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On simplicity of vacuum modules

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Abstract

We find necessary and sufficient conditions of irreducibility of vacuum modules over affine Lie algebras and superalgebras. From this we derive conditions of simplicity of minimal W-algebras. Moreover, in the case of the Virasoro and Neveu–Schwarz algebras we obtain explicit formulas for the vacuum determinants.

Keywords: Vacuum module; Vacuum determinant; Contragredient Lie superalgebra; W-algebra; Kazhdan–Lusztig polynomial

0. Introduction

0.1. One of the aims of the present paper is to find conditions of irreducibility of vacuum modules over the affine Lie superalgebra

$$\hat{\mathfrak{g}} = \mathfrak{g}\left[t, t^{-1}\right] + \mathbb{C}K,$$
$$\left[at^m, bt^n\right] = [a, b]t^{m+n} + m\delta_{m,-n}B(a|b)K,$$
$$\left[at^m, K\right] = 0,$$

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associated to a simple finite-dimensional Lie superalgebra $\mathfrak{g}$ with a non-degenerate even invariant bilinear form $B(.|.)$. Recall that the *vacuum module* is the induced module

$$V^k = \mathrm{Ind}_{\mathfrak{g}[t]+\mathbb{C}K}^{\hat{\mathfrak{g}}} \mathbb{C}_k$$

from the 1-dimensional module $\mathbb{C}_k$ with trivial action of $\mathfrak{g}[t]$ and $K = k \in \mathbb{C}$.

0.2. In order to state the result, let $2h_B^\vee$ be the eigenvalue of the Casimir operator $\sum_i a_i a^i$ in the adjoint representation of $\mathfrak{g}$, where $\{a_i\}$ and $\{a^i\}$ are dual bases of $\mathfrak{g}$, i.e. $B(a^i|a_j) = \delta_{ij}$. The numbers $h_B^\vee$ and k depends on the normalization of the bilinear form B: if B is multiplied by a non-zero number γ, then both $h_B^\vee$ and k get multiplied by γ^{-1}.

For a simple Lie algebra $\mathfrak{g}$ the standard normalization is $B(\alpha|\alpha) = 2$ for a long root α. In this case, $h_B^\vee$ is called the dual Coxeter number; it is a positive integer, denoted by $h^\vee$ (these integers are listed, e.g. in [13]). For simple Lie superalgebras a "standard" normalization of B was introduced, and the values of $h^\vee$ listed, in [17].

For a non-isotropic root α introduce

$$k_\alpha := \frac{k + h_B^\vee}{B(\alpha|\alpha)}.$$

Note that this number is independent on the normalization of B.

0.2.1. Theorem. *Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. The vacuum $\hat{\mathfrak{g}}$-module V^k is not irreducible if and only if $k_\alpha \in \mathbb{Q}_{\geqslant 0} \setminus \{\frac{1}{2m}\}_{m=1}^\infty$ for a short root α of $\mathfrak{g}$ (equivalently, if and only if $l(k + h^\vee)$ is a non-negative rational number which is not the inverse of an integer, where l is the ratio of the lengths squared of a long and a short root of $\mathfrak{g}$).*

0.2.2. Theorem. *Let $\mathfrak{g}$ be a simple Lie superalgebra $\mathfrak{osp}(1, 2n)$. The vacuum $\hat{\mathfrak{g}}$-module V^k is not irreducible if and only if $k_\alpha \in \mathbb{Q}_{\geqslant 0} \setminus \{\frac{1}{2m+1}\}_{m=0}^\infty$ where α is an odd root of $\mathfrak{g}$ (equivalently, if and only if $k + 2n + 1$ is a non-negative rational number which is not the inverse of an odd integer, if $B(\alpha|\alpha) = 1$ for an odd root α of $\mathfrak{g}$).*

0.2.3. Conjecture. *Let $\mathfrak{g}$ be an (almost) simple finite-dimensional Lie superalgebra of positive defect* [17], *i.e. one of the Lie superalgebras $\mathfrak{sl}(m, n)$ $(m, n \geqslant 1)$, $\mathfrak{osp}(m, 2n)$ $(m \geqslant 2,\ n \geqslant 1)$, $D(2, 1, a)$, $F(4)$ or $G(3)$. Then the $\hat{\mathfrak{g}}$-module V^k is not irreducible if and only if*

$$k_\alpha \in \mathbb{Q}_{\geqslant 0} \quad \textit{for some even root } \alpha \textit{ of } \mathfrak{g}. \tag{1}$$

Note that V^k is always reducible at the *critical* level $k = -h_B^\vee$.

0.2.4. Theorem. *Conjecture* 0.2.3 *holds for simple Lie superalgebras of defect* 1, *i.e.* $\mathfrak{g} = \mathfrak{sl}(1, n), \mathfrak{osp}(2, n), \mathfrak{osp}(n, 2), \mathfrak{osp}(3, n)$ *with $n \geqslant 2$, $D(2, 1, a)$, $F(4)$, $G(3)$, and for* $\mathfrak{g} = \mathfrak{gl}(2, 2)$.

More explicitly, in the standard normalization (see 10.1.2) for the Lie superalgebras $\mathfrak{g} = \mathfrak{sl}(1, n), \mathfrak{osp}(2, 2n)$, the module V^k is not irreducible if and only if $k + n - 1$ is a non-negative rational number. For the Lie superalgebras $\mathfrak{g} = \mathfrak{osp}(3, n), \mathfrak{osp}(n, 2)$ with $n > 2$, $F(4), G(3), D(2, 1, a)$ with $a \in \mathbb{Q}$, the module V^k is not irreducible iff $k + h^\vee$ is a rational number, where $h^\vee$ is given in the table in 10.1.2. For $D(2,1,a)$, $a \notin \mathbb{Q}$, the vacuum module V^k is not irreducible iff $k \in \mathbb{Q}_{\geqslant 0} \cup \mathbb{Q}_{>0}a \cup \mathbb{Q}_{>0}(-1-a)$. For $\mathfrak{g} = \mathfrak{gl}(2,2)$ the standard normalization is $B(\alpha|\alpha) = 2$ for an even root α; the module V^k is not irreducible iff k is a rational number.

0.2.5. In order to prove these results, we derive a formula for the determinant of the Shapovalov form on any generalized Verma module, induced from a 1-dimensional representation of a parabolic subalgebra of an arbitrary symmetrizable contragredient Lie superalgebra, using methods of [8] and [15]. Unfortunately, unlike in the Verma module case [15], the exponents of the factors of the determinant are rather complicated alternating sums, and it is a non-trivial problem to find when these sums are positive. It is a very interesting problem to find a determinant formula for a vacuum module over $\hat{\mathfrak{g}}$ with manifestly positive exponents.

0.3. We were unable to find such a formula for affine Lie superalgebras, but we did succeed in the case of the Virasoro algebra $\mathcal{V}ir$ and the Neveu–Schwarz superalgebra $\mathcal{NS}$.

Recall that $\mathcal{V}ir$ is a Lie algebra with a basis $\{L_n\ (n \in \mathbb{Z}), C\}$ and commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{m^3-m}{12}\delta_{m,-n}C, \qquad [C, L_m] = 0. \tag{2}$$

Given $c \in \mathbb{C}$, a *vacuum module* over $\mathcal{V}ir$ is the induced module

$$V^c = \operatorname{Ind}_{\mathcal{V}ir_+}^{\mathcal{V}ir} \mathbb{C}_c,$$

where $\mathcal{V}ir_+ = \mathbb{C}C + \sum_{n\geqslant -1} \mathbb{C}L_n$ and $\mathbb{C}_c$ is the 1-dimensional $\mathcal{V}ir_+$-module with trivial action of L_n's, and $C = c$. The problem is to compute the determinant of the Shapovalov form, restricted to the Nth eigenspace of L_0 in V^c, $N \in \mathbb{Z}_{\geqslant 0}$. This is a polynomial in c, which we denote by $\det'_N(c)$. (It is defined up to a non-zero constant factor, depending on a basis of the eigenspace.)

0.3.1. In the case of a Verma module over $\mathcal{V}ir$ the answer is given by the Kac determinant formula [12,16]:

$$\det_{N+h}(h,c) = const \prod_{r,s \in \mathbb{Z}_{\geqslant 1}} \varphi_{r,s}(h,c)^{p_{cl}(N-rs)},$$

where h is the eigenvalue of L_0 on the highest weight vector $|h,c\rangle$, $\varphi_{r,s}(h,c)$ are some (explicitly known) polynomials of c and h of degree $\leqslant 2$, and p_{cl} is the classical partition function. From this, using that a Verma module over $\mathcal{V}ir$ has no subsingular vectors (see [2] for a simple proof of this fact), one obtains immediately the roots of $\det'_N(c)$, but it is a non-trivial problem to compute exponents. We obtain the following formula (via checking a simple combinatorial identity):

$$\det{}'_N(c) = const \prod_{\substack{p>q\geqslant 2\\(p,q)=1}} \left(c - \left(1 - \frac{6(p-q)^2}{pq}\right)\right)^{\dim L_N^{p,q}}, \tag{3}$$

where $p, q \in \mathbb{Z}_{\geqslant 2}$, $L^{p,q} = L((p-1)(q-1); 1 - \frac{6(p-q)^2}{pq})$ is the irreducible highest weight $\mathcal{V}ir$-module with the lowest eigenvalue of L_0 equal $(p-1)(q-1)$ and $c = 1 - \frac{6(p-q)^2}{pq}$, and $L_N^{p,q}$ is the Nth eigenspace of L_0 in $L^{p,q}$. The dimensions of these eigenspaces are known explicitly [4]:

$$\dim L_N^{p,q} = \sum_{j \in \mathbb{Z} \setminus \{0\}} \bigl(p_{cl}\bigl(N - (jp+1)(jq+1)\bigr) - p_{cl}\bigl(N - (jp+1)(jq-1) - 1\bigr)\bigr). \tag{4}$$

Next, we prove the following fact (which can be deduced from [4], but our proof is simpler and can be extended to other cases).

0.3.2. Theorem. *Let $v \in V^c$ be an eigenvector of L_0, killed by all L_n with $n > 0$, and not proportional to the highest weight vector $|0, c\rangle$. Then $L_0 v = 2N v$ for some positive integer N, and in the decomposition*

$$v = \sum_{\substack{j_1 \geqslant j_2 \geqslant \cdots \geqslant j_s \geqslant 2 \\ j_1 + j_2 + \cdots + j_s = 2N}} c_{j_1 j_2 \ldots j_s} L_{-j_1} L_{-j_2} \ldots L_{-j_s} |0, c\rangle$$

the coefficient of $L_{-2}^N |0, c\rangle$ is non-zero.

The following corollary of formula (3) and Theorem 0.3.2 is well known.

0.3.3. Corollary. *The following conditions on the $\mathcal{V}ir$-module V^c are equivalent*:

(i) V^c *is not irreducible*;
(ii) $c = 1 - \frac{6(p-q)^2}{pq}$ *for some relatively prime integers $p, q \in \mathbb{Z}_{\geqslant 2}$*;
(iii) $\operatorname{span}\{L_{-n} v \mid n > 2,\ v \in V_c\}$, *where V_c is the irreducible factor module of the $\mathcal{V}ir$-module V^c, has finite codimension in V_c.*

We also obtain results, analogous to formula (3), Theorem 0.3.2, and Corollary 0.3.3, for the Neveu–Schwarz algebra, the simplest super extension of the Virasoro algebra.

0.4. Recall (see e.g. [14]) that the $\hat{\mathfrak{g}}$-module V^k (respectively, $\mathcal{V}ir$-module V^c) carries a canonical structure of a vertex algebra, and the irreducibility of these modules is equivalent to the simplicity of the associated vertex algebras, i.e. to the isomorphism $V_k \cong V^k$ (respectively, $V_c \cong V^c$). An important problem, coming from conformal field theory, is when a vertex algebra satisfies Zhu's C_2 condition [24]. In the case of the vertex algebra V_c, C_2 condition is property (iii) of Corollary 0.3.3; thus, this corollary says that V_c satisfies C_2 condition if and only if V^c is not simple. We also show that the same property holds for the Neveu–Schwarz algebra (but it does not hold for $N > 1$ superconformal algebras).

It is easy to see that the vertex algebras V^k and the vertex algebras $W^k(\mathfrak{g}, f)$, obtained from V^k by quantum Hamiltonian reduction (where f is a nilpotent even element of $\mathfrak{g}$) [18,19] never satisfy the C_2 condition. It is also not difficult to show that among their quotients only the simple ones have a chance to satisfy the C_2 condition, and a simple affine vertex algebra V_k satisfies the C_2 condition if and only if $\mathfrak{g}$ is either a simple Lie algebra, or $\mathfrak{g} = \mathfrak{osp}(1, 2n)$ (i.e. $\mathfrak{g}$ has defect zero), and the $\hat{\mathfrak{g}}$-module V_k is integrable.

Much more non-trivial is the problem for the simple quotients $W_k(\mathfrak{g}, f)$ of the vertex algebra $W^k(\mathfrak{g}, f)$, which includes the Virasoro, Neveu–Schwarz, and other superconformal algebras. It has been proved in many cases [1] that the image of a simple V^k-module under the quantum Hamiltonian reduction is either a simple $W^k(\mathfrak{g}, f)$-module, or 0. Using this, Theorem 0.2.1 and the Kazhdan–Lusztig theory, we were able to find the necessary and sufficient conditions on k for which $W^k(\mathfrak{g}, f)$ is simple in the case when $\mathfrak{g}$ is a simple Lie algebra and f is a minimal nilpotent element. Namely, for $\mathfrak{g} \neq \mathfrak{sl}_2$, the k for which $W^k(\mathfrak{g}, f)$ is simple are given by Theorem 0.2.1 (since $W^k(\mathfrak{sl}_2, f)$ is the Virasoro vertex algebra, in this case the answer is given by Corollary 0.3.3). Consequently, for these values of k the vertex algebra $W_k(\mathfrak{g}, f)$ does not satisfy the C_2 condition.

1. Preliminaries

Our base field is $\mathbb{C}$. We set $\mathbb{Z}_{\geqslant n} := \{m \in \mathbb{Z} \mid m \geqslant n\}$. If V is a superspace, we denote by $p(v)$ the parity of a vector $v \in V$. For a Lie superalgebra $\mathfrak{g}$ considered in this paper, any root space $\mathfrak{g}_\gamma$ is either pure even or pure odd; we denote by $p(\gamma) \in \mathbb{Z}/2\mathbb{Z}$ the parity of $\mathfrak{g}_\gamma$ and let $s(\gamma) = (-1)^{p(\gamma)}$. For a Lie (super)algebra $\mathfrak{g}$ we denote by $\mathcal{U}(\mathfrak{g})$ its universal enveloping (super)algebra.

1.1. Contragredient Lie superalgebras

Let J be a finite index set, and let $p: J \to \mathbb{Z}/2\mathbb{Z} = \{\bar{0}, \bar{1}\}$ be a map called the *parity map*. Consider a triple $\mathcal{A} = (\mathfrak{h}, \Pi, \Pi^\vee)$, where $\mathfrak{h}$ is a finite-dimensional vector space over $\mathbb{C}$, $\Pi = \{\alpha_i\}_{j\in J}$ is a linearly independent subset of $\mathfrak{h}^*$, and $\Pi^\vee = \{h_i\}_{j\in J}$ is a linearly independent set of vectors of $\mathfrak{h}$. One associates to the data $(\mathcal{A}, p)$ the *contragredient Lie superalgebra* $\mathfrak{g}(\mathcal{A}, p)$ as follows [11,13].

1.1.1. First, introduce an auxiliary Lie superalgebra $\tilde{\mathfrak{g}}(\mathcal{A}, p)$ with the generators e_j, f_j $(j \in J)$ and $\mathfrak{h}$, the parity defined by $p(e_j) = p(f_j) = p(j)$, $p(\mathfrak{h}) = \bar{0}$, and the following defining relations:

$$[e_i, f_j] = \delta_{ij} h_i \quad (i, j \in J), \qquad [h, h'] = 0 \quad (h, h' \in \mathfrak{h}),$$
$$[h, e_j] = \alpha_j(h) e_j, \qquad [h, f_j] = -\alpha_j(h) f_j \quad (h \in \mathfrak{h},\ j \in J).$$

The free abelian group Q on generators $\{\alpha_i\}_{j\in J}$ is called the *root lattice*. Denote by Q^+ the subset of Q, consisting of linear combinations of α_j with non-negative coefficients. Define the standard partial ordering on $\mathfrak{h}^*$: $\alpha \geqslant \beta$ for $\alpha - \beta \in Q^+$. Letting

$$\deg e_j = \alpha_j = -\deg f_j, \qquad \deg \mathfrak{h} = 0$$

defines a Q-grading of the Lie superalgebra $\tilde{\mathfrak{g}}(\mathcal{A}, p)$:

$$\tilde{\mathfrak{g}}(\mathcal{A}, p) = \bigoplus_{\alpha\in Q} \tilde{\mathfrak{g}}_\alpha.$$

It is clear that each $\tilde{\mathfrak{g}}_\alpha$ has parity $p(\alpha)$, where $p: Q \to \mathbb{Z}/2\mathbb{Z}$ is defined by additively extending $p: J \to \mathbb{Z}/2\mathbb{Z}$. One has the triangular decomposition

$$\tilde{\mathfrak{g}}(\mathcal{A}, p) = \tilde{\mathfrak{n}}_- \oplus \mathfrak{h} \oplus \tilde{\mathfrak{n}}_+,$$

where $\tilde{\mathfrak{n}}_-$ (respectively, $\tilde{\mathfrak{n}}_+$) is a subalgebra of $\tilde{\mathfrak{g}}(\mathcal{A}, p)$ generated by the f_j's (respectively, e_j's). Consequently, $\tilde{\mathfrak{g}}_0 = \mathfrak{h}$, $\tilde{\mathfrak{n}}_\pm = \bigoplus_{\alpha \in Q^+} \tilde{\mathfrak{g}}_{\pm\alpha}$.

1.1.2. Let $I(\mathcal{A}, p)$ be the sum of all Q-graded ideals of $\tilde{\mathfrak{g}}(\mathcal{A}, p)$, which have zero intersection with the subalgebra $\mathfrak{h}$, and let

$$\mathfrak{g}(\mathcal{A}, p) := \tilde{\mathfrak{g}}(\mathcal{A}, p)/I(\mathcal{A}, p).$$

The Lie superalgebra $\mathfrak{g}(\mathcal{A}, p)$ carries the induced root space decomposition

$$\mathfrak{g}(\mathcal{A}, p) = \bigoplus_{\alpha \in Q} \mathfrak{g}_\alpha,$$

and the induced triangular decomposition

$$\mathfrak{g}(\mathcal{A}, p) = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+, \quad \text{where } \mathfrak{g}_0 = \mathfrak{h},\ \mathfrak{n}_\pm = \bigoplus_{\alpha \in Q^+} \mathfrak{g}_{\pm\alpha}.$$

An element $\alpha \in Q$ is called a *root* (respectively, a *positive root*) if $\dim \mathfrak{g}_\alpha \neq 0$ and $\alpha \neq 0$ (respectively, $\alpha \in Q^+$). Denote by Δ the set of all roots α and by $\Delta^+ = \Delta \cap Q^+$ the set of all positive roots.

The Lie superalgebra $\mathfrak{g}(\mathcal{A}, p)$ carries an anti-involution σ (i.e. $\sigma([a, b]) = [\sigma(b), \sigma(a)]$ and $\sigma^2 = \mathrm{id}$) defined on the generators by

$$\sigma(e_j) = f_j, \qquad \sigma(f_j) = e_j, \qquad \sigma|_{\mathfrak{h}} = \mathrm{id}_{\mathfrak{h}}\,.$$

1.1.3. The matrix $A := (\alpha_j(h_i))_{i,j \in J}$ is called the *Cartan matrix* of the data $\mathcal{A}$ (one can show that A and $\dim \mathfrak{h}$ uniquely determine $\mathcal{A}$, and that, given A the triple $\mathcal{A}$ exists iff $\dim \mathfrak{h} \geqslant |J| + \operatorname{corank} A$). The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \mathit{diag}(d_j)_{j \in J}$, such that the matrix $DA = (b_{ij})$ is symmetric. It is easy to see that if A is symmetrizable then there exists a non-degenerate symmetric bilinear form $(.|.)$ on $\mathfrak{h}$ such that

$$d_j(h_j|h) = \alpha_j(h) \quad \text{for all } j \in J,\ h \in \mathfrak{h}. \tag{5}$$

This bilinear form induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$, defined by $\nu(h)(h') = (h, h')$, $h, h' \in \mathfrak{h}$, and we have:

$$\alpha_j = d_j \nu(h_j), \quad j \in J,$$

and, for the induced bilinear form $(.|.)$ on $\mathfrak{h}^*$ we have:

$$(\alpha_i|\alpha_j) = b_{ij}, \quad i, j \in J.$$

1.1.4. The following proposition is proved as in [13].

Proposition. *Suppose that the Cartan matrix A is symmetrizable, and let $(.|.)_{\mathfrak{h}}$ be a non-degenerate symmetric bilinear form on $\mathfrak{h}$. Then $\mathfrak{g}(\mathcal{A}, p)$ carries a unique invariant bilinear form $(.|.)$ (i.e. $([a,b]|c) = (a|[b,c])$), whose restriction to $\mathfrak{h}$ is the bilinear form $(.|.)_{\mathfrak{h}}$ if and only if $(.|.)_{\mathfrak{h}}$ satisfies* (5) *for some non-zero d_j's such that the matrix DA is symmetric. Moreover, this bilinear form has the following properties*:

(i) $(\mathfrak{g}_\alpha|\mathfrak{g}_\beta) = 0$ *if* $\alpha + \beta \neq 0$, $(.|.)_{\mathfrak{g}_\alpha + \mathfrak{g}_{-\alpha}}$ *is non-degenerate for* $\alpha \in \Delta$;
(ii) $[a,b] = (a|b)\nu^{-1}(\alpha)$, *if* $a \in \mathfrak{g}_\alpha$, $b \in \mathfrak{g}_{-\alpha}$, $\alpha \in \Delta$;
(iii) $(.|.)$ *is supersymmetric.*

1.1.5. Choose $\rho \in \mathfrak{h}^*$ in such a way that

$$\rho(h_j) = \alpha_j(h_j)/2 = a_{jj}/2 \quad \text{for any } j \in J.$$

One has $(\rho|\alpha_j) = (\alpha_j|\alpha_j)/2$ if A is symmetrizable. (Note that if $\det A = 0$ then ρ is not uniquely defined.)

Assume that A is symmetrizable. For each $\alpha \in \Delta^+ \cup \{0\}$ choose a basis $\{e_\alpha^i\}$ of $\mathfrak{g}_\alpha$ and the dual basis $\{e_{-\alpha,i}\}$ of $\mathfrak{g}_{-\alpha}$, i.e. $(e_\alpha^i|e_{-\alpha,j}) = \delta_{ij}$, and define the generalized Casimir operator

$$\Omega := 2\nu^{-1}(\rho) + \sum_i e_{0,i}e_0^i + 2\sum_{\alpha\in\Delta^+}\sum_i e_{-\alpha,i}e_\alpha^i.$$

This operator is well defined in any restricted $\mathfrak{g}(\mathcal{A}, p)$-module V, i.e. a module V such that for any $v \in V$, $\mathfrak{g}_\alpha v = 0$ for all but finitely many $\alpha \in \Delta^+$. The following proposition is proved as in [13]:

Proposition.

(i) *The operator Ω commutes with $\mathfrak{g}(\mathcal{A}, p)$ in any restricted $\mathfrak{g}(\mathcal{A}, p)$-module.*
(ii) *If N is a $\mathfrak{g}(\mathcal{A}, p)$-module and $v \in N$ is such that $e_j v = 0$ for all $j \in J$, and for some $\lambda \in \mathfrak{h}^*$ one has $hv = \lambda(h)v$ for all $h \in \mathfrak{h}$, then $\Omega(v) = (\lambda + 2\rho|\lambda)v$. Moreover, if v generates the module N then N is restricted and $\Omega = (\lambda + 2\rho|\lambda)\,\mathrm{Id}_N$.*

1.1.6. Let $s(\alpha) = (-1)^{p(\alpha)}$ for $\alpha \in Q$, and introduce the following (in general infinite) product:

$$R := \prod_{\alpha\in\Delta^+}\left(1 - s(\alpha)e^{-\alpha}\right)^{s(\alpha)\dim\mathfrak{g}_\alpha}.$$

Using the geometric series, we can expand the inverse of this product:

$$R^{-1} = \sum_{\alpha\in Q^+} K(\alpha)e^{-\alpha}, \quad \text{where } K(\alpha) \in \mathbb{Z}_{\geqslant 0}.$$

Set $K(\mu)=0$ for $\mu \in Q \setminus Q^+$. Note that $K(\alpha)$ (the Kostant partition function) is the number of partition of α into a sum of positive roots (counting multiplicities), where odd roots appear at most once.

1.2. Generalized Verma modules

Write $\mathfrak{g}(\mathcal{A}, p)$ as $\mathfrak{g}$. Given $I \subset J$, let Q_I be the $\mathbb{Z}$-span of $\{\alpha_i\}_{i\in I}$. Set

$$\mathfrak{n}_{\pm,I} := \bigoplus_{\alpha\in Q_I} \mathfrak{g}_{\pm\alpha}, \qquad \mathfrak{h}_I := \sum_{i\in I} \mathbb{C}h_i, \qquad \mathfrak{h}_I^{\perp} := \bigl\{\lambda \in \mathfrak{h}^* \bigm| \lambda(h_i)=0\ \forall i \in I\bigr\}.$$

Note that $\mathfrak{h}_I = [\mathfrak{n}_+, \mathfrak{n}_{-,I}] \cap \mathfrak{h} = [\mathfrak{n}_{+,I}, \mathfrak{n}_{-,I}] \cap \mathfrak{h}$.

1.2.1. Introduce the following $(\mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_{-,I})$-module structure on the symmetric algebra $S(\mathfrak{h}/\mathfrak{h}_I)$: the action of $(\mathfrak{n}_+ + \mathfrak{h}_I + \mathfrak{n}_{-,I})$ is trivial and $h \in \mathfrak{h}$ acts by the multiplication by the image $\bar{h}$ of the map: $\mathfrak{h} \to \mathfrak{h}/\mathfrak{h}_I$. Introduce the following $\mathfrak{g}$-module:

$$M_I := \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_{-,I}} S(\mathfrak{h}/\mathfrak{h}_I).$$

Note that M_I is a $\mathfrak{g}$–$\mathfrak{h}$ bimodule.

1.2.2. For $\lambda \in \mathfrak{h}_I^{\perp}$ denote by $\operatorname{Ker}_\lambda$ the kernel of λ in $S(\mathfrak{h}/\mathfrak{h}_I)$. The *generalized Verma module* $M_I(\lambda)$ is the evaluation of M_I at λ, that is

$$M_I(\lambda) := M_I/M_I \operatorname{Ker}_\lambda = \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_{-,I}} \mathbb{C}_\lambda,$$

where $\mathbb{C}_\lambda$ is an even one-dimensional space, $(\mathfrak{n}_+ + \mathfrak{h}_I + \mathfrak{h}_{-,I})$ acts trivially on $\mathbb{C}_\lambda$, and $\mathfrak{h}$ acts via the character λ.

1.2.3. If I is empty then $\mathfrak{h}_I^{\perp} = \mathfrak{h}$, $\mathfrak{n}_{-,I} = 0$ and $M_\emptyset(\lambda) = M(\lambda)$ is the usual Verma module with the highest weight λ; denote by $L(\lambda)$ its unique simple quotient. Clearly, for any $I \subset J$ the generalized Verma module $M_I(\lambda)$ is a quotient of $M(\lambda)$.

1.2.4. Identify the universal enveloping algebra $U(\mathfrak{h})$ with the symmetric algebra $S(\mathfrak{h})$. The triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ induces the following decomposition of the universal enveloping superalgebra: $U(\mathfrak{g}) = S(\mathfrak{h}) \oplus (\mathfrak{n}_- U(\mathfrak{g}) + U(\mathfrak{g})\mathfrak{n}_+)$; the corresponding projection $\mathrm{HC}: U(\mathfrak{g}) \to S(\mathfrak{h})$ is called the *Harish-Chandra projection*. Let $\mathrm{HC}_I : U(\mathfrak{g}) \to S(\mathfrak{h}/\mathfrak{h}_I)$ be the composition of HC and the canonical map $S(\mathfrak{h}) \to S(\mathfrak{h}/\mathfrak{h}_I) = S(\mathfrak{h})/S(\mathfrak{h})\mathfrak{h}_I$. Define the bilinear form $S(.,.)$, called the *Shapovalov form*, on M_I as follows:

$$S(u.1, u'.1) = \mathrm{HC}_I\bigl(\sigma(u)u'\bigr) \quad \text{for } u, u' \in U(\mathfrak{g}),$$

where dot denotes the action and 1 stands for the canonical generator of M_I.

It is easy to see that the bilinear form $S: M_I \otimes M_I \to S(\mathfrak{h}/\mathfrak{h}_I)$ satisfies the following properties, which determine it uniquely:

$$S(1,1)=1,$$
$$S(uv,v')=S\big(v,\sigma(u)v'\big)\quad \text{for } u\in\mathfrak{g},\ v,v'\in M_I,$$
$$S(v,v'h)=S(vh,v')=S(v,v')h\quad \text{for } h\in\mathfrak{h},\ v,v'\in M_I. \tag{6}$$

One easily deduces from the uniqueness that this bilinear form is symmetric.

