$ .

Let  $\{\theta_1, \dots, \theta_n\}$  be a basis of  $\text{Hom}_{\mathfrak{g}}(\tilde{V}(\nu), \mathcal{H})$  and  $\{v_1, \dots, v_n\}$  be a basis of  $\tilde{V}(\nu)|_0$ . Define a matrix  $PRV^\nu$  by the formula

$$(PRV^\nu)_{ij=1}^n := \left( \Upsilon(\theta_j(v_i)) \right)_{ij}.$$

Remark that the entries of the matrix  $PRV^\nu$  are elements of  $\mathcal{S}(\mathfrak{h})$ .

The matrix  $PRV^\nu$  has the following property:  $j^{\text{th}}$  column of this matrix is zero at the point  $\mu \in \mathfrak{h}^*$  iff  $\theta_j(\tilde{V}(\nu)) \subset \text{Ann } \tilde{V}(\mu)$ — see Lemma 7.2 in [J1].

For any change of the above bases (and of the space of harmonics  $\mathcal{H}$ ), the new matrix is of the form  $M(PRV^\nu)N$  where  $M, N$  are invertible matrices. Therefore, the corank of the  $PRV^\nu(\mu)$  is correctly defined for each  $\mu \in \mathfrak{h}^*$  and  $\det PRV^\nu \in \mathcal{S}(\mathfrak{h})$  is defined up to a non-zero scalar. Moreover

$$\text{corank } PRV^\nu(\mu) = [\text{Ann}_{\mathcal{H}} \tilde{V}(\mu) : \tilde{V}(\nu)] \quad (2)$$

**3.1.1.** In [GL] we decomposed  $\det PRV^\nu$  into linear factors and showed that

$$\det PRV^\nu = t^{r_\nu} P \quad (3)$$

where  $P \in \mathcal{S}(\mathfrak{h})$  is a polynomial (depending from  $\nu$ ) which has no zeroes in the set  $\mathfrak{h}_s^*$  and the polynomial  $t$  and the number  $r_\nu$  are given by the formulas

$$t(\mu) := \prod_{\beta \in \Delta_1^+} (\beta, \mu + \rho) \quad \forall \mu \in \mathfrak{h}^*$$

$$r_\nu := \sum_{n=1}^{\infty} (-1)^{n+1} \dim \tilde{V}(\nu)_{n\beta_1}.$$

Note that  $r_\nu$  is a non-negative number for any  $\nu \in P^+(\pi)$ . Introduce

$$\mathfrak{h}_d^* := \{\mu \in \mathfrak{h}^* \mid t(\mu) = 0\},$$

$$\mathfrak{h}_{sd}^* := \mathfrak{h}_d^* \cap \mathfrak{h}_s^*.$$

As a corollary of the formula (3) we obtained the following theorem.

**3.1.2. Theorem.** *The annihilator of the Verma module  $\tilde{M}(\mu)$  is generated by its intersection with the centre  $\mathcal{Z}(\mathfrak{g})$  iff  $\mu \notin \mathfrak{h}_d^*$ .*

**3.1.3.** We also proved in [GL] that for any  $\mu \in \mathfrak{h}_d^*$  and  $\nu \in P^+(\pi)$

$$\text{corank } PRV^\nu(\mu) \geq r_\nu. \quad (4)$$

For any  $\mu \in \mathfrak{h}^*$  set

$$d(\mu) := \text{Card}\{\beta \in \Delta_1^+, (\mu + \rho, \beta) = 0\}.$$

Then (3) implies that the order of zero of  $\det PRV^\nu$  at a point  $\mu \in \mathfrak{h}_s^*$  is equal to  $r_\nu d(\mu)$ . Hence (4) forces the equalities

$$\text{corank } PRV^\nu(\mu) = r_\nu \quad \forall \nu \in P^+(\pi) \quad (5)$$

for any  $\mu \in \mathfrak{h}_s^*$  such that  $d(\mu) = 1$ .

**3.2. Matrices  $PRV_\mu^\nu$ .** Fix  $\mu \in \mathfrak{h}^*$  and set

$$\mathfrak{m} := \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu).$$

Let  $\mathcal{L}$  be an ad  $\mathfrak{g}$ -submodule of  $\mathcal{U}(\mathfrak{g})$  such that

$$\mathcal{U}(\mathfrak{g}) = \mathcal{L} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{m}. \quad (6)$$

By Theorem 2.9, one has the following isomorphisms of  $\mathfrak{g}$ -modules

$$\mathcal{L} \cong \mathcal{U}(\mathfrak{g}) / (\mathcal{U}(\mathfrak{g})\mathfrak{m}) = (\mathcal{H} \otimes \mathcal{Z}(\mathfrak{g})) / (\mathcal{H} \otimes \mathfrak{m}) \cong \mathcal{H}.$$

Thus  $[\mathcal{L} : \widetilde{V}(\nu)] = \dim \widetilde{V}(\nu)|_0$  for all  $\nu \in P^+(\pi)$ . This allows us to define a matrix  $PRV_\mu^\nu$  as follows.

Fix  $\nu \in P^+(\pi)$ . Let  $\{\theta_1, \dots, \theta_n\}$  be a basis of  $\text{Hom}_{\mathfrak{g}}(\widetilde{V}(\nu), \mathcal{L})$  and  $\{v_1, \dots, v_n\}$  be a basis of  $\widetilde{V}(\nu)|_0$ . Define a matrix  $PRV_\mu^\nu$  by the formula

$$(PRV_\mu^\nu)_{ij=1}^n := \left( \Upsilon(\theta_j(v_i)) \right)_{ij}.$$

**3.2.1.** Let us compare matrices  $PRV^\nu$  and  $PRV_\mu^\nu$ .

Denote by  $\psi$  a canonical map from  $\mathcal{U}(\mathfrak{g})$  to  $\mathcal{U}(\mathfrak{g}) / (\mathcal{U}(\mathfrak{g})\mathfrak{m})$ . Since  $\mathfrak{m}$  is an ideal of  $\mathcal{Z}(\mathfrak{g})$ ,  $\psi$  is a homomorphism of  $\mathfrak{g}$ -modules and of algebras. By (6), the restriction of  $\psi$  on  $\mathcal{L}$  and on  $\mathcal{H}$  are isomorphisms onto  $\psi(\mathcal{U}(\mathfrak{g}))$ . Therefore there exists a basis  $\{\eta_1, \dots, \eta_n\}$  of  $\text{Hom}_{\mathfrak{g}}(\widetilde{V}(\nu), \mathcal{H})$  such that

$$\psi(\eta_j(v)) = \psi(\theta_j(v)), \quad \forall j = 1, \dots, n \quad (7)$$

where  $v$  is a highest weight vector of  $\widetilde{V}(\nu)$ . Since  $\psi$  is a homomorphism of  $\mathfrak{g}$ -modules, it implies that

$$\psi(\eta_j(v_i)) = \psi(\theta_j(v_i)), \quad \forall i, j = 1, \dots, n. \quad (8)$$

By Theorem 2.9, there exist elements  $z_{pj} \in \mathcal{Z}(\mathfrak{g})$  such that

$$\theta_j(v) = \sum_p \eta_p(v) z_{pj}, \quad \forall j = 1, \dots, n. \quad (9)$$

Every  $z_{pj}$  is ad-invariant so  $\theta_j(v_i) = \sum_p \eta_p(v_i) z_{pj}$  and therefore

$$\Upsilon(\theta_j(v_i)) = \sum_p \Upsilon(\eta_p(v_i)) \Upsilon(z_{pj}), \quad \forall i, j = 1, \dots, n$$

since the restriction of  $\Upsilon$  on  $\mathcal{U}(\mathfrak{g})|_0$  is an algebra homomorphism.



The matrix  $(\Upsilon(\eta_j(v_i)))_{ij}$  is a usual  $PRV^\nu$  matrix. Hence

$$PRV_\mu^\nu = PRV^\nu(\Upsilon(z_{pj})). \quad (10)$$

Since  $\mathcal{U}(\mathfrak{g})\mathfrak{m} \subset \text{Ann } \widetilde{M}(\mu)$  it follows that  $\Upsilon(\ker \psi) = \Upsilon(\mathcal{U}(\mathfrak{g})\mathfrak{m}) = \mathfrak{m}_\mu$  where  $\mathfrak{m}_\mu = \{Q \in \mathcal{S}(\mathfrak{h}) \mid Q(\mu) = 0\}$  is a maximal ideal in  $\mathcal{S}(\mathfrak{h})$ . Comparing (7) and (9), we conclude that  $\psi(z_{pj}) = \delta_p^j$  and so  $\Upsilon(z_{pj}) = \delta_p^j$  modulo  $\mathfrak{m}_\mu$  for all  $p, j = 1, \dots, n$ . Then the formula (10) implies that  $PRV_\mu^\nu(\mu) = PRV^\nu(\mu)$  and moreover  $\det PRV_\mu^\nu = \det PRV^\nu(1 + Q)$  where  $Q \in \mathfrak{m}_\mu$  that is  $Q(\mu) = 0$ . By (3),  $\det PRV^\nu = t^{r\nu}P$  where  $P$  is a polynomial which has no zeroes on the set  $\mathfrak{h}_d^*$ . We can summarize our conclusions as follows:

**3.2.2. Lemma.** *For any  $\nu \in P^+(\pi)$  one has*

- (i)  $\text{corank } PRV_\mu^\nu(\mu) = \text{corank } PRV^\nu(\mu)$ .
- (ii)  $\det PRV_\mu^\nu = t^{r\nu}P_1$  where  $P_1 \in \mathcal{S}(\mathfrak{h})$  is a polynomial such that  $P_1(\mu) \neq 0$ .

#### 4. THE CENTRALIZER OF $\mathcal{U}(\mathfrak{g})_0$ IN $\mathcal{U}(\mathfrak{g})$

**4.1.** In this section we describe the centralizer  $\mathcal{A}$  of  $\mathcal{U}(\mathfrak{g})_0$  in  $\mathcal{U}(\mathfrak{g})$  that is

$$\mathcal{A} := \{a \in \mathcal{U}(\mathfrak{g}) \mid \forall b \in \mathcal{U}(\mathfrak{g})_0 \ ab = ba\}.$$

This algebra admits another natural description:

**Lemma.** *The algebra  $\mathcal{A}$  is the set of elements in  $\mathcal{U}(\mathfrak{g})$  which act by scalars on each homogeneous component of any simple highest weight graded module that is*

$$\mathcal{A} = \{a \in \mathcal{U}(\mathfrak{g}) \mid \forall(\mu \in \mathfrak{h}^*, \bar{i}, \bar{j} \in \mathbb{Z}_2) \ \exists c \in \mathbb{C} \text{ s.t. } (a - c)\widetilde{V}(\mu, \bar{i})_{\bar{j}} = 0\}.$$

*Proof.* Assume that  $a \in \mathcal{U}(\mathfrak{g})$  acts by scalars on each homogeneous component of any simple highest weight graded module. Then for any  $b \in \mathcal{U}(\mathfrak{g})_0$ , the element  $(ab - ba)$  acts by zero on any simple highest weight module so  $ab - ba = 0$  by 2.10. Conversely, assume that  $a \in \mathcal{A}$ . Note that  $\mathfrak{h} \subset \mathcal{U}(\mathfrak{g})_0$ , so  $\mathcal{A} \subset \mathcal{U}(\mathfrak{g})^{\mathfrak{h}}$  and therefore any weight subspace of any  $\mathfrak{g}$ -module is  $\mathcal{A}$ -stable. Thus any weight subspace of any simple highest weight graded module contains an  $a$ -eigenvector, since it is finite dimensional. Taking into account, that each homogeneous component of any graded simple highest weight module is a simple  $\mathcal{U}(\mathfrak{g})_0$ -module, we conclude that  $a$  acts by a scalar on whole homogeneous component as required.  $\square$

**4.2. Lemma.**  $\Upsilon(\mathcal{A}) \subseteq \mathcal{S}(\mathfrak{h})^D$ .

*Proof.* Recall that  $D$  is the subgroup of  $W$  generated by the reflections  $s_\alpha, \alpha \in \overline{\Delta}_0^+$ . Fix  $\alpha \in \overline{\Delta}_0^+$  and  $a \in \mathcal{A}$ . The homogeneous component  $\widetilde{M}(\mu, \bar{i})_{\bar{i}}$  is generated as  $\mathcal{U}(\mathfrak{g})_0$ -module by a primitive highest weight vector of weight  $\mu$  and so  $(a - \Upsilon(a)(\mu))\widetilde{M}(\mu, \bar{i})_{\bar{i}} = 0$ . Taking into account that  $\widetilde{M}(\mu, \bar{i}) \supset \widetilde{M}(s_\alpha \cdot \mu, \bar{i})$  if  $\langle \mu + \rho, \alpha \rangle \in \mathbb{N}^+$  (see 2.7.2), we conclude that

$$\Upsilon(a)(\mu) = \Upsilon(a)(s_\alpha \cdot \mu) = s_\alpha \cdot (\Upsilon(a))(\mu), \quad \forall \mu \text{ s.t. } \langle \mu + \rho, \alpha \rangle \in \mathbb{N}^+.$$

Since both  $\Upsilon(a)$  and  $s_\alpha \cdot (\Upsilon(a))$  are elements of  $\mathcal{S}(\mathfrak{h})$  and the set  $\{\mu \in \mathfrak{h}^* \mid \langle \mu + \rho, \alpha \rangle \in \mathbb{N}^+\}$  is a Zariski dense subset of  $\mathfrak{h}^*$ , we conclude that  $\Upsilon(a) = s_\alpha \cdot (\Upsilon(a))$  as required.  $\square$

**4.3.** Since  $D$  is a subgroup of index 2 in  $W$ ,  $W$  acts on  $\mathcal{S}(\mathfrak{h})^D$  by the identity and by a non trivial involution denoted by  $\sigma$ . Note that for any odd root  $\beta$  one has

$$s_\beta \cdot \Upsilon(a) = \sigma(\Upsilon(a)) \quad \forall a \in \mathcal{A}.$$

The following proposition describes the action of  $\mathcal{A}$  on the graded highest weight simple modules  $\widetilde{V}(\mu, \bar{i})$ .

