DLA ON CAYLEY GRAPHS

Start with a single particle in the grid. Bring a second particle from infinity performing a simple random walk until it hits the first particle. Glue it there, that is at last place it visited before hitting the particle. Bring a third particle and glue it when it hits the existing particles. Watch the aggregate grow. This process, known as diffusion-limited aggregation, DLA for short, was suggested by physicists T. Witten and L. Sander [WS81] for the case of $\mathbb{Z}^2$. They ran simulations for a few thousand particles and discovered that a random fractal ensues. The elegance of the model immediately caught the eyes of both physicists and mathematicians.

Not much has been proven about this model rigorously. Kesten [K87] showed upper bounds for the growth rate, but these do not demonstrate the fractal nature of the model. Say the cluster grow arms iff the ratio between the diameter of the cluster and a ball of similar volume goes to infinity. It is wildly believed though that the DLA cluster in $\mathbb{Z}^d$ a.s. grow arms.

What about DLA on other Cayley graphs?

On a transient Cayley graph simple random walk starting far, will not likely to hit the finite cluster. To naturally define the DLA process do the following. Inductively, given a finite set in the graph, for each neighboring vertex look at the escape probability for random walk starting at the vertex. That is, the probability simple random walk starting there will never hit the set. Pick the next vertex in the cluster proportionally to the escape probability.

For the definition of arms in graphs of exponential volume growth it might be natural to assume maybe ratio do not go to 1? Still we will stick to the definition above.

If random walk has many different directions to escape to infinity, then one might not expect arms? This was established in particular in [BPP97] for the binary tree. This suggests the following speculations:

A graph has the Liouville property iff it admits no non constant bounded harmonic functions. A property related to having "only one direction of escaping to infinity", see e.g. [KV83].

Question: G a Cayley graph, is it the case that DLA on G grow arms iff G is Liouville?

It is not the case in the context of general graphs (two $\mathbb{Z}^3$ copies connected at the origin). Other possible thresholds for arm growing are exponential growth and non amenability. Note that subexponential growth groups are Liouville and Liouville groups are amenable.

It is of interest to study DLA on the lamplighter groups $\mathrm{LL}(\mathbb{Z}^d)$. These are all amenable and are Liouville iff $d \leq 2$, see [KV83]. A simulation to see if there is a difference between $d \leq 2$ and $d > 2$ is of interest too.

Proposition 0.1. *DLA on nonamenable graphs do not grow arms.*

Proof. By lemma 2.1 from [BNP09] for any finite set in a nonamenable graph, the escape probability is uniformly bounded from below by a constant, strictly bigger than 0, for a set of neighbors of order the set, both constants depending only on the expansion of the graph. Thus the maximal gluing probability by stage n is bounded from above by c/n, for some fixed $c < \infty$. By Kesten's argument [K87] it follows that there are no arms. □

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References

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Itai Benjamini