1.2.5. The module M_I is graded by Q^+: $M_I=\bigoplus_{\nu\in Q^+}M_{I,\nu}$, where

$$M_{I,\nu}=\big\{v\in M_I \bigm| hv-vh=-\nu(h)v \text{ for } h\in\mathfrak{h}\big\}.$$

The image of $M_{I,\nu}$ in $M_I(\lambda)$ is the weight space $M_{I,\nu}(\lambda)$ of weight $\lambda-\nu$.

It is easy to see that $S(M_{I,\nu},M_{I,\mu})=0$ for $\nu\neq\mu$. Let S_ν be the restriction of S to $M_{I,\nu}$. Each component $M_{I,\nu}$ is a free $S(\mathfrak{h}/\mathfrak{h}_I)$-module of rank not greater than $K(\nu)$. Therefore $\det S_\nu$ is an element of $S(\mathfrak{h}/\mathfrak{h}_I)$, defined up to a non-zero constant factor, depending on the basis of $M_{I,\nu}$. Clearly, $\det S_\nu=1$ for $\nu=0$.

1.2.6. For $\lambda\in\mathfrak{h}_I^\perp$ the evaluation of S at λ gives a bilinear form

$$S(\lambda): M_I(\lambda)\otimes M_I(\lambda)\to\mathbb{C},$$

whose restriction to $M_{I,\nu}(\lambda)$ is $S_\nu(\lambda)$. It is easy to show that the kernel of $S(\lambda)$ coincides with the maximal proper submodule of $M_I(\lambda)$. As a consequence,

$$M_I(\lambda) \text{ is simple} \quad\Longleftrightarrow\quad \det S_\nu(\lambda)\neq 0\quad \forall\nu\in Q^+.$$

1.2.7. Introduce the following linear function $\phi_\alpha(\lambda)$ on $\mathfrak{h}^*$ for each $\alpha\in Q$:

$$\phi_\alpha(\lambda)=(\lambda+\rho|\alpha)-\frac{1}{2}(\alpha|\alpha).$$

Set $\Delta_I^+:=\Delta\cap Q_I$ and introduce the following product:

$$R_I:=\prod_{\alpha\in\Delta_I^+}\big(1-s(\alpha)e^{-\alpha}\big)^{s(\alpha)\dim\mathfrak{g}_\alpha}.$$

Using the geometric series, we can expand this product:

$$R_I=\sum_{\alpha\in Q_I^+}k_I(\alpha)e^{-\alpha},\quad \text{where } k_I(\alpha)\in\mathbb{Z};$$

set $k_I(\alpha)=0$ for $\alpha\in Q\setminus Q_I^+$.

1.2.8. In Section 2 we will prove the following theorem.

Theorem. *Let $\mathfrak{g}(\mathcal{A}, p)$ be the contragredient Lie superalgebra, attached to the data $(\mathcal{A}, p)$ with a symmetrizable Cartan matrix, and let $I \subset J$. Then one has for $\lambda \in \mathfrak{h}_I^\perp$ (up to a non-zero constant factor depending on the basis of $M_{I,\nu}(\lambda)$):*

$$\det S_\nu(\lambda) = \prod_{r=1}^{\infty} \prod_{\gamma \in \Delta^+ \setminus Q_I^+} \prod_{\alpha \in Q_I^+} \phi_{r\gamma+\alpha}(\lambda)^{(-1)^{(r-1)p(\gamma)} k_I(\alpha) K(\nu - r\gamma - \alpha) \dim \mathfrak{g}_\gamma}.$$

1.2.9. Remark. For $I = \emptyset$, i.e. ordinary Verma modules, $Q_I^+ = \{0\}$, $k_I(\alpha) = \delta_{\alpha,0}$, and we recover the determinant formula from [15] in the non-super case, and from [12] in the super case.

1.2.10. Remark. Let $\mathfrak{h}'$ be a subspace of $\mathfrak{h}$, containing $\Pi^\vee$. Then $\mathfrak{g}'(\mathcal{A}, p) = \mathfrak{n}_- \oplus \mathfrak{h}' \oplus \mathfrak{n}_+$ is a subalgebra of $\mathfrak{g}(\mathcal{A}, p)$, and any generalized Verma module over $\mathfrak{g}'(\mathcal{A}, p)$ extends (non-uniquely) to that over $\mathfrak{g}(\mathcal{A}, p)$ by extending $\lambda \in (\mathfrak{h}')^*$ to a linear function on $\mathfrak{h}$. Defining the weight spaces of the former as that of the latter, it is clear that, by restriction, Theorem 1.2.8 still holds for the generalized Verma module $M_I(\lambda)$ over $\mathfrak{g}'(\mathcal{A}, p)$, and the formula for $\det S_\nu(\lambda)$ is independent of the extension of λ from $\mathfrak{h}'$ to $\mathfrak{h}$.

1.2.11. Example. The simple finite-dimensional Lie algebras carry, of course a unique, up to an isomorphism, structure of a contragredient Lie algebra. Simple Lie superalgebras $\mathfrak{sl}(m,n)$ for $m \neq n$, $\mathfrak{osp}(m,n)$, $D(2,1,a)$, $F(4)$ and $G(3)$ carry a structure of a contragredient Lie superalgebra as well, in fact, several non-isomorphic such structures (which depend on the choice of the set of positive roots) [11].

The Lie superalgebra $\mathfrak{sl}(m,m)$ is not quite a contragredient Lie superalgebra, but $\mathfrak{gl}(m,m)$ is. Hence, by Remark 1.2.10, Theorem 1.2.8 holds for $\mathfrak{sl}(m,m)$ and $\mathfrak{sl}(m,m)/\mathbb{C}I_{2m}$ as well.

1.2.12. Example. If $\mathfrak{g}$ is one of the simple Lie superalgebras from Example 1.2.11, then the affine Lie superalgebra $\hat{\mathfrak{g}} = \mathfrak{g}[t,t^{-1}] \oplus \mathbb{C}K$, described in 0.1 is not quite a contragredient Lie superalgebra, but $\mathbb{C}D \ltimes \hat{\mathfrak{g}}$, where $D = t\frac{d}{dt}$ on $\mathfrak{g}[t,t^{-1}]$ and $[D,K] = 0$, is [13].

Recall that the Cartan subalgebra of $\mathbb{C}D \ltimes \hat{\mathfrak{g}}$ is chosen to be

$$\hat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}D,$$

where $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$. Define the linear function δ on $\hat{\mathfrak{h}}$ by $\delta|_{\mathfrak{h}\oplus\mathbb{C}K} = 0$, $\delta(D) = 1$. Let $\theta \in \Delta^+$ be the highest root, and let $e_\theta \in \mathfrak{g}_\theta$, $f_\theta \in \mathfrak{g}_{-\theta}$ be such that $(f_\theta | e_\theta) = 1$.

Recall that if $\mathfrak{g} = \mathfrak{g}(\mathcal{A}, p)$, where $\mathcal{A} = (\mathfrak{h}, \Pi, \Pi^\vee)$ is a structure of a contragredient Lie superalgebra on $\mathfrak{g}$, with generators e_j, f_j $(j \in J)$ and $\mathfrak{h}$, then $\mathbb{C}D \ltimes \hat{\mathfrak{g}} = \mathfrak{g}(\hat{\mathcal{A}}, \hat{p})$, where $\hat{\mathcal{A}} = (\hat{\mathfrak{h}}, \hat{\Pi}, \hat{\Pi}^\vee)$ with

$$\hat{\Pi} = \Pi \cup \{\alpha_0 = \delta - \theta\}, \qquad \hat{\Pi}^\vee = \Pi^\vee \cup \{h_0 = K - \nu^{-1}(\theta)\},$$

and the generators

$$e_0 = f_\theta t, \qquad e_j \quad (j \in J); \qquad f_0 = e_\theta t^{-1}, \qquad f_j \quad (j \in J),$$

so that the index set is $\hat{J} = \{0\} \cup J$, and $\hat{p}(0) = p(\theta)$, $\hat{p}(j) = p(j)$ for $j \in J$ (see [13] for details in the Lie algebra case). By Remark 1.2.10, Theorem 1.2.8 applies to $\hat{\mathfrak{g}}$.

1.2.13. Example. If $\mathfrak{g} = \mathfrak{gl}(m, m)$, then $\mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$ contains an ideal $J = \sum_{n \neq 0} \mathbb{C}I_{2m}t^n$, which intersects $\mathfrak{h} \oplus \mathbb{C}K$ trivially. Let $\hat{\mathfrak{g}} = (\mathfrak{g}[t, t^{-1}]/J) \oplus \mathbb{C}K$. It is easy to see that $\hat{\mathfrak{g}}$ extends to a contragredient Lie superalgebra as in Remark 1.2.10, hence Theorem 1.2.8 again applies.The same is true for $\hat{\mathfrak{g}} = \mathfrak{g}[t, t^{-1}] \oplus \mathbb{C}K$, where $\mathfrak{g} = \mathfrak{sl}(m, m)/\mathbb{C}I_{2m}$.

2. Determinant of the Shapovalov form

Let $\mathfrak{g} := \mathfrak{g}(\mathcal{A}, p)$ be the Lie superalgebra, attached to the data $(\mathcal{A}, p)$, with a symmetrizable Cartan matrix. In this section we prove Theorem 1.2.8.

2.1. Linear factorization

Let

$$\mathrm{Irr} := \{\alpha \in Q^+ \setminus Q_I \mid \alpha/n \notin Q^+ \text{ for } n \in \mathbb{Z}_{\geqslant 2}\},$$
$$\widetilde{\mathrm{Irr}} := \{(m, \alpha) \in \mathbb{Z}_{\geqslant 1} \times \mathrm{Irr} \mid (\alpha|\alpha) \neq 0\} \cup \{(1, \alpha) \mid \alpha \in \mathrm{Irr} \ \& \ (\alpha|\alpha) = 0\}.$$

2.1.1. Any simple subquotient of $M_I(\lambda)$ is of the form $L(\lambda - \alpha)$ for $\alpha \in Q^+ \setminus Q_I$. From Proposition 1.1.5 we conclude that the Casimir element acts on $M(\lambda)$ by the scalar $(\lambda + 2\rho|\lambda)$, and that if $L(\lambda - \alpha)$ is a subquotient of $M_I(\lambda)$ then $(\alpha|2(\lambda + \rho) - \alpha) = 0$. Writing $\alpha = m\beta$ with $\beta \in \mathrm{Irr}$, $m \geqslant 1$ we obtain that

$$[M_I(\lambda): L(\lambda - m\beta)] \neq 0 \quad \Longrightarrow \quad \phi_{m\beta}(\lambda) = 0. \tag{7}$$

Observe that $\phi_{m\beta} = \phi_\beta$ if $(\beta|\beta) = 0$. Hence up to a non-zero constant factor one has

$$\det S_\nu(\lambda) = \prod_{(m,\beta) \in \widetilde{\mathrm{Irr}}} \phi_{m\beta}(\lambda)^{d_{m,\beta}(\nu)},$$

where $d_{m,\beta}(\nu)$ are some non-negative integers. Note that $d_{m,\beta}(\nu) \neq 0$ forces $m\beta \leqslant \nu$, i.e. $\nu - m\beta \in Q^+$.

2.2. Jantzen filtration

In 2.2.1 we recall the construction of the Jantzen filtration (see [8]). This filtration depends on a "generic element" $\rho' \in \mathfrak{h}^*$. For a semisimple Lie algebra one can take a sum of fundamental roots: $\rho' := \sum_{j \in J \setminus I} \omega_j$. It is known that for semisimple Lie algebras the Jantzen filtration does not depend on a choice of "generic" ρ', see [3, 5.3.1].

2.2.1. Fix $\rho' \in \mathfrak{h}_I^\perp$ such that $(\rho'|\alpha) \neq 0$ for any $\alpha \in Q^+ \setminus Q_I^+$. Let t be an indeterminate.

Take $\lambda \in \mathfrak{h}_I^\perp$. Introduce the generalized Verma module $M_I(\lambda + t\rho')$ as follows. Define the action of $(\mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_{-,I})$ on $\mathbb{C}[t]$: $(\mathfrak{n}_+ + \mathfrak{h}_I + \mathfrak{n}_{-,I})$ acts trivially and $h \in \mathfrak{h}$ acts by the multiplication to $(\lambda + t\rho')(h) = \lambda(h) + t\rho'(h)$. Now $M_I(\lambda + t\rho')$ is the following $\mathfrak{g} - \mathbb{C}[t]$ bimodule:

$$M_I(\lambda + t\rho') := \operatorname{Ind}^{\mathfrak{g}}_{\mathfrak{n}_+ + \mathfrak{h} + \mathfrak{n}_{-,I}} \mathbb{C}[t].$$

The module $M_I(\lambda + t\rho')$ admits a unique invariant $\mathbb{C}[t]$-bilinear form $S^{\lambda + t\rho'} : M_I(\lambda + t\rho') \otimes M_I(\lambda + t\rho') \to \mathbb{C}[t]$ which satisfies the properties (6); this form is symmetric. For $r \in \mathbb{Z}_{\geqslant 0}$, set

$$M_I^r(\lambda + t\rho') := \{ v \in M_I(\lambda + t\rho') \mid S^{\lambda + t\rho'}(v, v') \in t^r \mathbb{C}[t] \ \forall v' \}.$$

This defines a decreasing filtration. The second property of (6) insures that each $M_I^r(\lambda + t\rho')$ is a sub-bimodule of $M_I(\lambda + t\rho')$. The weight spaces of $M_I(\lambda + t\rho')$ are free of finite rank $\mathbb{C}[t]$-modules so we can define $\det S_\nu^{\lambda + t\rho'}$ (up to a non-zero constant factor). Clearly, $\det S_\nu^{\lambda + t\rho'} = \det S_\nu(\lambda + t\rho')$ and this is non-zero due to the linear factorization of $\det S_\nu$ and "genericity" of ρ'. As a result, $\bigcap_{r=0}^\infty M_I^r(\lambda + t\rho') = 0$.

Specializing this filtration at $t = 0$ we obtain the Jantzen filtration $\mathcal{F}^r(M_I(\lambda))$ on $M_I(\lambda)$. The weight spaces of $M_I^r(\lambda + t\rho')$ are free of finite rank $\mathbb{C}[t]$-modules. Thus $\mathcal{F}^r(M_I(\lambda))$ is just the image of $M_I^r(\lambda + t\rho')$ under the canonical map $M_I(\lambda + t\rho') \to M_I(\lambda + t\rho')/tM_I(\lambda + t\rho') \xrightarrow{\sim} M_I(\lambda)$. In particular the $\mathcal{F}^r(M_I(\lambda))$ form a decreasing filtration by submodules of $M_I(\lambda)$ having zero intersection. One readily sees that $\mathcal{F}^0(M_I(\lambda)) = M_I(\lambda)$ and that $\mathcal{F}^1(M_I(\lambda))$ coincides with the maximal proper submodule of $M_I(\lambda)$.

2.2.2. Define the sets $\tilde{R}_{m,\beta}$ ($m \in \mathbb{Z}_{\geqslant 1}$, $\beta' \in \mathrm{Irr}$) and $C(\lambda)$ ($\lambda \in \mathfrak{h}_I^\perp$) as follows:

$$\tilde{R}_{m,\beta} := \{ (m', \beta') \in \mathbb{Z}_{\geqslant 1} \times \mathrm{Irr} \mid \phi_{m\beta}|_{\mathfrak{h}_I^\perp} = \phi_{m'\beta'}|_{\mathfrak{h}_I^\perp} \},$$
$$C(\lambda) := \{ (m, \alpha) \in \mathbb{Z}_{\geqslant 1} \times \mathrm{Irr} \mid \phi_{m\alpha}(\lambda) = 0 \}.$$

2.2.3. The following "sum formula" is proven in [8]:

$$\sum_{i=1}^\infty \dim \mathcal{F}^i(M_I(\lambda)_{\lambda - \nu}) = \sum_{(m,\beta) \in C(\lambda) \cap \widetilde{\mathrm{Irr}}} d_{m,\beta}(\nu), \tag{8}$$

where $d_{m,\beta}(\nu)$ are exponents introduced in 2.1.1.

Proof. Note that the sum $\sum_{r=1}^\infty \dim \mathcal{F}^r(M_I(\lambda)_{\lambda - \nu})$ is equal to the order of zero of $\det S_\nu$ at the point $\lambda \in \mathfrak{h}_I^\perp$. Let A be the localization of $\mathbb{C}[t]$ by the maximal ideal generated by t: $A = \mathbb{C}[t]_{(t)}$. Let N be a free A-module of finite rank, endowed with a non-degenerate bilinear form $D : N \otimes N \to A$. Define a decreasing filtration

$$F^j(N) := \{ v \in N \mid D(v, v') \in At^j \text{ for any } v' \in N \}.$$

Taking N to be the localized module $M_I(\lambda)_{\lambda - \nu} \otimes_A \mathbb{C}[t]$ and D to be the bilinear form induced by $S^{\lambda + t\rho'}$, we see that the filtration on N, induced by the Jantzen filtration, is just $F^j(N)$. Now the sum formula follows from the following claim: *the order of zero of* $\det D$ *at the origin is equal to*

$$\sum_{j=1}^{\infty}\dim\bigl(F^j(N)/\bigl(F^j(N)\cap tN\bigr)\bigr).$$

In order to prove the claim, note that N has two systems of generators v_i and v_i' (for $i=1,\dots,r$) such that $D(v_i,v_j')=\delta_{ij}t^{s_i}$ (for $s_i\in\mathbb{Z}_{\geqslant 0}$). The order of zero of $\det D$ at the origin is $\sum_{i=1}^r s_i$ and

$$\dim F^j(N)/\bigl(F^j(N)\cap tN\bigr)=\bigl|\{i\mid s_i\geqslant j\}\bigr|.$$

The equality $\sum_i s_i=\sum_{j=1}^{\infty}|\{i\mid s_i\geqslant j\}|$ implies the claim. $\square$

2.2.4. Define the functions $d_{m,\beta},\tau_\alpha: Q\to\mathbb{Z}_{\geqslant 0}$ $(\alpha,\beta\in Q)$ by

$$\tau_\alpha:\nu\mapsto K(\nu-\alpha),\qquad d_{m,\beta}|_{(Q\setminus Q^+)\cup Q_I}=0,\qquad d_{m,\beta}:\nu\mapsto d_{m,\beta}(\nu)\quad\text{for }\nu\in Q^+\setminus Q_I.$$

The following lemma is proven in [10, 6.8] for simple Lie algebras; however, the proof is valid in our general setup.

2.2.5. Lemma. *For any* $\lambda\in\mathfrak{h}_I^\perp$, $m\geqslant 1$, $\beta\in\mathrm{Irr}$ *there exist integers* $a_{m,\beta}^\lambda$, $a_{m,\beta}$ *such that*

(i) $\sum_{i=1}^{\infty}\operatorname{ch}\mathcal{F}^i(M_I(\lambda))=\sum_{(m,\beta)\in C(\lambda)}a_{m,\beta}^\lambda\operatorname{ch}M(\lambda-m\beta)$,
(ii) $\sum_{(m',\beta')\in\tilde{R}_{m,\beta}\cap\widetilde{\mathrm{Irr}}}d_{m',\beta'}=\sum_{(m',\beta')\in\tilde{R}_{m,\beta}}a_{m',\beta'}\tau_{m'\beta'}$.

Proof. Combining the fact that $\mathcal{F}^i(M_I(\lambda))$ is a $\mathfrak{g}$-submodule of $M_I(\lambda)$ and formula (7), we deduce that $\operatorname{ch}\mathcal{F}^i(M_I(\lambda))=\sum_{(m,\beta)\in C(\lambda)}a_{m,\beta}^{\lambda,i}\operatorname{ch}M(\lambda-m\beta)$ for some integers $a_{m,\beta}^{\lambda,i}$; note that the sum is infinite, but "locally finite": for each $\nu\in Q^+$ only finitely many terms $M(\lambda-m\beta)_{\lambda-\nu}$ are non-zero. Thus we obtain (i) for $a_{m,\beta}^\lambda:=\sum_{i=1}^{\infty}a_{m,\beta}^{\lambda,i}$. For (ii) fix a pair (m,β). Let $\lambda\in\mathfrak{h}_I^\perp$ be a "generic point" of the hyperplane $\{\xi\colon \phi_{m\beta}(\xi)=0\}$ in the following sense: λ does not belong to the hyperplanes $\{\xi\colon \phi_{m'\beta'}(\xi)=0\}$ if $(m',\beta')\notin\tilde{R}_{m,\beta}$; in other words, $C(\lambda)=\tilde{R}_{m,\beta}$. Combining (i) and formula (8) one obtains

$$\sum_{(m',\beta')\in C(\lambda)}a_{m',\beta'}^\lambda\tau_{m'\beta'}=\sum_{(m',\beta')\in C(\lambda)\cap\widetilde{\mathrm{Irr}}}d_{m',\beta'}.$$

Since $C(\lambda)=\tilde{R}_{m,\beta}$ one gets (ii) for the integers $a_{m,\beta}:=a_{m,\beta}^\lambda$. $\square$

2.2.6. Corollary.

$$\prod_{(m',\beta')\in\tilde{R}_{m,\beta}\cap\widetilde{\mathrm{Irr}}}\phi_{m'\beta'}^{d_{m',\beta'}(\nu)}=\prod_{(m',\beta')\in\tilde{R}_{m,\beta}}\phi_{m'\beta'}^{a_{m',\beta'}(\nu)K(\nu-m'\beta')}.$$

Proof. By definition $\phi_{m'\beta'} = \phi_{m\beta}$ for $(m', \beta') \in \tilde{R}_{m,\beta}$. In the light of Lemma 2.2.5(ii), both sides of formula are equal to $\phi_{m\beta}^{\sum_{(m',\beta')\in\tilde{R}_{m,\beta}} a_{m',\beta'}(\nu)\tau_{m'\beta'(\nu)}}$. □

2.3. Leading term

Using the geometric series, we expand

$$R_I/R = \prod_{\alpha\in\Delta^+\setminus Q_I} \left(1 - s(\alpha)e^{-\alpha}\right)^{-s(\alpha)\dim\mathfrak{g}_\alpha} = \sum_{\alpha\in Q^+} K_I(\alpha)e^{-\alpha}, \quad \text{where } K_I(\alpha)\in\mathbb{Z}_{\geqslant 0}.$$

Set $K_I(\alpha) = 0$ for $\alpha \in Q \setminus Q^+$; note that $K_I(\alpha) = 0$ for $\alpha \in Q_I$, $\alpha \neq 0$.

Consider the natural grading on the symmetric algebra $\mathcal{S}(\mathfrak{h}/\mathfrak{h}_I) = \bigoplus_{r=0}^{\infty} \mathcal{S}^r(\mathfrak{h}/\mathfrak{h}_I)$. The following proposition is a particular case of [5, Theorem 3.1].

Proposition. *Up to a non-zero constant factor, the leading term of* $\det S_\nu$ *is*

$$\operatorname{gr}\det S_\nu = \prod_{\alpha\in\Delta^+\setminus Q_I} h_\alpha^{(\dim\mathfrak{g}_\alpha)\sum_{r\geqslant 1}(-1)^{(r-1)p(\alpha)}K_I(\nu - r\alpha)},$$

where $h_\alpha \in \mathfrak{h}/\mathfrak{h}_I$ *is such that* $\mu(h_\alpha) = (\mu|\alpha)$ *for any* $\mu \in \mathfrak{h}_I^\perp$.

Proof. We prove the proposition in 2.3.1–2.3.6 below. Denote by $\tilde{\Delta}_0^+$, $\tilde{\Delta}_1^+$ the corresponding multisets of roots, where the multiplicity of γ is equal to $\dim\mathfrak{g}_\gamma$. Set $\tilde{\Delta}^+ := \tilde{\Delta}_0^+ \cup \tilde{\Delta}_1^+$. Define similarly $\tilde{\Delta}_I^+$ (the multiset corresponding to Δ_I^+). Fix a total ordering on $\tilde{\Delta}^+$ such that $\gamma_1 \geqslant \gamma_2$ if $\gamma_1 - \gamma_2 \in Q^+$.

2.3.1. A vector $\mathbf{m} = \{m_\gamma\}_{\gamma\in\tilde{\Delta}^+\setminus\tilde{\Delta}_I^+}$ is called a *partition of* $\alpha \in Q^+ \setminus Q_I$ if $\alpha = \sum_{\gamma\in\tilde{\Delta}^+} m_\gamma\gamma$: $m_\gamma \in \mathbb{Z}_{\geqslant 0}$ for $\gamma \in \tilde{\Delta}_0^+ \setminus \tilde{\Delta}_I^+$, and $m_\gamma \in \{0, 1\}$ for $\gamma \in \tilde{\Delta}_1^+ \setminus \tilde{\Delta}_I^+$. Denote by $\mathcal{P}(\alpha)$ the set of all partitions of α. One has $|\mathcal{P}(\alpha)| = K_I(\alpha)$.

2.3.2. For $\gamma \in \tilde{\Delta}^+$ denote by $\overline{\gamma}$ the corresponding element in the set Δ. Choose bases $\{f_\gamma\}_{\gamma\in\tilde{\Delta}^+}$ of $\mathfrak{n}_-$ and $\{e_\gamma\}_{\gamma\in\tilde{\Delta}^+}$ of $\mathfrak{n}_+$ such that $f_\gamma \in \mathfrak{g}_{-\overline{\gamma}}$, $e_\gamma \in \mathfrak{g}_{\overline{\gamma}}$. In the light of Proposition 1.1.4(ii), for each $\alpha \in \Delta^+$ the entries of the matrix

$$D_\alpha = \left([f_{\gamma_i}, e_{\gamma_j}]\right)_{\overline{\gamma_i}=\overline{\gamma_j}=\alpha}$$

are proportional to h_α and $\det D_\alpha \neq 0$. Hence we can choose the bases in such a way that all matrices D_α are diagonal: $D_\alpha = (\delta_{ij}h_\alpha)_{i,j}$.

For every $\mathbf{m} \in \mathcal{P}(\alpha)$, define the monomial

$$\mathbf{f}^{\mathbf{m}} := \prod_\alpha f_\alpha^{m_\alpha}, \qquad \mathbf{e}^{\mathbf{m}} := \prod_\alpha e_\alpha^{m_\alpha},$$

where the order of factors is given by the total ordering fixed above. Take $\lambda \in \mathfrak{h}_I^{\perp}$ and let v_λ be the highest weight vector of $M_I(\lambda)$. The set $\{\mathbf{f}^{\mathbf{m}} v_\lambda \mid \mathbf{m} \in \mathcal{P}(\nu)\}$ forms a basis of $M_I(\lambda)_{\lambda-\nu}$. By definition given in 1.2.4,

$$\det S_\nu = \det\bigl(\mathrm{HC}_I\bigl(\sigma\bigl(\mathbf{f}^{\mathbf{m}}\bigr)\mathbf{f}^{\mathbf{s}}\bigr)\bigr)_{\mathbf{m},\mathbf{s}\in\mathcal{P}(\nu)}.$$

Since both $\{\sigma(\mathbf{f}^{\mathbf{m}})\}_{\mathbf{m}\in\mathcal{P}(\nu)}$ and $\{\mathbf{e}^{\mathbf{m}}\}_{\mathbf{m}\in\mathcal{P}(\nu)}$ are bases of the same vector space, one has

$$\det S_\nu = \det\bigl(\mathrm{HC}_I\bigl(\mathbf{e}^{\mathbf{m}}\mathbf{f}^{\mathbf{s}}\bigr)\bigr)_{\mathbf{m},\mathbf{s}\in\mathcal{P}(\nu)},$$

up to a non-zero constant factor.

2.3.3. Set $|\mathbf{k}| = \sum_{\alpha\in\Delta^+} k_\alpha$. For $u \in \mathcal{U}(\mathfrak{g})$ denote by $\mathrm{gr}\, u$ the image of u in the symmetric algebra $\mathcal{S}(\mathfrak{g})$.

Lemma. *For any $\mathbf{m}, \mathbf{s} \in \mathcal{P}(\nu)$, we have*

(i) $\deg \mathrm{HC}(\mathbf{e}^{\mathbf{m}}\mathbf{f}^{\mathbf{s}}) \leqslant \min(|\mathbf{m}|, |\mathbf{s}|)$;
(ii) *if* $|\mathbf{m}| = |\mathbf{s}|$, *we have*

$$\deg \mathrm{HC}\bigl(\mathbf{e}^{\mathbf{m}}\mathbf{f}^{\mathbf{s}}\bigr) = |\mathbf{m}| \quad\Longleftrightarrow\quad \mathbf{m} = \mathbf{s};$$

(iii) *up to a non-zero constant factor,*

$$\mathrm{gr}\, \mathrm{HC}\bigl(\mathbf{e}^{\mathbf{m}}\mathbf{f}^{\mathbf{m}}\bigr) = \prod_{\gamma\in\tilde{\Delta}^+} h_{\overline{\gamma}}^{m_\gamma}.$$

Proof is by induction on $\nu \in Q^+$ with respect to the partial order (see 1.1.1).

2.3.4. Corollary. *Up to a non-zero constant factor, the leading term of* $\det S_\nu$ *is equal to*

$$\prod_{\alpha\in\Delta^+\setminus\Delta_I} h_\alpha^{r_\alpha(\nu)},$$

where

$$r_\alpha(\nu) := \sum_{\gamma\in\tilde{\Delta}^+:\ \overline{\gamma}=\alpha}\ \sum_{\mathbf{m}\in\mathcal{P}(\nu)} m_\gamma.$$

2.3.5. Lemma. *For any $\gamma \in \tilde{\Delta}^+ \setminus \Delta_I$ one has*

$$\sum_{\mathbf{m}\in\mathcal{P}(\nu)} m_\gamma = \sum_{r=1}^{\infty} (-1)^{(r-1)p(\overline{\gamma})} K_I(\nu - r\overline{\gamma}).$$

Proof is by induction on $\nu \in Q^+ \setminus Q_I$ with respect to the partial order (see 1.1.1).