**4.3.1. Proposition.** *For any  $\mu \in \mathfrak{h}^*, i \in \mathbb{Z}_2$  and  $a \in \mathcal{A}$  one has*

$$a \text{ acts by the scalar } \begin{cases} \Upsilon(a)(\mu) & \text{on } \widetilde{V}(\mu, \bar{i})_{\bar{i}} \\ \sigma(\Upsilon(a))(\mu) & \text{on } \widetilde{V}(\mu, \bar{i})_{\bar{i}+1} \end{cases}$$

*Proof.* It is clear that  $(a - \Upsilon(a)(\mu))\widetilde{V}(\mu, \bar{i})_{\bar{i}} = 0$ . Let us verify that

$$(a - \sigma(\Upsilon(a)(\mu)))\widetilde{V}(\mu, \bar{i})_{\bar{i}+1} = 0.$$

Fix  $a \in \mathcal{A}$  and let  $\phi$  be the map  $\mathfrak{h}^* \setminus \{0\} \rightarrow \mathbb{C}$  such that  $a$  acts by the scalar  $\phi(\mu)$  on  $\widetilde{V}(\mu, \bar{i})_{\bar{i}+1}$ . We shall first show that  $\phi$  is a restriction of a polynomial function.

For each  $\mu \in \mathfrak{h}^*, \bar{i} \in \mathbb{Z}_2$  choose a highest weight vector  $v_\mu \in \widetilde{V}(\mu, \bar{i})$ . Fix  $\beta \in \Delta_1^+$  and choose  $y \in \mathfrak{g}_{-\beta} \setminus \{0\}$  and  $x \in \mathfrak{g}_\beta$  such that  $[y, x](\lambda) = (\beta, \lambda)$  for any  $\lambda \in \mathfrak{h}^*$ . Then  $yv_\mu \in \widetilde{V}(\mu, \bar{i})_{\bar{i}+1}$  so

$$xayv_\mu = \phi(\mu)xyv_\mu = \phi(\mu)(\beta, \mu)v_\mu.$$

Since  $xay \in \mathcal{U}(\mathfrak{g})|_0$  one has  $xayv_\mu = \Upsilon(xay)(\mu)v_\mu$ . Thus

$$\phi(\mu)(\beta, \mu) = \Upsilon(xay)(\mu). \tag{11}$$

This implies that  $\Upsilon(xay)(\mu)$  vanishes on whole hyperplane  $(\beta, \mu) = 0$ , which means that  $(\beta, \mu)$  divides  $\Upsilon(xay)(\mu)$  and therefore  $\phi$  is a restriction of the polynomial function  $\Upsilon(xay)(\mu)/(\beta, \mu)$ .

Assume now that  $\mu$  is such that  $\langle s_\beta \cdot \mu + \rho, \beta \rangle \in (2\mathbb{N} + 1)$  for some  $\beta \in \Delta_1^+$ . In that case, by 2.7.2, one has  $\Pi(\widetilde{M}(s_\beta \cdot \mu, \bar{i})) \supset \widetilde{M}(\mu, \bar{i})$  that is  $\widetilde{V}(\mu, \bar{i})$  is a subquotient of  $\Pi\widetilde{M}(s_\beta \cdot \mu, \bar{i})$  so

$$\phi(\mu) = \Upsilon(a)(s_\beta \cdot \mu) = (s_\beta \cdot \Upsilon(a))(\mu) = \sigma(\Upsilon(a))(\mu).$$

Since  $\{\mu \text{ s.t. } \exists \beta \in \Delta_1^+, \langle \mu + \rho, \beta \rangle \in (2\mathbb{N} + 1)\}$  is Zariski dense in  $\mathfrak{h}^*$  the proposition follows.  $\square$

**4.4.** For  $i = 1, \dots, l$  let  $x_i \in \mathcal{S}(\mathfrak{h})$  be the function which maps  $\mu$  to  $(\mu + \rho, \beta_i)$ . With these notations,  $W$  acts on  $\mathcal{S}(\mathfrak{h})$  by signed permutations of the  $x_i$ 's and  $D$  is the stabilizer of the element  $t = \prod_{i=1}^l x_i$ . (see 3.1.1).

Obviously  $t \notin \mathcal{S}(\mathfrak{h})^W$ ,  $t^2 \in \mathcal{S}(\mathfrak{h})^W$ . Moreover,

$$\mathcal{S}(\mathfrak{h})^D = \mathcal{S}(\mathfrak{h})^W \oplus t\mathcal{S}(\mathfrak{h})^W.$$

Indeed, write  $\mathcal{S}(\mathfrak{h})^D = \ker(\sigma - id) \oplus \ker(\sigma + id) = \mathcal{S}(\mathfrak{h})^W \oplus \ker(\sigma + id)$ . By definition of  $\sigma$ , for any  $x \in \mathcal{S}(\mathfrak{h})^D$ ,  $\sigma(x) = -x$  implies  $s_{\beta_i} \cdot x = \sigma(x) = -x$  for all  $i = 1 \dots l$ . On the other hand, observe that if  $x \in \mathcal{S}(\mathfrak{h})$  is such that  $s_{\beta_i} \cdot x = -x$  then necessary  $x_i | x$ . Hence any element of  $\ker(\sigma + id)$  is of the form  $ty$  with  $y \in \mathcal{S}(\mathfrak{h})^W$ .

#### 4.4.1. Proposition.

- (i) *The restriction of  $\Upsilon$  on  $\mathcal{A}$  is an algebra isomorphism from  $\mathcal{A}$  onto  $\mathcal{S}(\mathfrak{h})^D$ .*
- (ii) *There exists an element  $T$  in  $\mathcal{A}$  such that  $\Upsilon(T) = t$ ,  $T^2 \in \mathcal{Z}(\mathfrak{g})$  and*

$$\mathcal{A} = \mathcal{Z}(\mathfrak{g}) \oplus T\mathcal{Z}(\mathfrak{g}).$$

- (iii) *Let  $\sigma$  be the involution of  $\mathcal{A}$  defined by the involution  $\sigma$  on  $\mathcal{S}(\mathfrak{h})^D$  (see 4.3) through the identification (i), that is  $\sigma$  acts identically on  $\mathcal{Z}(\mathfrak{g})$  and maps  $T$  to  $(-T)$ . Then, for any  $u \in \mathcal{U}(\mathfrak{g})_1$  and  $a \in \mathcal{A}$  one has*

$$ua = \sigma(a)u.$$

*Proof.* Recall that the restriction of  $\Upsilon$  on  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is an algebra isomorphism from  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  onto  $\mathcal{S}(\mathfrak{h})^S$ . Let  $T$  be an element of  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  such that  $\Upsilon(T) = t$ . By [Mu1] 5.5,  $T$  commutes with the elements of  $\mathcal{U}(\mathfrak{g})_0$  and thus  $T \in \mathcal{A}$ . Since  $\mathcal{A}$  contains  $\mathcal{Z}(\mathfrak{g})$ , we conclude that  $\Upsilon(\mathcal{A})$  is a subalgebra of  $\mathcal{S}(\mathfrak{h})^S$  which contains  $t$  and  $\mathcal{S}(\mathfrak{h})^W$ . Therefore  $\Upsilon(\mathcal{A})$  contains  $\mathcal{S}(\mathfrak{h})^D$  and so  $\Upsilon(\mathcal{A}) = \mathcal{S}(\mathfrak{h})^D$ , by Lemma 4.2. Thus the restriction of  $\Upsilon$  on  $\mathcal{A}$  is an algebra isomorphism from  $\mathcal{A}$  onto  $\mathcal{S}(\mathfrak{h})^D$ . Now (ii) follows from 4.4.

According to [Mu1] 5.5,  $Tx + xT = 0$  for all  $x \in \mathfrak{g}_1$  so  $uT = \sigma(T)u$  for any  $u \in \mathcal{U}(\mathfrak{g})_1$ . Now (iii) follows from (ii).  $\square$

**4.4.2. Remark.** This element  $T$  was considered by Musson in [Mu1] in order to show that annihilators of certain Verma modules are not generated by their intersection with the centre. Also, an explicit construction of  $T$  is given in [ABF].

**4.5.** As a corollary we obtain another characterization of the algebra  $\mathcal{A}$ :

**Corollary.** *The algebra  $\mathcal{A}$  coincides with the set of elements of  $\mathcal{U}(\mathfrak{g})$  which act by scalars on each homogeneous component of any Verma module.*

*Proof.* Any element  $a \in \mathcal{A}$  acts on  $\widetilde{M}(\mu, \bar{i})_{\bar{i}}$  by  $\Upsilon(a)(\mu)$  and on  $\widetilde{M}(\mu, \bar{i})_{\bar{i}+1}$  by  $\Upsilon(\sigma(a))(\mu)$  (this follows from Proposition 4.4.1). Conversely, the assertion that any element which acts by scalars on each homogeneous components of any Verma module belongs to  $\mathcal{A}$ , follows from the fact that the intersection of annihilators of all Verma modules is zero (see 2.10).  $\square$

## 5. FREEDOM OF $\mathcal{U}(\mathfrak{g})$ OVER $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$

The goal of this section is to prove the

**5.1. Theorem.** *There exists an ad  $\mathfrak{g}_0$ -submodule  $\mathcal{E}$  of  $\mathcal{U}(\mathfrak{g})$  such that the multiplication map induces an isomorphism of  $\mathfrak{g}_0$ -modules  $\mathcal{E} \otimes \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$ .*

Subsections 5.5-5.9 are devoted to the proof of this theorem. In 5.10, we shall easily deduce from it the result we shall actually need in the proof of Theorem 6.2, that is the

**5.2. Theorem.** *There exists an ad  $\mathfrak{g}_0$ -submodule  $\mathcal{K}$  of  $\mathcal{U}(\mathfrak{g})$  such that the multiplication map induces an isomorphism of  $\mathfrak{g}_0$ -modules  $\mathcal{K} \otimes \mathcal{A} \rightarrow \mathcal{U}(\mathfrak{g})$ .*

**5.3. Remark.** For a complex semisimple Lie algebra  $\mathfrak{k}$ , one can show that the centre  $\mathcal{Z}(\mathfrak{k})$  is integrally closed in  $\mathcal{U}(\mathfrak{k})$ . As Pinczon observed in [Pi], this fails to be true for  $\mathfrak{g}$ . However, one can prove that the integral closure of  $\mathcal{Z}(\mathfrak{g})$  (resp.  $\mathcal{A}$ ) is precisely  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ .

**5.4. Remark.** Let  $\mathcal{H}$  be as in Theorem 2.9 and  $\mathcal{K}$  as in Theorem 5.2. Since  $\mathcal{A}$  is free of rank 2 over  $\mathcal{Z}(\mathfrak{g})$  (see Proposition 4.4.1 (ii)), one has the following multiplicity formula

$$[\mathcal{K} : V(\lambda)] = \frac{1}{2}[\mathcal{H} : V(\lambda)] \quad \forall \lambda \in P^+(\pi). \quad (12)$$

**5.5. Proof of theorem 5.1.** Musson proved in [Mu1], 5.3 that the restriction of  $\Upsilon$  on  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is an algebra isomorphism from  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  onto  $\mathcal{S}(\mathfrak{h})^S$ . Following the approach of [BL], our proof is based on this fact and the lemma below.

**Lemma.** (see [BL]) *Let  $D$  be a filtered algebra and  $M$  a filtered  $D$ -module. Let  $\{m_k\}$  be a family of elements of  $M$  such that  $\{\text{gr } m_k\}$  is a free basis of the  $\text{gr } D$ -module  $\text{gr } M$ . Then  $\{m_k\}$  is a free basis of  $M$  over  $D$ .*

However, the proof of Theorem 5.1 will not be an automatic generalization of [BL]. It will require the introduction of several different filtrations and the careful study of their respective associated gradings. The main reason for these difficulties is that the canonical filtration  $\mathcal{F}$  (see 2.4) is not adapted anymore to this situation. Indeed, let  $C, C_0$  be respectively the Casimir elements of  $\mathcal{Z}(\mathfrak{g})$  and  $\mathcal{Z}(\mathfrak{g})_0$ . It is easy to check that  $x := \text{gr}_{\mathcal{F}}(C - C_0) \in \wedge^2 \mathfrak{g}_1$ . Hence  $x^2 = 0$  and  $\text{gr}_{\mathcal{F}} \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is not a domain. Moreover, in the case  $\mathfrak{g} = \text{osp}(1, 2)$ , one can verify that  $\mathcal{S}(\mathfrak{g})$  is not graded-free over  $\text{gr}_{\mathcal{F}} \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ .

Throughout this section when we say that an algebra  $U$  is free over its subalgebra  $A$  the action of  $A$  on  $U$  is assumed to be right multiplication.

All filtrations of Lie superalgebras are assumed to be exhausting increasing  $\frac{1}{2}\mathbb{N}$ -filtrations such that each step of the filtration is  $\mathbb{Z}_2$ -graded space. If  $\mathcal{F}$  is such a filtration on a finite dimensional Lie superalgebra  $\mathfrak{t}$ , then the natural extension of  $\mathcal{F}$  to  $\mathcal{U}(\mathfrak{t})$  is also exhausting increasing  $\frac{1}{2}\mathbb{N}$ -filtration such that each step of the filtration is  $\mathbb{Z}_2$ -graded space. Moreover, the associated graded algebra  $\text{gr}_{\mathcal{F}} \mathcal{U}(\mathfrak{t})$  is canonically isomorphic to the universal enveloping algebra  $\mathcal{U}(\text{gr}_{\mathcal{F}} \mathfrak{t})$ . We will identify this two objects.

**5.6. Filtration  $\mathcal{F}_1$ .** Define a filtration  $\mathcal{F}_1$  on  $\mathfrak{g}$  by

$$\mathcal{F}_1^{(1/2)}(\mathfrak{g}) = \mathfrak{g}_1, \quad \mathcal{F}_1^{(1)}(\mathfrak{g}) = \mathfrak{g}_0$$

and extend it to  $\mathcal{U}(\mathfrak{g})$ . Since  $\mathcal{F}_1$  is  $\text{ad } \mathfrak{g}_0$ -invariant, the associated graded algebra  $\text{gr}_1 \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\text{gr}_1 \mathfrak{g})$  inherits the structure of  $\text{ad } \mathfrak{g}_0$ -module. Moreover each graded subspace  $\mathcal{U}^{(n)}(\text{gr}_1 \mathfrak{g})$  is finite dimensional.

**5.6.1.** In  $\text{gr}_1 \mathfrak{g}$  one has the following relations

$$[\text{gr}_1(a), \text{gr}_1(b)] = \text{gr}_1([a, b]) \quad \text{for } a, b \in \mathfrak{g}_1, \quad [\text{gr}_1(a), \text{gr}_1(b)] = 0 \quad \text{for } a \in \mathfrak{g}_0, b \in \mathfrak{g}.$$

**5.6.2.** As in the classical case one has the following easy implication

**Lemma.** *If  $\text{gr}_1 \mathcal{U}(\mathfrak{g})$  is graded-free over  $\text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  then there exists a  $\mathfrak{g}_0$ -submodule  $\mathcal{E}$  of  $\mathcal{U}(\mathfrak{g})$  such that the multiplication map induces the isomorphism  $\mathcal{E} \otimes \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$ .*

*Proof.* Denote by  $\text{gr}_1^+ \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  the ideal in  $\text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  generated by elements of positive degree.

Suppose that  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g})$  is graded-free over  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  that is there exists a graded subspace  $E$  of  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g})$  such that the multiplication map induces isomorphism

$$\psi : E \otimes \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \xrightarrow{\sim} \mathrm{gr}_1 \mathcal{U}(\mathfrak{g}).$$

In particular one has  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g}) = \mathrm{gr}_1 \mathcal{U}(\mathfrak{g}) \mathrm{gr}_1^+ \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \oplus E$ .

Since  $\mathfrak{g}_0$  acts trivially on the elements of  $\mathrm{gr}_1^+ \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ , the left ideal  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g}) \mathrm{gr}_1^+ \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is graded  $\mathfrak{g}_0$ -submodule of  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g})$ . Thus there exists a graded  $\mathfrak{g}_0$ -submodule  $E'$  of  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g})$  such that  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g}) = \mathrm{gr}_1 \mathcal{U}(\mathfrak{g}) \mathrm{gr}_1^+ \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \oplus E'$ . Denote by  $\psi'$  the map  $E' \otimes \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \xrightarrow{\sim} \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})$  induced by the multiplication. Using the induction on the degree of a homogeneous component, it is easy to check that  $\psi'$  is surjective. Indeed, since  $\mathrm{Im} \psi' \cdot \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \subset \mathrm{Im} \psi'$ ,

$$\mathrm{gr}_1^{(n+1)} \mathcal{U}(\mathfrak{g}) = \sum_{k=1}^{n+1} \mathrm{gr}_1^{n+1-k} \mathcal{U}(\mathfrak{g}) \mathrm{gr}_1^k \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \oplus (E' \cap \mathrm{gr}_1^{(n+1)} \mathcal{U}(\mathfrak{g}))$$

is contained in  $\mathrm{Im} \psi'$  if  $\mathrm{gr}_1^{(k)} \mathcal{U}(\mathfrak{g}) \subset \mathrm{Im} \psi'$  for  $k \leq n$ . Taking into account that each homogeneous component of  $\mathrm{gr}_1 \mathcal{U}(\mathfrak{g})$  is finite dimensional, we obtain from above that

$$\dim(E \cap \mathrm{gr}_1^{(n)} \mathcal{U}(\mathfrak{g})) = \dim(E' \cap \mathrm{gr}_1^{(n)} \mathcal{U}(\mathfrak{g}))$$

for all  $n \in \mathbb{N}/2$ . Since  $\psi$  is an isomorphism, it follows that  $\psi'$  is also an isomorphism.

For each  $n \in \mathbb{N}/2$  let  $\phi_n$  be the canonical map  $\mathcal{F}_1^{(n)}(\mathcal{U}(\mathfrak{g})) \rightarrow \mathrm{gr}_1^{(n)} \mathcal{U}(\mathfrak{g})$  and let  $\mathcal{E}^n$  be a  $\mathfrak{g}_0$ -submodule of  $\mathcal{F}_1^{(n)}(\mathcal{U}(\mathfrak{g}))$  such that  $\phi_n(\mathcal{E}^n) = E' \cap \mathrm{gr}_1^{(n)} \mathcal{U}(\mathfrak{g})$ . Set  $\mathcal{E} = \sum \mathcal{E}^n$ . Clearly, that  $\mathcal{E}$  is  $\mathfrak{g}_0$ -invariant. Moreover, by Lemma 5.5,  $\mathcal{E} \otimes \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \xrightarrow{\sim} \mathcal{U}(\mathfrak{g})$  as required.  $\square$

### 5.7. Filtration $\mathcal{F}_2$ . Set

$$\bar{\mathfrak{g}}_0 := \bigoplus_{\alpha \in \bar{\Delta}_0} (\mathfrak{g}_0)_\alpha, \quad \mathfrak{g}_r := \bigoplus_{i=1}^l (\mathfrak{g}_{2\beta_i} \oplus \mathfrak{g}_{-2\beta_i}).$$

Define a filtration  $\mathcal{F}_2$  on  $\mathrm{gr}_1 \mathfrak{g}$  by

$$\begin{aligned} \mathcal{F}_2^{(0)}(\mathrm{gr}_1 \mathfrak{g}) &= \mathrm{gr}_1 \bar{\mathfrak{g}}_0, \\ \mathcal{F}_2^{(1/2)}(\mathrm{gr}_1 \mathfrak{g}) &= \mathrm{gr}_1 \mathfrak{g}_1 \oplus \mathrm{gr}_1 \bar{\mathfrak{g}}_0, \\ \mathcal{F}_2^{(1)}(\mathrm{gr}_1 \mathfrak{g}) &= \mathrm{gr}_1 \mathfrak{g} \end{aligned}$$

and extend it to  $\mathcal{U}(\mathrm{gr}_1 \mathfrak{g})$ . Since  $\mathcal{F}_2$  is  $\mathfrak{h}$ -invariant filtration, the associated graded algebra  $\mathrm{gr}_2 \mathrm{gr}_1 \mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g})$  inherits the structure of  $\mathfrak{h}$ -module. The following relations (with all possible choices of signs) hold in  $\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g}$ :

$$\begin{aligned} [\mathrm{gr}_2 \mathrm{gr}_1 x_{\pm\beta_i}, \mathrm{gr}_2 \mathrm{gr}_1 x_{\pm\beta_j}] &= 0 && \text{iff } i \neq j, \ i, j \in \{1, \dots, l\}, \\ [\mathrm{gr}_2 \mathrm{gr}_1 x_{\pm\beta_i}, \mathrm{gr}_2 \mathrm{gr}_1 x_{\pm\beta_i}] &= \mathrm{gr}_2 \mathrm{gr}_1 [x_{\pm\beta_i}, x_{\pm\beta_i}], && \text{for } i = 1, \dots, l. \end{aligned} \tag{13}$$

Moreover, the even part of  $\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g}$  coincides with its centre.

The relations (13) imply that  $[x, x] \neq 0$  for any non-zero odd  $x \in \text{gr}_2 \text{gr}_1 \mathfrak{g}$ . Therefore, by [AL],  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})$  is a domain. In particular,  $\mathcal{U}(\text{gr}_1 \mathfrak{g})$  is also a domain.

**5.7.1.** Denote by  $\mathfrak{t} := \mathfrak{t}_0 \oplus \mathfrak{t}_1$  the subalgebra of  $\text{gr}_2 \text{gr}_1 \mathfrak{g}$  given by

$$\mathfrak{t}_1 := \text{gr}_2 \text{gr}_1 \mathfrak{g}_1, \quad \mathfrak{t}_0 := [\mathfrak{t}_1, \mathfrak{t}_1] = \text{gr}_2 \text{gr}_1 \mathfrak{h} \oplus \text{gr}_2 \text{gr}_1 \mathfrak{g}_r.$$

The superalgebra  $\mathfrak{t}$  is a graded  $\mathfrak{h}$ -invariant subalgebra of  $\text{gr}_2 \text{gr}_1 \mathfrak{g}$ : the non-zero elements of  $\mathfrak{t}_1$  have degree  $\frac{1}{2}$  and the non-zero elements of  $\mathfrak{t}_0$  have degree 1.

**5.7.2. Lemma.** *Assume that*

- (i)  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \subset \mathcal{U}(\mathfrak{t})$
- (ii)  $\mathcal{U}(\mathfrak{t})$  is graded-free over  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ .

Then  $\text{gr}_1 \mathcal{U}(\mathfrak{g})$  is graded-free over  $\text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ .

*Proof.* The graded Lie superalgebra  $\text{gr}_2 \text{gr}_1 \mathfrak{g}$  is the graded product of the commutative Lie algebra  $\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0$  (which lies in degree zero) and the Lie superalgebra  $\mathfrak{t}$ . Therefore  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0)$ ,  $\mathcal{U}(\mathfrak{t})$  are graded subalgebras of  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})$  and one has

$$\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g}) = \mathcal{U}(\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0) \otimes \mathcal{U}(\mathfrak{t}) \tag{14}$$

as graded algebras.

Recall that  $\text{gr}_1 \bar{\mathfrak{g}}_0$  is a Lie subalgebra of  $\text{gr}_1^{(1)} \mathfrak{g}$  and thus  $\mathcal{U}(\text{gr}_1 \bar{\mathfrak{g}}_0)$  is a graded subalgebra of  $\mathcal{U}(\text{gr}_1 \mathfrak{g})$ . Let  $n$  be a graded linear basis of  $\mathcal{U}(\text{gr}_1 \bar{\mathfrak{g}}_0)$ . Since  $\mathcal{U}(\text{gr}_1 \bar{\mathfrak{g}}_0) \subset \mathcal{F}_2^{(0)}(\mathcal{U}(\text{gr}_1 \mathfrak{g}))$ ,  $\text{gr}_2 n$  is a linear basis of  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0)$ . By (14), it follows that  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})$  is free over  $\mathcal{U}(\mathfrak{t})$  with the basis  $\text{gr}_2 n$ .

Observe that the  $\mathcal{F}_2$ -filtration degree of the elements of  $\text{gr}_1 \mathfrak{g}_1$  and  $\text{gr}_1(\mathfrak{g}_r \oplus \mathfrak{h})$  coincides with the degree with respect to the grading on  $\text{gr}_1 \mathfrak{g}$ . Thus there exists a map  $\iota : \mathcal{U}(\mathfrak{t}) \rightarrow \mathcal{U}(\text{gr}_1 \mathfrak{g})$  of graded linear space such that  $\text{gr}_2 \circ \iota(a) = a$  for every homogeneous element  $a \in \mathcal{U}(\mathfrak{t})$ . Let  $k$  be a graded basis of  $\mathcal{U}(\mathfrak{t})$  over  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ . Then  $k \times \text{gr}_2 n$  is a basis of  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})$  over  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ . Since  $\iota$  is a map of graded spaces,  $\iota(k)$  consists of homogeneous elements. Hence, by Lemma 5.5,  $\iota(k) \times n$  is a graded basis of  $\text{gr}_1 \mathcal{U}(\mathfrak{g})$  over  $\text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  as required.  $\square$

**5.8. Proof of the assumptions of Lemma 5.7.2.** In this subsection we prove the assumptions of Lemma 5.7.2. By the lemmas 5.6.2, 5.7.2 they imply Theorem 5.1.

