Isoperimetric Rigidity?

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Abstract

Are there restrictions on the possible isoperimetric profile of three dimensional simply connected bounded geometry Riemanian manifolds?

A Riemanian manifold admits isoperimetric dimension d, iff

$$d = \inf \big\{ k : \lim_{V(\Omega) > r} (\frac{A(\partial\Omega)}{V(\Omega)^{(k-1)/k}} = 0 \big\},$$

were the inf is over precompact domains, V denotes volume and A the area.

Question 1. *Is there a bounded geometry Riemanian metric on $\mathbb{R}^3$ with an isoperimetric dimension 5?*

- Bowditch following an assertion by Gromov, proved that any metric on $\mathbb{R}^2$ either has isoperimetric dimension at most two or admits a positive Cheeger constant, in particular infinite isoperimetric dimension. (There are also proofs by Olshanski, Papasoglu, and more recently Drutu among others). A manifold admits a bounded geometry iff all the sectional curvatures are pinched and the injectivity radius is strictly positive. D. Burago outlined an example for us, showing that the bounded geometry assumption is essential. At first one might be led to believe that all non-compact bounded geometry Riemanian manifolds are roughly isometric to a bounded geometry Riemanian metric on $\mathbb{R}^3$. This is not the case. In respond to our question, Yehuda Shalom showed us that no bounded geometric metric τ on $\mathbb{R}^3$ is roughly isometric to Euclidean $\mathbb{R}^5$. This is done using the Bursuk-Ulam theorem.

- Recall, two metric spaces, X and Y are *roughly isometric* iff there is a constant $K < \infty$ and a map f from X into Y, so that $f(X)$ intersects every ball of radius K in Y, and distances are distorted by multiplicative and additive constants smaller than K. That is, the two spaces enjoy the same coarse structure.

- $\mathbb{Z}^5$ of course can be embedded in $\mathbb{R}^3$, this can be maybe adapted to a metric on $\mathbb{R}^3$ which is rough isometric to Euclidean $\mathbb{R}^5$, but it is not a metric with bounded geometry on $\mathbb{R}^3$.