2.3.6. Combining Corollary 2.3.4 and Lemma 2.3.5, we obtain Proposition 2.3. $\square$

2.4. Computation of $a_{m,\beta}$

Since $\coprod \tilde{R}_{m,\beta} = \mathbb{Z}_{\geqslant 1} \times \mathrm{Irr}$, Lemma 2.2.5(ii) gives

$$\sum_{(m,\beta)\in\widetilde{\mathrm{Irr}}} d_{m,\beta} = \sum_{(m,\beta)\in\mathbb{Z}_{\geqslant 1}\times\mathrm{Irr}} a_{m,\beta}\tau_{m\beta}. \tag{9}$$

Both sides of the above formula are well-defined functions on Q: for each $\nu \in Q$ only summands indexed by the pairs (m, β), where $\nu - m\beta \in Q^+$, are non-zero at ν, and thus only finitely many summands are non-zero for each $\nu \in Q$.

2.4.1. From Proposition 2.3,

$$\sum_{(m,\beta)\in\widetilde{\mathrm{Irr}}} d_{m,\beta}(\nu) = \sum_{\alpha\in\Delta^+\setminus Q_I} \sum_{r=1}^{\infty} (-1)^{(r+1)p(\alpha)} (\dim \mathfrak{g}_\alpha) K_I(\nu - r\alpha),$$

and thus, using (9) and $\tau_{m\beta}(\nu) = K(\nu - m\beta)$, we get

$$\sum_{(m,\beta)\in\mathbb{Z}_{\geqslant 1}\times\mathrm{Irr}} a_{m,\beta} \sum_{\nu} K(\nu - m\beta)e^{-\nu} = \sum_{\alpha\in\Delta^+\setminus Q_I} \sum_{r=1}^{\infty} \sum_{\nu} (-1)^{(r+1)p(\alpha)} (\dim \mathfrak{g}_\alpha) K_I(\nu - r\alpha)e^{-\nu},$$

which can be rewritten as

$$\sum_{(m,\beta)\in\mathbb{Z}_{\geqslant 1}\times\mathrm{Irr}} a_{m,\beta} \sum_{\nu} K(\nu)e^{-\nu-m\beta} = \sum_{\alpha\in\Delta^+\setminus Q_I} \sum_{r=1}^{\infty} \sum_{\nu} (-1)^{(r+1)p(\alpha)} (\dim \mathfrak{g}_\alpha) K_I(\nu)e^{-\nu-r\alpha},$$

that is

$$R^{-1} \sum_{(m,\beta)\in\mathbb{Z}_{\geqslant 1}\times\mathrm{Irr}} a_{m,\beta}e^{-m\beta} = R^{-1}R_I \sum_{\alpha\in\Delta^+\setminus Q_I} \sum_{r=1}^{\infty} (-1)^{(r+1)p(\alpha)} (\dim \mathfrak{g}_\alpha)e^{-r\alpha}.$$

Therefore the integer $a_{m,\beta}$ is equal to the coefficient of $e^{-m\beta}$ in the expression

$$R_I \sum_{\gamma\in\Delta^+\setminus Q_I} \sum_{r=1}^{\infty} (-1)^{(r+1)p(\gamma)} (\dim \mathfrak{g}_\gamma)e^{-r\gamma}.$$

Hence

$$a_{m,\beta} = \sum_{\gamma\in\Delta^+\setminus Q_I} \sum_{r=1}^{\infty} (-1)^{(r+1)p(\gamma)} (\dim \mathfrak{g}_\gamma) k_I(m\beta - r\gamma).$$

2.4.2. Substituting the formula for $a_{m,\beta}$ into Corollary 2.2.6 we get

$$\begin{aligned}\det S_\nu &= \prod_{(m,\beta)\in\mathbb{Z}_{\geqslant 1}\times \mathrm{Irr}} \phi_{m\beta}^{K(\nu-m\beta)\sum_{\gamma\in\Delta^+\setminus Q_I}\sum_{r=1}^{\infty}(-1)^{(r+1)p(\gamma)}(\dim\mathfrak{g}_\gamma)k_I(m\beta-r\gamma)} \\ &= \prod_{\alpha\in Q^+}\phi_\alpha^{K(\nu-\alpha)\sum_{\gamma\in\Delta^+\setminus Q_I}\sum_{r=1}^{\infty}(-1)^{(r+1)p(\gamma)}(\dim\mathfrak{g}_\gamma)k_I(\alpha-r\gamma)} \\ &= \prod_{\alpha\in Q^+}\prod_{r=1}^{\infty}\prod_{\gamma\in\Delta^+\setminus Q_I}\phi_\alpha^{(-1)^{(r+1)p(\gamma)}(\dim\mathfrak{g}_\gamma)k_I(\alpha-r\gamma)K(\nu-\alpha)} \\ &= \prod_{r=1}^{\infty}\prod_{\alpha\in Q^+}\prod_{\gamma\in\Delta^+\setminus Q_I}\phi_{\alpha+r\gamma}^{(-1)^{(r+1)p(\gamma)}(\dim\mathfrak{g}_\gamma)k_I(\alpha)K(\nu-\alpha-r\gamma)}.\end{aligned}$$

Recalling that $k_I(\alpha)=0$ for $\alpha\notin Q_I^+$, this completes the proof of Theorem 1.2.8.

3. Vacuum determinant

Let $\mathfrak{g}=\mathfrak{g}(\mathcal{A},p)$ be a finite-dimensional contragredient Lie superalgebra and let $\mathfrak{g}(\hat{\mathcal{A}},\hat{p})=\mathbb{C}D\ltimes\hat{\mathfrak{g}}$ be its (untwisted) affinization described in Example 1.2.12, which notation we retain, except that here we take $\hat{\mathfrak{h}}=\mathfrak{h}+\mathbb{C}K$.

We denote $\Delta(\hat{\mathcal{A}},\hat{p})$ by $\hat{\Delta}$ and $\Delta_I=\Delta(\mathcal{A},p)$ by Δ, $Q^+(\hat{\mathcal{A}},\hat{p})$ by $\hat{Q^+}$ and $Q_I^+=Q^+(\mathcal{A},p)$ by Q^+, and so on.

Denote the Weyl group of $\mathfrak{g}$ (respectively, of $\hat{\mathfrak{g}}$) by W (respectively, by $\hat{W}$). Introduce the twisted action $\hat{W}$ on $\hat{\mathfrak{h}}^*$ as $w.\lambda:=w(\lambda+\hat{\rho})-\hat{\rho}$; notice that $w.\lambda:=w(\lambda+\rho)-\rho$ if $w\in W$. Recall that [13]

$$(\delta|\hat{\rho})=h_B^\vee=(\rho|\theta)+\frac{1}{2}(\theta|\theta). \tag{10}$$

Recall that $\hat{J}=\{0\}\cup J$. We apply Theorem 1.2.8 to $\hat{\mathfrak{g}}$ (see Remark 1.2.10) and $I=J\subset\hat{J}$.

3.1. Introduce $\Lambda_0\in\hat{\mathfrak{h}}^*$ such that $\Lambda_0(h)=0$ for $h\in\mathfrak{h}$, $\Lambda_0(K)=1$. Any $\lambda\in\hat{\mathfrak{h}}_I^\perp$ takes the form $\lambda=k\Lambda_0$ for some $k\in\mathbb{C}$. Thus, $\det S_\nu$ is a polynomial in one variable k.

3.2. The generalized Verma module $M_I(k\Lambda_0)$ is the vacuum module V^k. Using Theorem 1.2.8 for $\hat{\mathfrak{g}}$, we obtain the formula for the *vacuum determinant*:

$$\det S_\nu(k)=\prod_{r=1}^{\infty}\prod_{\gamma\in\hat{\Delta}^+\setminus\Delta}\prod_{\alpha\in Q^+}\phi_{r\gamma+\alpha}(k)^{(\dim\hat{\mathfrak{g}}_\gamma)\hat{K}(\nu-\alpha-r\gamma)(-1)^{(r+1)p(\gamma)}k_I(\alpha)}, \tag{11}$$

where

$$\sum_{\alpha\in Q^+}k_I(\alpha)e^{-\alpha}=\prod_{\alpha\in\Delta_0^+}\left(1-e^{-\alpha}\right)\prod_{\alpha\in\Delta_1^+}\left(1+e^{-\alpha}\right)^{-1}.$$

Recall that $\dim \hat{\mathfrak{g}}_\gamma = \dim \mathfrak{h}$ if $\gamma \in \mathbb{Z}\delta$ (unless $\mathfrak{g} = \mathfrak{gl}(m,m)$, when $\dim \hat{\mathfrak{g}}_{s\delta} = 2m-1$, see Example 1.2.13), and $\dim \hat{\mathfrak{g}}_\gamma = 1$ if $\gamma \notin \mathbb{Z}\delta$.

Write $\gamma \in \hat{\Delta}^+ \setminus \Delta$ as $\gamma = u\delta + \gamma'$, where $u \in \mathbb{Z}_{\geqslant 1}$ and $\gamma' \in \Delta$. Then, by (10):

$$\frac{1}{ru}\phi_{r\gamma+\alpha}(k) = k + h_B^\vee + \frac{(\rho - \alpha|\gamma') - r(\gamma'|\gamma')/2}{u} + \frac{(\alpha|\rho) - (\alpha|\alpha)/2}{ru}. \tag{12}$$

3.2.1. Remark. It is easy to see that $\phi_\xi = \phi_{w(\xi-\rho)+\rho}$ for any $w \in W$ since $(\lambda|w\xi) = (\lambda|\xi)$ if $\lambda \in \hat{\mathfrak{h}}_I^\perp$, for any $w \in W$.

3.2.2. Apart from the case $D(2,1,a)$ with irrational a, a finite-dimensional contragredient Lie superalgebra $\mathfrak{g}$ admits a symmetrizable Cartan matrix with integer entries. As a consequence, we can (and will) normalize the bilinear form $(.|.)$ in such a way that the scalar product of any two roots is rational. Unless otherwise stated, we will assume that $\mathfrak{g} \neq D(2,1,a)$ with irrational a.

Corollary. *If $\mathfrak{g}$ is not of the type $D(2,1,a)$ with irrational a then the vacuum module V^k is simple for $k \notin \mathbb{Q}$.*

3.2.3. Write $\det S_\nu(k) = \prod_{b\in\mathbb{C}}(k + h_B^\vee - b)^{m_b(\nu)}$ and set

$$M_b := R\sum_\nu m_b(\nu)e^{-\nu}.$$

Then

$$M_b = \sum_{(r;\gamma;\alpha)\in Y(b)} (-1)^{(r+1)p(\gamma)}(\dim \hat{\mathfrak{g}}_\gamma)k_I(\alpha)e^{-\alpha-r\gamma},$$

where $Y(b)$ is the set of triples $(r;\gamma;\alpha)$ with $\phi_{r\gamma+\alpha}$ proportional to $(k + h_B^\vee - b)$, that is

$$Y(b) := \left\{(r;\gamma;\alpha) \;\middle|\; r \in \mathbb{Z}_{\geqslant 1},\ \gamma \in \hat{\Delta}^+ \setminus \Delta,\ \alpha \in Q^+ \text{ such that } \frac{\phi_{r\gamma+\alpha}}{k + h_B^\vee - b} \in \mathbb{C}^*\right\}.$$

We know that $M_0 \neq 0$. By Corollary 3.2.2, for $\mathfrak{g} \neq D(2,1,a)$ with irrational a, one has $M_b = 0$ if $b \notin \mathbb{Q}$. In this case, we present a non-zero rational number b in the form $b = p/q$, where p, q are relatively prime non-zero integers and $q \geqslant 1$.

3.3. Consider the restriction of the bilinear form $(.|.)$ to the real vector space $\mathfrak{h}_\mathbb{R} := \sum_{\alpha\in\Delta} \mathbb{R}\alpha$; the dimension of a maximal isotropic subspace of $\mathfrak{h}_\mathbb{R}$ is called the *defect* of $\mathfrak{g}$.

A simple finite-dimensional contragredient superalgebra of defect zero is either a Lie algebra or $\mathfrak{osp}(1,2n)$.

3.3.1. Let $\kappa(.|.)$ denote the Killing form. If κ is non-zero, set $\Delta^\# = \{\alpha \in \Delta \mid \kappa(\alpha|\alpha) > 0\}$. Then $\Delta_0^\#$ is the root system of one of simple components of $\mathfrak{g}_0$. If $\kappa = 0$ then $\mathfrak{g}$ is of type $A(n,n)$, $D(n+1,n)$ or $D(2,1,a)$. In this case the root system is a union of two mutually orthogonal subsystems: $\Delta_0 = A_n \cup A_n$, $D_{n+1} \cup C_n$, $D_2 \cup C_1$, respectively; we let $\Delta^\# = \Delta_0^\#$ be the first subset. Let $W^\#$ be the Weyl group corresponding to $\Delta_0^\#$, that is the subgroup of W generated by the r_α with $\alpha \in \Delta_0^\#$.

3.3.2. A subset S of Δ is called *maximal isotropic* if it consists of the defect $\mathfrak{g}$ roots that span a maximal isotropic subspace of $\mathfrak{h}_{\mathbb{R}}$. The existence of S is proven in [17]; it is also shown that one can choose a set of simple roots Π is such a way that $S \subset \Pi$. We fix S and Π which contains S. We set

$$\mathbb{N}S := \left\{ \sum_{\beta\in S} n_\beta \beta,\ n_\beta \in \mathbb{Z}_{\geqslant 0} \right\}.$$

For $\alpha \in \mathbb{N}S$ denote by $\operatorname{ht}\alpha$ the *height* of α: $\operatorname{ht}\alpha = \sum n_\beta$ if $\alpha = \sum_{\beta\in S} n_\beta\beta$.

3.3.3. By a *regular exponential function* on $\hat{\mathfrak{h}}$ we mean a finite linear combination of exponentials e^λ: $\lambda \in \hat{\mathfrak{h}}^*$. A *rational exponential function* is a ratio P/Q, where P, Q are regular exponential functions and $Q \neq 0$. The Weyl group $\hat{W}$ acts on the field of rational exponential functions by the formulas $w(e^\lambda) = e^{w\lambda}$, $w.(e^\lambda) = e^{w.\lambda}$.

3.4. Retain notation of 3.2.3.

Theorem. *Assume that $W^\# S \subset \Delta^+$. Then*

$$\det S_\nu(k) = \prod_{r\geqslant 1} \prod_{\gamma\in\hat{\Delta}^+\setminus\Delta} \prod_{\alpha\in\mathbb{N}S} \phi_{r\gamma+\alpha}(k)^{(\dim\hat{\mathfrak{g}}_\gamma) d_{r,\gamma,\alpha}(\nu)},$$

where $\phi_{r\gamma+\alpha}(k) = (\Lambda_0|\gamma)k + (\hat{\rho} - \alpha|\gamma) - r(\gamma|\gamma)/2$, and $d_{r,\gamma,\alpha}(\nu)$ are integers, defined by

$$\sum_{\nu\in\hat{Q}} d_{r,\gamma,\alpha}(\nu) e^{-\nu} = (-1)^{\operatorname{ht}\alpha + (r-1)p(\gamma)} R^{-1} \sum_{w\in W^\#} (-1)^{l(w)} e^{w.(-r\gamma-\alpha)},$$

i.e.

$$M_b = \sum_{(r;\gamma;\alpha)\in Y_S(b)} (-1)^{(r+1)p(\gamma)+\operatorname{ht}\alpha} (\dim\hat{\mathfrak{g}}_\gamma) \sum_{w\in W^\#} (-1)^{l(w)} e^{w.(-r\gamma-\alpha)},$$

where

$$Y_S(b) := \left\{ (r;\gamma;\alpha) \,\middle|\, r \in \mathbb{Z}_{\geqslant 1},\ \gamma \in \hat{\Delta}^+ \setminus \Delta,\ \alpha \in \mathbb{N}S \text{ such that } \frac{\phi_{r\gamma+\alpha}}{k + h_B^\vee - b} \in \mathbb{C}^* \right\}.$$

Note that $r\gamma + \alpha$ uniquely determines a triple $(r;\gamma;\alpha)$, and that $\dim\hat{\mathfrak{g}}_\gamma = 1$, *apart from the case when $\phi_{r\gamma+\alpha}(k)$ is proportional to $k + h_B^\vee$.*

Proof. Since S consists of simple mutually orthogonal isotropic roots one has $(\alpha|\alpha) = (\alpha|\hat{\rho}) = 0$ for any $\alpha \in \mathbb{N}S$. This gives the formula for $\phi_{r\gamma+\alpha}$.

For $\alpha \in \mathbb{N}S$ and $\xi := -w(-\alpha)$ one has $\phi_{r\gamma+\xi} = \phi_{r\gamma+\rho-w\rho+w\alpha} = \phi_{rw^{-1}\gamma+l\beta}$ by Remark 3.2.1. Combining the formulas in 3.2 and Lemma 3.5 we obtain for each r:

$$\prod_{\gamma\in\hat{\Delta}^+\setminus\Delta}\prod_{\alpha\in Q^+}\phi_{r\gamma+\alpha}^{\hat{K}(\nu-\alpha-r\gamma)k_I(\alpha)}=\prod_{\gamma\in\hat{\Delta}^+\setminus\Delta}\prod_{w\in W^\#}\prod_{\alpha\in\mathbb{N}S}\phi_{rw^{-1}\gamma+\alpha}^{(-1)^{l(w)+\mathrm{ht}\,\alpha}\hat{K}(\nu-r\gamma+w\rho-\rho-w\alpha)}$$
$$=\prod_{\gamma\in\hat{\Delta}^+\setminus\Delta}\prod_{w\in W^\#}\prod_{\alpha\in\mathbb{N}S}\phi_{r\gamma+\alpha}^{(-1)^{l(w)+\mathrm{ht}\,\alpha}\hat{K}(\nu-rw\gamma+w\rho-\rho-w\alpha)}$$
$$=\prod_{\gamma\in\hat{\Delta}^+\setminus\Delta}\prod_{\alpha\in\mathbb{N}S}\prod_{w\in W^\#}\phi_{r\gamma+\alpha}^{(-1)^{l(w)+\mathrm{ht}\,\alpha}\hat{K}(\nu+w.(-r\gamma-\alpha))}.$$

Now, the formula for the integers $d_{r,\gamma,\alpha}(\nu)$ follows from 3.2. □

3.5. Lemma. *Assume that $W^\# S\subset\Delta^+$. Then for any $\alpha\in Q$ the orbit $W^\#.(-\alpha)$ meets $-\mathbb{N}S$ at most once and*

$$k_I(\alpha)=\begin{cases}0, & \text{if } W.(-\alpha)\cap(-\mathbb{N}S)=\emptyset,\\ (-1)^{l(w)+\mathrm{ht}(-w.(-\alpha))}, & \text{if } -w.(-\alpha)\in\mathbb{N}S.\end{cases}$$

Proof. First, let us show that $\mathbb{N}S\cap\Delta^+=S$. Indeed, if the defect of $\mathfrak{g}$ is not greater than one the assertion is trivial. The root systems of finite-dimensional contragredient Lie superalgebras are described in [11, 2.5.4]. All exceptional superalgebras have defect one. For non-exceptional superalgebras of non-zero defect, $\mathfrak{h}_{\mathbb{R}}$ has an orthogonal basis $\{\varepsilon_i|\delta_j\}_{i=1,n;\,j=1,m}$, where $(\varepsilon_i|\varepsilon_i)=-(\delta_j|\delta_j)$ for any i,j and $\Delta\subset\{\pm t\varepsilon_i,\pm t\delta_j\colon\ t=1,2;\ \pm\varepsilon_i\pm\delta_j;\ \pm\varepsilon_i\pm\varepsilon_{i'};\ \pm\delta_j\pm\delta_{j'}\}$. As a result, S is of the form $\{\pm\varepsilon_{i_l}\pm\delta_{j_l}\}$, where $i_l\neq i_{l'}$, $j_l\neq j_{l'}$ for $l\neq l'$. This implies $\mathbb{N}S\cap\Delta^+=S$.

Recall that all root spaces of $\mathfrak{g}$ are one-dimensional and thus

$$R:=\prod_{\alpha\in\Delta_0^+}\frac{(1-e^{-\alpha})}{(1+e^{-\alpha})}=\sum_{\alpha\in Q^+}k_I(\alpha)e^{-\alpha}.$$

Since $\mathbb{N}S\cap\Delta^+=S$ we have $k_I(\alpha)=(-1)^{\mathrm{ht}\,\alpha}$ for $\alpha\in\mathbb{N}S$. Theorem 2.1 of [17] states that

$$e^\rho R=\sum_{w\in W^\#}(-1)^{l(w)}w\left(\frac{e^\rho}{\prod_{\beta\in S}(1+e^{-\beta})}\right).$$

The assumption $W^\# S\subset\Delta^+$ forces $W^\#(\mathbb{N}S)\subset Q^+$ and the above formula gives

$$R=\sum_{w\in W^\#}(-1)^{l(w)}\sum_{\alpha\in\mathbb{N}S}(-1)^{\mathrm{ht}\,\alpha}e^{w.(-\alpha)}.\tag{13}$$

We see that $k_I(\alpha)=0$ if $W^\#.(-\alpha)$ does not meet $\mathbb{N}S$, and, moreover, $w(e^\rho R)=(-1)^{l(w)}e^\rho R$, that is

$$k_I(\alpha)=(-1)^{l(w)}k_I\big(-w.(-\alpha)\big)\quad\text{for any }\alpha\in Q,\ w\in W^\#.$$

We already know that $k_I(\alpha)=(-1)^{\mathrm{ht}\,\alpha}$ for $\alpha\in\mathbb{N}S$. It remains to verify that for any $\xi\in Q$ the orbit $W.(-\xi)$ meets $-\mathbb{N}S$ at most once or, equivalently, that for any $\alpha\in\mathbb{N}S$ one has

$W^{\#}.(-\alpha)\cap(-\mathbb{N}S)=\{-\alpha\}$. Indeed, from (13) for any $\alpha\in\mathbb{N}S$ one has

$$k_I(\alpha)=\sum_{w\in W^{\#}:\ -w.(-\alpha)\in\mathbb{N}S}(-1)^{l(w)+\mathrm{ht}(-w.(-\alpha))}.$$

However if $-w.(-\alpha)\in\mathbb{N}S$ then

$$k_I\big(-w.(-\alpha)\big)=(-1)^{\mathrm{ht}(-w.(-\alpha))}\quad\text{and}\quad k_I\big(-w.(-\alpha)\big)=(-1)^{l(w)}k_I(\alpha)$$

so $k_I(\alpha)=(-1)^{l(w)+\mathrm{ht}(-w.(-\alpha))}$. Hence $k_I(\alpha)=\sum_{w\in W^{\#}:-w.(-\alpha)\in\mathbb{N}S}k_I(\alpha)$ so $W^{\#}.(-\alpha)\cap(-\mathbb{N}S)=\{-\alpha\}$ as required. $\square$

4. Virasoro algebra

In this section we prove formula (3) and Theorem 0.3.2 (see Theorem 4.2.1 and Proposition 4.3.2, respectively).

4.1. Notation

Denote by $\mathcal{V}ir_{\geqslant k}$ (respectively, $\mathcal{V}ir_{<k}$) the subspace spanned by L_j, $j\geqslant k$ (respectively, L_j, $j<k$). Notice that $\mathcal{V}ir_{\geqslant -1}$, $\mathcal{V}ir_{<-1}$ are subalgebras. A *Verma module* $M(h;c)$ ($h,c\in\mathbb{C}$) over $\mathcal{V}ir$ is induced from the one-dimensional module $\mathbb{C}|h;c\rangle$ of $\mathcal{V}ir_{\geqslant 0}+\mathbb{C}C$, where $\mathcal{V}ir_{>0}$ acts trivially, L_0 acts by the scalar h and C acts by the scalar c. The weight spaces of $M(h;c)$ are eigenspaces of L_0 with eigenvalues $h+n$, $n\in\mathbb{Z}_{\geqslant 0}$.

A *vacuum module* V^c is induced from the one-dimensional module $\mathbb{C}|0;c\rangle$ of $\mathcal{V}ir_{\geqslant -1}+\mathbb{C}C$, where $\mathcal{V}ir_{\geqslant -1}$ acts trivially and C acts by the scalar c. Clearly, $V^c=M(0;c)/M(1;c)$.

In this section we use letters r,s,p,q,k,m for non-negative integers. For positive integers p,q we denote, as before, by (p,q) their greatest common divisor. We denote the maximal proper submodule of $M(h;c)$ by $\overline{M}(h;c)$ and the simple quotient of $M(h;c)$ by $L(h;c)$.

4.2. Main result

Introduce the anti-involution σ on $\mathcal{V}ir$ by the formulas $\sigma(L_n)=L_{-n}$, $\sigma(C)=C$. Define the triangular decomposition $\mathcal{V}ir=\mathcal{V}ir_{<0}\oplus(\mathbb{C}L_0+\mathbb{C}C)\oplus\mathcal{V}ir_{\geqslant 1}$, and introduce the Harish-Chandra projection with respect to this triangular decomposition. Define the contravariant forms on Verma modules and on vacuum modules as in 1.2.4. Define the Jantzen filtrations on these modules as in 2.2.1 and observe that the "sum formula" (8) holds in this setup. We denote the determinant of the contravariant form on the eigenspace of L_0 with the eigenvalue $h+N$ ($N\in\mathbb{Z}_{\geqslant 0}$) in $M(h;c)$ by $\det_{h+N}$ (respectively, on the eigenspace V^c_N of L_0 in V^c with the eigenvalue N by $\det'_N$). These are polynomials in h and c (respectively, c).

4.2.1. Theorem. *Let* $c_{p,q}=1-\frac{6(p-q)^2}{pq}$.

(i) *Up to a non-zero scalar factor, the vacuum determinant is as follows*:

$$\det{}'_N(c)=\prod_{p>q>1,\,(p,q)=1}(c-c_{p,q})^{\dim L((p-1)(q-1);c_{p,q})_N},$$

where $\dim L((p-1)(q-1);c_{p,q})_N$ *is given by the right-hand side of* (4).

(ii) *A vacuum module V^c is simple iff $c \notin \{c_{p,q}\}_{p,q\in\mathbb{Z}_{\geqslant 2},(p,q)=1}$.*

(iii) *If V^c is not simple, then $\mathcal{F}^1(V^c) = L((p-1)(q-1); c_{p,q})$, $\mathcal{F}^2(V^c) = 0$, where $c = c_{p,q}$, $p, q \in \mathbb{Z}_{\geqslant 2}$, $(p,q) = 1$.*

(iv) *The vertex algebra $\mathcal{V}ir_c$ satisfies Zhu's C_2 condition iff the vacuum module V^c is not simple.*

We prove (iv) in 4.3 and (i) in 4.4 below; (ii), (iii) follow from (i) and Jantzen sum formula (8).

4.3. Singular vectors in V^c

Since $L_{-1}|0;c\rangle = 0$ in V^c, it is clear that C_2 holds iff the vectors $L_{-2}^k|0;c\rangle$ $(k \geqslant 1)$ are linearly dependent over $C_2(\mathcal{V}ir_c) := \operatorname{span}\{L_{-k}v \mid k > 2,\ v \in \mathcal{V}ir_c\}$. If V^c is simple then $\mathcal{V}ir^c = \mathcal{V}ir_c$ and the vectors $L_{-2}^k|0;c\rangle$ are linearly independent over $C_2(\mathcal{V}ir_c)$ and thus $\mathcal{V}ir_c$ does not satisfy C_2 condition. Take c such that V^c is not simple. In order to check the C_2 condition, it is enough to verify that a singular vector in V^c is of the form $(L_{-2}^k + a)|0;c\rangle$, where $a \in \mathcal{U}(\mathcal{V}ir_{<-1})$ lies in the right ideal generated by L_{-i}, $i > 2$. This will be shown in Proposition 4.3.2.

4.3.1. A total ordering on monomials

For $v \in V^c$ call *the monomials of* v all the ordered monomials appearing in u, where $u \in \mathcal{U}(\mathcal{V}ir_{<-1})$ is such that $v = u|0;c\rangle$.

Introduce the following (lexicographic) total order (cf. [22]) on the ordered monomials of $\mathcal{U}(\mathcal{V}ir_{<-1})$ with given $\operatorname{ad} L_0$-eigenvalue: for $L_{-i_s}\dots L_{-i_1}$ and $L_{-j_r}\dots L_{-j_1}$ with $i_s \geqslant \dots \geqslant i_1 \geqslant 2$, $j_r \geqslant \dots \geqslant j_1 \geqslant 2$ and $\sum i_m = \sum j_n = N$, put $L_{-i_s}\dots L_{-i_1} < L_{-j_r}\dots L_{-j_1}$ if either $i_1 < j_1$, or $i_1 = j_1$, $i_2 = j_2$, $\dots$, $i_m = j_m$, $i_{m+1} < j_{m+1}$. For example, $L_{-4}^2 > L_{-5}L_{-3} > L_{-3}L_{-3}L_{-2} > L_{-2}^4$.