**5.8.1.** Consider the canonical filtration  $\mathcal{F}$  on  $\mathcal{U}(\mathfrak{g})$  defined in 2.4 Recall that the symmetric superalgebra  $\mathcal{S}(\mathfrak{g})$  is the associated graded algebra of  $\mathcal{U}(\mathfrak{g})$  with respect to the filtration  $\mathcal{F}_0$ . We denote the grading map corresponding to the canonical filtration by  $\text{gr}_0$ .

Note that the algebra  $\text{gr}_2 \text{gr}_1 \mathfrak{h} \cong \text{gr}_0 \mathfrak{h}$  as graded algebras. Identify the graded algebra  $\mathcal{S}(\mathfrak{h})$  with  $\mathcal{U}(\mathfrak{h})$  and  $\text{gr}_2 \text{gr}_1(\mathcal{U}(\mathfrak{h}))$ . The Lie superalgebra  $\text{gr}_2 \text{gr}_1 \mathfrak{g}$  admits a triangular

decomposition

$$\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g} = \mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{n}^+$$

Let  $\Upsilon_2 : \mathcal{U}(\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g}) \rightarrow \mathcal{S}(\mathfrak{h})$  be the Harish-Chandra projection with respect to the above triangular decomposition of  $\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g}$ . Define similarly  $\Upsilon_0 : \mathcal{S}(\mathfrak{g}) \rightarrow \mathcal{S}(\mathfrak{h})$ . By [Mu1], the restriction of  $\Upsilon_0$  on  $\mathrm{gr}_0 \mathcal{Z}(\mathfrak{g}) = \mathcal{S}(\mathfrak{g})^{\mathfrak{g}}$  gives an algebra isomorphism from  $\mathrm{gr}_0 \mathcal{Z}(\mathfrak{g})$  onto  $\mathcal{S}(\mathfrak{h})^W$ .

Remark that all elements of  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  are even so  $\mathrm{gr}_2^{(n)} \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} = 0$  for all  $n \in \frac{1}{2}\mathbb{N} \setminus \mathbb{N}$ .

### 5.8.2. Proposition.

- (i) *The restriction of the Harish-Chandra projection  $\Upsilon_2$  on  $\mathrm{gr}_2 \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is a graded algebra isomorphism from  $\mathrm{gr}_2 \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  onto  $\mathcal{S}(\mathfrak{h})^S$ .*
- (ii)  $\mathrm{gr}_2 \mathrm{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \subset \mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$ .

*Proof.* Fix a basis  $\{x_i\}_{i \in I}$  of  $\mathfrak{g}$  consisting of weight vectors and fix a total ordering on this basis compatible with the partial order on the set of weights  $\Omega(\mathfrak{g})$  given by  $\mu_1 > \mu_2$  iff  $\mu_1 - \mu_2 \in \mathbb{N}\pi$ .

Consider the corresponding PBW-basis of  $\mathcal{U}(\mathfrak{g})$ . Set

$$I_1 := \{i \in I : x_i \in \mathfrak{g}_1\}, \quad I_0 = \{i \in I : x_i \in \mathfrak{g}_0\}, \quad I_h = \{i \in I : x_i \in \mathfrak{h}\}, \quad I_r = \{i \in I : x_i \in \mathfrak{g}_r\}.$$

For an element  $e_\nu = \prod_{i \in I} x_i^{\nu_i}$  of the PBW-basis set

$$\begin{aligned} d_0(e_\nu) &= \sum_{i \in I} \nu_i, \\ d_1(e_\nu) &= \sum_{i \in I_0} \nu_i + \frac{1}{2} \sum_{i \in I_1} \nu_i, \\ d_2(e_\nu) &= \sum_{i \in I_h \cup I_r} \nu_i + \frac{1}{2} \sum_{i \in I_1} \nu_i \end{aligned}$$

For each element  $u \in \mathcal{U}(\mathfrak{g})$  let  $u_\nu$  be the coordinate of  $u$  corresponding to  $e_\nu$ , that is  $u = \sum u_\nu e_\nu$ . Set  $\mathrm{supp}(u) = \{\nu : u_\nu \neq 0\}$ . For  $i = 0, 1, 2$  set

$$d_i(u) = \max_{\nu \in \mathrm{supp}(u)} d_i(e_\nu), \quad d_i(0) = 0.$$

Note that  $d_0(u) \geq d_1(u) \geq d_2(u)$ .

Identify the PBW-basis with the corresponding PBW bases of the universal enveloping algebras  $\mathcal{U}(\mathfrak{g})$ ,  $\mathcal{S}(\mathfrak{g})$ ,  $\mathcal{U}(\mathrm{gr}_1 \mathfrak{g})$ ,  $\mathcal{U}(\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g})$  and extend these identifications to the identifications of  $\mathcal{U}(\mathfrak{g})$ ,  $\mathcal{S}(\mathfrak{g})$ ,  $\mathcal{U}(\mathrm{gr}_1 \mathfrak{g})$ ,  $\mathcal{U}(\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g})$  as vector spaces. Then the maps  $\mathrm{gr}_0, \mathrm{gr}_1$  and  $\mathrm{gr}_2 \mathrm{gr}_1$  are given by the following formulas

$$\begin{aligned} \mathrm{gr}_i(u) &= \sum_{\nu \in \mathrm{supp}(u) : d_i(e_\nu) = d_i(u)} u_\nu e_\nu, \quad i = 0, 1 \\ \mathrm{gr}_2 \mathrm{gr}_1(u) &= \sum_{\nu \in \mathrm{supp}(u) : d_2(e_\nu) = d_2(u)} u_\nu e_\nu \end{aligned} \tag{15}$$

Note that under the previous identification of  $\mathcal{U}(\mathfrak{g})$ ,  $\mathcal{S}(\mathfrak{g})$ ,  $\mathcal{U}(\mathrm{gr}_2 \mathrm{gr}_1 \mathfrak{g})$  as vector spaces, the Harish-Chandra maps  $\Upsilon$ ,  $\Upsilon_0$ ,  $\Upsilon_2$  coincide. Set

$$J_h = \{\nu : \nu_i = 0, \forall i \notin I_h\}.$$



Note that  $d_2(e_\nu) = d_0(e_\nu)$  for all  $\nu \in J_h$ . One has

$$\Upsilon_0(\text{gr}_0(u)) = \begin{cases} \text{gr}_0\Upsilon(u), & \text{if } \exists \nu \in (\text{supp}(u) \cap J_h) : d_0(u) = d_0(e_\nu) \\ 0, & \text{otherwise} \end{cases} \quad (16)$$

$$\Upsilon_2(\text{gr}_2 \text{gr}_1(u)) = \begin{cases} \text{gr}_0\Upsilon(u), & \text{if } \exists \nu \in (\text{supp}(u) \cap J_h) : d_2(u) = d_2(e_\nu) \\ 0, & \text{otherwise} \end{cases} \quad (17)$$

Since  $\mathcal{F}_1, \mathcal{F}_2$  are  $\mathfrak{h}$ -invariant filtrations, one has

$$\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \subset \mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})^{\mathfrak{h}}.$$

The restriction of the Harish-Chandra projection  $\Upsilon_2$  on  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})^{\mathfrak{h}}$  is a graded algebra homomorphism from  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})^{\mathfrak{h}}$  to  $\mathcal{S}(\mathfrak{h})$  (see 2.6). Hence to prove (i) it is sufficient to show that

- (a)  $\Upsilon_2(\text{gr}_2 \text{gr}_1((a))) \neq 0, \quad \forall a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \setminus \{0\}$
- (b) for any homogeneous  $f \in \mathcal{S}(\mathfrak{h})^S \quad \exists a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  s.t.  $\Upsilon_2(\text{gr}_2 \text{gr}_1((a))) = f$ .

Let us show that the condition (a) implies (b). In fact, let  $f \in \mathcal{S}(\mathfrak{h})^S$  be a homogeneous element and let  $f' \in \mathcal{S}(\mathfrak{h})$  be such that  $f'(\lambda) = f(\lambda + \rho)$  for any  $\lambda \in \mathfrak{h}^*$ . Then  $f' \in \mathcal{S}(\mathfrak{h})^S$  and moreover  $\text{gr}_0(f') = f$ . By 2.6, there exists  $a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  such that  $\Upsilon(a) = f'$ . Then (a) implies that  $\Upsilon_2(\text{gr}_2 \text{gr}_1(a)) \neq 0$ . Therefore, by (17),

$$\Upsilon_2(\text{gr}_2 \text{gr}_1(a)) = \text{gr}_0\Upsilon(a) = \text{gr}_0(f') = f.$$

Hence (a) implies (b) and it remains to verify the condition (a).

Consider the case  $z \in \mathcal{Z}(\mathfrak{g}) \setminus \{0\}$ . Recall that the restriction of the Harish-Chandra projection  $\Upsilon_0$  on  $\text{gr}_0 \mathcal{Z}(\mathfrak{g})$  is an injection. By (16), it follows that  $d_0(z) = d_0(e_\nu)$  for some  $\nu \in (\text{supp}(z) \cap J_h)$ . One has

$$d_0(e_\nu) = d_2(e_\nu) \leq d_2(z) \leq d_0(z) = d_0(e_\nu)$$

so  $d_2(z) = d_2(e_\nu)$ . By (17),  $\Upsilon_2(\text{gr}_2 \text{gr}_1(a)) \neq 0$  and this proves (a) for  $a \in \mathcal{Z}(\mathfrak{g})$ .

Let  $\mu$  be such that  $e_\mu \notin \mathcal{U}(\mathfrak{t})$  that is  $\mu_i \neq 0$  for some  $i \in I_0 \setminus (I_h \cup I_r)$ . Then  $d_2(e_\mu) < d_0(e_\mu) \leq d_0(z)$  if  $\mu \in \text{supp}(z)$ . Thus  $d_2(e_\mu) < d_2(z)$  and so  $(\text{gr}_2 \text{gr}_1(z))_\mu = 0$  by (15). Hence  $(\text{gr}_2 \text{gr}_1(z))_\mu = 0$  for all  $\mu$  be such that  $e_\mu \notin \mathcal{U}(\mathfrak{t})$ . Therefore  $\text{gr}_2 \text{gr}_1(z) \in \mathcal{U}(\mathfrak{t})$  and thus  $\text{gr}_2 \text{gr}_1(\mathcal{Z}(\mathfrak{g})) \subset \mathcal{U}(\mathfrak{t})$ .

Fix an arbitrary non-zero element  $a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ . By [Mu1], 2.5 and 5.3,  $a$  is algebraic over  $\mathcal{Z}(\mathfrak{g})$  that is  $\sum_{i=0}^n a^i z_i = 0$  for some elements  $z_i \in \mathcal{Z}(\mathfrak{g})$ ,  $z_n \neq 0$ . Recall from 7.2.2 that  $\text{gr}_1(\mathcal{U}(\mathfrak{g}))$ ,  $\text{gr}_2 \text{gr}_1(\mathcal{U}(\mathfrak{g}))$  are domains. Hence,  $\text{gr}_2 \text{gr}_1(a^i z_i) = (\text{gr}_2 \text{gr}_1(a))^i \text{gr}_2 \text{gr}_1(z_i)$  for  $i = 0, \dots, n$ . Then there exist  $0 < k \leq n$  and  $z'_i \in \{\text{gr}_2 \text{gr}_1 z_i, 0\}$  for  $i = 0, \dots, k$  such that  $\sum_{i=0}^k (\text{gr}_2 \text{gr}_1(a))^i z'_i = 0$  and  $z'_k \neq 0$ . Hence  $\text{gr}_2 \text{gr}_1(a)$  is algebraic over  $\text{gr}_2 \text{gr}_1 \mathcal{Z}(\mathfrak{g})$ .

Let  $\sum_{i=0}^k (\text{gr}_2 \text{gr}_1(a))^i z'_i = 0$  be a minimal polynomial of  $\text{gr}_2 \text{gr}_1(a)$  over  $\text{gr}_2 \text{gr}_1 \mathcal{Z}(\mathfrak{g})$ . Since  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})$  is a domain,  $z'_0 \neq 0$ . Since the restriction of  $\Upsilon_2$  on  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})^{\mathfrak{h}}$  is an

algebra homomorphism one has

$$\sum_{i=0}^k (\Upsilon_2(\text{gr}_2 \text{gr}_1(a)))^i \Upsilon_2(z'_i) = 0$$

and  $\Upsilon_2(z'_0) \neq 0$  as we already showed. Therefore  $\Upsilon_2(\text{gr}_2 \text{gr}_1(a)) \neq 0$ . Hence the condition (a) holds and this proves (i).

