A Flexible, Scalable and Provably Tight Relaxation for Matching Problems

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Matching
Representing Correspondences
Quadratic Matching

penalty for a pair of matches = $W_{qr,st}$
Quadratic Matching

\[ W_{qr,st} = |d_{qs} - d_{rt}| \]
Quadratic Matching

$$\min_{X \in \Pi} \sum_{q,r,s,t} W_{qr,st} X_{qr} X_{st}$$
Quadratic Matching

$$\min_{X \in \Pi} \sum_{q,r,s,t} W_{qr,st} X_{qr} X_{st}$$

$$\min_{X \in \Pi} [X]^T W [X]$$
The Challenge

- Non-convex objective
- Non-convex domain
- NP-hard problem

\[
\min_{X} \quad [X]^T W [X] \\
\text{subject to} \quad X \in \Pi
\]
Doubly Stochastic Relaxation

\[
\min_x [X]^T W [X] \\
\text{subject to } X \in \Pi
\]

- Tractable for \( W \succeq 0 \)

\[ X \in \text{conv}\Pi \]
Doubly Stochastic Relaxation

\[
\min_X \quad [X]^T W [X]
\]

subject to \( X \in \text{conv} \Pi \)
Spectral Relaxation

\[
\begin{align*}
\min_{X} & \quad [X]^T W [X] \\
\text{subject to} & \quad \|X\|_F^2 = n
\end{align*}
\]

- Eigenvector problem

[Leordeanu & Hebert 2005]
Spectral Relaxation

\[
\min_X \quad [X]^T W [X]
\]
subject to \[\|X\|_F^2 = n\]
SDP Relaxation

- Tight!
- Not scalable - $O(n^4)$ variables

[Zhao et al. 1998, Kezurer et al. 2015]
**Question:**
Can we find a tight relaxation without compromising scalability?

[Kezurer et al. 2015]
Our approach

• Construct a parametric family of equivalent problems

• Choose optimal parameter value for relaxation

• Place in relaxation hierarchy
Equivalent formulations

$$[X]^T W [X] - a \left( \| X \|_F^2 - n \right)$$

$$X \in \Pi_n$$
Relaxation

\[
[X]^T W [X] - a \left( \|X\|_F^2 - n \right) \geq E(X, \alpha)
\]

\(X \in \text{conv}(\Pi_n)\)
Goal: Find convex relaxation that generates maximal lower bound
Optimal parameter value

Lemma

For $X \in \text{conv}(\Pi_n)$, $b > a$ we have $E(X, b) > E(X, a)$

⇒ Take maximal $a$ s.t. problem is convex
Optimal parameter value

Solution (1)

Take $a = \lambda_{min}(W)$

$\Rightarrow$ Hessian is $W - \lambda_{min}I \Rightarrow$ convex

Can we do better?

[Fogel et al. 2013, 2015]
Optimal parameter value

Solution (2)

Take $a = \overline{\lambda}_{min} = \lambda_{min}(\mathcal{W}|_{aff(\Pi_n)})$

$\Rightarrow$ convex on $aff(\Pi_n)$
Recap

• We have found an optimal relaxation in the family we proposed. We call it $\text{DS}++$:

$$\min_x [X]^TW[X] - \frac{\lambda_{\min}}{\lambda_{\max}} (\|X\|_F^2 - n)$$

subject to

$$x \in \text{conv}(\Pi_n)$$

• Is it a good relaxation?
  • We show it is
  • Method: compare all relaxations by “embedding” them in a high dim space
Relaxation hierarchy

- Establish a partial order on relaxations

- In our case: partial order == relaxation domain inclusion

- Need to move to a common domain!

[Kezurer et al. 2015]
Relaxation hierarchy

- New variable \((X, Y)\)
- \(Y\) represents quadratic monomials in \(X\)
Relaxation hierarchy

• The **doubly stochastic relaxation** as SDP:

  \[
  \min_{X,Y} \quad \text{tr}(WY) \\
  \text{subject to} \quad Y \succeq [X][X]^T \\
  A[X] = b, \quad [X] \geq 0
  \]

  Similar to \( \text{tr}(WY) = \text{tr}(W[X][X]^T) = [X]^T W[X] \)

  SDP constraint

  Equivalent to \( X \in \text{conv}\Pi_n \)
Relaxation hierarchy

- The **spectral relaxation** as SDP:

$$\begin{align*}
\min_{X,Y} & \quad \text{tr}(WY) \\
\text{subject to} & \quad Y \succeq [X][X]^T \\
& \quad \text{tr}Y = n
\end{align*}$$
Relaxation hierarchy

**Theorem**

DS++ is equivalent to the following:

\[
\min_{X,Y} \quad \text{tr}(WY)
\]

subject to

\[
Y \succeq [X][X]^T
\]

\[
Ax = b
\]

\[
[X] \succeq 0
\]

\[
\text{tr}Y = n
\]

\[
AY = bx^T
\]

**doubly stochastic constraint**

**Spectral constraint**

**Additional n^3 constraints!**
Corollary (1)

DS++ is more accurate than both the DS and Spectral relaxations!

Corollary (2)

DS++ is less accurate than [Kezurer 15’]
SDP relaxation in $n^4$ variables

\[
\begin{align*}
\min_{X,Y} & \quad \text{tr}(WY) \\
\text{subject to} & \quad Y \succeq [X][X]^T \\
& \quad Ax = b \\
& \quad [X] \geq 0 \\
& \quad \text{tr}Y = n \\
& \quad AY = bx^T
\end{align*}
\]

Fast!

quadratic program in $n^2$ variables

\[
\begin{align*}
\min_{X} & \quad [X]^T W [X] \\
& \quad - \lambda_{\min} (\|X\|_F^2 - n) \\
\text{subject to} & \quad x \in \text{conv}(\Pi_n)
\end{align*}
\]
Projection
Natural projection

- **Problem:** what if $X^*$ is not a permutation matrix?
Natural projection

• **Problem:** what if $X^*$ is not a permutation matrix?
• **Common solution:** $L_2$ projection – does not take functional into account
Natural projection

• Our solution:
  • Solve convex relaxation $E(X, a)$ for optimal $a_0 = \lambda_{\text{min}}$.
  • gradually deform objective from convex to concave by increasing $a$

Concave objective – guaranteed to get a permutation!
• We use [Solomon et al. 2016] for optimization

[Ogier and Beyer 1990; Zaslavskiy et al. 2009].
Natural projection

\[ E(X, a_0) \quad \ldots \quad E(X, a_n) \]
Applications
Applications: shape matching
Applications: shape matching

Inter-Model

Intra-Model

% Correspondences

Error

Error

DS++

[Chen et al. 2015]
Applications: shape matching
Applications: image arrangement

Image metric space

Euclidean grid
Applications: image ordering
Applications: image ordering

Icebergs

Oil rafts

Classrooms

Temples
Applications: image ordering

Before

after
Conclusion

• More accurate relaxation at the same complexity
• Natural projection method
• Works on all convex and concave energies

Limitations / future work

• Best relaxation in $n^2$ variables?
• Partial matching
• Optimization with Frank-Wolfe scheme
The End

• Code is available online:
  http://www.wisdom.weizmann.ac.il/~haggaim/

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• Thanks for listening!