4.3.2. Proposition. *The minimal monomial of a singular vector v of V^c, not proportional to $|0;c\rangle$, is L_{-2}^m, where m is a positive integer.*

Proof. Observe that for $u \in \mathcal{V}ir_{<-1}$ and $k \geqslant -1$, one has $L_k u|0;c\rangle = [L_k, u]|0;c\rangle$. In particular, if $v = u|0;c\rangle$ then the monomials of $L_k v$ for $k \geqslant -1$ are the monomials of $[L_k, u]$, which lie in $\mathcal{U}(\mathcal{V}ir_{<-1})$.

The minimal monomial of $[L_1, L_{-i_s}\dots L_{-i_1}]$ is $L_{-i_s}\dots L_{-i_2}L_{-i_1+1}$. In particular, if X, Y are monomials in $\mathcal{V}ir_{<-1}$ and $X < Y$ then the minimal monomial of $[L_1, X]$ is less than the minimal monomial of $[L_1, Y]$. If $i_1 > 2$ then $L_{-i_s}\dots L_{-i_2}L_{-i_1+1}$ belongs to $\mathcal{V}ir_{<-1}$. As a consequence, the minimal monomial of a singular vector v is of the form $L_{-i_s}\dots L_{-i_1}$, where $i_1 = 2$. Indeed, suppose that $i_1 > 2$; then the minimal monomial of $L_1 v$ is $L_{-i_s}\dots L_{-i_2}L_{-i_1+1}$, which belongs to $\mathcal{V}ir_{<-1}$, and thus $L_1 v \neq 0$, so v is not singular.

Now it remains to show that the minimal monomial of v is not of the form $X = X' L_{-r}^s L_{-2}^m$ for some $r > 2$, $s > 0$.

Let $X = X' L_{-r}^s L_{-2}^m$ be a monomial ($r > 2$ and X' does not contain L_{-2} and L_{-r}). Then the minimal monomial of $L_{r-2}X|0;c\rangle$ is $X' L_{-r}^{s-1} L_{-2}^{m+1}$. Suppose that X is the minimal monomial of a singular vector v (we have shown that $m \geqslant 1$). Since $L_{r-2}v = 0$, the monomial

$$Z := X' L_{-r}^{s-1} L_{-2}^{m+1}$$

should appear as a monomial in $L_{r-2}Y|0;c\rangle$ for some $Y > X$. Write $Y = Y'' L_{-2}^k$, where Y'' does not contain L_{-2}. Recall that

$$Y > X \iff \left[\begin{array}{l} k < m, \\ k = m \text{ and } Y'' > X' L_{-r}^s. \end{array}\right.$$

The degree of L_{-2} in any monomial of $[L_{r-2}, Y]$ is at most $k+1$. Hence $L_{r-2}Y|0;c\rangle$ does not contain Z if $k < m$.

In the remaining case $Y = Y'' L_{-2}^m$ for $Y' > X' L_{-r}^s$, write $Y = Y' L_{-t}^p L_{-2}^m$, where Y' does not contain L_{-t}. Then $t > 2$ and the condition $Y > X$ forces that either $t > r$ or $t = r, Y' L_{-r}^p > X' L_{-r}^s$.

If $Y = Y' L_{-t}^p L_{-2}^m$ for some $t > r$ then the degree of L_{-2} in any monomial of $[L_{r-2}, Y]$ is at most m so $L_{r-2}Y|0;c\rangle$ does not contain the monomial Z.

If $Y = Y' L_{-r}^p L_{-2}^m$, then the only monomial of $[L_{r-2}, Y]$, having a factor L_{-2}^{m+1}, is $Y' L_{-r}^{p-1} L_{-2}^{m+1}$. Since $Y' L_{-r}^p > X' L_{-r}^s$, one has $Y' L_{-r}^{p-1} > X' L_{-r}^{s-1}$ and so $Y' L_{-r}^{p-1} L_{-2}^{m+1} > Z$. Hence $L_{r-2}Y|0;c\rangle$ does not contain Z, a contradiction. The assertion follows. □

4.4. Proof of Theorem 4.2.1(ii)

4.4.1. Outline of the proof

In Lemma 4.4.2 we will show that V^c has a subquotient, isomorphic to $L((p-1)(q-1);c)$ if $c = c_{p,q}$, where $p > q \geqslant 2$, $(p,q) = 1$. Using the sum formula (8) we conclude that $\det'_N(c)$ is divisible by the polynomial

$$P_N(c) := \prod_{p>q\geqslant 2,\,(p,q)=1} (c - c_{p,q})^{\dim L((p-1)(q-1);c_{p,q})_N}.$$

In 4.4.3–4.4.5 we will show that the degree of $\det'_N(c)$ coincides with the degree of $P_N(c)$ so $\det'_N(c) = aP_N(c)$ for $a \in \mathbb{C}^*$. This proves Theorem 4.2.1(ii).

4.4.2. Lemma. *Let $c = c_{p,q}$, where $p > q \geqslant 2$ are relatively prime integers. Then V^c has a subquotient isomorphic to $L((p-1)(q-1);c)$.*

Proof. Recall that $V^c = M(0;c)/M(1;c)$. We will show that $M(0;c)$ has a singular vector of weight $(p-1)(q-1)$, whereas $M(1;c)$ does not have such a vector.

Recall (see [12], [16, 8.1–8.4]) that the determinant of the contravariant (= Shapovalov) form for a Verma module over the Virasoro algebra is, up to a non-zero constant factor:

$$\det_{N+h}(c,h) = \prod_{r,s\in\mathbb{Z}_{\geqslant 1}} \bigl(h - h_{r,s}(c)\bigr)^{p_{cl}(N-rs)},$$

where $p_{cl}(m)$ is the classical partition function:

$$\prod_{k=1}^{\infty}\bigl(1-x^k\bigr)^{-1} = \sum_{m\in\mathbb{Z}} p_{cl}(m)x^m,$$

and the functions $h_{r,s}(c)$ can be described as follows:

$$h_{r,r}(c)=\frac{(r^2-1)(c-1)}{24},\qquad h_{r,s}(c)=\frac{(p'r-q's)^2-(p'-q')^2}{4p'q'},$$

where $p',q'\in\mathbb{C}$ are such that $c=1-\frac{6(p'-q')^2}{p'q'}$.

Let $c=c_{p,q}$, where $p>q\geqslant 2$ are relatively prime integers. One has $h_{r,r}(c)\neq 0$ for $r\geqslant 2$ and $h_{r,s}(c)=0$ iff $pr-qs=\pm(p-q)$; $h_{r,r}(c)\neq 1$ for $r\geqslant 1$ and $h_{r,s}(c)=1$ iff $pr-qs=\pm(p+q)$. As a result, $h_{p-1,q-1}(p,q)=0$ and $h_{r,s}(p,q)\neq 1$ if $rs<(p+1)(q-1)$. Hence $M(0;c)$ has a singular vector of weight $(p-1)(q-1)$, whereas the minimal weight of a singular vector in $M(1;c)$ is $1+(p+1)(q-1)=pq-p+q$. The claim follows. $\square$

4.4.3. The leading term of $\det'_N(c)$ is $c^{d(N)}$, where

$$d(n)=\sum_{\lambda\vdash n,1\notin\lambda} l(\lambda).$$

Here $\lambda\vdash n$ stands for a partition of n (we will write $|\lambda|=n$), $l(\lambda)$ is the number of parts of λ, and $1\in\lambda$ means that λ contains a part equal to 1. One has

$$\sum_{\lambda:1\notin\lambda} t^{l(\lambda)}x^{|\lambda|}=\prod_{m=2}^{\infty}\left(1-tx^m\right)^{-1}$$

and this allows to express the generating function $\sum_n d(n)x^n$ as follows:

$$\sum_n d(n)x^n=\sum_{1\notin\lambda} l(\lambda)x^{|\lambda|}=\left.\frac{\partial\prod_{m=2}^{\infty}(1-tx^m)^{-1}}{\partial t}\right|_{t=1}.$$

Therefore

$$\sum_n d(n)x^n=\prod_{m=2}^{\infty}\left(1-x^m\right)^{-1}\sum_{r=2}^{\infty}\frac{x^r}{1-x^r}=\prod_{m=2}^{\infty}\left(1-x^m\right)^{-1}\sum_{r\geqslant 2,\,s\geqslant 1}x^{rs},$$

which can be rewritten as

$$\prod_{m=1}^{\infty}\left(1-x^m\right)\sum_n d(n)x^n=\sum_{r\geqslant 2,s\geqslant 1}\left(x^{rs}-x^{rs+1}\right).$$

4.4.4. Take $c=c_{p,q}$, where $p,q\in\mathbb{Z}$ are such that $p>q\geqslant 2$, $(p,q)=1$. One has

$$\prod_{k\geqslant 1}\left(1-x^k\right)\operatorname{ch}L\big((p-1)(q-1);c\big)=\sum_{k\in\mathbb{Z}\setminus\{0\}}\left(x^{(1+kq)(1+kp)}-x^{(kq-1)(1+kp)+1}\right),$$

see [2,4]. In order to prove that the degree of $\det'_N(c)$ coincides with the degree of $P_N(c)$ (see 4.4.1) it remains to verify the following identity of formal power series in x:

$$\sum_{r\geqslant 2,s\geqslant 1}\left(x^{rs}-x^{rs+1}\right)=\sum_{p>q\geqslant 2,\,(p,q)=1}\ \sum_{k\in\mathbb{Z}\setminus\{0\}}\left(x^{(1+kq)(1+kp)}-x^{(kq-1)(1+kp)+1}\right). \tag{14}$$

4.4.5. One has

$$\sum_{r\geqslant 2,s\geqslant 1}x^{rs}=\frac{1}{2}\sum_{k,l\geqslant 1}x^{(k+1)(l+1)}+\frac{1}{2}\sum_{k,l\geqslant 2}x^{(k-1)(l-1)}-x/2.$$

Writing $k=jp$, $l=jq$, where $j:=(k,l)$ and $(p,q)=1$, we obtain

$$\begin{aligned}\sum_{k,l\geqslant 1}x^{(k+1)(l+1)}&=\sum_{j\geqslant 1}\ \sum_{p,q\geqslant 1,(p,q)=1}x^{(jp+1)(jq+1)}\\&=\sum_{j\geqslant 1}\left(\sum_{p>q\geqslant 2,(p,q)=1}2x^{(jp+1)(jq+1)}+2\sum_{p\geqslant 2}x^{(j+1)(jp+1)}+x^{(j+1)^2}\right)\end{aligned}$$

and similarly

$$\begin{aligned}\sum_{k,l\geqslant 2}x^{(k-1)(l-1)}&=\sum_{j\geqslant 2}\ \sum_{p,q\geqslant 1,(p,q)=1}x^{(jp-1)(jq-1)}+\sum_{p,q\geqslant 2,(p,q)=1}x^{(p-1)(q-1)}\\&=2\sum_{j\geqslant 1}\ \sum_{(p,q)=1,p>q\geqslant 2}x^{(jp-1)(jq-1)}+\sum_{j\geqslant 2}\left(2\sum_{p\geqslant 2}x^{(j-1)(jp-1)}+x^{(j-1)^2}\right)\\&=2\sum_{j\geqslant 1}\ \sum_{(p,q)=1,p>q\geqslant 2}x^{(jp-1)(jq-1)}+\sum_{j\geqslant 2}\left(2\sum_{p\geqslant 2}x^{(j-1)(jp-1)}+x^{j^2}\right)+x.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{r\geqslant 2,s\geqslant 1}x^{rs}&=\sum_{j\geqslant 1}\ \sum_{(p,q)=1,p>q\geqslant 2}\left(x^{(jp+1)(jq+1)}+x^{(jp-1)(jq-1)}\right)\\&\quad+\sum_{j\geqslant 2}x^{j^2}+\sum_{j\geqslant 1}\sum_{p\geqslant 2}x^{(j+1)(jp+1)}+\sum_{j\geqslant 2}\sum_{p\geqslant 2}x^{(j-1)(jp-1)}\\&=\sum_{j\in\mathbb{Z}\setminus\{0\}}\ \sum_{(p,q)=1,p>q\geqslant 2}x^{(jp+1)(jq+1)}+\sum_{j\in\mathbb{Z}\setminus\{0,-1\}}\ \sum_{p\geqslant 2}x^{(j+1)(jp+1)}+\sum_{j\geqslant 2}x^{j^2}.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{r\geqslant 2,s\geqslant 1}x^{rs+1}&=\sum_{k\geqslant 2,l\geqslant 1}x^{(k-1)(l+1)+1}\\&=\sum_{j\geqslant 2}\ \sum_{p,q\geqslant 1,(p,q)=1}x^{(jp-1)(jq+1)+1}+\sum_{p\geqslant 2,q\geqslant 1,(p,q)=1}x^{(p-1)(q+1)+1}\end{aligned}$$

$$= \sum_{j\geqslant 1}\sum_{p>q\geqslant 2,\,(p,q)=1}\big(x^{(jp-1)(jq+1)+1}+x^{(jp+1)(jq-1)+1}\big)+\sum_{j\geqslant 2}x^{j^2}$$
$$+\sum_{j\geqslant 1}\sum_{p\geqslant 2}x^{(jp-1)(j+1)+1}+\sum_{j\geqslant 2}\sum_{p\geqslant 2}x^{(jp+1)(j-1)+1}$$
$$=\sum_{j\in\mathbb{Z}\setminus\{0\}}\sum_{(p,q)=1,\,p>q\geqslant 2}x^{(jp+1)(jq-1)+1}+\sum_{j\in\mathbb{Z}\setminus\{0,-1\}}\sum_{p\geqslant 2}x^{(j+1)(jp-1)+1}$$
$$+\sum_{j\geqslant 2}x^{j^2}.$$

Now (14) follows from the following identities:

$$\sum_{j\in\mathbb{Z}\setminus\{0,-1\}}x^{(jp-1)(j+1)+1}=\sum_{j\in\mathbb{Z}\setminus\{0,-1\}}x^{(-j)(-(j+1)p+1)}=\sum_{j\in\mathbb{Z}\setminus\{0,-1\}}x^{(i+1)(ip+1)}.$$

5. Neveu–Schwarz algebra

The Neveu–Schwarz superalgebra $\mathcal{NS}$ is a Lie superalgebra with the basis $\{C; L_i\}_{i\in\frac{1}{2}\mathbb{Z}}$, such that its even part is the Virasoro algebra (with the basis $\{C; L_i\}_{i\in\mathbb{Z}}$), the element C is central, and apart from the relations (2) of the Virasoro algebra, the following commutation relations for $m\in\mathbb{Z}$ and $i, j\in\frac{1}{2}\mathbb{Z}\setminus\mathbb{Z}$ hold:

$$[L_i, L_j]=2L_{i+j}+\delta_{0,i+j}\frac{4i^2-1}{12}C,\qquad [L_m, L_j]=\left(\frac{m}{2}-j\right)L_{j+m}.$$

5.1. Notation

Denote by $\mathcal{NS}_{>k}$ the subspace spanned by L_j, $j>k$. A *Verma module* $M(h;c)$ $(h,c\in\mathbb{C})$ over $\mathcal{NS}$ is induced from the one-dimensional module $\mathbb{C}|h;c\rangle$ of $\mathcal{NS}_{\geqslant 0}+\mathbb{C}C$, where $\mathcal{NS}_{>0}$ acts trivially, L_0 acts by the scalar h and C acts by the scalar c. The weight spaces of $M(h;c)$ are eigenspaces of L_0 with eigenvalues $h+n$, $n\in\frac{1}{2}\mathbb{Z}_{\geqslant 0}$.

Notice that $\mathcal{NS}_{\geqslant -1}$ is a subalgebra. A *vacuum module* V^c over $\mathcal{NS}$ is induced from the one-dimensional module $\mathbb{C}|0;c\rangle$ of $\mathcal{NS}_{\geqslant -1}+\mathbb{C}C$, where $\mathcal{NS}_{\geqslant -1}$ acts trivially and C acts by the scalar c. Clearly, $V^c=M(0;c)/M(1/2;c)$. Recall that it carries a canonical structure of a vertex algebra, denoted by $\mathcal{NS}^c$. Its unique simple quotient is denoted by $\mathcal{NS}_c$.

Let

$$Y:=\left\{(p,q)\in\mathbb{Z}_{\geqslant 1}\times\mathbb{Z}_{\geqslant 1}\;\middle|\; p\equiv q \bmod 2,\ \left(\frac{p-q}{2},q\right)=1\right\}. \tag{15}$$

Set

$$\psi(x,t):=\prod_{n=0}^{\infty}\big(1+tx^{n+1/2}\big)^{-1}\prod_{n=1}^{\infty}\big(1-tx^n\big).$$

The function $\psi(x,1)$ is the super analogue of the Virasoro denominator $\prod_{n=1}^{\infty}(1-x^n)$, namely one has: $\psi(x,1)^{-1}=\sum_{N\in\frac{1}{2}\mathbb{Z}}\dim M(h;c)_{h+N}x^N$.

5.2. Main result

Introduce the anti-involution σ on $\mathcal{NS}$ by the formulas $\sigma(L_n) = L_{-n}$, $\sigma(C) = C$. Define the triangular decomposition $\mathcal{NS} = \mathcal{NS}_{<0} \oplus (\mathbb{C}L_0 + \mathbb{C}C) \oplus \mathcal{NS}_{>0}$, and introduce the Harish-Chandra projection with respect this triangular decomposition. Define the contravariant forms and the Jantzen filtrations on Verma modules and on vacuum modules as in 1.2.4, 2.2.1; observe that the "sum formula" (8) holds in this setup. We denote the determinant of the contravariant form on the eigenspace of L_0 with the eigenvalue $h + N$ ($N \in \frac{1}{2}\mathbb{Z}_{\geqslant 0}$) in $M(h; c)$ by $\det_{h+N}$ (respectively, on the eigenspace V^c_N of L_0 in V^c with the eigenvalue N by $\det'_N$). These are polynomials in h and c (respectively, c).

5.2.1. Theorem. *Let $c^S_{p,q} = \frac{3}{2}(1 - \frac{2(p-q)^2}{pq})$, and recall notation* (15).

(i) *Up to a non-zero scalar factor, the vacuum determinant for $\mathcal{NS}$ is as follows*:

$$\det'_N(c) = \prod_{p>q\geqslant 2,\,(p,q)\in Y} \bigl(c - c^S_{p,q}\bigr)^{\dim L((p-1)(q-1)/2;\, c^S_{p,q})_N}. \tag{16}$$

(ii) *A vacuum $\mathcal{NS}$-module V^c is simple iff $c \neq c^S_{p,q}$, where $p > q \geqslant 2$, $(p,q) \in Y$. If V^c is not simple then its unique proper submodule is $L((p-1)(q-1)/2; c^S_{p,q})$ and*

$$\operatorname{ch} L\bigl((p-1)(q-1)/2; c^S_{p,q}\bigr) = \psi(x,1)^{-1} \sum_{k\in\mathbb{Z}\setminus\{0\}} \bigl(x^{(kp+1)(kq+1)/2} - x^{((kp+1)(kq-1)+1)/2}\bigr).$$

(iii) *If the $\mathcal{NS}$-module V^c is not simple, i.e. $c = c^S_{p,q}$, where $p > q \geqslant 2$, $(p,q) \in Y$, then*

$$\mathcal{F}^1\bigl(V^c\bigr) = L\bigl((p-1)(q-1)/2; c^S_{p,q}\bigr), \qquad \mathcal{F}^2\bigl(V^c\bigr) = 0.$$

(iv) *The vertex algebra $\mathcal{NS}_c$ satisfies Zhu's C_2 condition iff the vacuum module V^c is not simple.*

5.3. Superpartitions

Let us call $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ a *superpartition* of N if $\sum_{i=1}^m \lambda_i = N$, $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_m$, $2\lambda_i \in \mathbb{Z}_{\geqslant 1}$ for any i, and $\lambda_i \neq \lambda_{i\pm 1}$ if λ_i is not integer (i.e., any half-integer appears at most once in the multiset $\{\lambda_i\}_{i=1}^m$). Write $\lambda \vdash N$ if λ is a superpartition of N; set $|\lambda| = N$ and $l(\lambda) = m$ if $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$. For $b \in \frac{1}{2}\mathbb{Z}$ write $b \in \lambda$ if $\lambda_i = b$ for some i.

Note that $\psi(x,t)^{-1}$ is the generating function for superpartitions: $\psi(x,t)^{-1} = \sum_\lambda t^{l(\lambda)} x^{|\lambda|}$.

5.4. Proof of Theorem 5.2.1(i)–(iii)

It is straightforward to deduce (ii) from [2, 8.2] (which relies on the Kac determinant formula for $\mathcal{NS}$ [12]). It follows that if $c = c^S_{p,q}$, where $p > q \geqslant 2$, $(p,q) \in Y$, then $\mathcal{F}^1(V^c) = L((p-1)(q-1)/2; c^S_{p,q})$. From the sum formula (8), it follows that $\det'_N(c)$ is divisible by the right-hand side of (16) and, moreover, that (i) implies (iii). We prove (16) by showing that the degree of $\det'_N(c)$ is equal to the degree of the right-hand side of (16). Let $d(N)$ be the degree of $\det'_N(c)$. In terms of generating functions, we need to show that

$$\sum_{n\in\frac{1}{2}\mathbb{Z}} d(n)x^n = \sum_{p>q\geqslant 2,(p,q)\in Y} \dim L\big((p-1)(q-1)/2; c^S_{p,q}\big)_n x^n,$$

which can be rewritten as

$$\psi(x,1)\sum_{n\in\frac{1}{2}\mathbb{Z}} d(n)x^n = \sum_{p>q\geqslant 2,(p,q)\in Y}\ \sum_{k\in\mathbb{Z}\setminus\{0\}} \big(x^{(kp+1)(kq+1)/2} - x^{((kp+1)(kq-1)+1)/2}\big).$$

5.4.1. One has

$$d(n) = \sum_{\lambda\vdash n, 1\notin\lambda, \frac{1}{2}\notin\lambda} l(\lambda).$$

Observe that

$$\psi(x,t)^{-1}\frac{(1-tx)}{1+tx^{1/2}} = \sum_{\lambda\vdash n, 1\notin\lambda, \frac{1}{2}\notin\lambda} t^{l(\lambda)}x^{|\lambda|},$$

and this allows to express the generating function $\sum_{n\in\frac{1}{2}\mathbb{Z}} d(n)x^n$ as follows:

$$\sum_{n\in\frac{1}{2}\mathbb{Z}} d(n)x^n = \sum_{\lambda: 1\notin\lambda, \frac{1}{2}\notin\lambda} l(\lambda)x^{|\lambda|} = \frac{\partial}{\partial t}\frac{(1-tx)}{(1+tx^{1/2})\psi(x,t)}\bigg|_{t=1}.$$

Since

$$\frac{\partial}{\partial t}\frac{(1-tx)}{(1+tx^{1/2})\psi(x,t)} = \frac{(1-tx)}{(1+tx^{1/2})\psi(x,t)}\left(\sum_{n=1}^{\infty}\frac{x^{n+1/2}}{1+tx^{n+1/2}} + \sum_{n=2}^{\infty}\frac{x^n}{1-tx^n}\right),$$

we obtain

$$\psi(x,1)\sum_{n\in\frac{1}{2}\mathbb{Z}} d(n)x^n = \big(1-x^{1/2}\big)\left(\sum_{n=1}^{\infty}\frac{x^{n+1/2}}{1+x^{n+1/2}} + \sum_{n=2}^{\infty}\frac{x^n}{1-x^n}\right) = \big(1-x^{1/2}\big)\sum_{\substack{r\geqslant 2, s\geqslant 1\\ r\equiv s \bmod 2}} x^{\frac{rs}{2}}.$$

Put $y := x^{1/2}$. Let a be a rational number greater than 1 which is not an odd integer. Then a can be uniquely written as $a = p/q$, where $p > q \geqslant 2$ are integers of the same parity and q is the smallest one with this property, i.e., $(\frac{p-q}{2}, q) = 1$.

5.4.2. One has

$$\sum_{\substack{r\geqslant 2,s\geqslant 1\\ r\equiv s \bmod 2}} y^{rs}=\frac{1}{2}\sum_{\substack{k,l\geqslant 1\\ k\equiv l \bmod 2}} y^{(k+1)(l+1)}+\frac{1}{2}\sum_{\substack{k,l\geqslant 2\\ k\equiv l \bmod 2}} y^{(k-1)(l-1)}-y/2.$$

Here and further, r, s, k, l, p, q are integers.

For $k\equiv l \bmod 2$ we can write $k=jp$, $l=jq$, where $j:=(\frac{k-l}{2}, l)$ and $(p,q)\in Y$. We get

$$\begin{aligned}
&\sum_{\substack{k,l\geqslant 1\\ k\equiv l \bmod 2}} y^{(k+1)(l+1)}\\
&\quad=\sum_{j\geqslant 1}\sum_{p,q\geqslant 1,(p,q)\in Y} y^{(jp+1)(jq+1)}\\
&\quad=\sum_{j\geqslant 1}\Bigl(\sum_{p>q\geqslant 2,(p,q)\in Y} 2y^{(jp+1)(jq+1)}+2\sum_{p\geqslant 2,(p,1)\in Y} y^{(j+1)(pj+1)}+y^{(j+1)^2}\Bigr)
\end{aligned}$$

and similarly

$$\begin{aligned}
\sum_{\substack{k,l\geqslant 2\\ k\equiv l \bmod 2}} y^{(k-1)(l-1)}&=\sum_{j\geqslant 2}\sum_{p,q\geqslant 1,(p,q)\in Y} y^{(jp-1)(jq-1)}\\
&\quad+\sum_{p,q\geqslant 2,(p,q)\in Y} y^{(p-1)(q-1)}\\
&=2\sum_{j\geqslant 1}\sum_{p>q\geqslant 2,(p,q)\in Y} y^{(jp-1)(jq-1)}\\
&\quad+\sum_{j\geqslant 2}\Bigl(2\sum_{p\geqslant 2,(p,1)\in Y} y^{(j-1)(pj-1)}+y^{(j-1)^2}\Bigr)\\
&=2\sum_{j\geqslant 1}\sum_{p>q\geqslant 2,(p,q)\in Y} y^{(jp-1)(jq-1)}\\
&\quad+\sum_{j\geqslant 2}\Bigl(2\sum_{p\geqslant 2,(p,1)\in Y} y^{(j-1)(pj-1)}+y^{j^2}\Bigr)+y.
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{\substack{r\geqslant 2,s\geqslant 1\\ r\equiv s \bmod 2}} y^{rs}&=\sum_{j\geqslant 1}\sum_{p>q\geqslant 2,(p,q)\in Y}\bigl(y^{(jp+1)(jq+1)}+y^{(jp-1)(jq-1)}\bigr)\\
&\quad+\sum_{j\geqslant 2} y^{j^2}+\sum_{j\geqslant 1}\sum_{m=1}^{\infty} y^{(j+1)((2m+1)j+1)}+\sum_{j\geqslant 2}\sum_{m=1}^{\infty} y^{(j-1)((2m+1)j-1)}
\end{aligned}$$

$$= \sum_{j\in\mathbb{Z}\setminus\{0\}} \sum_{p>q\geqslant 2,(p,q)\in Y} y^{(jp+1)(jq+1)} + \sum_{j\in\mathbb{Z}\setminus\{0,-1\}} \sum_{m=1}^{\infty} y^{(j+1)((2m+1)j+1)} + \sum_{j\geqslant 2} y^{j^2}.$$

Moreover,

$$\begin{aligned}
\sum_{\substack{r\geqslant 2,s\geqslant 1\\ r\equiv s \bmod 2}} y^{rs+1} &= \sum_{\substack{k\geqslant 2,l\geqslant 1\\ k\equiv l \bmod 2}} y^{(k-1)(l+1)+1}\\
&= \sum_{j\geqslant 2} \sum_{p,q\geqslant 1,(p,q)\in Y} y^{(jp-1)(jq+1)+1} + \sum_{p\geqslant 2,q\geqslant 1,(p,q)\in Y} y^{(p-1)(q+1)+1}\\
&= \sum_{j\geqslant 1} \sum_{p>q\geqslant 2,(p,q)=1} \left(y^{(jp-1)(jq+1)+1} + y^{(jp+1)(jq-1)+1}\right) + \sum_{j\geqslant 2} y^{j^2}\\
&\quad + \sum_{p\geqslant 2,(p,1)\in Y} \left(\sum_{j\geqslant 1} y^{(jp-1)(j+1)+1} + \sum_{j\geqslant 2} y^{(jp+1)(j-1)+1}\right)\\
&= \sum_{j\in\mathbb{Z}\setminus\{0\}} \sum_{p>q\geqslant 2,(p,q)\in Y} y^{(jp+1)(jq-1)+1} + \sum_{j\in\mathbb{Z}\setminus\{0,-1\}} \sum_{p\geqslant 2,(p,1)\in Y} y^{(jp+1)(j-1)+1}\\
&\quad + \sum_{j\geqslant 2} y^{j^2}.
\end{aligned}$$

One has

$$\sum_{j\in\mathbb{Z}\setminus\{0,-1\}} y^{(jp-1)(j+1)+1} = \sum_{j\in\mathbb{Z}\setminus\{0,-1\}} y^{(-j)(-(j+1)p+1)} = \sum_{j\in\mathbb{Z}\setminus\{0,-1\}} y^{(i+1)(ip+1)}.$$

Hence we obtain the required equality:

$$\sum_{\substack{r\geqslant 2,s\geqslant 1\\ r\equiv s \bmod 2}} \left(y^{rs} - y^{rs+1}\right) = \sum_{j\geqslant 1} \sum_{p>q\geqslant 2,\,(p,q)\in Y} y^{(jp+1)(jq+1)} - y^{(jp-1)(jq+1)+1}.$$

5.5. Proof of Theorem 5.2.1(iv)

Let $C_2(\mathcal{NS}_c) := \operatorname{span}\{L_{-k}v \mid k > 2,\ v \in \mathcal{NS}_c\}$. Recall that the C_2 condition for $\mathcal{NS}_c$ means that $C_2(\mathcal{NS}_c)$ has finite codimension in V_c. Since $L_{-1}|0;c\rangle = L_{-1/2}|0;c\rangle = 0$ in V^c, it is clear that the C_2 condition holds iff the vectors $L_{-2}^k|0;c\rangle$ $(k \geqslant 1)$ are linearly dependent over $C_2(\mathcal{NS}_c)$.