For (ii), recall that  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g}) = \mathcal{U}(\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0) \otimes \mathcal{U}(\mathfrak{t})$  (see (14)). Since  $\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0$  is a commutative Lie algebra,  $\mathcal{U}(\text{gr}_2 \text{gr}_1 \mathfrak{g})$  is isomorphic to the algebra of polynomials in  $\text{gr}_2 \text{gr}_1 \bar{\mathfrak{g}}_0$  with coefficients in  $\mathcal{U}(\mathfrak{t})$ . As we already showed,  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is algebraic over  $\text{gr}_2 \text{gr}_1 \mathcal{Z}(\mathfrak{g})$  and  $\text{gr}_2 \text{gr}_1 \mathcal{Z}(\mathfrak{g}) \subset \mathcal{U}(\mathfrak{t})$ . Thus  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is algebraic over  $\mathcal{U}(\mathfrak{t})$  and so  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \subset \mathcal{U}(\mathfrak{t})$  as required.  $\square$

**5.8.3.** For  $i = 1, \dots, l$  set

$$x_i := \text{gr}_2 \text{gr}_1 x_{\beta_i}, \quad y_i := \text{gr}_2 \text{gr}_1 x_{-\beta_i}, \quad h_i := [y_i, x_i].$$

**Lemma.**

- (i) The algebra  $\mathcal{U}(\mathfrak{t})$  is graded free over  $\mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$ .
- (ii) The algebra  $\mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  is an algebra of polynomials in variables  $\{h_i, y_i x_i\}_{i=1}^l$ .
- (iii) The algebra  $\mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  is graded free over  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ .

*Proof.* For  $i = 1, \dots, l$  denote by  $\mathfrak{p}_i$  the subalgebra of  $\mathfrak{t}$  generated by  $y_i, x_i$ . By (13),  $\mathfrak{p}_i$  are pairwise isomorphic graded Lie superalgebras and  $\mathfrak{t} = \mathfrak{p}_1 \times \dots \times \mathfrak{p}_l$  as graded algebras. Thus

$$\mathcal{U}(\mathfrak{t}) = \mathcal{U}(\mathfrak{p}_1) \otimes \dots \otimes \mathcal{U}(\mathfrak{p}_l).$$

Since  $\Omega(\mathcal{U}(\mathfrak{p}_i)) \subset \mathbb{Z}\beta_i$ , it follows that

$$\mathcal{U}(\mathfrak{t})^{\mathfrak{h}} = \mathcal{U}(\mathfrak{p}_1)^{\mathfrak{h}} \otimes \dots \otimes \mathcal{U}(\mathfrak{p}_l)^{\mathfrak{h}}.$$

Therefore in order to prove (i) and (ii) it is sufficient to show that for  $i = 1, \dots, l$ , the algebra  $\mathcal{U}(\mathfrak{p}_i)$  is graded free over  $\mathcal{U}(\mathfrak{p}_i)^{\mathfrak{h}}$  and that  $\mathcal{U}(\mathfrak{p}_i)^{\mathfrak{h}}$  is an algebra of polynomials in two variables  $h_i$  and  $y_i x_i$ .

Fix  $i \in \{1, \dots, l\}$  and omit the index  $i$ . The elements  $x$  and  $y$  have degree  $\frac{1}{2}$  and form a basis of the odd part of  $\mathfrak{p}$ ; the elements  $h = [y, x]$ ,  $[x, x] = 2x^2$  and  $[y, y] = 2y^2$  have degree 1 and form a basis of the even part of  $\mathfrak{p}$ . Moreover  $x$  has weight  $\beta$  and  $y$  has weight  $-\beta$ .

By PBW-theorem, for any  $m \in \mathbb{N}$  the weight subspace  $\mathcal{U}(\mathfrak{p})|_{m\beta}$  has a basis  $\{h^r y^k x^{k+m}\}_{r,k \in \mathbb{N}}$  and therefore  $\mathcal{U}(\mathfrak{p})|_{m\beta}$  is graded-free over  $\mathcal{U}(\mathfrak{p})|_0 = \mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$  with the basis consisting of the one element  $x^m$ . Similarly, for any  $m \in \mathbb{N}$  the weight subspace  $\mathcal{U}(\mathfrak{p})|_{-m\beta}$  is graded-free over  $\mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$  with the basis consisting of the one element  $y^m$ . Hence  $\mathcal{U}(\mathfrak{p}) = \bigoplus_{m \in \mathbb{Z}} \mathcal{U}(\mathfrak{p})|_{m\beta}$  is graded-free over  $\mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$ . This proves (i).

Recall that even part of  $\mathfrak{t}$  lies in its centre. Thus  $x^2, y^2$  and  $h$  lies in the centre of  $\mathfrak{p}$ . Therefore

$$y^2 x^2 = (yx)(xy) = (xy)(yx)$$

and, by induction, for any  $k > 0$  one has

$$\begin{aligned} y^{2k} x^{2k} &= (xy^2 x)^k = (xy)^k (yx)^k = (yx)^k (xy)^k = (yx)^k (h - yx)^k \\ y^{2k+1} x^{2k+1} &= y(y^{2k} x^{2k})x = y(xy)^k (yx)^k x = (yx)^k y^2 (xy)^{k-1} x^2 = (yx)^{k+1} (xy)^k \\ &= (yx)^{k+1} (h - yx)^k. \end{aligned}$$

Since  $\mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$  has a basis  $\{h^r y^k x^k\}_{r,k \in \mathbb{N}}$ , it follows that  $\mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$  is generated as an algebra by  $h$  and  $yx$ .

Since both  $h$  and  $yx$  have degree 1, the homogeneous component of degree  $n$  of  $\mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$  is a span of  $\{h^r (yx)^k\}_{r,k \in \mathbb{N}, r+k=n}$ . On the other hand, this component has a basis  $\{h^r x^k y^k\}_{r,k \in \mathbb{N}, r+k=n}$  and so has the dimension equal to  $n+1$ . Therefore the elements  $\{h^r (yx)^k\}_{r,k \in \mathbb{N}, r+k=n}$  are linearly independent. Since  $h$  and  $yx$  are homogeneous, it implies that the elements  $\{h^r (yx)^k\}_{r,k \in \mathbb{N}}$  are linearly independent. Hence  $\mathcal{U}(\mathfrak{p})^{\mathfrak{h}}$  is an algebra of polynomials in  $h$  and  $yx$ . This proves (ii).

(iii) Define a filtration  $\mathcal{F}_3$  on  $\mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  setting the degree of  $y_i x_i$  equal to zero and the degree of  $h_i$  equal to one for all  $i = 1, \dots, l$ . Since, by (ii),  $\mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  is an algebra of polynomials in  $\{h_i, y_i x_i\}_{i=1}^l$  the filtration  $\mathcal{F}_3$  is correctly defined. Identify  $\mathcal{S}(\mathfrak{h})$  with its image in  $\text{gr}_3 \mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$ .

One has the following isomorphism of graded algebras:

$$\text{gr}_3 \mathcal{U}(\mathfrak{t})^{\mathfrak{h}} \cong \mathbb{C}[y_1 x_1, \dots, y_l x_l] \otimes \mathcal{S}(\mathfrak{h}), \quad (18)$$

where the grading of the algebra  $\mathbb{C}[y_1 x_1, \dots, y_l x_l]$  is trivial one and  $\mathcal{S}(\mathfrak{h})$  has the usual grading of the symmetric algebra.

Set  $e_{k,r} = \prod_{i=1}^l h_i^{r_i} (yx)^{k_i}$ ,  $|k| = \sum_{i=1}^l k_i$ ,  $|r| = \sum_{i=1}^l r_i$ . Fix  $a \in \mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$ . Taking into account that  $\text{gr}_2 \text{gr}_1(a) \in \mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  is a homogeneous element of  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})$ , we conclude from (ii) that

$$\text{gr}_2 \text{gr}_1(a) = \sum_{|k|+|r|=n} a_{k,r} e_{k,r}$$

where the coefficients  $a_{k,r}$  are scalars and  $n$  is the degree of  $\text{gr}_2 \text{gr}_1(a)$  as element of the graded algebra  $\text{gr}_2 \text{gr}_1 \mathcal{U}(\mathfrak{g})$ . By Proposition 5.8.2 (i), one has

$$\Upsilon_2(\text{gr}_2 \text{gr}_1(a)) = \sum_{|k|=n} a_{k,0} e_{k,0} \neq 0.$$

Since the degree of  $e_{k,r}$  with respect to the filtration  $\mathcal{F}_3$  is equal to  $|k|$ , it implies that  $\text{gr}_2 \text{gr}_1(a)$  has degree  $n$  with respect to  $\mathcal{F}_3$  and moreover

$$\text{gr}_3 \text{gr}_2 \text{gr}_1(a) = \Upsilon_2(\text{gr}_2 \text{gr}_1(a)).$$

By Proposition 5.8.2 (i), it implies that

$$\text{gr}_3 \text{gr}_2 \text{gr}_1(\mathcal{U}(\mathfrak{g}))^{\mathfrak{g}_0} = \mathcal{S}(\mathfrak{h})^S.$$

Let  $m$  be a graded basis of  $\mathcal{S}(\mathfrak{h})$  over  $\mathcal{S}(\mathfrak{h})^S$ . Then, by (18),  $\{\text{gr}_3 e_{0,r}\} \times m$  is a basis of  $\text{gr}_3 \mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  over  $\mathcal{S}(\mathfrak{h})^S$ . Therefore, by Lemma 5.5,  $\mathcal{U}(\mathfrak{t})^{\mathfrak{h}}$  is graded-free over  $\text{gr}_2 \text{gr}_1(\mathcal{U}(\mathfrak{g}))^{\mathfrak{g}_0}$  with the basis  $\{e_{0,r}\} \times m$ . This completes the proof of (iii).  $\square$

**5.9.** Lemma 5.7.2, Proposition 5.8.2 (ii) and Lemma 5.8.3 (iii) imply Theorem 5.1.

**5.10. Freeness of  $\mathcal{U}(\mathfrak{g})$  over  $\mathcal{A}$ .** Consider the grading on  $\mathcal{S}(\mathfrak{h})$  given by the total degree. Then  $\mathcal{S}(\mathfrak{h})$  is graded-free over  $\mathcal{S}(\mathfrak{h})^S$  and over  $\mathcal{S}(\mathfrak{h})^D$ . Using the reason of 5.6.2, one can show the existence of a  $D$ -invariant graded subspace  $N$  of  $\mathcal{S}(\mathfrak{h})$  such that the multiplication map induces an isomorphism of  $D$ -modules  $N \otimes \mathcal{S}(\mathfrak{h})^D \xrightarrow{\sim} \mathcal{S}(\mathfrak{h})$ . Since  $S$  is a subgroup of  $D$  one obtain the isomorphism  $N^S \otimes \mathcal{S}(\mathfrak{h})^D \xrightarrow{\sim} \mathcal{S}(\mathfrak{h})^S$ . Hence  $\mathcal{S}(\mathfrak{h})^S$  is free over  $\mathcal{S}(\mathfrak{h})^D$ . Using the isomorphisms from  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  to  $\mathcal{S}(\mathfrak{h})^S$  and from  $\mathcal{A}$  to  $\mathcal{S}(\mathfrak{h})^D$  (see 7.2.2) given by  $\Upsilon$  one deduces that  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0} \simeq \mathcal{A} \otimes \Upsilon^{-1}(N^S)$ . Finally, setting  $\mathcal{K} = \mathcal{E} \otimes \Upsilon^{-1}(N^S)$ , we get theorem 5.2.

*Remark.* One can also prove theorem 5.2 directly exactly as it is done for  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  throughout 5.5— 5.9. One has just to replace  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  by  $\mathcal{A}$  in all steps and to replace  $\mathcal{S}(\mathfrak{h})^S$  by  $\mathcal{S}(\mathfrak{h})^D$  in 5.8.2.

## 6. VERMA MODULES ANNIHILATORS

**6.1.** The aim of this section is to prove that the annihilator of any Verma module is generated by its intersection with the algebra  $\mathcal{A}$ . We start by giving a precise description of the annihilators of Verma modules in  $\mathcal{A}$ . These ideals of  $\mathcal{A}$  play exactly the same role as the central characters in the theory of semisimple Lie algebras.

Recall that  $\sigma$  is the involution of  $\mathcal{A}$  mapping  $T$  to  $(-T)$  and acts identically on  $\mathcal{Z}(\mathfrak{g})$ .