If V^c is simple then $\mathcal{NS}^c = \mathcal{NS}_c$ and the vectors $L_{-2}^k|0;c\rangle$ are linearly independent over $C_2(\mathcal{NS}_c)$, and thus $\mathcal{NS}_c$ does not satisfy the C_2 condition.

Take c such that V^c is not simple. Then V^c has a unique proper submodule $\overline{V}^c$. In order to check the C_2 condition, it is enough to verify that $\overline{V}^c$ contains a vector of the form $(L_{-2}^k + a)|0;c\rangle$, where $a \in \mathcal{U}(\mathcal{NS}_{<-1})$ lies in the right ideal generated by L_{-i}, $i > 2$. Let v be a singular vector of $\overline{V}^c$ (it is unique up to a scalar). In Corollary 5.5.4 we will show that either v or $L_{-1/2}v$ is of the form $(L_{-2}^k + a)|0; c\rangle$. This will prove (iv).

5.5.1. Consider the PBW basis of $\mathcal{U}(\mathcal{NS}_{<-1})$ which consists of the monomials of the form $L_{-i_s}^{m_s} \dots L_{-i_1}^{m_1}$, where $i_j \in \frac{1}{2}\mathbb{Z}$, $1 < i_1 < i_2 < \cdots < i_s$, $m_j \geqslant 1$ and $m_j = 1$ if $i_j \notin \mathbb{Z}$.

Define the (lexicographic) total order on the PBW basis of $\mathcal{U}(\mathcal{NS}_{<-1})$ with given $\operatorname{ad} L_0$-eigenvalue in the same way as in 4.3.1, and retain conventions of 4.3.1.

5.5.2. Lemma. *Let $v|0; c\rangle \in V^c$, where $v \in \mathcal{U}(\mathcal{NS}_{<-1})$, be a singular vector, not proportional to $|0; c\rangle$. Then v contains either a monomial L_{-2}^k $(k > 0)$ or a monomial $L_{-2}^m L_{-3/2}$ $(m \geqslant 0)$ with a non-zero coefficient.*

Proof. For $u \in \mathcal{U}(\mathcal{NS}_{<-1})$ denote by $\operatorname{supp} u$ the set of monomials, which u contains. In this proof the letters U, X, Y, Z stand for monomials in $\mathcal{U}(\mathcal{NS}_{<-1})$.

Let X, Y be monomials in $\mathcal{U}(\mathcal{NS}_{<-1})$ and let Y does not contain $L_{-3/2}$ and L_{-2}. Then for $r > 0$ one has:

$$\text{(i)}\quad YL_{-3/2} \in \operatorname{supp}[L_r, X] \quad\Longrightarrow\quad \left[\begin{array}{l} X = X'L_{-3/2},\ Y \in \operatorname{supp}[L_r, X'], \\ X = YL_{-(r+3/2)}; \end{array}\right.$$

$$\text{(ii)}\quad YL_{-2}^s L_{-3/2} \in \operatorname{supp}[L_r, X],\quad s > 0 \quad\Longrightarrow\quad \left[\begin{array}{l} X = X'L_{-2}^s L_{-3/2},\ Y \in \operatorname{supp}[L_r, X'], \\ X = YL_{-(r+2)}L_{-2}^{s-1}L_{-3/2}, \\ X = YL_{-(r+3/2)}L_{-2}^s; \end{array}\right.$$

$$\text{(iii)}\quad YL_{-2}^s \in \operatorname{supp}[L_r, X],\quad s > 0 \quad\Longrightarrow\quad \left[\begin{array}{l} X = X'L_{-2}^s,\ Y \in \operatorname{supp}[L_r, X'], \\ X = YL_{-(r+2)}L_{-2}^{s-1}, \\ X = YL_{-(r+1/2)}L_{-2}^{s-1}L_{-3/2}. \end{array}\right.$$

Let M be the minimal element in $\operatorname{supp} v$. Arguing as in Proposition 4.3.2, we see that M contains either $L_{-3/2}$ or L_{-2}.

Assume that M does not contain $L_{-3/2}$. Write $M = YL_{-2}^s$, where Y does not contain $L_{-3/2}, L_{-2}$; by the above $s > 0$. Then $YL_{-2}^{s-1}L_{-3/2} \in \operatorname{supp}[L_{1/2}, M]$. Since v is singular, $L_{1/2}v = 0$ and thus the monomial $YL_{-2}^{s-1}L_{-3/2} \in \operatorname{supp}[L_{1/2}, X]$ for some $X \in \operatorname{supp} v$, $X \neq M$. Since M is minimal in $\operatorname{supp} v$, X does not contain $L_{-3/2}$. From (i), (ii) above we conclude that $X = M$, a contradiction.

Hence M contains $L_{-3/2}$. If $M = L_{-2}^n L_{-3/2}$ the assertion of the lemma holds so we assume that

$$M = X'L_{-r}^s L_{-2}^n L_{-3/2},$$

where $r \in 1/2\mathbb{Z}$, $r > 2$, $s \geqslant 1$, $n \geqslant 0$ and X' does not contain L_{-r}.

Note that the minimal monomial of $[L_{r-2}, M]$, which belongs to $\mathcal{U}(\mathcal{NS}_{<-1})$ is $Z := X'L_{-r}^{s-1}L_{-2}^{n+1}L_{-3/2}$. Hence Z should appear as $[L_{r-2}, U]$ for some monomial $U \in \operatorname{supp} v$, $U > M$. From (ii) above we conclude that $U = X'L_{-r}^{s-1}L_{-(r-1/2)}L_{-2}^{n+1}$.

Let $\operatorname{supp}_0 v$ consists of the monomials in $\operatorname{supp} v$, which contain L_{-2} and do not contain $L_{-3/2}$. By the above, $U \in \operatorname{supp}_0 v$ and so $\operatorname{supp}_0 v$ is not empty. Let M_0 be the minimal element in $\operatorname{supp}_0 v$. By the above, $M_0 < U$, i.e.

$$M_0 < X' L_{-r}^{s-1} L_{-(r-1/2)} L_{-2}^{n+1}.$$

It remains to show that $M_0 = L_{-2}^k$ for some $k > 0$.

Suppose that $M_0 \neq L_{-2}^k$ and write $M_0 = Y L_{-j}^p L_{-2}^k$, where $j \in 1/2\mathbb{Z}$, $j > 2$, $p, k \geqslant 1$ and Y' does not contain L_{-j}. Observe that $k > n$. The minimal monomial of $[L_{j-2}, M_0]$, which belongs to $\mathcal{U}(\mathcal{NS}_{<-1})$ is $Z := Y L_{-j}^{p-1} L_{-2}^{k+1}$. Therefore Z should appear as $[L_{j-2}, Y]$ for some monomial $U \in \operatorname{supp} v$, $U \neq M_0$. By (iii) above, U is either of the form $U_1 = X' L_{-2}^{k+1}$ or $U_2 = X' L_{-2}^k L_{-3/2}$. Since $k > n$ one has $U_2 < M$ and thus $U_2 \notin \operatorname{supp} v$. Moreover, $U_1 < M_0$ and thus $U_1 \notin \operatorname{supp} v$ as well. Hence $M_0 = L_{-2}^k$ as required. □

5.5.3. Corollary. *Let $c = c_{p,q}^S$, $p > q \geqslant 2$, $(p, q) \in Y$. The minimal monomial of a singular vector v of V^c, which is not proportional to $|0; c\rangle$, is $L_{-2}^m L_{-3/2}$ if p, q are even, and is $L_{-5/2} L_{-2}^m L_{-3/2}$ if p, q are odd ($m \geqslant 0$).*

Proof. Note that if v contains $L_{-2}^m L_{-3/2}$ or $L_{-5/2} L_{-2}^m L_{-3/2}$, then the corresponding monomial is the minimal monomial in v. Moreover, for $c = c_{p,q}^S$ the weight of v is $(p-1)(q-1)/2$ and thus if v contains $L_{-2}^m L_{-3/2}$, then p, q are even and if v contains $L_{-5/2} L_{-2}^m L_{-3/2}$, then p, q are odd.

Suppose that v does not contain $L_{-2}^m L_{-3/2}$. Then, by Lemma 5.5.2, v contains L_{-2}^k. Since $[L_{1/2}, L_{-2}^k]$ contains $L_{-2}^{k-1} L_{-3/2}$, we conclude, using (ii) above, that $k > 1$ and that v contains $L_{-5/2} L_{-2}^m L_{-3/2}$. □

5.5.4. Corollary. *Let J be the right ideal in $\mathcal{U}(\mathcal{NS}_{<-1})$ generated by L_{-i}, $i > 2$. Let $c = c_{p,q}^S$, $p > q \geqslant 2$, $(p, q) \in Y$ and v be the singular vector of a proper submodule of V^c. If p, q are odd, then $v = (L_{-2}^{m+1} + a)|0; c\rangle$ for some $a \in J$, $m \geqslant 0$. If p, q are even, then $L_{-1/2} v = (L_{-2}^{m+1} + a)|0; c\rangle$ for some $a \in J$, $m \geqslant 0$.*

Proof. The monomials which do not lie in the right ideal generated by L_{-i}, $i > 2$ are of the form $L_{-2}^m L_{-3/2}$, L_{-2}^m for $m \geqslant 0$. Therefore if $v' \in V^c$ has integer weight then either $v' \in J|0; c\rangle$ or $v' = (L_{-2}^{m+1} + a)|0; c\rangle$ for $a \in J$, $m \geqslant 0$; in other words, if $v' \in V^c$ contains the monomial L_{-2}^{m+1} then $v' = (L_{-2}^{m+1} + a)|0; c\rangle$ for $a \in J$, $m \geqslant 0$. Now the assertion for p, q odd immediately follows from Lemma 5.5.2. If p, q are even, then v contains $L_{-2}^m L_{-3/2}$ and $L_{-1/2} v$ contains L_{-2}^{m+1}, and the assertion follows. □

6. Lie algebra case

In this section we will prove Theorem 0.2.1.

Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra. In this section we will use the following (non-standard) normalization of the form $B = (.|.)$ on $\mathfrak{h}^*$: $(\alpha|\alpha) = 2$ if α is a *short* root. This normalization is convenient since $(\beta|\beta)/2, (\rho|\beta)$ are positive integers for any root β. In this normalization Theorem 0.2.1 takes the form:

$$V^k \text{ is not irreducible} \quad \Longleftrightarrow \quad \big(k+h_B^\vee\big) \in \{0\} \cup \left\{ \frac{p}{q} \;\middle|\; p \in \mathbb{Z}_{\geqslant 2},\ q \in \mathbb{Z}_{\geqslant 1},\ (p,q)=1 \right\}.$$

In the notation of 3.2.3 this can be written as

$$M_{p/q} \neq 0 \quad \Longleftrightarrow \quad p \geqslant 2 \quad \text{or} \quad p=0.$$

We will check the last equivalence in 6.1–6.5 below.

6.1. Retain notation of 3.3. Since $\mathfrak{g}_1 = 0$ the set S (introduced in 3.3.2) is empty and the group $W^{\#}$ (introduced in 3.3.1) coincides with W. Theorem 3.4 gives

$$\det S_\nu(k) = \prod_{r\geqslant 1} \prod_{\gamma \in \hat{\Delta}^+ \setminus \Delta} \phi_{r\gamma}(k)^{(\dim \hat{\mathfrak{g}}_\gamma) d_{r,\gamma}(\nu)},$$

$$\phi_{r\gamma}(k) = (\Lambda_0|\gamma)k + (\hat{\rho}|\gamma) - r(\gamma|\gamma)/2, \qquad \sum_\nu d_{r,\gamma}(\nu) e^{-\nu} = R^{-1} \sum_{w\in W} (-1)^{l(w)} e^{w.(-r\gamma)}.$$

Using notation of 3.2.3 we obtain

$$M_b = \sum_{(r,\gamma):\ \phi_{r\gamma} = k + (\hat{\rho}|\delta) - b} (\dim \hat{\mathfrak{g}}_\gamma) E(-r\gamma),$$
$$\text{where } \ E(\lambda) := \sum_{w\in W} (-1)^{l(w)} e^{w.\lambda}.$$

Note that all $\dim \hat{\mathfrak{g}}_\gamma = 1$ in M_b if $b \neq 0$.

6.2. Recall that $(\hat{\rho}|\alpha) = (\rho|\alpha)$ for $\alpha \in \Delta$. For $r, s \geqslant 1$ and $\alpha \in \Delta^+$ one has

$$\phi_{r(m\delta)}(k) = k + (\hat{\rho}|\delta),$$
$$\phi_{r(m\delta-\alpha)}(k) = \big(k + (\hat{\rho}|\delta)\big)m - \left(\frac{r(\alpha|\alpha)}{2} + (\rho|\alpha) \right),$$
$$\phi_{s(m\delta+\alpha)}(k) = \big(k + (\hat{\rho}|\delta)\big)m - \left(\frac{s(\alpha|\alpha)}{2} - (\rho|\alpha) \right).$$

Therefore for $p \neq 0$ the factor $k + (\hat{\rho}|\delta) - p/q$ appears as

(i) $\phi_{r(m\delta-\alpha)}(k)$ for $r = \frac{2(pl-(\rho|\alpha))}{(\alpha|\alpha)}$, $m = ql$, where l is such that $r \in \mathbb{Z}_{\geqslant 1}$,
(ii) $\phi_{s(m\delta+\alpha)}(k)$ for $s = \frac{2(pl+(\rho|\alpha))}{(\alpha|\alpha)}$, $m = ql$, where l is such that $s \in \mathbb{Z}_{\geqslant 1}$.

Taking into account that

$$E(-sm\delta - s\alpha) = -E\left(-sm\delta + \left(s - \frac{2(\rho|\alpha)}{(\alpha|\alpha)}\right)\alpha\right),$$

we obtain

$$\begin{aligned} M_{p/q} = {} & \sum_{\alpha\in\Delta^+} \sum_{l:\ \frac{2pl-2(\rho|\alpha)}{(\alpha|\alpha)}\in\mathbb{Z}_{\geqslant 1}} E\left(-\frac{2pl-2(\rho|\alpha)}{(\alpha|\alpha)}(ql-\alpha)\right) \\ & - \sum_{\alpha\in\Delta^+} \sum_{l:\ l\geqslant 1,\ \frac{2pl+2(\rho|\alpha)}{(\alpha|\alpha)}\in\mathbb{Z}_{\geqslant 1}} E\left(-ql\frac{2pl+2(\rho|\alpha)}{(\alpha|\alpha)}\delta + \frac{2pl}{(\alpha|\alpha)}\alpha\right). \end{aligned} \tag{17}$$

Observe that $2(\rho|\alpha)/(\alpha|\alpha) \in \mathbb{Z}_{\geqslant 1}$ for $\alpha \in \Delta^+$.

6.3. For $p = 1$ we get

$$\begin{aligned} M_{1/q} = {} & \sum_{\alpha\in\Delta^+} \sum_{l:\ \frac{2(l-(\rho|\alpha))}{(\alpha|\alpha)}\in\mathbb{Z}_{\geqslant 1}} E\left(-ql\frac{2(l-(\rho|\alpha))}{(\alpha|\alpha)}\delta + \frac{2(l-(\rho|\alpha))}{(\alpha|\alpha)}\alpha\right) \\ & - \sum_{\alpha\in\Delta^+} \sum_{n:\ \frac{2n}{(\alpha|\alpha)}\in\mathbb{Z}_{\geqslant 1}} E\left(-qn\frac{2(n+(\rho|\alpha))}{(\alpha|\alpha)}\delta + \frac{2n}{(\alpha|\alpha)}\alpha\right) = 0 \end{aligned}$$

via the substitution $n := l - (\rho|\alpha)$.

6.4. Let us show that $M_{p/q} = 0$ for $p < 0$.

The above formulas show that for $\alpha \in \Delta^+$ one has $\phi_{r(m\delta+\alpha)}(k) = k + (\hat{\rho}|\delta) - a$ for $a \geqslant 1$; $\phi_{s(m\delta-\alpha)}(k) = k + (\hat{\rho}|\delta) - a$ for $a < 0$ iff $(\rho - s\alpha|\rho - s\alpha) < (\rho|\rho)$. By Lemma A.1 from Appendix A, $E(-sm\delta - s\alpha) = 0$. Hence $M_{p/q} = 0$.

6.5. Finally, let us show that $M_{p/q} \neq 0$ if $p > 1$. Let α be a simple root satisfying $(\alpha|\alpha) = 2$. Take $l \gg 0$ and introduce $r := \frac{2pl-2(\rho|\alpha)}{(\alpha|\alpha)} = pl - 1$. Then $r \gg 0$ and, in the light of Lemma A.2 from Appendix A, the only term in the expression (17) which can be canceled with $E(-rm\delta + r\alpha)$ is the term

$$E\left(-ql'\frac{2pl'+2(\rho|\alpha)}{(\alpha|\alpha)}\delta + \frac{2pl'}{(\alpha|\alpha)}\alpha\right) = E\bigl(-ql'(pl'+1)\delta + pl'\alpha\bigr),$$

where

$$-rm\delta + r\alpha = -ql'(pl'+1)\delta + pl'\alpha.$$

The last formula gives $pl - 1 = pl'$, which is impossible since $p \geqslant 2$.

7. The case $\mathfrak{osp}(1,2n)$

In this section we will prove Theorem 0.2.2.

Let $\mathfrak{g}=\mathfrak{osp}(1,2n)$. In this section we will normalize the form $(.|.)$ on $\mathfrak{h}^*$ by the condition: $(\alpha|\alpha)=2$ for $\alpha\in\Delta_1$. In this normalization Theorem 0.2.2 takes the form

$$M_{p/q}\neq 0 \quad\Longleftrightarrow\quad p\geqslant 0,\quad p\neq 2, \tag{18}$$

where $M_{p/q}$ is introduced in 3.2.3, p,q are relatively prime integers and $q>0$. In this section we will check the above equivalence.

7.1. Set

$$\overline{\hat{\Delta}}{}_0^+ := \{\alpha\in\hat{\Delta}_0^+ \mid \alpha/2\notin\hat{\Delta}_1^+\},\qquad \overline{\Delta}{}_0^+ := \{\alpha\in\Delta_0^+ \mid \alpha/2\notin\Delta_1^+\}.$$

Proposition.

$$\det S_\nu(k)=\prod_{(r,\gamma)\in\Omega}\phi_{r\gamma}(k)^{(\dim\hat{\mathfrak{g}}_\gamma)d_{r,\gamma}(\nu)},$$

where $\Omega := \{(r,\gamma)\mid r\in\mathbb{Z}_{\geqslant 1},\ \gamma\in\overline{\hat{\Delta}}{}_0^+\setminus\Delta\}\cup\{(2j-1,\gamma)\mid j\in\mathbb{Z}_{\geqslant 1},\ \gamma\in\hat{\Delta}_1^+\setminus\Delta\}$,

$$\phi_{r\gamma}(k)=(\Lambda_0|\gamma)k+(\hat{\rho}|\gamma)-r(\gamma|\gamma)/2,$$

$$\sum_\nu d_{r,\gamma}(\nu)e^{-\nu}=R^{-1}\sum_{w\in W}(-1)^{l(w)}e^{w.(-r\gamma)}. \tag{19}$$

Proof. For $\mathfrak{g}=\mathfrak{osp}(1,2n)$ the set S (introduced in 3.3.2) is empty and the group $W^\#$ (introduced in 3.3.1) coincides with W. Theorem 3.4 gives

$$\det S_\nu(k)=\prod_{\gamma\in\hat{\Delta}^+\setminus\Delta,\, r\geqslant 1}\phi_{r\gamma}(k)^{(\dim\hat{\mathfrak{g}}_\gamma)d_{r,\gamma}(\nu)},$$

where

$$\sum_\nu d_{r,\gamma}(\nu)e^{-\nu}=(-1)^{(r-1)p(\gamma)}R^{-1}\sum_{w\in W}(-1)^{l(w)}e^{w.(-r\gamma)}.$$

Now the assertion follows from the following observations: $\phi_{(2r)\gamma}=\phi_{r(2\gamma)}$, and $d_{2r,\gamma}=-d_{r,2\gamma}$ if γ is odd. $\square$

7.2. Using notation of 3.2.3 we have for $p\neq 0$

$$M_{p/q}=\sum_{(r,\gamma)\in\Omega:\ \phi_{r\gamma}=k+(\hat{\rho}|\delta)-p/q}E(-r\gamma),$$

$$\text{where } E(-r\gamma):=\sum_{w\in W}(-1)^{l(w)}e^{w.(-r\gamma)}. \tag{20}$$

7.3. The formulas for $\phi_{r(l\delta)}$, $\phi_{r(l\delta\pm\alpha)}$ have the same form as for Lie algebra case, see 6.2.

Note that a root $\gamma \in \hat{\overline{\Delta}}_0$ has the form $\gamma = l\delta \pm \alpha$ if $\alpha \in \overline{\Delta}_0^+$, or $\gamma = l\delta \pm 2\beta$ if l is odd and $\beta \in \Delta_1^+$. Taking into account that $(\beta|\beta) = 2$ for $\beta \in \Delta_1^+$ and $(\alpha|\alpha) = 4$ for $\alpha \in \overline{\Delta}_0^+$, we see that the factor $k + (\hat{\rho}|\delta) - p/q$ for $p \neq 0$ appears as

(i) $\phi_{r(l\delta\pm\alpha)}$ for $\alpha \in \overline{\Delta}_0^+$ if $r := \frac{pm\pm(\rho|\alpha)}{2} \geqslant 1$,
(ii) $\phi_{r(l\delta\pm\beta)}$ for $\beta \in \Delta_1^+$ if $r := pm \pm (\rho|\alpha) \geqslant 1$ and r is odd,
(iii) $\phi_{r(l\delta\pm2\beta)}$ for $\beta \in \Delta_1^+$ if $l := qm$ is odd and $r := \frac{pm\pm2(\rho|\beta)}{4} \in \mathbb{Z}_{\geqslant 1}$.

In all cases $l = qm$.

7.3.1. Remark. Observe that $(\rho|\alpha)$ is even for $\alpha \in \overline{\Delta}_0^+$ and $(\rho|\beta)$ is odd for $\beta \in \Delta_1^+$. As a result, pm is even in the cases (i), (ii); in the case (iii) both q, m are odd and $pm \equiv 2 \bmod 4$, so p is even.

7.4. Let us show that $M_{p/q} \neq 0$ for coprime integers p, q iff $p \geqslant 0$, $p \neq 2$.
We will use the letters l, l', m, q, r, s for positive integers.

7.4.1. Identify $\overline{\Delta}_0 \cup \Delta_1$ with the root system of B_n; notice that W identifies with the Weyl group of B_n. Now repeating the arguments of 6.4 we obtain $M_{p/q} = 0$ for $p < 0$.

7.4.2. Let us show that $M_{p/q} \neq 0$ if $p \geqslant 1$, $p \neq 2$.
Let β be a simple odd root; then $(\beta, \rho) = 1$. In the light of Lemma A.2 from Appendix A, for $r \gg 1$ the only term in the expression (20), which can be canceled with $E(-r(l\delta + \beta))$, is $E(-s(l'\delta - \alpha'))$ satisfying $-sl'\delta + s\alpha' = s_\beta.(-rl\delta - r\beta)$, that is

$$s\alpha' = (r-1)\beta \quad \text{and} \quad sl' = rl.$$

If $\alpha' = \beta$, then $s = r - 1$, which is impossible since both r and s should be odd (7.3, (ii)). If $\alpha' = 2\beta$, then $s = \frac{r-1}{2}$. From 7.3, (ii), (iii) we conclude that $r - 1$ and $4s + 2 = 2r$ are divisible by p. Moreover, by Remark 7.3.1, p is even. Hence $p = 2$ as required.

7.4.3. Finally, let us show that $M_{\frac{2}{q}} = 0$ if q is odd. In this case, for $\beta \in \Delta_1$ the term $E(-r(l\delta + \beta))$ appears if $l = qm$, $r = 2m + (\rho, \beta)$ for some $m \geqslant 1$; thus $s_\beta.(-r(l\delta + \beta)) = -rqm\delta + 2m\beta$ and so $E(-r(l\delta + \beta))$ cancels with $E(-m(rq\delta - 2\beta))$. For $\alpha \in \overline{\Delta}_0^+$ the term $E(-r(l\delta + \alpha))$ appears if $l = qm$, $r = m + (\rho|\alpha)/2$ for some $m \geqslant 1$; thus $s_\alpha(-r(l\delta + \alpha)) = -rqm\delta + m\alpha$ and so $E(-r(l\delta + \alpha))$ cancels with $E(-m(rq\delta - \alpha))$.

8. Lie superalgebras of non-zero defect

In this section we prove Theorem 0.2.4:

Theorem. *Let $\mathfrak{g}$ be $\mathfrak{gl}(2,2)$ or a simple Lie superalgebra of defect one, i.e., $\mathfrak{g} = \mathfrak{sl}(1,n)$, $\mathfrak{osp}(2,n)$, $\mathfrak{osp}(3,n)$, $\mathfrak{osp}(n,2)$ with $n > 2$, $D(2,1,a)$, $F(4)$, $G(3)$. Then*

$$V^k \text{ is not simple} \iff \exists \alpha \in \Delta_0^+ \text{ such that } \frac{k + (\hat{\rho}|\delta)}{(\alpha|\alpha)} \in \mathbb{Q}_{\geqslant 0}.$$

In other words, in the standard normalization of the invariant bilinear form, for $\mathfrak{g} = \mathfrak{sl}(1,n), \mathfrak{osp}(2,2n)$ the vacuum module V^k is not simple iff $k + h^\vee$ is a non-negative rational number; for all other Lie superalgebras of defect one, except for $D(2,1,a), a \notin \mathbb{Q}$, and for $\mathfrak{gl}(2,2)$ the vacuum module V^k is not simple iff $k + h^\vee \in \mathbb{Q}$. For $D(2,1,a), a \notin \mathbb{Q}$ the vacuum module V^k is not simple iff $k \in \mathbb{Q}_{\geqslant 0} \cup \mathbb{Q}_{>0}a \cup \mathbb{Q}_{>0}(-1-a)$, in the standard normalization of $(.|.)$.

Retain notation of 3.3. In this section $p, q, r \in \mathbb{Z}_{\geqslant 1}, s \in \mathbb{Z}_{\geqslant 0}, \gamma \in \hat{\Delta}^+ \setminus \Delta$.

8.1. *Case* $\mathfrak{g} = \mathfrak{gl}(2,2)$

Let us show that V^k is not simple iff $k \in \mathbb{Q}$, in the standard normalization of $(.|.)$.

8.1.1. Choose a set of simple roots $\Pi = \{\beta_1, \alpha, \beta_2\}$ which contains two odd roots $\beta_1 := \varepsilon_3 - \varepsilon_1, \beta_2 := \varepsilon_2 - \varepsilon_4$ and the even root $\alpha := \varepsilon_1 - \varepsilon_2$. The form is given by $(\varepsilon_i|\varepsilon_j) = 0$ for $i \neq j$, $(\varepsilon_i|\varepsilon_i) = -(\varepsilon_j|\varepsilon_j) = 1$ for $i = 1, 2,\ j = 3, 4$. Then

$$\Delta_0^+ = \{\alpha, \alpha+\beta_1+\beta_2\}, \qquad \Delta_1^+ = \{\beta_1, \beta_2, \alpha+\beta_1, \alpha+\beta_2\}, \qquad \rho = -\frac{\beta_1+\beta_2}{2}.$$

One has $S = \{\beta_1, \beta_2\}$, $W^\# = \{\mathrm{id}, s_\alpha\}$ and $W^\# S \subset \Delta^+$. Theorem 3.4 gives

$$\det S_\nu(k) = \prod_{r=1}^{\infty} \prod_{\gamma \in \hat{\Delta}^+ \setminus \Delta} \prod_{j_1, j_2 \geqslant 0} \phi_{r\gamma + j_1\beta_1 + j_2\beta_2}(k)^{(\dim \hat{\mathfrak{g}}_\gamma) d_{r\gamma; j_1; j_2}(\nu)},$$

where $d_{r\gamma; j_1; j_2} = (-1)^{(r+1)p(\gamma)+j_1+j_2} R^{-1} e^{-\rho}(1 - s_\alpha)(e^{\rho - j_1\beta_1 - j_2\beta_2 - r\gamma})$.

Note that $(\beta_1, \gamma') = (\beta_2, \gamma')$ for any $\gamma' \in \Delta$. Therefore $\phi_{r\gamma + j_1\beta_1 + j_2\beta_2}$ depends on $r\gamma$ and the sum $j_1 + j_2$ and thus

$$\det S_\nu = \prod_{r=1}^{\infty} \prod_{\gamma \in \hat{\Delta}^+ \setminus \Delta} \prod_{m \geqslant 0} \phi_{r\gamma + m\beta_1}^{\dim \hat{\mathfrak{g}}_\gamma d_{r\gamma; m}(\nu)},$$

where the new exponents $d_{r\gamma; m}$ are given by

$$R d_{r\gamma; m} = (-1)^{(r+1)p(\gamma)+m} e^{-\rho}(1 - s_\alpha)\big(e^{\rho - r\gamma} J(m)\big), \tag{21}$$

where

$$J(m) := \sum_{j=0}^{m} e^{-j\beta_1 - (m-j)\beta_2}.$$

8.1.2. Set $\alpha' := \alpha + \beta_1 + \beta_2$. Write $\gamma = l\delta + \gamma'$, where $\gamma' \in \Delta$. The factor $\phi_{r\gamma + m\beta_1}$ is proportional to $k - b(r; \gamma; m)$, where

$$b(r; \gamma; m) := \frac{r(\gamma'|\gamma')/2 - (\rho - m\beta_1|\gamma')}{l} = \frac{r(\gamma'|\gamma')/2 + (m+1)(\beta_1|\gamma')}{l}.$$

We have the following table:

γ'	$b(r;\gamma;m)$
$0;\ \pm\beta_i$	0
$\pm(\alpha+\beta_i)$	$\mp\frac{m+1}{l}$
$\pm\alpha$	$\frac{\mp(m+1)+r}{l}$
$\pm\alpha'$	$\frac{\mp(m+1)-r}{l}$

8.1.3. In the space $\mathcal{E}$ of regular exponential functions let $\mathcal{E}_{y;\beta}$ (respectively, $\mathcal{E}_{y;\delta}$) be the subspace generated by e^{λ}, $\lambda = x_\delta\delta + x_\alpha\alpha + x_1\beta_1 + x_2\beta_2 \in \mathfrak{h}$, where $x_1 + x_2 = y$ (respectively, $x_\delta = y$). Clearly, $\mathcal{E}_{y;\beta}, \mathcal{E}_{y;\delta}$ are invariant under the linear operator $Q \mapsto e^{-\rho}s_\alpha(e^\rho Q)$. We denote by $P_{i;\beta}$ (respectively, $P_{+;\beta}$) the projection $\mathcal{E} \to \mathcal{E}_{i;\beta}$ (respectively, $\mathcal{E} \to \sum_{x>0}\mathcal{E}_{x;\beta}$) with the kernel $\sum_{x\neq i}\mathcal{E}_{x;\beta}$ (respectively, $\mathcal{E} \to \sum_{x\geqslant 0}\mathcal{E}_{-x;\beta}$) and by $P_{l;\delta}$ the projection $\mathcal{E} \to \mathcal{E}_{l;\delta}$ with the kernel $\sum_{y\neq l}\mathcal{E}_{y;\delta}$. Recall that $J(m) \in \mathcal{E}_{-m;\beta}$.

8.1.4. Case $k = -p/q$

Let us show that $M_{-p/q} \neq 0$ if p, q are positive integers. Retain notation of 3.2.3, 8.1.3. Notice that $b(r;\gamma;m) = -p/q$ forces that $\gamma' \in \Delta^+ \cup \{-\alpha'\}$. The formula (21) shows that $Rd_{r\gamma;m} \in \mathcal{E}_{i;\beta}$ for some $i \leqslant 0$ if $\gamma' \in \Delta^+$, and for $\gamma' = -\alpha'$ we have $Rd_{r\gamma;m} \in \mathcal{E}_{2r-m;\beta} \cap \mathcal{E}_{rl;\delta}$, where $2r - m > 0$, because $\frac{r-m-1}{l} = \frac{p}{q} > 0$. Therefore

$$P_{i;\delta} \circ P_{+;\beta}(M_{-p/q}) = \sum_{(m,r,l)\in X_i} x_{m;r;l},$$

$$X_i := \left\{(m,r,l) \,\middle|\, r,l \geqslant 1,\ m \geqslant 0,\ \frac{r-m-1}{l} = \frac{p}{q},\ rl = i\right\},$$

$$x_{m;r;l} := Rd_{r(l\delta-\alpha');m} = (-1)^m e^{-r(l\delta-\alpha')}e^{-\rho}(1-s_\alpha)\big(e^\rho J(m)\big),$$

where the last equality uses $(\alpha'|\alpha) = 0$. Observe that $x_{m;r;l} \neq 0$ since

$$s_\alpha\big(e^\rho J(m)\big) = e^{\rho-(m+1)\alpha}J(m).$$

One readily sees that $X_{q(p+1)}$ contains a unique triple $(0; p+1; q)$ and thus $P_{q(p+1);\delta} \circ P_{+;\beta}(M_{-p/q}) = x_{0;p+1;q}$. Hence $M_{-p/q} \neq 0$ as required.

8.1.5. Case $k = p/q$

Let us show that $M_{p/q} \neq 0$ if p, q are positive integers. Retain notation of 3.2.3, 8.1.3. We will use the following formula:

$$J(m)\big(e^{-k\beta_1} + e^{-k\beta_2}\big) = J(m+k) + e^{-k(\beta_1+\beta_2)}J(m-k) \quad \text{for } m \geqslant k \geqslant 0. \tag{22}$$

Note that

$$b(r; l\delta - \alpha - \beta_i; m + r) = b(r; l\delta - \alpha'; m + 2r) = b(r; l\delta - \alpha; m).$$

Combining (21) and (22) we obtain

$$d_{r(l\delta-\alpha-\beta_1);m+r} + d_{r(l\delta-\alpha-\beta_2);m+r} + d_{r(l\delta-\alpha);m} + d_{r(l\delta-\alpha');m+2r} = 0.$$

Then

$$M_{p/q} = \sum_{j=1}^{2} \sum_{m,r,l:\ \frac{m+1}{l}=\frac{p}{q},\, m<r} Rd_{r(l\delta-\alpha-\beta_j);m} + \sum_{m,r,l:\ \frac{m+1-r}{l}=\frac{p}{q},\, m<2r} Rd_{r(l\delta-\alpha');m} + \sum_{m,r,l:\ \frac{r-(m+1)}{l}=\frac{p}{q}} Rd_{r(l\delta+\alpha);m}.$$

One has $Rd_{r(l\delta-\alpha-\beta_j);m} \in \mathcal{E}_{r-m;\beta}$ and $Rd_{r(l\delta-\alpha');m} \in \mathcal{E}_{2r-m;\beta}$, whereas $Rd_{r(l\delta+\alpha);m} \in \mathcal{E}_{-m;\beta}$. Therefore

$$P_{0;\beta}(M_{p/q}) = \sum_{r,l:\ \frac{r-1}{l}=\frac{p}{q}} Rd_{r(l\delta+\alpha);0}.$$

Hence $P_{q(p+1);\delta} \circ P_{0;\beta}(M_{p/q}) = Rd_{(p+1)(q\delta+\alpha);0} \neq 0$, and this gives $M_{p/q} \neq 0$.

8.2. Superalgebras of defect one

Let $\mathfrak{g}$ be a basic classical Lie superalgebra of defect one: $\mathfrak{g} = \mathfrak{sl}(1, n)$, $\mathfrak{osp}(3, 2n)$, $\mathfrak{osp}(N, 2)$, $\mathfrak{osp}(2, 2m)$, $F(4)$, $G(3)$, $D(2, 1, a)$. The root systems of these Lie superalgebras are described in Section 10; in particular, the group $W^\#$ is explicitly written there. We retain notation of Section 10 and for each algebra fix a system of simple roots Π described there. One has $S = \{\beta\} \subset \Pi$, where β is an isotropic root given there.

8.2.1. Write $\gamma = l\delta + \gamma'$, where $\gamma' \in \Delta \cup \{0\}$. The factor $\phi_{r\gamma+s\beta}(k)$ is proportional to $k + h^\vee - b(r; \gamma; s)$, where

$$b(r; \gamma; s) := \frac{r(\gamma'|\gamma')/2 - (\rho - s\beta|\gamma')}{l}. \tag{23}$$

For $b \neq 0$ Theorem 3.4 gives

$$M_b = \sum_{(r;s;\gamma):\ b(r;\gamma;s)=b} E(r; \gamma; s),$$

$$\text{where } E(r; \gamma; s) := (-1)^{s+(r-1)p(\gamma)} \sum_{w \in W^\#} (-1)^{l(w)} e^{w.(-r\gamma-s\beta)}. \tag{24}$$

Using the W-invariance of δ and $(\hat{\rho}-\rho)$, we get

$$E(r;\gamma;s)=0 \quad\Longleftrightarrow\quad \mathrm{Stab}_{W^\#}(\rho-r\gamma'-s\beta)\neq \mathrm{id}.$$

8.2.2. If the term $E(r;\gamma;s)$ is non-zero, it is a sum of the form $\sum_{i=1}^{|W^\#|}e^{\lambda_i}$, where all summands are distinct and there exists a unique index i such that λ_i is dominant with respect to $\Pi^\#$ (i.e., $(\lambda_i|\alpha)\geqslant 0$ for any $\alpha\in\Pi^\#$). As a result, $\sum_{i\in I}E(r_i;\gamma_i;s_i)=0$ iff the index set I admits an involution $\sigma: I\to I$ such that $E(r_i;\gamma_i;s_i)+E(r_{\sigma(i)};\gamma_{\sigma(i)};s_{\sigma(i)})=0$.

We will prove that $M_b\neq 0$ by exhibiting the triple $(r;s;\gamma)$ such that

$$\begin{aligned} &\text{(i)}\quad b(r;\gamma;s)=b \quad\text{and}\quad E(r;\gamma;s)\neq 0,\\ &\text{(ii)}\quad b(r';\mu;s')=b,\ \mu\in\hat{\Delta}\setminus\Delta \quad\Rightarrow\quad E(r;\gamma;s)+E(r';\mu;s')\neq 0. \end{aligned}\tag{25}$$

8.2.3. Cancelation

Suppose that $\gamma,\gamma_1\in\hat{\Delta}$ are such that $\gamma=\gamma_1+\beta$ and $\dim\hat{\mathfrak{g}}_\gamma=\dim\hat{\mathfrak{g}}_{\gamma_1}=1$. Then $\phi_{r\gamma+s\beta}(k)=\phi_{r\gamma_1+(s+r)\beta}(k)$ and $d_{r;s;\gamma}=-d_{r;s+r;\gamma_1}$ since $(-1)^{(r-1)p(\gamma)+s}=-(-1)^{(r-1)p(\gamma_1)+r+s}$. As a result, $\phi_{r\gamma+s\beta}^{d_{r;\gamma;s}}$ cancels with $\phi_{r(\gamma_1)+(s+r)\beta}^{d_{r;\gamma_1;s+r}}$:

$$\prod_{r\geqslant 1,\,s\geqslant 0}\phi_{r\gamma+s\beta}^{d_{r;\gamma;s}(\nu)}\phi_{r\gamma_1+s\beta}^{d_{r;\gamma_1;s}(\nu)}=\prod_{r>s\geqslant 0}\phi_{r\gamma_1+s\beta}^{d_{r;\gamma_1;s}(\nu)}.$$

8.3. Case $D(2,1,a)$

Retain notation of 10.10 and note that $(\hat{\rho}|\delta)=0$. We will show that V^k is not simple iff $k\in\mathbb{Q}_{\geqslant 0}\cup\mathbb{Q}_{>0}a\cup\mathbb{Q}_{<0}(1+a)$. If a is rational then V^k is not simple iff $k\in\mathbb{Q}$.

8.3.1. Take $\gamma\in\hat{\Delta}^+$. Note that $b(r;\gamma;s)=0$ if $\gamma'=0,\pm\beta$. Take γ such that $\gamma'\neq 0,\pm\beta$. In the light of 8.2.3 if $\gamma-\beta$ is a root then $\phi_{r\gamma+s\beta}^{r_{r;\gamma;s}}$ cancels with $\phi_{r(\gamma-\beta)+(s+r)\beta}^{d_{r;\gamma-\beta;s+r}}$. Observe that exactly one of the elements $\gamma-\beta,\gamma+\beta$ is a root. Theorem 3.4 gives

$$\det S_\nu(k)=k^{d(\nu)}\prod_{l\geqslant 1}\prod_{r>s\geqslant 0}\prod_{\substack{\gamma'\in\Delta,\,\gamma-\beta\notin\Delta\\ \gamma'\neq\pm\beta}}\phi_{r(l\delta+\gamma')+s\beta}(k)^{d_{r;s;l\delta+\gamma'}(\nu)}$$

for some $d(\nu)\in\mathbb{Z}_{\geqslant 0}$.

Set $P':=\{m\delta+\sum_{j=0}^{2}m_j\varepsilon_j:\ m_1,m_2\geqslant 0\}$. Clearly, for any $\mu\in\hat{Q}$ the orbit $W^\#\mu\cap P'$ contains a unique element $\sum_{j=0}^{2}m_j\varepsilon_j$ and $\mathrm{Stab}_{W^\#}\mu\neq\mathrm{id}$ iff $m_1m_2=0$.

8.3.2. Write $\gamma = l\delta + \gamma'$. For $\gamma' = -\beta$ one has $\phi_{r\gamma+s\beta} = k + (\hat{\rho}|\delta)$. For the remaining values of γ' (i.e., $\gamma' \neq -\beta$ and $\gamma - \beta \notin \Delta$) we have

γ'	$b(r;\gamma;s)$	$W^{\#}(\rho - r\gamma - s\beta) \cap P'$
$-2\varepsilon_0$	$(1+a)\frac{-r+s+1}{l}$	$-rl\delta + (2r-s-1)\varepsilon_0 + (s+1)\varepsilon_1 + (s+1)\varepsilon_2$
$-(\varepsilon_0+\varepsilon_1-\varepsilon_2)$	$a\frac{s+1}{l}$	$-rl\delta + (r-s-1)\varepsilon_0 + (r+s+1)\varepsilon_1 + (r-s-1)\varepsilon_2$
$-(\varepsilon_0-\varepsilon_1+\varepsilon_2)$	$\frac{s+1}{l}$	$-rl\delta + (r-s-1)\varepsilon_0 + (r-s-1)\varepsilon_1 + (r+s+1)\varepsilon_2$
$2\varepsilon_1$	$a\frac{r-s-1}{l}$	$-rl\delta - (s+1)\varepsilon_0 + (2r-s-1)\varepsilon_1 + (s+1)\varepsilon_2$
$2\varepsilon_2$	$\frac{r-s-1}{l}$	$-rl\delta - (s+1)\varepsilon_0 + (s+1)\varepsilon_1 + (2r-s-1)\varepsilon_2$
$\varepsilon_0+\varepsilon_1+\varepsilon_2$	$(1+a)\frac{-s-1}{l}$	$-rl\delta - (r+s+1)\varepsilon_0 + (r-s-1)\varepsilon_1 + (r-s-1)\varepsilon_2$

8.3.3. Set $X := \mathbb{Q}_{\geqslant 0} \cup \mathbb{Q}_{>0}a \cup \mathbb{Q}_{>0}(-1-a)$ and let us show that $M_b \neq 0$ iff $b \in X$.

From the above table we see that for $r > s \geqslant 0$ the term $b(r;\gamma;s) \in X$. Hence $M_b = 0$ if $b \notin X$. Moreover, we see that for $r > s$ the vector $\rho - r\gamma - s\beta$ has a non-trivial stabilizer in $W^{\#}$ iff $r = s+1$ and $\gamma' \in \{-(\varepsilon_0 - \varepsilon_1 + \varepsilon_2), -(\varepsilon_0 + \varepsilon_1 - \varepsilon_2), \varepsilon_0 + \varepsilon_1 + \varepsilon_2\}$. It is easy to see that the entries of last column are pairwise distinct, i.e. $W^{\#}(\rho - r\gamma - s\beta) \cap P' = W^{\#}(\rho - r_1\gamma_1 - s_1\beta) \cap P'$ forces $(r;\gamma;s) = (r_1;\gamma_1;s_1)$. In the light of 8.2.2 we obtain $M_b \neq 0$ for $b \in X$, as required.

8.4. In the remaining part of the section $\mathfrak{g}$ has defect one and $\mathfrak{g} \neq D(2,1,a)$.

8.4.1. Notation

In all cases, $\mathfrak{h}^*$ has a basis $\varepsilon_0; \varepsilon_1, \ldots, \varepsilon_n$ and $W^{\#}$ stabilizes ε_0 and leaves invariant the space $\mathfrak{h}^{\#} := \sum_{i \geqslant 1} \mathbb{C}\varepsilon_i$. For $\mu \in \hat{\mathfrak{h}}$ we denote by μ^{δ}, $\mu^{(i)}$ the corresponding coordinates of μ and by $\mu^{\#}$ the projection of μ on $\mathfrak{h}^{\#}$:

$$\mu =: \mu^{\delta}\delta + \sum_{i=0}^{n} \mu^{(i)}\varepsilon_i, \qquad \mu^{\#} := \sum_{i=1}^{n} \mu^{(i)}\varepsilon_i.$$

8.4.2. Recall that $W^{\#}$ stabilizes δ and ε_0. As a result, for $\gamma, \mu \in \Delta$ we have

$$E(r;\gamma;s) + E(r';\mu;s') = 0$$
$$\Longrightarrow \quad r\gamma^{\delta} = r'\mu^{\delta} \quad \text{and} \quad \begin{cases} r\gamma^{(0)} + s = r'\mu^{(0)} + s' & \text{for } \mathfrak{g} \neq F(4), \\ r\gamma^{(0)} + s/2 = r'\mu^{(0)} + s'/2 & \text{for } \mathfrak{g} = F(4). \end{cases}$$

8.5. Proof that $M_{p/q} \neq 0$ for $\mathfrak{g} = \mathfrak{osp}(3,2)$

We will deduce that $M_{p/q} \neq 0$ from (25). Observe that $b(2p+1; q\delta + \varepsilon_1; 0) = p/q$. One has

$$\rho - r\gamma - s\beta = -r\gamma^{(\delta)}\delta - \bigl(s + 1/2 + r\gamma^{(0)}\bigr)\varepsilon_0 + \bigl(s + 1/2 - r\gamma^{(1)}\bigr)\varepsilon_1.$$

Therefore $E(r;\gamma;s) \neq 0$ for any $\gamma \in \Delta$. Let us show that $E(r;\gamma;s) + E(2p+1; q\delta + \varepsilon_1; 0) \neq 0$ for any triple (r,s,γ) such that $b(r;\gamma;s) \neq 0$. Assume that $E(r;\gamma;s) + E(2p+1; q\delta + \varepsilon_1; 0) = 0$. Then $(\rho - r\gamma - s\beta) \in W^{\#}(\rho - (2p+1)(q\delta + \varepsilon_1))$, that is $r\gamma^{(\delta)} = (2p+1)q$, $r\gamma^{(0)} + s = 0$, $1/2 - r\gamma^{(1)} = \pm(2p + 1/2)$. The second formula gives $\gamma^{(0)} \leqslant 0$. Since $b(r;\gamma;s) = 0$ for $\gamma' \in \{0, \pm\beta\}$, we have the following cases: $\gamma' \in \{\pm\varepsilon_1, \pm 2\varepsilon_1\}$, $s = 0$ or $\gamma' \in \{-\varepsilon_0, -\varepsilon_0 - \varepsilon_1\}$,

$s = r$. By 8.2.3 the terms corresponding to $\gamma' = -\varepsilon_1, -2\varepsilon_1, s = 0$ cancel with the terms corresponding to $\gamma' = -\varepsilon_0, -\varepsilon_0 - \varepsilon_1, r = s$. It remains to show that $E(r;\gamma;s) + E(2p+1; l\delta + \varepsilon_1; 0) \neq 0$ for $\gamma' \in \{\varepsilon_1, 2\varepsilon_1\}, s = 0$. If $\gamma' = 2\varepsilon_1$ we get $1/2 - 2r = \pm(2p+1/2)$ which is impossible. Finally, for $\gamma' = \varepsilon_1$ the formulas $1/2 - r = \pm(2p+1/2), rl = (2p+1)q$ give $r = 2p+1$, $q = l$ and thus $E(r;\gamma;s) + E(2p+1; l\delta + \varepsilon_1; 0) = 2E(2p+1; l\delta+\varepsilon_1; 0) \neq 0$. Now the inequality $M_{p/q} \neq 0$ follows from (25).

8.6. Proof that $M_{p/q} \neq 0$ for $\mathfrak{g} \neq \mathfrak{osp}(3,2)$

Recall that there exists an isotropic root $\alpha \in \Delta^+$ satisfying $(\alpha|\beta) = -1$, $(\alpha|\rho) = 1$, $\alpha^{(0)} = \beta^{(0)}$. One has $b(m; q\delta - \alpha; p-1) = p/q$ for any $m \geqslant 1$. It is easy to see that $E(m; q\delta - \alpha; p-1) \neq 0$ for $m \gg 0$. It remains to verify the condition (ii) of (25) for some $m \gg 0$, i.e. to show that

$$\exists m \gg 0 \quad \text{such that} \quad b(r;\gamma;s) = p/q \quad \Longrightarrow \quad E(m; q\delta - \alpha; p-1) + E(r;\gamma;s) \neq 0. \tag{26}$$

We claim that this holds if $m \gg 0$ is a prime number.

Indeed, assume that $b(r;\gamma;s) = p/q$ and $E(m; q\delta - \alpha; p-1) + E(r;\gamma;s) = 0$. Write $\gamma = l\delta + \gamma'$. By 8.4.2 one has $rl = mq$.

The assumption gives

$$\frac{r(\gamma'|\gamma')/2 - (\rho - s\beta|\gamma')}{l} = \frac{p}{q}.$$

Notice that the numerator of the left-hand side is an integer and thus l is divisible by q except the case $\mathfrak{g} = \mathfrak{osp}(3,2n), \mathfrak{osp}(2n+1,2), \gamma' = \pm\varepsilon_i$. Using $rl = mq$ we get $(r;l) \in \{(m,q),(1,mq)\}$ if $\mathfrak{g} \neq \mathfrak{osp}(3,2n), \mathfrak{osp}(2n+1,2)$ or $\gamma' \neq \pm\varepsilon_i$.

If $\mathfrak{g} = \mathfrak{osp}(3,2n), \mathfrak{osp}(2n+1,2), \gamma' = \pm\varepsilon_i$, then $r(\gamma'|\gamma') - 2(\rho - s\beta|\gamma')$ is an integer, so $2l$ is divisible by q. Therefore $rl = mq$ gives that $(r;l) \in \{(m,q),(1,mq),(2m,q/2),(2,mq/2)\}$.

Now 8.4.2 gives

$$s = p - 1 - m - \frac{r\gamma^{(0)}}{\beta^{(0)}}.$$

Since $m \gg 0$ we obtain $r\gamma^{(0)} \ll 0$ so $\gamma^{(0)} < 0$ and $r \gg 0$. Examining root systems, we see that $\gamma^{(0)} < 0$ implies that either $\gamma^{(0)} = -\beta^{(0)}$ or $\gamma' = -2\beta^{(0)}\varepsilon_0$. Finally, we obtain the following cases:

(i) $\gamma^{(0)} = -\beta^{(0)}$ and $(r,l,s) = (m,q,p-1)$;
(ii) $\gamma' = -2\beta^{(0)}\varepsilon_0$ and $(r,l,s) = (m,q,m+p-1)$;
(iii) $\mathfrak{g} = \mathfrak{osp}(3,2n), \mathfrak{osp}(2n+1,2), \gamma' = -\varepsilon_0$ and $(r,l,s) = (2m,q/2,m+p-1)$.

We will show that the cases (ii), (iii) do not hold and (i) implies $(r;s;\gamma) = (m;p-1;q\delta-\alpha)$, that is $E(m;q\delta-\alpha;p-1) + E(r;\gamma;s) = 2E(m;q\delta-\alpha;p-1) \neq 0$.

8.6.1. Case (i)

Substituting in (23) we get $m(\gamma'|\gamma')/2 - (\rho - s\beta|\gamma') = p$. The condition $m \gg 0$ forces $(\gamma'|\gamma') = 0$ and thus $(\rho - (p-1)\beta|-\gamma') = p$. Since $\gamma^{(0)} < 0$ the root $-\gamma'$ is positive and isotropic. From Lemma 10.1.1, $-\gamma' = \alpha$. Hence $b(m;\gamma;s) = p/q$ forces $(r;s;\gamma) = (m; p-1; l\delta - \alpha)$.

8.6.2. Case (ii)

One has $m\gamma + s\beta = m(\gamma+\beta) + (p-1)\beta$ and $\gamma+\beta \in \hat{\Delta}$, since $\gamma' + \beta = s_{\varepsilon_0}\beta \in \Delta$ ($2\beta^{(0)}\varepsilon_0$ is a root, so Δ is invariant with respect to the reflection s_{ε_0}). Clearly, $(\gamma+\beta)^{(0)} = -\beta^{(0)}$. By 8.6.1, $b(m;\gamma+\beta;p-1) = p/q$ implies $\gamma+\beta = l\delta - \alpha$, which contradicts $\gamma' = -2\beta^{(0)}\varepsilon_0$.

8.6.3. Case (iii)

In this case $\mathfrak{g} = \mathfrak{osp}(3,2n), \mathfrak{osp}(2n+1,2), n > 1$. Substituting in (23), we get $p - 1 + (\rho|\varepsilon_0) = p/2$, which is impossible since $(\rho|\varepsilon_0) = n - 1/2 > 1$.

8.7. Case $M_{-p/q}$

We claim that

$$b(r;\gamma;s) < 0 \quad \Longrightarrow \quad \gamma' \in \Delta^+ \setminus \{\beta\} \quad \text{or} \quad \gamma' = \mathbb{Z}\varepsilon_0 \cap \Delta. \tag{27}$$

Indeed, take $\gamma' \in \Delta^-$. Since all simple roots, except β, have positive lengths squared, both $(\gamma'|\rho), -(\gamma'|\beta)$ are non-positive. From (23) we see that $b(r;\gamma;s) > 0$ forces $(\gamma'|\gamma') < 0$. Examining the root systems we see that $(\gamma'|\gamma') < 0$ iff $\gamma' = \mathbb{Z}\varepsilon_0 \cap \Delta$.

8.8. Proof that $M_{-p/q} = 0$ for $\mathfrak{g} = \mathfrak{sl}(1,n), C(n) = \mathfrak{osp}(2,2n-2)$

8.8.1. Take $\gamma' \in \Delta_0^+$ such that $\gamma' + \beta \notin \Delta$. Let us show that $E(r;\gamma;s) = 0$.

Indeed, $\gamma' + \beta \notin \Delta$ forces $(\beta|\gamma') \geqslant 0$. Since β is the only isotropic root in Π, $(\alpha|\beta) \leqslant 0$ for any $\alpha \in \Delta^+$. Hence $(\gamma'|\beta) = 0$. Set $\Pi_2 := \{\alpha \in \Pi^{\#}\colon (\alpha|\beta) = 0\}$ and define Δ_2^+, ρ_2, W_2 corresponding to Π_2. It is easy to check that $(\beta|\gamma') = 0$ forces $\gamma' \in \Delta_2^+$. As a result, $(\rho_2|\gamma') = (\rho|\gamma')$. Since $b(r;\gamma;s) < 0$, (23) gives $(\rho_2 - r\gamma'|\rho_2 - r\gamma') < (\rho_2|\rho_2)$. Then, by A.1, $\rho_2 - r\gamma'$ has a non-trivial stabilizer in W_2. Observe that $\beta, \rho - \rho_2$ are W_2-invariant. Hence $\rho - r\gamma' - s\beta$ has a non-trivial stabilizer in $W_2 \subset W^{\#}$. Therefore $E(r;\gamma;s) = 0$, as required.

8.8.2. Retain notation of 10.3, 10.4. For $\mathfrak{sl}(1,n), C(n)$ (27) gives $b(r;\gamma;s) < 0 \Rightarrow \gamma' \in \Delta^+ \setminus \{\beta\}$. Notice that for $\gamma' \in \Delta_1^+ \setminus \{\beta\}$ one has $\gamma' - \beta \in \Delta$. Combining 8.2.3 and 8.8.1, we conclude that $M_{-p/q}$ is the sum of $E(r;\gamma;s)$, where $b(r;\gamma;s) = -p/q$, $\gamma', \gamma'+\beta \in \Delta_0^+$ and $r > s \geqslant 0$.

8.8.3. $\mathfrak{g} = \mathfrak{sl}(1,n)$

The conditions $\gamma' \in \Delta_0^+, \gamma'+\beta \in \Delta^+$ mean that $\gamma' = \varepsilon_1 - \varepsilon_m$ $(1 < m \leqslant n)$. Since $b(r;\gamma;s) < 0$, (23) gives $m - 1 + s > r$. Using the condition $s < r$ we see that the permutation $(1; r-s+1)$ stabilizes the vector $\rho - r\gamma' - s\beta = (-n/2 - s)\varepsilon_0 + (n/2 - r + s)\varepsilon_1 + \sum_{2\leqslant i\leqslant n, i\neq m}(n/2 + 1 - i)\varepsilon_i + (n/2 + 1 - m + r)\varepsilon_m$. Hence $M_{-p/q} = 0$.

8.8.4. $\mathfrak{g} = C(n)$

In this case $\rho^{(i)} = n + 1 - i$ for $i \geqslant 1$. The conditions $\gamma' \in \Delta_0^+, \gamma' + \beta \in \Delta^+$ mean that $\gamma' = 2\varepsilon_1; \varepsilon_1 \pm \varepsilon_m$ $(1 < m \leqslant n)$.

For $\gamma' = \varepsilon_1 - \varepsilon_m$ the permutation $(1; r - s + 1)$ stabilizes $\rho - r\gamma' - s\beta$ as in 8.8.3.