### 6.1.1. Proposition.

- (i) *The map  $\iota : \mu \rightarrow \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  induces a bijection between the set of  $W$ -orbits of  $\mathfrak{h}^*$  and the set of maximal  $\sigma$ -invariant ideals of  $\mathcal{A}$ . Moreover  $\iota$  maps  $\mathfrak{h}_d^*$  to the set of  $\sigma$ -invariant ideals of  $\mathcal{A}$  which are maximal.*
- (ii) *Let  $N \not\cong \widetilde{V}(0)$  be a subquotient of a Verma module  $M$ . Then  $\text{Ann}_{\mathcal{A}} M = \text{Ann}_{\mathcal{A}} N$ .*

*Proof.* As we saw in Corollary 4.5, any element  $a \in \mathcal{A}$  acts on  $\widetilde{M}(\mu, \bar{i})_{\bar{i}}$  by  $\Upsilon(a)(\mu)$  and on  $\widetilde{M}(\mu, \bar{i})_{\bar{i}+1}$  by  $\Upsilon(\sigma(a))(\mu)$ . Consequently,  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu, \bar{i})_{\bar{i}+1} = \sigma(\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu, \bar{i})_{\bar{i}})$  and both  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu, \bar{i})_{\bar{i}+1}, \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu, \bar{i})_{\bar{i}}$  are maximal ideals in  $\mathcal{A}$ . Therefore their intersection  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  is a maximal  $\sigma$ -invariant ideal of  $\mathcal{A}$ .

From 4.3 and Proposition 4.4.1 (i) it follows that for any  $a \in \mathcal{A}$ ,  $\Upsilon(a)(w.\mu) = \Upsilon(a)(\mu)$  for all  $w \in D$  and  $\Upsilon(a)(w.\mu) = \Upsilon(\sigma(a))(\mu)$  for all  $w \in W \setminus D$ . Therefore  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) = \text{Ann}_{\mathcal{A}} \widetilde{M}(w.\mu)$  for any  $w \in W$ . Conversely, assume that  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) = \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu')$  for some  $\mu, \mu' \in \mathfrak{h}^*$ . Since  $\mathcal{Z}(\mathfrak{g}) \subset \mathcal{A}$  it implies that  $\text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu) = \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu')$  and so

$\mu \in W\mu'$ . Hence  $\iota$  induces an injective map from the set of  $W$ -orbits of  $\mathfrak{h}^*$  into the set of maximal  $\sigma$ -invariant ideals of  $\mathcal{A}$ .

Let  $J$  be a maximal  $\sigma$ -invariant ideal of  $\mathcal{A}$ . Then there exists a maximal ideal  $I$  of  $\mathcal{A}$  such that  $J = I \cap \sigma(I)$ . Since the restriction of  $\Upsilon$  on  $\mathcal{A}$  is an injection, there exists  $\mu \in \mathfrak{h}^*$  such that  $I = \{a - \Upsilon(a)(\mu) \mid a \in \mathcal{A}\}$ . Then  $J = \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  and this proves the surjectivity of  $\iota$ .

The ideal  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  is maximal iff  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu, \bar{i})_{\bar{i}+1} = \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu, \bar{i})_{\bar{i}}$  that is  $\Upsilon(a)(\mu) = \Upsilon(\sigma(a))(\mu)$  for any  $a \in \mathcal{A}$ . This is equivalent to the condition that  $\Upsilon(T)(\mu) = 0$  that is  $\mu \in \mathfrak{h}_d^*$ . This ends the proof of (i).

For (ii) note that the action of  $\mathfrak{h}$  separates the homogeneous components of a graded Verma module since it separates  $\mathcal{U}(\mathfrak{g})_0$  and  $\mathcal{U}(\mathfrak{g})_1$ . Consequently, one can define on  $N$  and  $M$  structures of graded  $\mathfrak{g}$ -modules such that endowed with these structures  $N$  becomes a graded subquotient  $M$ . Since  $N \neq \widetilde{V}(0)$ , both homogeneous components of  $N$  are non-zero. Since  $N$  is a subquotient of  $M$ , it implies that  $\text{Ann}_{\mathcal{A}} M_{\bar{i}} \subseteq \text{Ann}_{\mathcal{A}} N_{\bar{i}}$  for  $\bar{i} \in \mathbb{Z}_2$ . Taking into account that both  $\text{Ann}_{\mathcal{A}} M_{\bar{i}}$  are maximal ideals, we conclude that  $\text{Ann}_{\mathcal{A}} M_{\bar{i}} = \text{Ann}_{\mathcal{A}} N_{\bar{i}}$  and so  $\text{Ann}_{\mathcal{A}} M = \text{Ann}_{\mathcal{A}} N$  as required.  $\square$

**6.1.2. Remark.** Since  $\widetilde{V}(0)$  is one-dimensional,  $\text{Ann}_{\mathcal{A}} \widetilde{V}(0)$  contains the element  $T - t(0) = T - \prod_{i=1}^l (l - i + 1/2)$ . Thus this ideal is not  $\sigma$ -invariant.

We shall need later the following

### 6.1.3. Lemma.

- (i)  $\bigcap_{\mu \in \mathfrak{h}_d^*} \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) = \mathcal{A}T$
- (ii) For every  $\mu \in \mathfrak{h}_d^*$ , one has

$$\dim(\mathcal{A}T / (\mathcal{A}T \cap \mathcal{A} \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu))) = 1.$$

*Proof.* From Proposition 6.1.1 (i), it follows that  $\{\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) \mid \mu \in \mathfrak{h}_d^*\}$  coincides the set of maximal ideals in  $\mathcal{A}$  which contain  $T$ . This gives (i).

For (ii), fix  $\mu \in \mathfrak{h}_d^*$ . Since  $T^2 \in \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu)$  it follows that  $\mathcal{A}T \cap \mathcal{A} \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu) = (T\mathcal{Z}(\mathfrak{g}) \oplus \mathcal{Z}(\mathfrak{g})T^2) \cap \mathcal{A} \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu) = T \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu) \oplus \mathcal{Z}(\mathfrak{g})T^2$ . This proves (ii).  $\square$

**6.2.** Here is the main theorem of our paper.

**Theorem.** For any  $\mu \in \mathfrak{h}^*$ , one has

$$\text{Ann} \widetilde{M}(\mu) = \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) \tag{19}$$

The proof of Theorem 6.2 is given in 6.5—6.7.

### 6.3. Corollary.

- (i) *The annihilator of any Verma module coincides with the annihilator of its socle. In particular, it is a primitive ideal.*
- (ii) *Any minimal primitive ideal of  $\mathcal{U}(\mathfrak{g})$  is generated by its intersection with the algebra  $\mathcal{A}$ .*

*Proof.* Recall that the socle of a Verma module is a simple Verma module. Now (i) follows from Proposition 6.1.1 (ii) and the above theorem.

(ii) By [Mu2], any primitive ideal  $P$  of  $\mathcal{U}(\mathfrak{g})$  is the annihilator of a simple highest weight module. Thus  $P$  contains annihilator of some Verma module which is also a primitive ideal by (i). Hence the assertion follows from the above theorem.  $\square$

### 6.4. Corollary. For any $\lambda \in P^+(\pi)$

$$[\mathcal{U}(\mathfrak{g}) / \text{Ann } \widetilde{M}(\mu) : \widetilde{V}(\lambda)] = \begin{cases} \dim \widetilde{V}(\lambda)|_0 & \text{if } \mu \notin \mathfrak{h}_d^* \\ \sum_{m=0}^{\infty} (-1)^m \dim \widetilde{V}(\lambda)_{m\beta} & \text{if } \mu \in \mathfrak{h}_d^* \end{cases}$$

where  $\beta$  is any odd root.

Since the proof uses a step of the proof of Theorem 6.2 we put it in 6.8.

**6.5. The overview of the proof of Theorem 6.2.** The first step of the proof is to express the statement of the theorem in terms of PRV-matrices. This is done in the subsection 6.6. As a result, we obtain that Theorem 6.2 holds if and only if the corank of any matrix  $PRV^\nu$  ( $\nu \in P^+(\pi)$ ) is constant on the set  $\mathfrak{h}_{sd}^*$ — see Claim 6.6.6. In order to verify this last condition we shall construct in the subsection 6.7 for each  $\mu \in \mathfrak{h}_{sd}^*$  new matrices  $PRV_{\mathcal{L}}^\nu$  ( $\nu \in P^+(\pi)$ ) with entries in  $\mathcal{S}(\mathfrak{h})$ . These matrices are of the type  $PRV_\mu^\nu$  (see 3.2 for definition). Therefore, the  $PRV_{\mathcal{L}}^\nu$  are related to the matrices  $PRV^\nu$  as it is explained in 3.2. Namely, for each  $\nu \in P^+(\pi)$ , the entries of the matrix  $PRV_{\mathcal{L}}^\nu$  are equal to the corresponding entries of an appropriate matrix  $PRV^\nu$  modulo  $\mathfrak{m}_\mu$ , where  $\mathfrak{m}_\mu = \{P \in \mathcal{S}(\mathfrak{h}), P(\mu) = 0\}$ . In particular, for any  $\nu \in P^+(\pi)$  the coranks of the scalar matrices  $PRV^\nu(\mu)$  and  $PRV_{\mathcal{L}}^\nu(\mu)$  are equal. Moreover from the formula (3), we conclude that  $\det PRV_{\mathcal{L}}^\nu = t^{r_\nu} Q$  where  $Q \in \mathcal{S}(\mathfrak{h})$  is such that  $Q(\mu) \neq 0$ . The matrices  $PRV_{\mathcal{L}}^\nu$  are more convenient than  $PRV^\nu$  since the entries of the first  $r_\nu$  columns of  $PRV_{\mathcal{L}}^\nu$  are divisible by  $t$ . Using this fact, we obtain that the corank of  $PRV_{\mathcal{L}}^\nu(\mu)$  is equal to  $r_\nu$ . Therefore the corank of  $PRV^\nu(\mu)$  is equal to  $r_\nu$  for any  $\mu \in \mathfrak{h}_{sd}^*$ . This will complete the proof of Theorem 6.2.

### 6.6. Another formulation of theorem 6.2.

Let  $M$  be a Verma module and  $M'$  be its simple Verma submodule. Assume that  $\mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} M' = \text{Ann} M'$ . By Proposition 6.1.1 (ii),  $\text{Ann}_{\mathcal{A}} M = \text{Ann}_{\mathcal{A}} M'$  and so

$$\mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} M \subseteq \text{Ann} M \subseteq \text{Ann} M' = \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} M$$

that is  $\mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} M = \text{Ann} M$ .

Consequently, it is enough to verify the statement of Theorem 6.2 only for the case of simple Verma modules.

Since  $\mathcal{A}$  contains  $\mathcal{Z}(\mathfrak{g})$ , the annihilator of  $\widetilde{M}(\mu)$  is generated by its intersection with  $\mathcal{A}$  for the case  $\mu \in (\mathfrak{h}^* \setminus \mathfrak{h}_d^*)$  (see Theorem 3.1.2). Hence it is enough to verify the statement of Theorem 6.2 for the case  $\mu \in \mathfrak{h}_{sd}^*$ .

**6.6.1.** Fix  $\mu \in \mathfrak{h}_d^*$ . By Proposition 6.1.1 (i),  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  is a maximal  $\sigma$ -invariant ideal. Using Proposition 4.4.1 (iii), we conclude that  $\mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  is a two-sided ideal and in particular it is  $\text{ad } \mathfrak{g}$ -invariant. Consider the natural  $\mathfrak{g}$ -map

$$\phi_{\mu} : \mathcal{U}(\mathfrak{g}) / (\mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)) \rightarrow \mathcal{U}(\mathfrak{g}) / \text{Ann} \widetilde{M}(\mu).$$

It is clear that  $\phi_{\mu}$  is a surjective map and that  $\text{Ann} \widetilde{M}(\mu) = \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  iff  $\phi_{\mu}$  is an isomorphism.

By Theorem 5.2, one has the following isomorphisms of  $\mathfrak{g}_0$ -modules

$$\mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) \simeq (\mathcal{K} \otimes \mathcal{A}) / (\mathcal{K} \otimes \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)) \simeq \mathcal{K}$$

since  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  is a maximal ideal. Using the multiplicity formula (12), we obtain for any  $\lambda \in P^+(\pi)$

$$[\mathcal{U}(\mathfrak{g}) / \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu) : V(\lambda)] = [\mathcal{K} : V(\lambda)] = \frac{1}{2} [\mathcal{H} : V(\lambda)]. \quad (20)$$

On the other hand, Theorem 2.9 implies that  $\mathcal{U}(\mathfrak{g}) / \text{Ann} \widetilde{M}(\mu)$  is isomorphic to  $\mathcal{H} / \text{Ann}_{\mathcal{H}} \widetilde{M}(\mu)$  as  $\mathfrak{g}$ -modules. Therefore for any  $\nu \in P^+(\pi)$  one has

$$\begin{aligned} [\mathcal{U}(\mathfrak{g}) / \text{Ann} \widetilde{M}(\mu) : \widetilde{V}(\nu)] &= [\mathcal{H} : \widetilde{V}(\nu)] - [\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : \widetilde{V}(\nu)] \\ [\mathcal{U}(\mathfrak{g}) / \text{Ann} \widetilde{M}(\mu) : V(\nu)] &= [\mathcal{H} : V(\nu)] - [\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : V(\nu)] \end{aligned} \quad (21)$$

**6.6.2.** Since  $\phi_{\mu}$  is a surjective map, the equalities (21) and (20) imply that for any  $\mu \in \mathfrak{h}_d^*$

$$[\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : V(\lambda)] \geq \frac{1}{2} [\mathcal{H} : V(\lambda)], \quad \forall \lambda \in P^+(\pi). \quad (22)$$

Moreover  $\phi_{\mu}$  is an isomorphism iff for all  $\lambda \in P^+(\pi)$  the inequality (22) is an equality.

**6.6.3.** Let  $\mu \in \mathfrak{h}_{sd}^*$ . Combining (2) and (4) we conclude that

$$[\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : \widetilde{V}(\nu)] = \text{corank } PRV^\nu(\mu) \geq r_\nu \quad \nu \in P^+(\pi) \quad (23)$$

where  $r_\nu = \sum_{n=1}^{\infty} (-1)^{n+1} \dim \widetilde{V}(\lambda)_{n\beta_1}$  is a non-negative integer.

Recall also that by (5)

$$[\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : \widetilde{V}(\nu)] = \text{corank } PRV^\nu(\mu) = r_\nu \quad (24)$$

for any  $\mu \in \mathfrak{h}_s^*$  such that  $d(\mu) = 1$ .

**6.6.4. Lemma.** For any  $\mu \in \mathfrak{h}_s^*$  such that  $d(\mu) = 1$  one has

$$[\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : V(\lambda)] = \frac{1}{2}[\mathcal{H} : V(\lambda)].$$

*Proof.* Fix  $\mu \in \mathfrak{h}_s^*$  such that  $d(\mu) = 1$ . For any  $\lambda \in P^+(\pi)$  denote by  $\text{Ind } V(\lambda)$  the induced module  $\mathfrak{g}$ -module  $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} V(\lambda)$ . Frobenius reciprocity gives

$$[\text{Ind } V(\lambda) : \widetilde{V}(\nu)] = [\widetilde{V}(\nu) : V(\lambda)] \quad \forall \lambda, \nu \in P^+(\pi).$$

Therefore, using (24), one has for any  $\lambda \in P^+(\pi)$

$$\begin{aligned} [\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : V(\lambda)] &= \sum_{\nu \in P^+(\pi)} r_\nu [\widetilde{V}(\nu) : V(\lambda)] \\ &= \sum_{\nu \in P^+(\pi)} \left( \sum_{n=1}^{\infty} (-1)^{n+1} \dim \widetilde{V}(\nu)_{n\beta_1} \right) [\widetilde{V}(\nu) : V(\lambda)] \\ &= \sum_{\nu \in P^+(\pi)} \left( \sum_{n=1}^{\infty} (-1)^{n+1} \dim \widetilde{V}(\nu)_{n\beta_1} \right) [\text{Ind } V(\lambda) : \widetilde{V}(\nu)] \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} \dim \text{Ind } V(\lambda)_{n\beta_1} \end{aligned}$$

and moreover

$$\begin{aligned} [\mathcal{H} : V(\lambda)] &= \sum_{\nu \in P^+(\pi)} [\mathcal{H} : \widetilde{V}(\nu)] [\widetilde{V}(\nu) : V(\lambda)] \\ &= \sum_{\nu \in P^+(\pi)} \dim \widetilde{V}(\nu)_0 [\text{Ind } V(\lambda) : \widetilde{V}(\nu)] \\ &= \dim \text{Ind } V(\lambda)_0. \end{aligned}$$

Hence the lemma is equivalent to the equality

$$2 \sum_{n=0}^{\infty} (-1)^n \dim \text{Ind } V(\lambda)_{n\beta_1} = \dim \text{Ind } V(\lambda)_0.$$



One has

$$\text{ch Ind } V(\lambda) = \text{ch } V(\lambda) \cdot \text{ch } \mathfrak{g}_1 = \text{ch } V(\lambda) \prod_{i=1, \dots, l} (1 + e^{\beta_i})(1 - e^{-\beta_i}).$$

Set  $\pi_0(\sum_{\nu \in \mathbb{N}\pi} a_\nu e^{\lambda - \nu}) := a_0$  for any series of such type. Then

$$\begin{aligned} \dim \text{Ind } V(\lambda)_0 &= \pi_0 \left( \text{ch } V(\lambda) \prod_{i=1, \dots, l} (1 + e^{\beta_i})(1 - e^{-\beta_i}) \right) \\ &= \pi_0 \left( (2 + e^{\beta_1} + e^{-\beta_1}) \underbrace{\text{ch } V(\lambda) \prod_{i=2, \dots, l} (1 + e^{\beta_i})(1 - e^{-\beta_i})}_a \right) \\ &= 2\pi_0(a) + \pi_{\beta_1}(a) + \pi_{-\beta_1}(a) \\ &= 2\pi_0(a) + 2\pi_{-\beta_1}(a) \text{ since } a \text{ is invariant under } s_{\beta_1} \in W \\ &= 2\pi_0 \left( (1 + e^{-\beta_1}) a \right) \\ &= 2\pi_0 \left( (1 + e^{\beta_1})^{-1} \text{ch Ind } V(\lambda) \right) \\ &= 2 \sum_{n=0}^{\infty} (-1)^n \dim \text{Ind } V(\lambda)_{n\beta_1} \end{aligned}$$

This completes the proof of the Lemma.  $\square$

Taking into account 6.6.2, we get

**6.6.5. Corollary.** *For any  $\mu \in \mathfrak{h}_s^*$  such that  $d(\mu) = 1$ , one has*

$$\text{Ann } \widetilde{M}(\mu) = \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu).$$

**6.6.6. Claim.** *The annihilator of any Verma module  $\widetilde{M}(\mu)$  ( $\mu \in \mathfrak{h}^*$ ) is generated by its intersection with  $\mathcal{A}$  iff for any  $\nu \in P^+(\pi)$  the corank of the matrix  $PRV^\nu$  is constant on the set  $\mathfrak{h}_{sd}^*$ .*

*Proof.* Assume that for any  $\nu \in P^+(\pi)$  the corank of the matrix  $PRV^\nu$  is constant on the set  $\mathfrak{h}_{sd}^*$ . Fix  $\mu \in \mathfrak{h}_{sd}^*$  and  $\mu' \in \mathfrak{h}_s^*$  such that  $d(\mu') = 1$ . Then for all  $\nu \in P^+(\pi)$   $\text{corank } PRV^\nu(\mu) = \text{corank } PRV^\nu(\mu')$ . Then, by 6.6.3, (23)  $\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) \cong \text{Ann}_{\mathcal{H}} \widetilde{M}(\mu')$  as  $\mathfrak{g}$ -module. Therefore  $[\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : V(\lambda)] = \frac{1}{2}[\mathcal{H} : V(\lambda)]$  by Lemma 6.6.4. Hence  $\text{Ann } \widetilde{M}(\mu) = \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  by 6.6.2. By 6.6, this proves the implication  $\Leftarrow$ .

Conversely, assume that for some  $\nu_0 \in P^+(\pi)$  the corank of the matrix  $PRV^{\nu_0}$  is not constant on the set  $\mathfrak{h}_{sd}^*$ . This means, by 6.6.3, (23), that for some  $\mu \in \mathfrak{h}_{sd}^*$   $\text{corank } PRV^{\nu_0}(\mu) > r_{\nu_0}$ . Since the numbers  $r_\nu$  are non-negative and  $\text{corank } PRV^\nu(\mu) \geq r_\nu$  for any  $\nu \in P^+(\pi)$ , Lemma 6.6.4 and (24) imply that  $[\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) : V(\lambda)] > \frac{1}{2}[\mathcal{H} : V(\lambda)]$  for all  $\lambda$  such that  $[\widetilde{V}(\nu_0) : V(\lambda)] > 0$ . Therefore  $\text{Ann } \widetilde{M}(\mu) \neq \mathcal{U}(\mathfrak{g}) \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  by 6.6.2. This proves the implication  $\Rightarrow$ .  $\square$

### 6.7. The corank of $PRV^\nu$ is constant on $\mathfrak{h}_{sd}^*$ .

Retain notation of 6.6.3. In view of Claim 6.6.6, we should verify that for any  $\mu \in \mathfrak{h}_{sd}^*$  and  $\nu \in P^+(\pi)$  the corank of  $PRV^\nu(\mu)$  is equal to  $r_\nu$ . In order to do this, we shall use the matrices  $PRV_\mu^\nu$  introduced in 3.2.

Throughout this subsection  $\mu \in \mathfrak{h}_{sd}^*$  is fixed.

**6.7.1. Notation.** Introduce some new notations. Set

$$\mathfrak{m} := \text{Ann}_{\mathcal{Z}(\mathfrak{g})} \widetilde{M}(\mu).$$

Denote by  $I$  the intersection

$$I := \bigcap_{\mu' \in \mathfrak{h}_d^*} \text{Ann} \widetilde{M}(\mu').$$

By Lemma 6.1.3 (i),  $I \supseteq \mathcal{KAT}$ . On the other hand, by Corollary 6.6.5

$$I \subseteq \bigcap_{\mu' \in \mathfrak{h}_d^*, d(\mu')=1} \text{Ann} \widetilde{M}(\mu') = \mathcal{K} \bigcap_{\mu' \in \mathfrak{h}_d^*, d(\mu')=1} \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu').$$

Since the set  $\{\mu' \in \mathfrak{h}_d^*, d(\mu') = 1\}$  is a Zariski dense subset of  $\mathfrak{h}_d^*$ , one has by Lemma 6.14 in [GL]

$$\bigcap_{\mu' \in \mathfrak{h}_d^*, d(\mu')=1} \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu') = \bigcap_{\mu' \in \mathfrak{h}_d^*} \text{Ann}_{\mathcal{A}} \widetilde{M}(\mu') = \mathcal{AT}.$$

Finally

$$I = \mathcal{KAT}. \tag{25}$$

Choose  $\mathfrak{g}$ -submodules  $L$  and  $L'$  of  $\mathcal{U}(\mathfrak{g})$  such that

$$I = L \oplus (I \cap \mathcal{U}(\mathfrak{g})\mathfrak{m}) \quad \mathcal{U}(\mathfrak{g}) = L' \oplus (I + \mathcal{U}(\mathfrak{g})\mathfrak{m}). \tag{26}$$

Setting  $\mathcal{L} := L + L'$  one has

$$\mathcal{L} = L \oplus L', \quad \mathcal{U}(\mathfrak{g}) = \mathcal{L} \oplus \mathcal{U}(\mathfrak{g})\mathfrak{m}. \tag{27}$$

**6.7.2.** Fix  $\nu \in P^+(\pi)$ . Set  $n := \dim(\widetilde{V}(\nu)|_0)$  and  $s(\nu) := [L : \widetilde{V}(\nu)]$ . From (27) it follows that

$$\dim \widetilde{V}(\nu)|_0 = [\mathcal{H} : \widetilde{V}(\nu)] = [\mathcal{L} : \widetilde{V}(\nu)] = [L : \widetilde{V}(\nu)] + [L' : \widetilde{V}(\nu)].$$

Let  $\{\theta_1, \dots, \theta_{s(\nu)}\}$  and  $\{\theta_{s(\nu)+1}, \dots, \theta_n\}$  be bases of respectively  $\text{Hom}_{\mathfrak{g}}(\widetilde{V}(\nu), L)$  and  $\text{Hom}_{\mathfrak{g}}(\widetilde{V}(\nu), L')$ . Let  $\{v_1, \dots, v_n\}$  be a basis of  $\widetilde{V}(\nu)|_0$ . Then the matrix  $PRV_{\mathcal{L}}^\nu$  given by the formula

$$(PRV_{\mathcal{L}}^\nu)_{ij=1}^n := \left( \Upsilon(\theta_j(v_i)) \right)_{ij}$$

is a  $PRV_\mu^\nu$  matrix in the sense of 3.2.

We will take advantage of the fact that  $\{\theta_i(\widetilde{V}(\nu))\} \subset L \subset I$  for  $i = 1, \dots, s(\nu)$ . Indeed, since  $I$  is the intersection of  $\text{Ann} \widetilde{M}(\mu')$  for  $\mu' \in \mathfrak{h}_d^*$ , the projection  $\Upsilon(I)$  lies in the set of polynomials in  $\mathcal{S}(\mathfrak{h})$  which vanish on the set  $h_d^*$ . The set of such polynomials is the ideal

in  $\mathcal{S}(\mathfrak{h})$  generated by  $t$ . Hence the entries of the first  $s(\nu)$  columns of the matrix  $PRV_{\mathcal{L}}^{\nu}$  are divisible by  $t$ .

Retain notations of 6.6.3. One has

**6.7.3. Lemma.** *For any  $\nu \in P^+(\pi)$ ,  $r_{\nu} = s(\nu)$ .*

*Proof.* By Corollary 3.2.2 (ii),  $\det PRV_{\mathcal{L}}^{\nu} = t^{r_{\nu}} P_1$  where  $P_1(\mu) \neq 0$ . Recall that the entries of the first  $s(\nu)$  columns of the matrix  $PRV_{\mathcal{L}}^{\nu}$  are divisible by  $t$  and  $t(\mu) = 0$  since  $\mu \in \mathfrak{h}_d^*$ . It implies that  $s(\nu) \leq r_{\nu}$  for all  $\nu \in P^+(\pi)$ .