Fix $m \in \mathbb{Z}$ such that $1 < m \leqslant n$ and set $\gamma' := \varepsilon_1 + \varepsilon_m$. One has $b := b(r; \gamma; s) = \frac{2n+1+s-m-r}{l}$. $x_1 = n + s - r, x_j = n + 1 - j$ for $j \neq 0, 1, m$, and $x_m = n + 1 - m - r$. Since $b > 0$, one has $x_1 > -(n+1-m)$; if $x_1 \neq n+1-m$ then $(\rho - r\gamma' - s\beta)$ has a non-trivial stabilizer (since either $x_1 = 0$ or $x_1 = \pm x_j$ for $1 < j < m$). If $x_1 = n + 1 - m$, then $r - s = m - 1$, so $b = \frac{2(n+1-m)}{l}$ and

$$s_{\varepsilon_1-\varepsilon_m}(\rho - r\gamma' - s\beta) = (n + 1 - m - r)\varepsilon_1 + \sum_{j=2}^{n}(n + 1 - j)\varepsilon_n. \tag{28}$$

For $\gamma' := 2\varepsilon_1$ one has $b := b(r; \gamma; s) = \frac{2(n-r+s)}{l}$, and so $b > 0$ forces $m \leqslant n$ for $m := r - s + 1$; since $s < r$, we have $1 < m \leqslant n$. One has $\rho - r\gamma' - s\beta = (n + 1 - m - r)\varepsilon_1 + \sum_{j=2}^{n}(n + 1 - j)\varepsilon_n$. Using (28) we get $E(r; l\delta + 2\varepsilon_1; s) + E(r; l\delta + \varepsilon_1 - \varepsilon_{r-s+1}; s) = 0$, and this completes the proof for $\mathfrak{g} = C(n)$.

8.9. Proof that $M_{-p/q} \neq 0$ for $\mathfrak{g} \neq \mathfrak{sl}(n, 1), C(n)$

Our proof is based on (25).

8.9.1. $\mathfrak{g} = \mathfrak{osp}(2n, 2)$. One has $b(r; l\delta - 2\varepsilon_0; s) = 2\frac{n+s-r-1}{l}$. Clearly, $b(r; l\delta - 2\varepsilon_0; s)$ can be any negative rational number. Fix $(r; s; l)$ such that $b(r; l\delta - 2\varepsilon_0; s) = -p/q$. The term $(\rho - r\gamma - s\beta)^\# = (n - 1 + s)\varepsilon_1 + \sum_{i=2}^{n}(n - i)\varepsilon_i$ is dominant with respect to $\Pi^\#$. It is easy to see that this implies $E(r; l\delta - 2\varepsilon_0; s) \neq 0$ and $E(r; l\delta - 2\varepsilon_0; s) + E(r'; l'\delta - 2\varepsilon_0; s') \neq 0$. The inequality $b(r; l\delta - 2\varepsilon_0; s) < 0$ gives $r \geqslant n + s$. Then, by 8.4.2, $E(r; l\delta - 2\varepsilon_0; s) + E(r_1; \gamma; s_1) \neq 0$ if $\gamma' \in \Delta^+$. Now (25) follows from (27). Hence $M_{-p/q} \neq 0$.

8.9.2. One has

$$b(r; l\delta - \varepsilon_0; s) = \begin{cases} \frac{2n+2s-r-1}{2l} & \text{for } \mathfrak{g} = \mathfrak{osp}(2n + 1, 2), \mathfrak{osp}(3, 2n); \\ 3\frac{s+3-r}{l} & \text{for } \mathfrak{g} = F(4); \\ \frac{2s+5-r}{l} & \text{for } \mathfrak{g} = G(3). \end{cases}$$

Fix $(r; s; l)$ such that $b(r; l\delta - \varepsilon_0; s) = -p/q$ and r is odd. Then $r \geqslant s$ and thus, by 8.4.2, $E(r; l\delta - 2\varepsilon_0; s) + E(r_1; \gamma; s_1) \neq 0$ if $\gamma' \in \Delta^+$. Using (27) we reduce (25) to the formulas

$$\begin{aligned} &\text{(i)} \quad E(r; l\delta - \varepsilon_0; s) \neq 0, \\ &\text{(ii)} \quad E(r; l\delta - \varepsilon_0; s) + E(r_1; \gamma; s_1) \neq 0 \text{ if } b(r_1; \gamma; s_1) = -p/q \;\&\; \gamma' = -t\varepsilon_0. \end{aligned} \tag{29}$$

Take $\gamma = l_1\delta - t\varepsilon_0$ $(t = 1, 2)$. We have: $(\rho - s_1\beta - r_1\gamma)^\# = (\rho - s_1\beta)^\#$ is dominant with respect to $\Pi^\#$, and this gives (29)(i). To verify (29)(ii) assume that $\rho - s\beta - r(l\delta - \varepsilon_0) = w(\rho - s_1\beta - r_1\gamma)$ for some $w \in W^\#$. Then $w = \mathrm{id}$, $s_1 = s$ and thus $r(l\delta - \varepsilon_0) = r_1\gamma$, that is $r = r_1 t, rl = r_1 l_1$. Since r is odd, we have $t = 1$ and $(r_1, l_1) = (r, l)$. Hence $E(r; l\delta - \varepsilon_0; s) + E(r_1; \gamma; s_1) = 2E(r; l\delta - \varepsilon_0; s)$ and 29(ii) follows from 29(i).

9. Simplicity of minimal W-algebras

Let $\mathfrak{g}$ be a simple contragredient finite-dimensional Lie superalgebra and let f_θ be a root vector attached to the lowest root $-\theta$, which assumed to be even. Let $(.|.)$ be the invariant bilinear form on $\mathfrak{g}$, normalized by the condition $(\theta|\theta) = 2$. This normalization may differ in the super case from the standard normalization (due to inequivalent choices of θ). The corresponding dual Coxeter numbers $h^\vee$ are listed in [18]. In Section 10 we list them in the standard normalization of $(.|.)$. For each $k \in \mathbb{C}$, one attaches to the above data a vertex algebra $W^k(\mathfrak{g}, f_\theta)$, as described in [18,19], called the minimal W-algebra. We denote by $W_k(\mathfrak{g}, f_\theta)$ its (unique if $k \neq -h^\vee$) simple quotient. Our goal is to determine when $W^k(\mathfrak{g}, f_\theta)$ is simple. We assume that $k \neq -h^\vee$, since in the "critical" case, when $k = -h^\vee$, $W^k(\mathfrak{g}, f_\theta)$ is never simple. We shall also exclude the case $\mathfrak{g} = \mathfrak{sl}_2$, since $W^{k+2}(\mathfrak{sl}_2, f_\theta)$ is isomorphic to the Virasoro vertex algebra V^c with $c = 1 - 6\frac{(k-1)^2}{k}$.

9.1. Main results

In [18,19] a functor H from the category of restricted $\hat{\mathfrak{g}}$-modules of level k to the category of $\mathbb{Z}$-graded $W^k(\mathfrak{g}, f_\theta)$-modules is described. The image of the vacuum $\hat{\mathfrak{g}}$-module V^k is the vertex algebra $W^k(\mathfrak{g}, f_\theta)$, viewed as a module over itself. The vertex algebra $W^k(\mathfrak{g}, f_\theta)$ is simple iff $H(V^k)$ is an irreducible module.

9.1.1. According to [1] the functor H is exact and $H(L(\lambda))$ is either zero or irreducible; one has [1,18]: $H(L(\lambda)) = 0$ iff f_{α_0} acts locally nilpotently on $L(\lambda)$.

9.1.2. Theorem.

(i) *The vertex algebra $W^k(\mathfrak{g}, f_\theta)$ is simple iff the $\hat{\mathfrak{g}}$-module V^k is irreducible, or $k \in \mathbb{Z}_{\geqslant 0}$ and V^k has length two (i.e., the maximal proper submodule of the $\hat{\mathfrak{g}}$-module V^k is irreducible).*
(ii) *If $\mathfrak{g}$ is a simple Lie algebra, $\mathfrak{g} \neq \mathfrak{sl}_2$, then the vertex algebra $W^k(\mathfrak{g}, f_\theta)$ is simple iff the $\hat{\mathfrak{g}}$-module V^k is irreducible.*

Proof. By 9.1.1, $W^k(\mathfrak{g}, f_\theta)$ is simple if the $\hat{\mathfrak{g}}$-module V^k is irreducible. Let N be the maximal proper submodule of V^k. If $k \in \mathbb{Z}_{\geqslant 0}$ and N is simple then, by 9.1.1, $H(V^k/N) = H(L(k\Lambda_0)) = 0$ and $H(N)$ is simple. Hence $H(V^k)$ is simple. Now assume that $W^k(\mathfrak{g}, f_\theta)$ is simple and V^k is not irreducible. Since $\mathbb{C}[f_{\alpha_0}]$ acts freely on V^k, f_{α_0} does not act locally nilpotently on N. Therefore $H(N)$ is a non-zero submodule of $W^k(\mathfrak{g}, f_\theta)$. Hence $H(V^k/N) = H(L(k\Lambda_0)) = 0$. This gives $k \in \mathbb{Z}_{\geqslant 0}$. It remains to show that N is simple.

Recall that $f_{\alpha_0}, e_{\alpha_0}$ generate a Lie algebra $\mathfrak{s}$ isomorphic to $\mathfrak{sl}(2)$. Let v be a singular vector such that $\mathbb{C}[f_{\alpha_0}]v$ is a simple Verma module over $\mathfrak{s}$. Let N' be a $\hat{\mathfrak{g}}$-submodule of V^k generated by v and N'' be the maximal proper submodule of N'. Since $\mathbb{C}[f_{\alpha_0}]v$ is a simple Verma module over $\mathfrak{s}$, N'' does not meet $\mathbb{C}[f_{\alpha_0}]v$ and thus $H(N'/N'') \neq 0$ by 9.1.1.

Now let N' be any non-zero submodule of V^k and v be a singular vector in N'. Note that $\mathbb{C}[f_{\alpha_0}]v$ is a Verma module over $\mathfrak{s}$, which is either simple or has a unique proper submodule with an $\mathfrak{s}$-singular vector v'. Since $[f_{\alpha_0}, e_\alpha] = 0$ for any $\alpha \in \hat{\Pi} \setminus \{\alpha_0\}$, v' is singular. Therefore either v or v' is a singular vector, which generates a simple Verma module over $\mathfrak{s}$. By above, $H(N') \neq 0$.

Let N' be the maximal proper submodule of N. By above, $H(V^k/N) = 0$ and $H(N/N') \neq 0$. Since $H(V^k)$ is simple, this gives $N' = 0$ and establishes (i).

Finally, (ii) will be proven in 9.2–9.5 below. □

Now Theorem 0.2.1 gives

9.1.3. Corollary. *Let $\mathfrak{g}$ be a simple Lie algebra, $\mathfrak{g} \neq \mathfrak{sl}_2$. Then the vertex algebra $W^k(\mathfrak{g}, f_\theta)$ is not simple iff $l(k+h^\vee)$ is a non-negative rational number, which is not the inverse of an integer (here l is the "lacety" of $\mathfrak{g}$).*

9.1.4. Remark. Recall (see [18]) that $W^k(\mathfrak{g}, f_\theta)$ has central charge

$$c = \frac{k \,\mathrm{sdim}\, \mathfrak{g}}{k+h^\vee} - 6k + h^\vee - 4, \tag{30}$$

if $(\theta|\theta) = 2$. For example, if $k+n = p/q$, where $p, q \in \mathbb{Z}_{\geqslant 1}$, the vertex algebra $W^k(\mathfrak{sl}_n, f_\theta)$ has the same central charge for $k_1 = -n + p/q$ and $k_2 = -n + \frac{n(n^2-1)}{6} q/p$. For $n > 2$ we obtain pairs of non-isomorphic W-algebras of the same central charge: if $p = 1$ and $q > 1$, the vertex algebra $W^{k_1}(\mathfrak{sl}_n, f_\theta)$ is simple, but the vertex algebra $W^{k_2}(\mathfrak{sl}_n, f_\theta)$ is not simple. (For $n = 2$ these W-algebras are isomorphic.) Note, that in contrast to the case of $\mathfrak{g}$ of rank > 1, $W^k(\mathfrak{sl}_2, f_\theta)$ is simple for $k \in \mathbb{Z}_{\geqslant 0}$.

9.1.5. The following corollary follows from the above results and the description of the $N = 1, 2, 3, 4$ and big $N = 4$ vertex algebras, given in [18], in terms of the minimal W-algebras.

Corollary.

(i) *The Neveu–Schwarz ($N = 1$) vertex algebra is simple iff its central charge c is not of the form $\frac{3}{2}(1 - \frac{2(p-q)^2}{pq})$, where p and q are relatively prime positive integers such that $p > q$ and p/q is not an odd integer. (The latter set coincides with the set of central charges of $N = 1$ minimal models, cf. e.g.* [19, (6.3)].)

(ii) *The $N = 2$ vertex algebra is simple iff its central charge c is not of the form $3 - 6p/q$, where p and q are relatively prime positive integers and $q \geqslant 2$. (The subset with $p = 1$ of the latter set coincides with the set of central charges of $N = 2$ minimal models.)*

(iii) *The $N = 3$ vertex algebra with central charge c is simple if c is not a rational number. For all other values of c, except, possibly, for $c = -3b$, where b is a positive odd integer, this vertex algebra is not simple.*

(iv) *The $N = 4$ vertex algebra with central charge c is simple if c is not a rational number. For all other values of c, except, possibly, for $c = -6b$, where b is a positive integer, this vertex algebra is not simple.*

(v) *The big $N = 4$ vertex algebra with central charge c is simple if $c \notin \mathbb{Q}_{\geqslant 0} \cup \mathbb{Q}_{>0}a \cup \mathbb{Q}_{>0}(-1-a)$. For all other values of c, except, possibly, for $c = -3b$, where b is a positive odd integer, this vertex algebra is not simple.*

Proof. Combining Theorems 9.1.2, 0.2.4, and formula (30), we obtain (iii)–(v).

(i) follows from Theorem 5.2.1. We will give another proof by deducing (i) from Theorem 0.2.2. Indeed, set $a(k) := 2k + 3$. Formula (30) gives $c = \frac{15}{2} - 3(a + \frac{1}{a})$. By Theorem 0.2.2 for $\mathfrak{g} = \mathfrak{osp}(1,2)$ with the standard normalization $(\theta|\theta) = 2$, V^k is simple iff $a \notin \mathbb{Q} \setminus \{\frac{1}{2m+1}\}_{m=0}^{\infty}$. By Theorem 9.1.2, $W^k(\mathfrak{osp}(1,2), f_\theta)$ is simple for $a \notin \mathbb{Q} \setminus \{\frac{1}{2m+1}\}_{m=0}^{\infty}$, and is not simple for $a \in \mathbb{Q} \setminus \{\frac{1}{2m+1}; 2m+3\}_{m=0}^{\infty}$. Since $c(a) = c(1/a)$ and the Neveu–Schwarz vertex algebra $W^k(\mathfrak{osp}(1,2), f_\theta)$ is determined by its central charge, the vertex algebras $W^k(\mathfrak{osp}(1,2), f_\theta)$ and

$W^{k'}(\mathfrak{osp}(1,2), f_\theta)$ are isomorphic if $a(k)a(k')=1$. Hence, since $W^k(\mathfrak{osp}(1,2), f_\theta)$ is simple for $a \in \{\frac{1}{2m+1}\}_{m=0}^\infty$, it is also simple for $a \in \{2m+1\}_{m=0}^\infty$. Hence $W^k(\mathfrak{osp}(1,2), f_\theta)$ is not simple for $c = \frac{15}{2} - 3(a + \frac{1}{a})$, where $a \in \mathbb{Q} \setminus \{\frac{1}{2m+1}; 2m+1\}_{m=0}^\infty$. Since $c(a) = c(1/a)$, we can take $a \in \mathbb{Q} \setminus \{\frac{1}{2m+1}; 2m+1\}_{m=0}^\infty$ such that $a > 1$ and write $a = p/q$, where p and q are relatively prime positive integers. This proves (i).

(ii) The $N=2$ vertex algebra is isomorphic to the minimal W-algebra $W^k(\mathfrak{sl}(2,1), f_\theta)$ [18], and by formula (30) one has $c = -3 - 6k$. Combining Theorems 9.1.2(i) and 0.2.4, we see that $W^k(\mathfrak{sl}(2,1), f_\theta)$ is simple if c is not of the form $3 - 6p/q$, where p and q are relatively prime positive integers, and that for all other values of c, except, possibly, for $c = -3b$, where b is a positive odd integer, this vertex algebra is not simple.

By Theorem 9.1.2(i), it remains to verify that if k is a non-negative integer, then the vacuum $\hat{\mathfrak{sl}}(2,1)$-module V^k has length two, i.e., the only singular vectors in V^k have weights $k\Lambda_0$ and $s_{\alpha_0}.k\Lambda_0$. Consider the natural embedding $\hat{\mathfrak{sl}}(2)$ into $\hat{\mathfrak{sl}}(2,1)$. We will describe the weights of $\hat{\mathfrak{sl}}(2)$-singular vectors in V^k and then deduce the required assertion from the fact that $(k\Lambda_0 + \rho, k\Lambda_0 + \rho) = (\mu + \rho, \mu + \rho)$ if μ is the weight of singular vector in V^k.

Let α be an even root for $\mathfrak{sl}(2,1)$ and $\beta, \alpha+\beta$ be odd roots. Choose the following set of simple roots for $\mathfrak{sl}(2,1)$: $\{\alpha+\beta, -\beta\}$; then $\theta = \alpha$ and the set of simple roots for $\hat{\mathfrak{sl}}(2,1)$ is $\{\alpha+\beta, -\beta, \alpha_0 := \delta - \alpha\}$. Note that $\hat{\mathfrak{sl}}(2,1)$ contains a copy of $\hat{\mathfrak{sl}}(2)$ with a common simple root α_0. As a result, the shifted actions of the reflection $s_0 := s_{\alpha_0}$ with respect to $\hat{\mathfrak{sl}}(2,1)$ and $\hat{\mathfrak{sl}}(2)$ coincide, i.e., $s_0.\mu = s_0(\mu + \hat{\rho}) - \hat{\rho} = s_0(\mu + \hat{\rho}') - \hat{\rho}'$, where $\hat{\rho}$ corresponds to $\hat{\mathfrak{sl}}(2|1)$ and $\hat{\rho}'$ corresponds to $\hat{\mathfrak{sl}}(2)$.

For $S \subset \hat{\Delta}^+$ set $|S| = \sum_{\gamma \in S} \gamma$. A Verma module $M(\lambda)$ over $\hat{\mathfrak{sl}}(2|1)$ has a filtration by $\hat{\mathfrak{sl}}(2)$-modules $M'(\lambda - |S|)$: $S \subset \hat{\Delta}_1^+ \setminus \Delta_1^+$. Let V^k be the vacuum module over $\hat{\mathfrak{sl}}(2,1)$. Since $\mathfrak{sl}(2)$ acts locally finitely on V^k, V^k has a filtration by generalized Verma $\hat{\mathfrak{sl}}(2)$-modules $M_I'(\lambda - |S|)$, where $I = \{\alpha\}$. One has $M_I'(\lambda) = M'(\mu)/M'(s_\alpha.\mu)$. Since the Weyl group of $\hat{\mathfrak{sl}}(2)$ is the infinite dihedral group generated by $s_\alpha, s_0 = s_{\alpha_0}$, $M_I'(\mu)$ has at most two $\hat{\mathfrak{sl}}(2)$-singular vectors: of weight μ and of weight $s_0.\mu$ if $s_0.\mu < \mu$. Thus the weight of a $\hat{\mathfrak{sl}}(2)$-singular vector of V^k is of the form $k\Lambda_0 - |S|$ or $s_0.(k\Lambda_0 - |S|)$, where $S \subset \hat{\Delta}_1^+ \setminus \Delta_1^+$. For $S = \emptyset$ we have $k\Lambda_0$ and $s_0.k\Lambda_0$. Let us show that there are no other $\hat{\mathfrak{sl}}(2|1)$-singular vectors.

Indeed, if μ is a weight of $\hat{\mathfrak{sl}}(2)$-singular vector then $(\lambda + \hat{\rho}, \lambda + \hat{\rho}) = (\mu + \hat{\rho}, \mu + \hat{\rho})$. Since $(s_0.\mu + \hat{\rho}, s_0.\mu + \hat{\rho}) = (\mu + \hat{\rho}, \mu + \hat{\rho})$, it is enough to show that for $S \neq \emptyset$ one has $(k\Lambda_0 + \hat{\rho}, k\Lambda_0 + \hat{\rho}) > (k\Lambda_0 - |S| + \hat{\rho}, k\Lambda_0 - |S| + \hat{\rho})$, which can be rewritten as

$$2\big(k\Lambda_0 + \hat{\rho}, |S|\big) > \big(|S|, |S|\big). \tag{31}$$

For $\mu \in \Delta_1 = \{\pm\alpha \pm \beta\}$ set $S_\mu := S \cap \{k\delta + \mu\}_{k>0}$, $s_\mu := \#S_\mu$. Then $S = \coprod_{\mu \in \Delta_1} S_\mu$, so $|S| = \sum_{\mu \in \Delta_1} |S_\mu|$. Observe that $S_\mu = \{r_i\delta + \mu\}_{i=1}^{s_\mu}$, where $1 \leqslant r_1 < r_2 < \cdots < r_{s_\mu}$ and thus $|S_\mu| = r\delta + s_\mu\mu$, where $r \geqslant \frac{s_\mu(s_\mu+1)}{2}$. Hence

$$S = m\delta + (s_{\alpha+\beta} - s_{-\alpha-\beta})(\alpha+\beta) + (s_\beta - s_{-\beta})\beta$$

for some $m \geqslant \sum_{\mu \in \Delta_1} s_\mu(s_\mu + 1)$. Taking into account $(\hat{\rho}, \beta) = (\hat{\rho}, \alpha+\beta) = 0$, $(\hat{\rho}, \delta) = 1$ we get

$$2\big(k\Lambda_0 + \hat{\rho}, |S|\big) = 2(k+1)m \geqslant (k+1) \sum_{\mu \in \{\pm\beta, \pm(\alpha+\beta)\}} s_\mu(s_\mu + 1).$$

On the other hand,

$$\big(|S|,|S|\big) = -(s_{\alpha+\beta} - s_{-\alpha-\beta})(s_\beta - s_{-\beta}).$$

For $S \neq \emptyset$ at least one of the quantities $s_\beta, s_{-\beta}, s_{\alpha+\beta}, s_{-\alpha-\beta}$ is non-zero (and all of them are non-negative integers) and thus

$$(k+1) \sum_{\mu \in \{\pm\beta, \pm(\alpha+\beta)\}} s_\mu(s_\mu + 1) > (s_{-\alpha-\beta} - s_{\alpha+\beta})(s_\beta - s_{-\beta}),$$

since $k \geqslant 0$. This establishes (31) and (ii). $\square$

We believe that in all questionable cases in (iii)–(v) the vertex algebra is not simple, but we do not know how to prove this.

9.2. Outline of the proof of Theorem 9.1.2(ii)

In 9.2–9.5 we assume that $\mathfrak{g}$ is a finite-dimensional semisimple *Lie algebra* and k is a non-negative integer.

9.2.1. In 9.3 we will show that for $k \in \mathbb{Z}_{\geqslant 0}$, $H(V^k)$ is not simple iff $Q_{s_0,w} \neq 1$ for some $w \in \hat{W}$, where Q stands for the inverse Kazhdan–Lusztig polynomial. (This condition does not depend on the non-negative integer k and thus $H(V^k)$ is simple iff $H(V^0)$ is simple.)

Remark that for $\mathfrak{g} = \mathfrak{sl}_2$, the Weyl group is the infinite dihedral group; by [7, 7.12] one has $P_{x,z} = 1$ for $x \leqslant z$ which implies $Q_{x,z} = 1$ for $x \leqslant z$. This implies the simplicity of $H(V^k)$ (which is well known).

9.2.2. Let Θ be the set of pairs (α, X_n), where α is a node of a Dynkin diagram X_n satisfying the property:

$$Q_{s_\alpha,w} \neq 1 \quad \text{for some } w \in W(X_n),$$

where $W(X_n)$ is the Coxeter group of type X_n. From 9.3 we see that $(\alpha, X_n) \notin \Theta$ iff for $\mathfrak{p}$ being the maximal parabolic not containing α, the generalized Verma module $\operatorname{Ind}_{\mathfrak{p}}^{\mathfrak{g}} L'$ has length two, where L' is the trivial one-dimensional $\mathfrak{p}$-module.

Geometrically, if X_n is of finite type, $(\alpha, X_n) \in \Theta$ is equivalent to the fact that a certain codimension one Schubert variety is not rationally smooth. It is well known (see, for example, [23, 12.2.E]) that these Schubert varieties are rationally smooth in rank two cases ($n = 2$) and for the pairs (α_n, C_n) (in enumeration below). We need to study the case when X_n is an affine diagram.

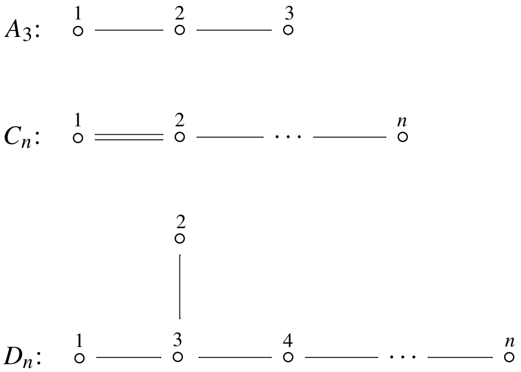
9.2.3. We have to show that $(\alpha_0, X_n^{(1)}) \in \Theta$, where $X_n^{(1)}$ is the affinization of a finite type diagram X_n ($n > 1$) and α_0 is the affine simple root.

One has:

(i) $(\alpha, X'_n) \in \Theta$, X'_n is a subdiagram of $X_m \implies (\alpha, X_m) \in \Theta$;

(ii) $\alpha \in X_n$ is connected to α' only, $\big(\alpha', X_n \setminus \{\alpha\}\big) \in \Theta \implies (\alpha, X_n) \in \Theta$, (32) where "$X'_n$ is a subdiagram of X_m" in (i) means that the set of nodes of X'_n is a subset of the nodes of X_m and the set of edges of X'_n consists of all edges between these nodes in X_m; and in (ii) the diagram $X_n \setminus \{\alpha\}$ is obtained from X_n by removing the extremal node α and the edge between α and α'. We will prove (ii) in 9.4.5; (i) follows from the fact that the inverse Kazhdan–Lusztig polynomials are the same for a diagram and its subdiagram.

Taking into account (32), the verification $(\alpha_0, X_n^{(1)}) \in \Theta$ for $n > 2$ reduces to the cases (α_2, A_3), (α_1, C_3). Here and further we use the following enumeration of the vertices of X_n:



Indeed, the pair $(\alpha_0, A_n^{(1)})$ for $n > 2$ has a subdiagram (α_2, A_3); the pair $(\alpha_0, C_n^{(1)})$ for $n > 2$ has a subdiagram (α_1, C_3); applying 32(ii) to (α_2, A_3) $n-3$ times we obtain the pair (α_n, D_n) and now using 32(i) we obtain the pairs $(\alpha_0, X_n^{(1)})$ for $X = B, D, E$; finally, applying 32(ii) to (α_1, C_3) twice we get $(\alpha_0, F_4^{(1)})$.

It is easy to verify that for A_3 one has $Q_{s_2, s_2 s_1 s_3 s_2} = 1 + q$, and that for C_3 one has $Q_{s_1, s_1 s_2 s_1 s_3 s_2 s_1} = 1 + q^2$. As a result, $(\alpha_2, A_3), (\alpha_1, C_3) \in \Theta$. The remaining cases $(\alpha_0, X_2^{(1)})$ are verified in 9.5.

9.3. Multiplicity formula

Let $\hat{W}$ be the Weyl group of $\hat{\mathfrak{g}}$.

9.3.1. Let N be the maximal proper submodule of V^k. By Theorem 9.1.2(i), it is enough to show that N is not simple. Recall that $k\Lambda_0$ is a dominant integral weight and so all subquotients of V^k are of the form $L(w.k\Lambda_0)$, $w \in \hat{W}$, where $\hat{W}$ is the Weyl group of $\hat{\mathfrak{g}}$. The highest weight of N is $s_0.k\Lambda_0$ and so $L(s_0.k\Lambda_0)$ is a quotient of N. It remains to verify that

$$\bigl[N\colon\, L(w.k\Lambda_0)\bigr] \neq 0 \quad \text{for some } w \neq s_0. \tag{33}$$

Let us describe the multiplicity $[N\colon\, L(w.k\Lambda_0)]$ in terms of Kazhdan–Lusztig polynomials.

9.3.2. One has

$$\operatorname{ch} V^k = \frac{R_I}{R} e^{k\Lambda_0}, \qquad \operatorname{ch} L(k\Lambda_0) = R^{-1} \sum_{w \in \hat{W}} (-1)^{l(w)} e^{w.k\Lambda_0},$$

where $R = \prod_{\alpha \in \hat{\Delta}^+} (1 - e^{-\alpha})$, $R_I = \prod_{\alpha \in \Delta^+} (1 - e^{-\alpha})$.

Using the well-known formula $R_I = \sum_{w\in W}(-1)^{l(w)} e^{w.0}$ we get

$$\operatorname{ch} V^k = R^{-1}\sum_{w\in W}(-1)^{l(w)} e^{w.k\Lambda_0},$$

that is

$$\operatorname{ch} N = R^{-1}\sum_{w\in \hat{W}\setminus W}(-1)^{l(w)+1} e^{w.k\Lambda_0} = \sum_{w\in \hat{W}\setminus W}(-1)^{l(w)+1}\operatorname{ch} M(w.k\Lambda_0).$$

9.3.3. In [20] the Kazhdan–Lusztig conjecture was established for the symmetrizable, hence affine, Kac–Moody Lie algebras. This gives

$$\operatorname{ch} M(w.k\Lambda_0) = \sum_{z\in\hat{W}} P_{w,z}(1)\operatorname{ch} L(z.k\Lambda_0),$$

where $P_{w,z}$ are the Kazhdan–Lusztig polynomials defined in [21] (we describe the polynomials in 9.4). One has $P_{w,z}\neq 0$ iff $w\leqslant z$. We obtain

$$\big[N\colon L(z.k\Lambda_0)\big] = \sum_{w\in\hat{W}\setminus W}(-1)^{l(w)+1}P_{w,z}(1) = \sum_{w\in\hat{W}:\ s_0\leqslant w\leqslant z}(-1)^{l(w)+1}P_{w,z}(1).$$

Now the condition (33) can be rewritten as

$$\sum_{w\in\hat{W}:\ s_0\leqslant w\leqslant z}(-1)^{l(w)+1}P_{w,z}(1)\neq\delta_{s_0,z}\quad\text{for some } z\in\hat{W}. \tag{34}$$

One has $\sum_{w\in\hat{W}:\ s_0\leqslant w\leqslant z}(-1)^{l(w)+1}Q_{s_0,w}P_{w,z}=\delta_{s_0,z}$, where $Q_{s_0,w}$ are the inverse Kazhdan–Lusztig polynomials. Hence (34) is equivalent to $Q_{s_0,w}(1)\neq 1$ for some $w\in\hat{W}$. Using 38(i) we conclude that (34) is equivalent to

$$Q_{s_0,w}\neq 1\quad\text{for some } w\in\hat{W}.$$

9.4. Kazhdan–Lusztig polynomials

Let W be a Coxeter group; denote the unit element in W by e. For the elements x, y of W set

$$[x,y] := \{w\colon x\leqslant w\leqslant y\}.$$

9.4.1. For $x, y\in W$ the Kazhdan–Lusztig polynomials $P_{x,y}(q)$ can be computed recursively using the following properties: the polynomial $P_{x,y}$ has degree $\leqslant \frac{l(y)-l(x)-1}{2}$ and

$$P_{x,y} = \begin{cases} 0, & x\nleqslant y,\\ 1, & x\leqslant y \text{ and } l(y)-l(x)\leqslant 2,\\ \sum_{x\leqslant w\leqslant y}(-1)^{l(w)-l(x)}R_{x,w}\overline{P_{w,y}}q^{l(y)-l(w)}, & \end{cases}$$

where $\overline{P}$ is the image of P under the algebra involution $q \mapsto q^{-1}$ and the polynomials $R_{x,y}(q)$ can be defined recursively by the formulas

$$R_{x,y} = \begin{cases} 0, & x \nleqslant y, \\ R_{sx,sy}, & sx < x,\ sy < y, \\ (q-1)R_{sx,y} + qR_{sx,sy}, & sx > x,\ sy < y, \end{cases}$$

where s is a simple reflection. One has:

$$\begin{aligned} &\text{(i)} \quad R_{x,y} = R_{x^{-1},y^{-1}}; \\ &\text{(ii)} \quad R_{x,y} = (q-1)^{l(y)-l(x)} \quad \text{if } x \leqslant y,\ l(y)-l(x) \leqslant 2; \\ &\text{(iii)} \quad \overline{R_{x,y}} = (-q)^{l(x)-l(y)} R_{x,y}. \end{aligned} \tag{35}$$

By [21, 2.3.g], one has:

$$P_{x,y} = P_{sx,y} \quad \text{if } sy < y. \tag{36}$$

9.4.2. The inverse Kazhdan–Lusztig polynomials $Q_{y,w}(q)$ are defined by the formula

$$\sum_w (-1)^{l(w)-l(y)} Q_{y,w} P_{w,z} = \delta_{y,z}. \tag{37}$$

A geometric meaning of the inverse Kazhdan–Lusztig polynomials is discussed in [20]. Their results imply that $Q_{x,y}$ have non-negative integer coefficients in the case of a symmetrizable Kac–Moody Lie algebra.

One has: $Q_{x,z} \neq 0$ iff $x \leqslant z$; for $x \leqslant z$ the polynomials $Q_{x,z}$ have the following properties:

$$\begin{aligned} &\text{(i)} \quad Q_{x,z} = 1 + a_1 q + a_2 q^2 + \cdots + a_k q^k, \quad a_i \in \mathbb{Z}_{\geqslant 0},\ k \leqslant \frac{l(z)-l(x)-1}{2}; \\ &\text{(ii)} \quad Q_{x,z} = 1 \quad \text{if } l(z)-l(x) \leqslant 2; \\ &\text{(iii)} \quad Q_{x,z} = \sum_{w \in [x,z]} (-1)^{l(z)-l(w)} q^{l(w)-l(x)} \overline{Q_{x,w}} R_{w,z}; \\ &\text{(iv)} \quad Q_{e,z} = 1, \quad \text{for all } z. \end{aligned} \tag{38}$$

The first property follows from [20]; (ii) follows from (i). From [20, Lemma 5.2.1, 5.3] we obtain, using (35):

$$q^{l(x)} Q_{x,z} = q^{l(z)} \sum_{w \in [x,z]} \overline{Q_{x,w} R_{w^{-1},z^{-1}}} = q^{l(z)} \sum_{w \in [x,z]} \overline{Q_{x,w}} (-q)^{l(w)-l(z)} R_{w,z},$$

and this gives (iii). Finally, combining (37) and (36) we get (iv).

9.4.3. Let

$$M(x,z) := \sum_{w\in[x,z]} (-1)^{l(z)-l(w)} q^{l(w)-l(x)} R_{w,z}.$$

One has

$$Q_{x,w} = 1 \quad \forall w \in [x,z] \quad \Longleftrightarrow \quad M(x,w) = 1 \quad \forall w \in [x,z]. \tag{39}$$

Indeed, assume that $Q_{x,w} = 1$ for all $w \in [x,z]$. Then for any $y \in [x,z]$ one has $Q_{x,w} = 1$ for all $w \in [x,y]$ and 38(iii) gives $M(x,y) = 1$. For the inverse implication assume that $M(x,w) = 1$ for all $w \in [x,z]$. We prove that $Q_{x,w} = 1$ by induction on $w \in [x,z]$ with respect to $l(w)$ (note that $l(x) \leqslant l(w) \leqslant l(z)$). If $l(w) = l(x)$, for $w \in [x,z]$, then $w = x$ and $Q_{x,x} = 1$. Suppose that $Q_{x,w} = 1$ for all $w \in [x,z]$ with $l(w) < m$. Take $y \in [x,z]$ such that $l(y) = m$. Then $Q_{x,w} = 1$ for all $w \in [x,y]$ and 38(iii) gives

$$Q_{x,y} = M_{x,y} - q^{l(y)-l(x)} R_{y,y} + q^{l(y)-l(x)} \overline{Q}_{x,y} R_{y,y} = 1 + q^{l(y)-l(x)} (\overline{Q}_{x,y} - 1),$$

that is $Q_{x,y} - 1 = q^{l(y)-l(x)} \overline{Q_{x,y} - 1}$. If $Q_{x,y} - 1 = 0$, then $Q_{x,y} - 1 = b_1 q^{i_1} + b_2 q^{i_2} + \cdots + b_s q^{i_s}$, where $i_1 < i_2 < \cdots < i_s$ and $i_1 + i_s = l(y) - l(x)$. However, by 38(i) $2i_s < l(y) - l(x)$, a contradiction.

9.4.4. In [21] there is the following definition:

Definition. Given $y, w \in W$ we say that $y \prec w$ if the following conditions are satisfied: $y < w$, $l(w) - l(y)$ is odd and $P_{y,w}$ is a polynomial in q of degree exactly $\frac{l(w)-l(y)-1}{2}$.

Lemma. *Assume that $y \prec z$, $l(z) - l(y) \geqslant 3$. Then $Q_{y,w} \neq 1$ for some $w \in [y,z]$.*

Proof. Suppose that $Q_{y,w} = 1$ for all $w \in [y,z]$. Then (37) gives

$$Q_{y,z} = 1 + (-1)^{l(z)} \sum_{w\in[y,z]} (-1)^{l(w)+1} P_{w,z}.$$

The condition $y \prec z$ implies that the degree of $P_{w,z}$ is less than the degree of $P_{y,z}$ if $w \in]y,z]$. Hence $Q_{y,z}$ has degree $\frac{l(z)-l(y)-1}{2}$ and, in particular, $Q_{y,z} \neq 1$, a contradiction. □

9.4.5. Proof of 32(ii)

Let α_0 be an extremal node of a Dynkin diagram X_n, α_1 be the only node which is connected to α_0 and X'_n be the Dynkin diagram obtained from X_n by removing the extremal node α_0 and the edge between α_0 and α_1. Let W (respectively, W') be the Coxeter group of X_n (respectively, X'_n). Assume that $(\alpha_1, X'_n) \in \Theta$, that is $Q_{s_1,z} \neq 1$ for some $z \in W'$; let z be a shortest element with this property.

Note that $Q_{s_1,w} = 1$ for all $w \in [s_1, z[$. Formulas 38(iv) and (39) give

$$M(s_1,w) = 1 \quad \text{for all } w \in [s_1, z[, \qquad M(s_1,z) \neq 1, \qquad M(e,z) = 1. \tag{40}$$

Let us show that $Q_{s_0,z'} \neq 1$ for some $z' \in [s_0, s_0 z s_0]$. By (39) it is enough to verify that $M(s_0, s_0 z s_0) \neq 1$. Observe that the elements of $[s_0, s_0 z s_0]$ are of the form $s_0 w s_0, s_0 w, w s_0$ if $w \in [s_1, z]$ and $s_0 w$ if $w \leqslant z$, $w \notin [s_1, z]$. Since w does not contain s_0 (i.e., $w \not\geqslant s_0$), one has $l(s_0 w s_0) = l(w) + 2$ if $w \in [s_1, z]$ and $l(s_0 w) = l(w) + 1$ if $w \leqslant z$, $w \notin [s_1, z]$. The properties of $R_{x,y}$ imply $R_{s_0 w, s_0 z s_0} = R_{w s_0, s_0 z s_0} = (q-1) R_{w,z}$ (since $z \not\geqslant s_0$) and $R_{s_0 w s_0, s_0 z s_0} = R_{w,z}$. We obtain

$$\begin{aligned}
M(s_0, s_0 z s_0) &= \sum_{y \in [s_0, s_0 z s_0]} (-1)^{l(s_0 z s_0) - l(y)} q^{l(y)-1} R_{y, s_0 z s_0} \\
&= \sum_{w \in [s_1, z]} (-1)^{l(z) - l(w)} q^{l(w)+1} R_{w,z} + (q-1) \sum_{w \in [s_1, z]} (-1)^{l(z)+1-l(w)} q^{l(w)} R_{w,z} \\
&\quad + (q-1) \sum_{w \leqslant z} (-1)^{l(z)+1-l(w)} q^{l(w)} R_{w,z} \\
&= q^2 M(s_1, z) - q(q-1) M(s_1, z) + (1-q) M(e, z) \\
&= 1 - q + q M(s_1, z) \neq 1, \quad \text{by (40).}
\end{aligned}$$

Hence $Q_{s_0,z'} \neq 1$ for some $z' \leqslant s_0 z s_0$.

Using similar arguments and the fact that $M(s_1, y) = 1$ for all $y \in [s_1, z[$, we can show that $Q_{s_0,z'} = 1$ for $z' < s_0 z s_0$ and thus $Q_{s_0, s_0, s_0 z s_0} \neq 1$. □

9.5. Rank 2 cases

9.5.1. Case A_2

In this case the Weyl group $\hat{W}$ is generated by s_0, s_1, s_2, where the relations are $(s_0 s_1)^3 = (s_0 s_2)^3 = (s_0 s_2)^2 = e$. It is easy to see that $Q_{s_0, s_0 s_1 s_2 s_0} = 1 + q$.

9.5.2. Case C_2

In this case the Weyl group $\hat{W}$ is generated by s_0, s_1, s_2, where the non-trivial relations are $(s_0 s_1)^4 = (s_1 s_2)^4 = (s_0 s_2)^2 = e$. It is not hard to compute that $Q_{s_0, s_0 s_1 s_0 s_2 s_1 s_0} = 1 + q^2$. We can also check that $Q_{s_0,w} \neq 1$ for some w using the tables of Kazhdan–Lusztig polynomials by M. Goresky: one has $s_0 \prec s_0 s_1 s_0 s_2 s_1 s_0$ [6, the case $\tilde{B}_2$, No. 57] and then Lemma 9.4.4 implies the required assertion.

9.5.3. Case G_2

Here we use the tables [6]. Take $z := s_0 (s_1 s_2)^2 s_0 s_1 s_2 s_1 s_0$ (No. 133 in the tables [6]; in their notation the affine root is the third one). We have $s_0 \prec z$ and Lemma 9.4.4 gives $Q_{s_0,w} \neq 1$ for some $w \in [s_0, z]$ (it is easy to see that, in fact, $Q_{s_0,w} = 1$ for $w < z$ and $Q_{s_0,z} = 1 + q^4$).

10. Root systems of defect one

The list of simple Lie superalgebras $\mathfrak{g}$ of defect one consists of Lie superalgebras $A(0,n) = \mathfrak{sl}(1, n+1)$, $C(n+1) = \mathfrak{osp}(2, 2n)$, $B(1,n) = \mathfrak{osp}(3, 2n)$, $B(n,1) = \mathfrak{osp}(2n+1, 2)$, $D(n+1, 1) = \mathfrak{osp}(2n+2, 2)$, where $n \geqslant 1$, and the exceptional Lie superalgebras $D(2,1,a)$, $F(4)$, $G(3)$ [11,17]. In this section we will describe some properties of the root systems of defect one, which we use in the paper.

10.1. We choose a set of simple roots of $\mathfrak{g}$ which contains a unique isotropic root β. We will describe $\Delta^\#$ and $W^\#$ (see 3.3). Let $\Pi^\#$ be the system of simple roots for $\Delta^\# \cap \Delta^+$. We have:

(i) $\Pi^\# = \Pi \setminus \{\beta\}$;
(ii) $W^\#\beta \subset \Delta^+$;
(iii) if $\mathfrak{g} \neq D(2,1,a)$, then there exists a unique simple root α_1 such that $(\alpha_1|\beta) \neq 0$; for $\mathfrak{g} \neq \mathfrak{osp}(3,2)$ one has $(\alpha_1|\beta) = -1$, $(\alpha_1|\alpha_1) = 2$.

The following lemma is used in 8.6.

10.1.1. Lemma. *Let* $\mathfrak{g} \neq D(2,1,a), B(1,1)$. *If* α *is a positive isotropic root satisfying* $(\rho - t\beta, \alpha) = t + 1$ *for some* t *then* $\alpha = \beta + \alpha_1$.

Proof. Write $\Pi = \{\beta, \alpha_1, \alpha_2, \ldots, \alpha_m\}$ and $\alpha = m_0\beta + \sum m_i\alpha_i$. Since α is isotropic, $\alpha \notin \Delta^\#$. The properties (i), (iii) imply that $m_0, m_1 \geqslant 1$. One has $(\rho|\alpha) \geqslant m_1 \geqslant 1$ and $-(\beta|\alpha) = m_1 \geqslant 1$. The assumption gives $(\rho|\alpha) = -(\beta|\alpha) = 1$ and thus $m_1 = 1$. Then $(\rho|\alpha) = 1$ forces $m_i = 0$ for $i > 1$. Finally, $(\alpha|\alpha) = 2 - 2m_0 = 0$ and thus $\alpha = \beta + \alpha_1$ as required. □

10.1.2. The standard normalization of the invariant form B, introduced in [17], is given by $(\alpha|\alpha) = 2$ for an even root $\alpha \in \Delta^\#$. In this normalization the dual Coxeter number $h^\vee$ is given by the following table:

$\mathfrak{g}$	$A(0,n-1)$	$C(n)$	$B(1,n)$	$B(n,1), n>1$	$D(n+1,1)$	$F(4)$	$G(3)$	$D(2,1,a)$
$h^\vee$	$n-1$	$n-1$	$n-1/2$	$2n-3$	$2n-2$	3	2	0

10.2. Non-exceptional case ($\mathfrak{g} \neq F(4), G(3), D(2,1,a)$*).*

The root system is described in terms of a basis $\{\varepsilon_i\}_{i=0,1,\ldots}$. We use the following bilinear form $(.|.)$, which is a multiple of the standard invariant form: $(\varepsilon_i|\varepsilon_j) = 0$ if $i \neq j$ and $(\varepsilon_0|\varepsilon_0) = -1$, $(\varepsilon_i|\varepsilon_i) = 1$ for $i > 0$. Then in all cases $(\gamma_1|\gamma_2) \in \mathbb{Z}$ for all roots γ_1, γ_2. We choose $\beta := \varepsilon_0 - \varepsilon_1$. One has $\Delta_0^\# = \Delta_0 \setminus \mathbb{Z}\varepsilon_0$ and $W^\#$ is the subgroup of W which stabilizes ε_0. One has $\hat{\Delta}_0^\# = \Delta_0$ for $A(0,n-1), C(n)$; in all other cases, except for $B(1,1)$ and $D(2,1)$, $\hat{\Delta}_0^\#$ corresponds to a simple component of $\mathfrak{g}_0$ which is not isomorphic to $\mathfrak{sl}(2)$.

10.3. Case $A(0,n-1)$

In this case the even part is $\mathfrak{sl}(n) \oplus \mathbb{C}$. Let $\{\pm(\varepsilon_i - \varepsilon_j)\colon 1 \leqslant i < j \leqslant n\}$ be the root system for $\mathfrak{sl}(n)$ and $\{\pm(\varepsilon_0 - \varepsilon_i)\}_{i=1}^n$ be the set of odd roots of $\mathfrak{sl}(1,n)$. One has $\Delta_0^\# = \Delta_0$, $W^\# = W = S_n$. Take

$$\Pi := \{\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \ldots, \varepsilon_{n-1} - \varepsilon_n\}, \qquad \Pi^\# := \Pi \cap \Delta_0 = \Pi \setminus \{\beta\},$$
$$\Delta_1^+ = \{\varepsilon_0 - \varepsilon_i\}_{i=1}^n, \qquad \Delta_0^+ = \{\varepsilon_i - \varepsilon_j\}_{1 \leqslant i < j \leqslant n}.$$

The highest root is $\theta = \varepsilon_0 - \varepsilon_n$ and $\rho = -\frac{n}{2}\varepsilon_0 + \sum_{i=1}^n (\frac{n}{2} + 1 - i)\varepsilon_i$.

10.4. Case $C(n+1)$

In this case the even part is $C_n \oplus \mathbb{C}$. Let $\{\pm 2\varepsilon_i; \pm\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i < j \leqslant n\}$ be the root system for the Lie algebra of type C_n and $\{\pm\varepsilon_0 \pm \varepsilon_i\}_{i=1}^n$ be the set of odd roots of $C(n)$. One has $\Delta_0^\# = \Delta_0$, $W^\# = W$. Take $\Pi := \{\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, 2\varepsilon_n\}$. Then

$$\Delta_1^+ = \{\varepsilon_0 \pm \varepsilon_i\}_{i=1}^n, \qquad \Delta_0^+ = \{2\varepsilon_i, \varepsilon_i \pm \varepsilon_j\}_{1 \leqslant i < j \leqslant n}.$$

One has $\theta = \varepsilon_0 + \varepsilon_1$ and $\rho = -n\varepsilon_0 + \sum_{i=1}^n (n+1-i)\varepsilon_i$.

10.5. Case $B(1,1)$, $n > 1$

Take $\Pi := \{\varepsilon_0 - \varepsilon_1; \varepsilon_1\}$. Then $\Delta_0^+ = \{\varepsilon_0; 2\varepsilon_1\}$, $\Delta_1^+ = \{\varepsilon_0 \pm \varepsilon_1; \varepsilon_1\}$. One has $\Delta_0^\# = \pm 2\varepsilon_1$, $W^\# \cong \mathbb{Z}_2$ is the corresponding Weyl group. One has $\theta = \varepsilon_0 + \varepsilon_1$ and $2\rho = -\varepsilon_0 + \varepsilon_1$.

10.6. Cases $B(n,1)$, $B(1,n)$: $n > 1$

Take $\Pi := \{\varepsilon_0 - \varepsilon_1; \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_n\}$. One has

$$\begin{aligned} &\Delta_1^+ = \{\varepsilon_0 \pm \varepsilon_i; \varepsilon_0\}_{i=1}^n, \qquad \Delta_0^+ = \{\varepsilon_i, \varepsilon_i \pm \varepsilon_j; 2\varepsilon_0\}_{1 \leqslant i < j \leqslant n} \quad \text{if } \mathfrak{g} = B(n,1);\\ &\Delta_1^+ = \{\varepsilon_0 \pm \varepsilon_i; \varepsilon_i\}_{i=1}^n, \qquad \Delta_0^+ = \{2\varepsilon_i, \varepsilon_i \pm \varepsilon_j; \varepsilon_0\}_{1 \leqslant i < j \leqslant n} \quad \text{if } \mathfrak{g} = B(1,n). \end{aligned}$$

The group $W^\#$ is the group of signed permutations of $\{\varepsilon_i\}_{i=1}^n$ and $\rho = -(n - \frac{1}{2})\varepsilon_0 + \sum_{i=1}^n (n - i + \frac{1}{2})\varepsilon_i$. One has $\theta = 2\varepsilon_0$ if $\mathfrak{g} = B(n,1)$ and $\theta = \varepsilon_0 + \varepsilon_1$ if $\mathfrak{g} = B(1,n)$.

10.7. Case $D(n,1)$, $n > 1$

Take $\Pi := \{\varepsilon_0 - \varepsilon_1, \varepsilon_1 - \varepsilon_2, \dots, \varepsilon_{n-1} - \varepsilon_n, \varepsilon_{n-1} + \varepsilon_n\}$. Then

$$\Delta_1^+ = \{\varepsilon_0 \pm \varepsilon_i\}_{i=1}^n, \qquad \Delta_0^+ = \{\varepsilon_i \pm \varepsilon_j; 2\varepsilon_0\}_{1 \leqslant i < j \leqslant n};$$

$W^\#$ is the group of signed permutations of $\{\varepsilon_i\}_{i=1}^n$ which change the even number of signs. One has $\theta = 2\varepsilon_0$ and $\rho = -(n-1)\varepsilon_0 + \sum_{i=1}^n (n-i)\varepsilon_i$.

10.8. Case $F(4)$

The even part of $F(4)$ is $B_3 \oplus \mathfrak{sl}(2)$. Let $\{\pm\varepsilon_i; \pm\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant i < j \leqslant 3\}$ be the root system for the Lie algebra of type B_3 and ε_0 be a root corresponding to $\mathfrak{sl}(2)$. Take

$$\beta := \frac{1}{2}(\varepsilon_0 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3), \qquad \Pi := \{\beta, -\varepsilon_1, \varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3\},$$

$$\Delta_1^+ = \left\{\frac{1}{2}(\varepsilon_0 \pm \varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3)\right\}, \qquad \Delta_0^+ = \{\varepsilon_0; -\varepsilon_i, -\varepsilon_i \pm \varepsilon_j\colon 1 \leqslant j < i \leqslant 3\}.$$

Normalize the form in such a way that $(\varepsilon_0, \varepsilon_0) = -6$; then $(\varepsilon_i, \varepsilon_j) = 2\delta_{i,j}$ if $i \geqslant 0$, $j > 0$. One has $\theta = \varepsilon_0$ and $\rho = -(3\varepsilon_0 + \varepsilon_1 + 3\varepsilon_2 + 5\varepsilon_3)/2$.

10.9. Case $G(3)$

The even part of $G(3)$ is $G_2 \oplus \mathfrak{sl}(2)$, $\Delta^{\#}$ is the root system for G_2 and $W^{\#}$ is the Weyl group of G_2. The roots are expressed in terms of $\varepsilon_1, \varepsilon_2, \varepsilon_3$: $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$ corresponding to G_2 and ε_0 corresponding to $\mathfrak{sl}(2)$. We take $\Pi := \{\varepsilon_0 + \varepsilon_1, \varepsilon_2, \varepsilon_3 - \varepsilon_2\}$, $\beta := \varepsilon_0 + \varepsilon_1$. Then

$$\Delta_1^+ = \{\varepsilon_0; \varepsilon_0 \pm \varepsilon_i\colon i = 1, 2, 3\}, \qquad \Delta_0^+ = \{2\varepsilon_0; -\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_3 - \varepsilon_2, \varepsilon_2 - \varepsilon_1, \varepsilon_3 - \varepsilon_1\}.$$

Normalize the form in such a way that $(\varepsilon_i|\varepsilon_i) = 2$ for $i > 0$; then $(\varepsilon_i|\varepsilon_j) = -1$ for $0 < i < j$ and $(\varepsilon_0|\varepsilon_i) = -2\delta_{0,i}$. One has $\theta = 2\varepsilon_0$ and $\rho = (-5\varepsilon_0 - 3\varepsilon_1 + \varepsilon_2 + 3\varepsilon_3)/2$.

10.10. Case $D(2,1,a)$

In this case the even part is $\mathfrak{sl}(2) \oplus \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. We take

$$\Delta_1^+ = \{\varepsilon_0 \pm \varepsilon_1 \pm \varepsilon_2\}, \qquad \Delta_0^+ = \{2\varepsilon_0, 2\varepsilon_1, 2\varepsilon_2\}$$

and $\beta := \varepsilon_0 - \varepsilon_1 - \varepsilon_2$, $\Pi := \{\beta, 2\varepsilon_1, 2\varepsilon_2\}$. One has $\theta = 2\varepsilon_0$ and $\rho = -\beta$.

We take $\Delta_0^{\#} := \{\pm 2\varepsilon_1, \pm 2\varepsilon_2\}$; then $W^{\#}\beta = \Delta_1^+$. We normalize the form as follows:

$$(\varepsilon_0|\varepsilon_0) = \frac{-1-a}{2}, \qquad (\varepsilon_1|\varepsilon_1) = a/2, \qquad (\varepsilon_2|\varepsilon_2) = 1/2, \qquad (\varepsilon_i|\varepsilon_j) = 0, \quad i \neq j.$$

Appendix A

We will prove two lemmas used in the main text.

Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra, Δ^+ the set of positive roots, P the weight lattice, and W the Weyl group of $\mathfrak{g}$.

A.1. Lemma. *If $\lambda \in P$ is such that $(\lambda|\lambda) < (\rho|\rho)$, then $\sum_{w \in W} (-1)^{l(w)} e^{w\lambda} = 0$.*

Proof. First, $\mathrm{Stab}_W \lambda \neq \mathrm{id} \Leftrightarrow \sum_{w \in W} (-1)^{l(w)} e^{w\lambda} = 0$, since $\mathrm{Stab}_W \lambda$ is generated by reflections it contains (see, for instance, [9, A.1.1]). Hence we may assume that λ has a trivial stabilizer in W. Let λ' be the maximal element in the orbit $W\lambda$; then for any simple root α one has $s_\alpha \lambda' < \lambda'$, hence $(\lambda'|\alpha) \geqslant (\rho|\alpha)$. Therefore $\lambda' = \rho + \xi$, where $\xi \in P^+$ and we obtain

$$(\lambda|\lambda) = (\rho + \xi|\rho + \xi) = (\rho|\rho) + (\xi|\xi) + (2\rho|\xi) \geqslant (\rho|\rho),$$

since $2\rho \in Q^+$. $\square$

A.2. Lemma. *For each $\alpha \in \Delta^+$ and all $r \gg 0$ one has*

$$r\alpha = w.(r'\alpha'), \quad \textit{for some } \alpha' \in \Delta \cup \{0\},\ r' \geqslant 1,\ w \in W \quad \Longrightarrow \quad w = s_\alpha \quad \textit{or} \quad w = \mathrm{id}.$$

Proof. Since W is a finite group, it is enough to show that for each $w \in W$, $w \neq \mathrm{id}, s_\alpha$ one has

$$r \gg 0 \quad \Longrightarrow \quad r\alpha \neq w.(r'\alpha') \quad \text{for } \alpha' \in \Delta \cup \{0\},\ r' \geqslant 1.$$

Assume that $r\alpha = w.(r'\alpha')$, that is $\rho - w\rho + r\alpha = r'(w\alpha')$. Write $\rho - w\rho =: \sum m_i\beta_i$, $w\alpha' =: \sum k_i\beta_i$, where $\{\beta_i\} \subset \Delta$ is a set of simple roots such that $\beta_1 = \alpha$. The condition $w \neq \mathrm{id}, s_\alpha$ implies that $\rho - w\rho$ is not proportional to α so $m_i \neq 0$ for some $i > 1$. One has $\rho - w\rho + r\alpha = (m_1 + r)\beta_1 + \sum_{i>2} m_i\beta_i$ and thus $\frac{m_1+r}{m_i} = \frac{k_1}{k_i}$. Since $w\alpha'$ lies in a finite set $\Delta \cup \{0\}$, the set of possible values for $\frac{m_1+r}{m_i}$ is finite so the set of possible values for r is finite as well, which is a contradiction. $\square$

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