By formula (25) and lemma 6.1.3 (ii) one has the following isomorphisms of  $\mathfrak{g}_0$ -modules

$$L \cong I/(I \cap \mathcal{U}(\mathfrak{g})\mathfrak{m}) = (\mathcal{K}AT)/(\mathcal{K}(AT \cap \mathfrak{m})) \cong \mathcal{K}.$$

Since  $s(\nu) = [L : \tilde{V}(\nu)]$ , it follows that

$$[\mathcal{K} : V(\lambda)] = [L : V(\lambda)] = \sum_{\nu \in P^+(\pi)} s(\nu) [\tilde{V}(\nu) : V(\lambda)] \quad \forall \lambda \in P^+(\pi).$$

On the other hand combining (24), Lemma 6.6.4 and the multiplicity formula (12) we obtain

$$[\mathcal{K} : V(\lambda)] = \sum_{\nu \in P^+(\pi)} r_{\nu} [\tilde{V}(\nu) : V(\lambda)].$$

Comparing two last equalities and taking into account that  $0 \leq s(\nu) \leq r_{\nu}$  for all  $\nu \in P^+(\pi)$ , we conclude that  $s(\nu) = r_{\nu}$  for all  $\nu \in P^+(\pi)$ . The lemma is proven.  $\square$

**6.7.4.** Summarizing 6.7.2 and Lemma 6.7.3, we obtain that the entries of the first  $r_{\nu}$  columns of the matrix  $PRV_{\mathcal{L}}^{\nu}$  are divisible by  $t$ . Using Lemma 3.2.2 (ii), we conclude that the corank of the matrix  $PRV_{\mathcal{L}}^{\nu}(\mu)$  is equal to  $r_{\nu}$ . Therefore the corank of the matrix  $PRV^{\nu}(\mu)$  is also equal to  $r_{\nu}$  by Lemma 3.2.2 (i). Hence Theorem 6.2 follows from Claim 6.6.6.

**6.8. The proof of Corollary 6.4.** If  $\mu \in \mathfrak{h}^* \setminus \mathfrak{h}_d^*$  the Corollary follows directly from Theorem 3.1.2 and Theorem 2.9.

Consider the case  $\mu \in \mathfrak{h}_d^*$ . By Corollary 6.3 (i), it is enough to verify the claim in the case when  $\tilde{M}(\mu)$  is simple that is  $\mu \in \mathfrak{h}_{sd}^*$ . Combining Claim 6.6.6, (5) and (2), we obtain

$$[\text{Ann}_{\mathcal{H}} \tilde{M}(\mu) : \tilde{V}(\lambda)] = r_{\lambda}$$

for any  $\mu \in \mathfrak{h}_{sd}^*$ . It is easy to deduce from Theorem 2.9 that  $\mathcal{U}(\mathfrak{g})/\text{Ann} \tilde{M}(\mu)$  is isomorphic to  $\mathcal{H}/\text{Ann}_{\mathcal{H}} \tilde{M}(\mu)$  as  $\mathfrak{g}$ -modules. Hence for any  $\mu \in \mathfrak{h}_{sd}^*$

$$\begin{aligned} [\mathcal{U}(\mathfrak{g})/\text{Ann} \tilde{M}(\mu) : \tilde{V}(\lambda)] &= [\mathcal{H} : \tilde{V}(\lambda)] - [\text{Ann}_{\mathcal{H}} \tilde{M}(\mu) : \tilde{V}(\lambda)] = \\ \dim \tilde{V}(\lambda)|_0 - \sum_{m=1}^{\infty} (-1)^{m+1} \dim \tilde{V}(\lambda)_{m\beta} &= \sum_{m=0}^{\infty} (-1)^m \dim \tilde{V}(\lambda)_{m\beta} \end{aligned}$$

as required.

## 7. DECOMPOSITION OF VERMA MODULES

Theorem 6.2 allows us to give an answer to the following question posed by Musson in [Mu1]: for which  $\mu \in \mathfrak{h}^*$ ,  $\widetilde{M}(\mu)$  is a direct sum of  $\mathfrak{g}_0$ -Verma modules?

**7.1.** Retain notations of 2.7. Since as a  $\mathfrak{g}_0$ -module,  $\widetilde{M}(\mu, \bar{i}) = \widetilde{M}(\mu, \bar{i})_0 \oplus \widetilde{M}(\mu, \bar{i})_1$  the question is to know for which  $\mu$  both  $\widetilde{M}(\mu, \bar{i})_0$  and  $\widetilde{M}(\mu, \bar{i})_1$  are direct sum of  $\mathfrak{g}_0$ -Verma modules. Set

$$\begin{aligned}\Gamma &= \left\{ \sum_{i=1}^l \delta_i \beta_i, \delta_i \in \{0, 1\} \right\} \\ \Gamma_0 &= \left\{ \sum_{i=1}^l \delta_i \beta_i, \delta_i \in \{0, 1\}, \sum_{i=1}^l \delta_i \in 2\mathbb{N} \right\} \\ \Gamma_1 &= \left\{ \sum_{i=1}^l \delta_i \beta_i, \delta_i \in \{0, 1\}, \sum_{i=1}^l \delta_i \in 2\mathbb{N} + 1 \right\}.\end{aligned}$$

By [Mu1] 3.2,  $\widetilde{M}(\mu, \bar{i})_{\bar{i}+\bar{j}}$  considered as a  $\mathfrak{g}_0$ -module, has a filtration whose factors are  $\{M(\mu - \gamma)\}_{\gamma \in \Gamma_{\bar{j}}}$ .

For all  $\mu \in \mathfrak{h}^*$ ,  $\gamma \in \Gamma$ , denote by  $\chi_{\mu-\gamma}^0$  the  $\mathcal{Z}(\mathfrak{g}_0)$ -character of the Verma module  $M(\mu - \gamma)$ . If both multisets  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_0\}$  and  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_1\}$  consist of distinct central characters then  $\mathcal{Z}(\mathfrak{g}_0)$  separates the modules  $\{M(\mu - \gamma), \gamma \in \Gamma_{\bar{j}}\}$  so

$$\widetilde{M}(\mu) = \bigoplus_{\gamma \in \Gamma} M(\mu - \gamma).$$

We will show in Proposition 7.2.2 below that the above condition is also necessary.

**7.2.** Consider the adjoint action of  $\mathfrak{g}$  on the algebra of endomorphisms of  $\widetilde{M}(\mu)$ . Let  $F(\widetilde{M}(\mu))$  be the  $\mathfrak{g}$ -locally finite part of the algebra of endomorphisms of  $\widetilde{M}(\mu)$ . Denote by  $\psi$  the natural  $\mathfrak{g}$ -map  $\mathcal{U}(\mathfrak{g}) \rightarrow F(\widetilde{M}(\mu))$ . The  $\mathfrak{g}_0$ -invariance of  $\psi$  forces  $\psi(\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}) = (\psi(\mathcal{U}(\mathfrak{g})))^{\mathfrak{g}_0}$ . On the other hand, by [Mu1], 5.3 the algebra  $\mathcal{U}(\mathfrak{g})^{\mathfrak{g}_0}$  is generated by  $\mathcal{Z}(\mathfrak{g})$  and  $\mathcal{Z}(\mathfrak{g}_0)$ . Finally

$$(\psi(\mathcal{U}(\mathfrak{g})))^{\mathfrak{g}_0} = \psi(\mathcal{Z}(\mathfrak{g}_0)).$$

**7.2.1.** Let  $\mu \in \mathfrak{h}^*$  be such that

$$\widetilde{M}(\mu) = \bigoplus_{\gamma \in \Gamma} M(\mu - \gamma).$$

Observe that  $\dim(\psi(\mathcal{U}(\mathfrak{g})))^{\mathfrak{g}_0} = \dim \psi(\mathcal{Z}(\mathfrak{g}_0))$  is equal to the number of distinct elements in the set  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma\}$ .

*Case  $\mu \notin \mathfrak{h}_d^*$ .* In this case,  $\text{Ann}_{\mathcal{H}} \widetilde{M}(\mu) = 0$  so, reasoning as in the proof of 6.6.4, one has

$$\dim(\psi(\mathcal{U}(\mathfrak{g})))^{\mathfrak{g}_0} = \dim \mathcal{H}^{\mathfrak{g}_0} = [\mathcal{H} : V(0)] = \dim \text{Ind } V(0)|_0$$

where  $\text{Ind}_{\mathfrak{g}_0}^{\mathfrak{g}} V(\lambda) = \mathcal{U}(\mathfrak{g}) \otimes_{\mathcal{U}(\mathfrak{g}_0)} V(\lambda)$ . From the PBW-theorem,  $\mathcal{U}(\mathfrak{g}) = \mathcal{U}(\mathfrak{g}_0) \otimes \Lambda(\mathfrak{g}_1)$  as  $\mathfrak{h}$ -modules so  $\dim \text{Ind } V(0)_0 = \dim \Lambda(\mathfrak{g}_1)_0 = 2^l$ .

Hence the set  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma\}$  contains  $2^l$  distinct elements that is all its elements are distinct.

*Case  $\mu \in \mathfrak{h}_d^*$ .* In this case, by Theorem 6.2 and Proposition 6.1.1 (i),  $\text{Ann}_{\mathcal{K}} \widetilde{M}(\mu) = 0$  and  $\text{Ann}_{\mathcal{A}} \widetilde{M}(\mu)$  is a maximal ideal. Therefore the image of  $\psi$  is isomorphic to  $\mathcal{K}$  as  $\mathfrak{g}_0$ -modules. Using (12) we get

$$\dim \psi(\mathcal{Z}(\mathfrak{g}_0)) = \dim(\psi(\mathcal{U}(\mathfrak{g})))^{\mathfrak{g}_0} = \frac{1}{2} \dim \mathcal{H}^{\mathfrak{g}_0} = 2^{l-1}.$$

Hence the set  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma\}$  contains  $2^{l-1}$  distinct elements.

Let  $\beta$  be an odd root such that  $(\mu + \rho, \beta) = 0$  (it exists since  $\mu \in \mathfrak{h}_d^*$ ). For any  $\gamma \in \Gamma$  the central character  $\chi_{\mu-\gamma}^0$  coincides with the central character  $\chi_{s_\beta^*(\mu-\gamma)}^0$ , where

$$s_\beta * \nu = s_\beta(\nu + \rho_0) - \rho_0, \quad \nu \in \mathfrak{h}^*.$$

One has

$$s_\beta^*(\mu - \gamma) = \mu - \gamma - 2(\mu - \gamma + \rho_0, \beta)\beta = \mu - \gamma - 2(\mu - \gamma + \rho + \rho_1, \beta)\beta = \mu - \gamma - 2(\rho_1 - \gamma, \beta)\beta.$$

This implies that for any  $\gamma \in \Gamma_0$  (resp.  $\gamma \in \Gamma_1$ ) there exists  $\gamma' \in \Gamma_1$  (resp.  $\gamma' \in \Gamma_0$ ) such that  $s_\beta^*(\mu - \gamma) = \mu - \gamma'$ . Consequently, the sets of the central characters  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_0\}$  and  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_1\}$  coincide. Since the union of these sets contains  $2^{l-1}$  distinct elements, we conclude that each of them contains  $2^{l-1}$  distinct elements. Hence both sets  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_0\}$  and  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_1\}$  consist of distinct central characters.

We can summarize our conclusions as follows

**7.2.2. Corollary.** *The module  $\widetilde{M}(\mu)$  is a direct sum of  $\mathfrak{g}_0$ -Verma modules iff both sets  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_0\}$  and  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_1\}$  consist of distinct central characters.*

*Remark.* By [Mu1], 3.11 both sets  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_0\}$  and  $\{\chi_{\mu-\gamma}^0, \gamma \in \Gamma_1\}$  consist of distinct central characters iff  $(\mu + \rho, \alpha) \neq 0$  for all  $\alpha \in \overline{\Delta}_0^+$ .

## 8. APPENDIX: INDEX OF NOTATIONS

Symbols used frequently are given below under the section number where they are first defined.

- 2  $\Delta_0^+, \Delta_1^+, \Delta_{irr}, \rho, W$
- 2.2  $\beta_i$
- 2.3  $D, S$
- 2.5  $M_\lambda$
- 2.6  $\Upsilon$
- 2.7  $\widetilde{M}(\lambda, \bar{i})$

- 2.7.1  $\widetilde{M}(\lambda), \widetilde{V}(\lambda), \mathfrak{h}_s^*$
- 2.8  $V(\lambda)$
- 2.9  $\mathcal{H}$
- 3.1  $PRV^\nu$
- 3.1.1  $t, r_\nu, \mathfrak{h}_d^*, \mathfrak{h}_{sd}^*$
- 3.1.3  $d(\mu)$
- 3.2  $PRV_\mu^\nu$
- 4.1  $\mathcal{A}$
- 4.4.1  $\sigma, T$
- 5.2  $\mathcal{K}$

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